Linear HoTT and Quipper

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CQTS

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The Goal

- Add linear type formers $\otimes, I, \multimap$ to DTT.
- Leave the rest of DTT exactly the same.

The Symmetry Proof We Want

Proposition

sym : $A \otimes B \simeq B \otimes A$

Proof.

To define sym : $A \otimes B \to B \otimes A$, suppose we have $p : A \otimes B$. Then $\otimes$-induction allows us to assume $p \equiv x \otimes y$, and we have $y \otimes x$.

$$\mathsf{sym} :\equiv \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ y \otimes x$$

Then to prove $\prod_{(p:A\otimes B)} \mathsf{sym}(\mathsf{sym}(p)) = p$, use $\otimes$-induction again: the goal reduces to $x \otimes y = x \otimes y$ for which we have reflexivity.

$$\mathsf{inv} :\equiv \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{refl}_{x\otimes y}$$

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Colourful Variables

We need to prevent terms like $\lambda x.x \otimes x : A \to A \otimes A$, so variable use needs to be restricted somehow.

- Every term a has a *colour* $\mathfrak{C}$.
- Every variable binding $x :^{\mathfrak{C}} A$ also has a colour $\mathfrak{C}$.
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Building a term, we need to keep track of the current colour. A context has the form

$$\Gamma \vdash_{\mathfrak{C}} a : A$$

where $\Gamma \vdash \mathfrak{C}\ \mathsf{colour}$ is an iterated tensor of 'primitive colours' bound in the context.

Each variable is a term of its own colour:

$$\mathfrak{r}, \mathfrak{b}, \mathfrak{p}, x :^{\mathfrak{r}} A, y :^{\mathfrak{b}} B, z :^{\mathfrak{p}} C \vdash_{\mathfrak{r}} x : A$$
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$$\mathfrak{r}, \mathfrak{b}, \mathfrak{p}, x :^{\mathfrak{r}} A, y :^{\mathfrak{b}} B, z :^{\mathfrak{p}} C \vdash_{\mathfrak{p}} z : C$$

Each colour has a copy of all ordinary type formers.

$$\sum_{(x:A)} B(x) \qquad \prod_{(x:A)} B(x) \qquad (\lambda x.b)$$

$$\mathsf{ind}_{+}(z.C, x.c_1, y.c_2, p) \qquad \mathsf{ind}_{=}(x.x'.p.C, x.c, p)$$

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- **Formation:** For closed* $A : \mathcal{U}$ and $B : \mathcal{U}$, there is a type $A \otimes B : \mathcal{U}$.
- **Introduction:** Given $a : A$ with colour $\mathfrak{R}$ and $b : B$ with colour $\mathfrak{B}$, there is a term

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  with colour $\mathfrak{R} \otimes \mathfrak{B}$.
- **Elimination:** Any term $p : A \otimes B$ of colour $\mathfrak{P}$ may be assumed to be of the form $x \otimes y$ for some variables $x^{\mathfrak{r}} : A, y^{\mathfrak{b}} : B$ where $\mathfrak{r}$ and $\mathfrak{b}$ are fresh colours, when constructing some other term $c : C$.

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Eg: Symmetry

Proposition

There is a function $\mathsf{sym} : A \otimes B \to B \otimes A$

Proof.

Suppose $p : A \otimes B$. Then $\otimes$-induction on p gives $x :^{\mathfrak{r}} A$ and $y :^{\mathfrak{b}} B$.

We can form $y \otimes x : B \otimes A$ of colour $\mathfrak{b} \otimes \mathfrak{r}$... □

But now we are stuck, the term $y \otimes x$ has colour $\mathfrak{b} \otimes \mathfrak{r}$ rather than $\mathfrak{p}$ so we can't write

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2-cells

We need a judgement describing how colours relate to each other. There will be an admissible principle

$$\text{REWRITE}\ \frac{\Gamma \vdash s : \mathfrak{C} \Rightarrow \mathfrak{D} \qquad \Gamma \vdash_{\mathfrak{D}} a : A}{\Gamma \vdash_{\mathfrak{C}} s^*(a) : s^*(A)}$$

