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Rings of Smooth Functions and Their Localizations, I

I. MOERDIJK AND G. E. REYES

Mathematisch Instituut, Univ. van Amsterdam, Roetterssstraat 15, Amsterdam 1, 1018WB, The Netherlands and Département de Mathématiques, Université de Montréal, Montréal, Canada

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Several types of rings of smooth functions, such as differentiable algebras and formal algebras, occupy a central position in singularity theory and related subjects. In this series of papers we will be concerned with a larger class of rings of smooth functions, which would play a role in Differential Geometry similar to the role played by commutative rings or k-algebras in Algebraic Geometry. This larger class of rings is obtained from rings of smooth functions on manifolds by dividing by ideals and taking filtered colimits.

The original motivation to introduce and study C^∞-rings was to construct topos-models for synthetic differential geometry (SDG). The program of SDG (see, e.g., Kock [1]) was proposed by F. W. Lawvere, and it was in this context that C^∞-rings first appeared explicitly in the literature (see, e.g., Reyes and Wraith [14] and Dubuc [2]).

These toposes which provide models for SDG are constructed in a way similar to the toposes occurring in algebraic geometry, but with k-algebras replaced by C^∞-rings. In particular, the C^∞-analogue of the Zariski topos, the so-called smooth Zariski topos of Moerdijk and Reyes [11] contains a category of "smooth schemes," just as the usual Zariski topos contains the schemes of algebraic geometry (see Demazure and Gabriel [1]).

Despite this original motivation from SDG, C^∞-rings and their schemes can be studied by themselves, and independently from topos theory in general, and topos-models for SDG in particular. In these two papers, we will start to explore this independent line of development of the theory of C^∞-rings. This can make the connection with algebraic geometry stronger, since the usual presentation of the relation between algebra and geometry takes place at the level of schemes, rather than toposes.

The organization of this paper and its sequel, part II, is as follows.

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In the first section of this paper, we recall the definition of the category of C^∞-rings and C^∞-homomorphisms, we introduce some notation, and collect some basic facts.

In Section 2, we study C^∞-rings which are local (i.e., have a unique maximal ideal). It will be shown, for example, that any C^∞-domain is a local ring, that every local C^∞-ring is Henselian, and that every C^∞-field is real closed.

In part II, written with Ngo Van Quê, these results will be used to define and study the spectrum of a C^∞-ring. The two main ingredients are the theorem of Muñoz and Ortega (Theorem 1.3 of this paper), which will enable us to give a coherent axiomatization of the notion of a localization of a C^∞-ring (in an arbitrary Grothendieck topos), and the notion of a C^∞-radical prime ideal (introduced in Section 2 of this paper), which allows us to give an explicit description of the spectrum of a C^∞-ring (in the case of Sets).

1. Basic Properties of C^∞-Rings

As we said above, the notion of a C^∞-ring stems from the program of synthetic differential geometry. As such, C^∞-rings do not occur explicitly in the classical literature, but the main examples do. Consequently, although the statements of some of the basic facts about C^∞-rings seem new, their proofs are either known or easily derivable from known techniques in classical analysis (see e.g., Malgrange [9] and Tougeron [15]). In this section, we will introduce the notation, and list a few basic facts about C^∞-rings that we will need later on. For more information about C^∞-rings, the reader is referred to Dubuc [3], Kock [7], Moerdijk & Reyes [12].

In this paper, *ring* means *commutative ring with unit element.* Let C^∞ be the category whose objects are the euclidean spaces $\mathbb{R}^n$, $n \geq 0$, and morphisms are all smooth maps. A *C^∞-ring* is a finite product-preserving functor $A: C^\infty \to \text{Sets}$. More generally, a C^∞-ring in a topos $\mathscr{E}$ is a finite product preserving functor $C^\infty \to \mathscr{E}$. Homomorphisms of C^∞-rings, or *C^∞-homomorphisms*, are just natural transformations.

If $A: C^\infty \to \text{Sets}$ (or $C^\infty \to \mathscr{E}$) is a C^∞-ring, we will also write A for "the underlying set" $A(\mathbb{R})$. So a C^∞-ring is a set (or an object of $\mathscr{E}$) A in which we can interpret every smooth map $\mathbb{R}^n \to^f \mathbb{R}^m$ as a map $A^n \to^{A(f)} A^m$ (in a functorial way), and a C^∞-homomorphism $\varphi: A_1 \to A_2$ is a function φ of the underlying sets which preserves these interpretations, i.e., $A_2(f) \circ \varphi^n = \varphi^m \circ A_1(f)$.

Every C^∞-ring is in particular an $\mathbb{R}$-algebra, and every C^∞-homomorphism is a morphism of $\mathbb{R}$-algebras.

The free C^∞-ring on n generators is the ring $C^\infty(\mathbb{R}^n)$ of smooth functions $\mathbb{R}^n \to \mathbb{R}$ (the projections $x_1, ..., x_n$ being the generators), and the C^∞-structure on $C^\infty(\mathbb{R}^n)$ is defined by composition. By Hadamard's lemma, any (ring-theoretic) ideal I in $C^\infty(\mathbb{R}^n)$ is a C^∞-congruence, i.e., there is a well-defined C^∞-ring structure on the quotient $C^\infty(\mathbb{R}^n)/I$ which makes the projection $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/I$ into a C^∞-homomorphism.

Filtered colimits of C^∞-rings are constructed as filtered colimits of commutative rings. So if E is any set, the free C^∞-ring with E as a set of generators is

$$C^\infty(\mathbb{R}^E) := \varinjlim \{C^\infty(\mathbb{R}^D) \mid D \text{ a finite subset of } E\},$$

that is, $C^\infty(\mathbb{R}^E)$ is the ring of functions $\mathbb{R}^E \to \mathbb{R}$ which smoothly depend on finitely many variables only. So any C^∞-ring is isomorphic to one of the form $C^\infty(\mathbb{R}^E)/I$. Observe that from this representation it is clear that every C^∞-ring A is formally real (i.e., $\forall a_1, ..., a_n \in A$: $1 + \sum a_i^2$ is invertible).

Let us recall a lemma of Whitney's:

1.1. Lemma. *Every closed set $F \subset \mathbb{R}^n$ is the zeroset of a smooth function $f: \mathbb{R}^n \to [0, 1]$, i.e, $F = Z(f) = \{x \mid f(x) = 0\}$.*

The complement of $Z(f)$ will be denoted by U_f. If $U \subset \mathbb{R}^n$ is open and $f \in C^\infty(\mathbb{R}^n)$ is such that $U = U_f$, then f is said to be a *characteristic function* for U.

If E is any set, a subset $F \subset \mathbb{R}^E$ is called a *zeroset* if there exists and $f \in C^\infty(\mathbb{R}^E)$ such that $F = Z(f)$. Thus $F \subset \mathbb{R}^E$ is a zeroset iff there exists a finite $D \subset E$ and a closed $\tilde{F} \subset \mathbb{R}^D$ such that $F = \pi_D^{-1}(\tilde{F})$, where $\pi_D: \mathbb{R}^E \to \mathbb{R}^D$ is the projection.

Coproducts of C^∞-rings exist, and the coproduct of A and B in the category of C^∞-rings is denoted by $A \otimes_\infty B$. In fact, it suffices to show that coproducts of finitely generated C^∞-rings exist, and here we have the formula

$$C^\infty(\mathbb{R}^n)/I \otimes_\infty C^\infty(\mathbb{R}^m)/J \cong C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I, J),$$

where (I, J) is the ideal generated by $\{f \circ \pi_1 \mid f \in I\} \cup \{g \circ \pi_2 \mid g \in J\}$.

If A is a C^∞-ring, $A[t]$ denotes the ring of polynomials with coefficients in A, i.e., the solution of freely adjoining an element to A in the category of rings. There is also the construction of freely adjoining an element to A in the category of C^∞-rings, which will be denoted by $A\{t\}$. So

$$A\{t\} \cong A \otimes_\infty C^\infty(\mathbb{R})$$

since $C^\infty(\mathbb{R})$ is free on one generator, and if $A \cong C^\infty(\mathbb{R}^E)/I$,