With axioms:

$$\mathsf{sym} : \mathfrak{C} \otimes \mathfrak{D} \Rightarrow \mathfrak{D} \otimes \mathfrak{C} \qquad \mathsf{assoc} : (\mathfrak{C} \otimes \mathfrak{D}) \otimes \mathfrak{E} \Rightarrow \mathfrak{C} \otimes (\mathfrak{D} \otimes \mathfrak{E})$$

$$s;t : \mathfrak{C} \Rightarrow \mathfrak{E} \text{ for } s : \mathfrak{C} \Rightarrow \mathfrak{D}, t : \mathfrak{D} \Rightarrow \mathfrak{E}$$

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Rules for $\otimes$, Take 2

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- **Elimination:** Any term $p : A \otimes B$ of colour $\mathfrak{P}$ may be assumed to be of the form $x \otimes_s y$ for some variables $x :^{\mathfrak{r}} A, y :^{\mathfrak{b}} B$ where $\mathfrak{r}$ and $\mathfrak{b}$ are fresh colours and there is a fresh 2-cell $s : \mathfrak{P} \Rightarrow \mathfrak{r} \otimes \mathfrak{b}$, when constructing some other term $c : C$.

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$$\mathsf{sym} :\equiv \lambda p.\mathsf{let}\ x \otimes_s y\ \mathsf{be}\ p\ \mathsf{in}\ y \otimes_{s;\mathsf{sym}} x$$

To prove $\prod_{(p:A\otimes B)} \mathsf{sym}(\mathsf{sym}(p)) = p$, use $\otimes$-induction again: the goal reduces to $x \otimes_s y = x \otimes_s y$ for which we have reflexivity.

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□

- Linear HoTT is a *bunched* type theory.
- There is no notion of 'linear variable' which may only be used once. Instead, colours denote permission to use a resource.
- When we have access to a variable, we can use it any ordinary way we like.

$$f : A \otimes B \to A \otimes (B \times B \times B)$$
$$f :\equiv \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ x \otimes (y, y, y)$$

$$g : A \otimes B \to (A \otimes B) \times (A \otimes B)$$
$$g :\equiv \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ (x \otimes y, x \otimes y)$$

Formal Rules and a Stupid Trick

$$\otimes\text{-FORM}\ \frac{\Gamma, \mathfrak{l} \vdash_{\mathfrak{l}} A \text{ type} \qquad \Gamma, \mathfrak{r} \vdash_{\mathfrak{r}} B \text{ type}}{\Gamma \vdash_{\mathfrak{C}} A \otimes B \text{ type}}$$

$$\otimes\text{-INTRO}\ \frac{\begin{array}{c}\Gamma \vdash s : \mathfrak{C} \Rightarrow \mathfrak{L} \otimes \mathfrak{R} \\ \Gamma \vdash_{\mathfrak{L}} a : A[\mathfrak{L}/\mathfrak{l}] \qquad \Gamma \vdash_{\mathfrak{R}} b : B[\mathfrak{R}/\mathfrak{r}]\end{array}}{\Gamma \vdash_{\mathfrak{C}} a \otimes_s b : A \otimes B}$$

$$\otimes\text{-ELIM}\ \frac{\begin{array}{c}\Gamma, z :^{\mathfrak{D}} A \otimes \vdash_{\mathfrak{C}} C \text{ type} \\ \Gamma, \mathfrak{l}, \mathfrak{r}, s : \mathfrak{D} \Rightarrow \mathfrak{l} \otimes \mathfrak{r}, x :^{\mathfrak{l}} A, y :^{\mathfrak{r}} B \vdash_{\mathfrak{C}} c : C[x \otimes_s y/z] \\ \Gamma \vdash_{\mathfrak{D}} p : A \otimes B\end{array}}{\Gamma \vdash_{\mathfrak{C}} \text{let } x^{\mathfrak{l}} \otimes_s y^{\mathfrak{r}} \text{ be } p \text{ in } c : C[p/z]}$$

Comparison with Old Rules

Pros:

- No more baroque rules for “splits”.
- Simpler crisp induction principles without complicated pattern matching.
- No context clearing/manipulation in modal rules.