$$A\{t\} \cong C^\infty(\mathbb{R}^E \times \mathbb{R})/(I(x)),$$

where $(I(x))$ is the ideal of functions $f(x, t) \in C^\infty(\mathbb{R}^E \times \mathbb{R})$ generated by the functions $g(x) \in I$. An element $p(t) \in A\{t\}$ can indeed be regarded as a "smooth polynomial" $A \to A$, i.e., $p(t)$ induces a map $A \to A$ by composition: given $a \in A$, a corresponds to a map $C^\infty(\mathbb{R}) \to^a A$, and $p(a)$ is defined as the composite

$$C^\infty(\mathbb{R}) \xrightarrow{p} A \otimes_\infty C^\infty(\mathbb{R}) \xrightarrow{[1,a]} A,$$

or rather as the element of A corresponding to $[1, a] \circ p$. Of course, this is just substituting a for t: if $A = C^\infty(\mathbb{R}^E)/I$, and p is represented by $p(x, t) \in C^\infty(\mathbb{R}^E \times \mathbb{R})$, a by $a(x) \in C^\infty(\mathbb{R}^E)$, then $p(a)$ is represented by $p(x, a(x)) \in C^\infty(\mathbb{R}^E)$.

If $a \in A$, $A\{a^{-1}\}$ denotes the solution of universally inverting a in the category of C^∞-rings. So $A\{a^{-1}\} \cong A\{t\}/(t \cdot a - 1)$. This is *not* the ring of fractions with some power of a as denominator, but the implicit function theorem yields

1.2. Proposition. *If $A = C^\infty(\mathbb{R}^E)/I$ and $f \in C^\infty(\mathbb{R}^E)$ represents an element $f \in A$, then*

$$A\{f^{-1}\} \cong C^\infty(U_f)/(I \mid U_f).$$

Here $U_f = \{x \in \mathbb{R}^E \mid f(x) \neq 0\}$ and $C^\infty(U_f)$ is the ring of smooth functions on U_f depending on finitely many coordinates, while $(I \mid U_f)$ is the ideal of functions generated by the restrictions $g \mid U_f$, $g \in I$.

In particular, if $f \in C^\infty(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n)\{f^{-1}\} \cong C^\infty(U_f)$, the ring of smooth functions $U_f \to \mathbb{R}$. As said, not every smooth function $g: U_f \to \mathbb{R}$ is of the form h/f^m for some $h \in C^\infty(\mathbb{R}^n)$ and some m (i.e., adjoining an inverse for f is *not* the same for C^∞-rings and for commutative rings). A result that will play a key role, especially in part II (Moerdijk, Quê, Reyes, to appear) is the following theorem, due to Muñoz and Ortega [10]:

1.3. Theorem. *Let $U \subseteq \mathbb{R}^n$ be open, and $g \in C^\infty(U)$. Then there are $h, k \in C^\infty(\mathbb{R}^n)$ with $U_k = U$ and $g \cdot k \mid U \equiv h \mid U$.*

Proof. (sketch) Let $\{\alpha_n\}$ be a sequence of (smooth) functions such that

$\bar{U}_{\alpha_n} \subset U$ (each n) and $U = U_n U_{\alpha_n}$. Define g_n by $g_n(x) = \alpha_n(x) \cdot g(x)$ if $x \notin U$, $g_n(x) = 0$ if $x \in U$. Now if $\{p_n\}$ is an increasing sequence of seminorms defining the (Fréchet-) topology on $C^\infty(\mathbb{R}^n)$, we can put $h = \sum_{n \geq 0} g_n \cdot 2^{-n}(1 + p_n(\alpha_n))^{-1}(1 + p_n(g_n))^{-1}$ and $k = \sum_{n \geq 0} \alpha_n 2^{-n}(1 + p_n(\alpha_n))^{-1}(1 + p_n(g_n))^{-1}$. ∎

For a C^∞-ring A (*not* necessarily finitely generated) this can be rephrased as 1.4(i), while (ii) follows from Lemma 1.1.

1.4. THEOREM. (Algebraic reformulation of 1.1 and 1.3). *Let A be any C^∞-ring, and $a \in A$. Let $\eta: A \to A\{a^{-1}\}$ be the universal C^∞-homomorphism, Then*

(i) $\forall b \in A\{a^{-1}\}\ \exists c, d \in A(b \cdot \eta(c) = n(d)\ \&\ \eta(c) \in U(A\{a^{-1}\}))$, *where for any ring R, $U(R) = \{r \in R \mid r$ is invertible$\}$;*

(ii) $\forall b \in A(\eta(b) = 0 \Rightarrow \exists c \in A\ (\eta(c) \in U(A\{a^{-1}\})\ \&\ c \cdot b = 0$ *in* $A))$.