Cons:

- Explicit 2-cell manipulation.
  - Mostly inferable?
- Conversion checking seems harder.
  - E.g., $(\mathsf{sym} \otimes \mathsf{id})^*((a \otimes b) \otimes c) \equiv \mathsf{sym}^*(a \otimes b) \otimes c$
- Bad judgement states.
  - E.g. $x :^{\mathfrak{r}} A \vdash_{\mathfrak{r} \otimes \mathfrak{r}} x \otimes x : A \otimes A$
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This Only Works Sometimes

$$\flat\text{-INTRO}\ \frac{\Gamma \vdash s : \mathfrak{C} \Rightarrow \mathsf{f}(\mathfrak{D}) \qquad \Gamma \vdash_{\mathfrak{D}} a : A}{\Gamma \vdash_{\mathfrak{C}} \flat(a) : \flat A}$$

$$\flat\text{-ELIM}\ \frac{\begin{array}{c}\Gamma, z^{\mathfrak{D}} : \flat A \vdash_{\mathfrak{C}} C \ \mathsf{type} \\ \Gamma, \mathfrak{l}, s : \mathfrak{D} \Rightarrow \mathsf{f}(\mathfrak{l}), x :^{\mathfrak{l}} A \vdash_{\mathfrak{C}} c : C[\flat(x)/z] \\ \Gamma \vdash_{\mathfrak{D}} p : \flat A\end{array}}{\Gamma \vdash_{\mathfrak{C}} \mathsf{let}\ \flat(x^{\mathfrak{l}})\ \mathsf{be}\ p\ \mathsf{in}\ c : C[p/z]}$$

This "$\flat$" is *not* left-exact! Variables created by $\flat$-ELIM cannot interact.

$$\mathsf{func} : \flat(A \to B) \to \flat A \to \flat B$$

$$\mathsf{func}(h, u) :\equiv \mathsf{let}\ \flat(f^{\mathfrak{l}})\ \mathsf{be}\ h\ \mathsf{in}\ \mathsf{let}\ \flat(a^{\mathfrak{r}})\ \mathsf{be}\ u\ \mathsf{in}\ \flat(f(a))$$

A Fragment of Quipper

$$\frac{}{\Phi, x : A \vdash x : A}\ \textsc{var}$$

$$\frac{\Phi, \Gamma_1 \Vdash M : A \qquad \Phi, \Gamma_2 \Vdash N : B}{\Phi, \Gamma_1, \Gamma_2 \Vdash M \otimes N : A \otimes B}\ \otimes\text{-}\textsc{intro}$$

$$\frac{\Phi, \Gamma_1 \Vdash M : A \otimes B \qquad \Phi, \Gamma_2, x : A, y : B \Vdash N : C}{\Phi, \Gamma_1, \Gamma_2 \Vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N : C}\ \otimes\text{-}\textsc{elim}$$

$$\frac{\Phi \Vdash M : A}{\Phi \Vdash \mathbf{lift}\, M : !A}\ !\text{-}\textsc{intro}$$

$$\frac{\Phi, \Gamma \Vdash M : !A}{\Phi, \Gamma \Vdash \mathbf{force}\, M : A}\ !\text{-}\textsc{elim}$$

A Fragment of Quipper

VAR

$$\frac{}{x : A \vdash x : A}$$

⊗-INTRO

$$\frac{\Gamma_1 \Vdash M : A \qquad \Gamma_2 \Vdash N : B}{\Gamma_1, \Gamma_2 \Vdash M \otimes N : A \otimes B}$$

⊗-ELIM

$$\frac{\Gamma_1 \Vdash M : A \otimes B \qquad \Gamma_2, x : A, y : B \Vdash N : C}{\Gamma_1, \Gamma_2 \Vdash \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N : C}$$

!-INTRO

$$\frac{\cdot \Vdash M : A}{\cdot \Vdash \textbf{lift}\, M : !A}$$

!-ELIM

$$\frac{\Gamma \Vdash M : !A}{\Gamma \Vdash \textbf{force}\, M : A}$$

Translating from Quipper

A Quipper term $x : A, y : B \Vdash c : C$ is translated to

$$\phi :^{\mathfrak{c}_\phi} I, x :^{\mathfrak{c}_x} [\![A]\!], y :^{\mathfrak{c}_y} [\![B]\!] \vdash_{\mathfrak{c}_\phi \otimes \mathfrak{c}_x \otimes \mathfrak{c}_y} [\![c]\!] : [\![C]\!]$$

$$[\![x]\!] :\equiv \mathsf{let}\ ¤_i\ \mathsf{be}\ \phi\ \mathsf{in}\ \mathsf{unitor}_i(x)$$

$$\begin{aligned}
[\![A \otimes B]\!] &:\equiv [\![A]\!] \otimes [\![B]\!] \\
[\![(M, N)]\!] &:\equiv [\![M]\!] \otimes_{\mathsf{id}} \mathsf{unitorinv}_i([\![N]\!][¤_i/\phi]) \\
[\![\mathsf{let}\ (x, y)\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] &:\equiv \mathsf{let}\ x^{\mathfrak{c}_x} \otimes_s y^{\mathfrak{c}_y}\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ (\mathsf{id} \otimes s)^*([\![N]\!][¤_i/\phi])
\end{aligned}$$

$$\begin{aligned}
[\![!A]\!] &:\equiv I \times \natural(I \to [\![A]\!]) \\
[\![\mathbf{lift}\, M]\!] &:\equiv (\phi, (\lambda s. [\![M]\!][s/\phi])^\natural) \\
[\![\mathbf{force}\, M]\!] &:\equiv \mathsf{let}\ (\phi', f)\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ f_\natural(\phi')
\end{aligned}$$

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$$[\![A \otimes B]\!] :\equiv [\![A]\!] \otimes [\![B]\!]$$
$$[\![(M, N)]\!] :\equiv [\![M]\!] \otimes_{\mathsf{id}} \mathsf{unitorinv}_i([\![N]\!][\maltese_i/\phi])$$
$$[\![\mathsf{let}\ (x, y)\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] :\equiv \mathsf{let}\ x^{\mathfrak{c}_x} \otimes_s y^{\mathfrak{c}_y}\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ (\mathsf{id} \otimes s)^*([\![N]\!][\maltese_i/\phi])$$

$$[\![!A]\!] :\equiv I \times \natural(I \to [\![A]\!])$$
$$[\![\mathbf{lift}\, M]\!] :\equiv (\phi, (\lambda s.[\![M]\!][s/\phi])^\natural)$$
$$[\![\mathbf{force}\, M]\!] :\equiv \mathsf{let}\ (\phi', f)\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ f_\natural(\phi')$$

A Quipper term $x : A, y : B \Vdash c : C$ is translated to

$$\phi :^{\mathfrak{c}_\phi} I, x :^{\mathfrak{c}_x} [\![A]\!], y :^{\mathfrak{c}_y} [\![B]\!] \vdash_{\mathfrak{c}_\phi \otimes \mathfrak{c}_x \otimes \mathfrak{c}_y} [\![c]\!] : [\![C]\!]$$

$$[\![x]\!] :\equiv \mathsf{let}\ \natural_i\ \mathsf{be}\ \phi\ \mathsf{in}\ \mathsf{unitor}_i(x)$$

$$\begin{aligned}
[\![A \otimes B]\!] &:\equiv [\![A]\!] \otimes [\![B]\!] \\
[\![(M, N)]\!] &:\equiv [\![M]\!] \otimes_{\mathsf{id}} \mathsf{unitorinv}_i([\![N]\!][\natural_i/\phi]) \\
[\![\mathsf{let}\ (x, y)\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] &:\equiv \mathsf{let}\ x^{\mathfrak{c}_x} \otimes_s y^{\mathfrak{c}_y}\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ (\mathsf{id} \otimes s)^*([\![N]\!][\natural_i/\phi])
\end{aligned}$$

$$\begin{aligned}
[\![!A]\!] &:\equiv I \times \natural(I \to [\![A]\!]) \\
[\![\mathbf{lift}\ M]\!] &:\equiv (\phi, (\lambda s.[\![M]\!][s/\phi])^\natural) \\
[\![\mathbf{force}\ M]\!] &:\equiv \mathsf{let}\ (\phi', f)\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ f_\natural(\phi')
\end{aligned}$$

Translating from Quipper

A Quipper term $x : A, y : B \Vdash c : C$ is translated to

$$\phi :^{\mathfrak{c}_\phi} I, x :^{\mathfrak{c}_x} [\![A]\!], y :^{\mathfrak{c}_y} [\![B]\!] \vdash_{\mathfrak{c}_\phi \otimes \mathfrak{c}_x \otimes \mathfrak{c}_y} [\![c]\!] : [\![C]\!]$$

$$[\![x]\!] :\equiv \mathsf{let}\ \diamond_i\ \mathsf{be}\ \phi\ \mathsf{in}\ \mathsf{unitor}_i(x)$$

$$\begin{aligned}
[\![A \otimes B]\!] &:\equiv [\![A]\!] \otimes [\![B]\!] \\
[\![(M, N)]\!] &:\equiv [\![M]\!] \otimes_{\mathsf{id}} \mathsf{unitorinv}_i([\![N]\!][\diamond_i/\phi]) \\
[\![\mathsf{let}\ (x, y)\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] &:\equiv \mathsf{let}\ x^{\mathfrak{c}_x} \otimes_s y^{\mathfrak{c}_y}\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ (\mathsf{id} \otimes s)^*([\![N]\!][\diamond_i/\phi])
\end{aligned}$$

$$\begin{aligned}
[\![!A]\!] &:\equiv I \times \natural(I \to [\![A]\!]) \\
[\![\mathbf{lift}\ M]\!] &:\equiv (\phi, (\lambda s.[\![M]\!][s/\phi])^\natural) \\
[\![\mathbf{force}\ M]\!] &:\equiv \mathsf{let}\ (\phi', f)\ \mathsf{be}\ [\![M]\!]\ \mathsf{in}\ f_\natural(\phi')
\end{aligned}$$

The Simplest Circuit

```
object Qubit
gate H  : ! (Qubit -> Qubit)
circuit : ! (Qubit * Qubit -> Qubit * Qubit)
circuit n p = let (x, y) = p
              in (H x, y)
```

Then

$$[\![\mathbf{H}]\!] : I \times \natural(I \to (\mathsf{Qubit} \multimap \mathsf{Qubit}))$$
$$[\![\mathbf{circuit}]\!] : I \times \natural(I \to (\mathsf{Qubit} \otimes \mathsf{Qubit} \multimap \mathsf{Qubit} \otimes \mathsf{Qubit}))$$

The Simplest Translation

Generally, there is a map $\mathsf{e} : I \times \natural(I \to (A \multimap B)) \to (A \to B)$

$$\mathsf{e}(\llbracket \mathbf{circuit} \rrbracket) = \ldots = \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x) \otimes y$$

Then:

$$\begin{aligned}
&\mathsf{e}(\llbracket \mathbf{circuit} \rrbracket) \circ \mathsf{e}(\llbracket \mathbf{circuit} \rrbracket) \\
&= \ldots \\
&= (\lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x) \otimes y) \\
&\quad \circ (\lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x) \otimes y) \\
&\equiv \lambda p.\mathsf{let}\ x \otimes y\ \mathsf{be}\ (\mathsf{let}\ x' \otimes y'\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x') \otimes y')\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x) \otimes y \\
&= \lambda p.\mathsf{let}\ x' \otimes y'\ \mathsf{be}\ p\ \mathsf{in}\ (\mathsf{let}\ x \otimes y\ \mathsf{be}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x') \otimes y'\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x) \otimes y) \\
&\equiv \lambda p.\mathsf{let}\ x' \otimes y'\ \mathsf{be}\ p\ \mathsf{in}\ \mathsf{e}(\llbracket \mathbf{H} \rrbracket)(\mathsf{e}(\llbracket \mathbf{H} \rrbracket)(x')) \otimes y' \\
&= \lambda p.\mathsf{let}\ x' \otimes y'\ \mathsf{be}\ p\ \mathsf{in}\ x' \otimes y' \\
&= \lambda p.p
\end{aligned}$$

Thanks!

References I