If X is an arbitrary subset of $\mathbb{R}^n$, a function $X \to \mathbb{R}$ by definition is *smooth* if it is the restriction of a smooth function defined on some open set containing X. The ring $C^\infty(X)$ of smooth functions on X is a C^∞-ring. If X is *closed*, we find that every smooth function $X \to \mathbb{R}$ is the restriction of a smooth function defined on all of $\mathbb{R}^n$ (*smooth Tietze*), i.e., $C^\infty(\mathbb{R}^n) \to C^\infty(X)$ is a surjective C^∞-homomorphism. Consequently, if I is an ideal in $C^\infty(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ is an open set such that $\exists f \in I\ Z(f) \subset U$ then any $g \in C^\infty(U)$ defines a unique element of the ring $A = C^\infty(\mathbb{R}^n)/I$ (and analogously for $A = C^\infty(\mathbb{R}^E)/I$ not necessarily finitely generated, and U the complement of a zeroset in $\mathbb{R}^E$). We will often use this tacitly, or refer to this as smooth Tietze.

Every C^∞-ring A has a canonical preorder $\prec$ defined by

$$a \prec b \qquad \text{iff } \exists c \in U(A), \quad c^2 = b - a.$$

If $A = C^\infty(\mathbb{R}^E)/I$ and $f, g \in C^\infty(\mathbb{R}^E)$ then as elements of A,

$$f \prec g \qquad \text{iff } \exists \varphi \in I, \forall x \in Z(\varphi), \quad f(x) \prec g(x).$$

So $\prec$ is compatible with the ringstructure in the sense that $f, g \succ 0 \Rightarrow f \cdot g \succ 0$, $f + g \succ 0$, etc.

2. LOCAL C^∞-RINGS

In this section we will discuss some general properties of local C^∞-rings. Our main purpose will be to give a direct proof of the fact that every local

C^∞-ring is separably real closed (definitions will be given below). This result was first proved by different methods in Joyal and Reyes [5].

Any C^∞-ring is in particular a commutative ring. A *local* C^∞-ring is a C^∞-ring which is a local ring. (This definition differs from the often used, but confusing, terminology introduced in Dubuc [3]!) Important examples are rings of germs of smooth functions: if M is a manifold and $p \in M$, the C^∞-ring $C_p^\infty(M)$ of germs of smooth functions at p is local. Other examples can be obtained by taking quotients of rings of germs, such as formal power series and quotients of such (formal algebras). Indeed, the "Taylor series at 0"-map

$$T_0: \quad C_0^\infty(\mathbb{R}^n) \to \mathbb{R}[[X_1, ..., X_n]]$$

is a surjective C^∞-homomorphism by Borel's theorem. Not every (finitely generated) local C^∞-ring is quotient of a ring of germs. For instance, if $\mathscr{F}$ is a maximal (nonprincipal) filter on $\mathbb{N}$ and $I = \{f \in C^\infty(\mathbb{N}) \mid Z(f) \in \mathscr{F}\}$, then $C^\infty(\mathbb{N})/\mathscr{F}$ is a local ring which is not a quotient of a ring of germs.

2.1. Definition. A C^∞-ring is called *reduced* if for every $a \in A$, if $a \neq 0$ then $A\{a^{-1}\}$ is nontrivial.

Of course, for C^∞-rings it is not true that if $A\{a^{-1}\}$ is trivial then a is nilpotent.

If I is an ideal in A, we define the C^∞-radical $\sqrt[\infty]{I}$ of I by

$$\begin{aligned} a \in \sqrt[\infty]{I} \quad & \text{iff } (A/I)\{a^{-1}\} \quad \text{is trivial} \\ & \text{iff } \exists b \in I, \quad b \in U(A\{a^{-1}\}). \end{aligned}$$

I is called a C^∞*-radical ideal* if $I = \sqrt[\infty]{I}$. For a C^∞-ring A we write A_{red} for $A/\sqrt[\infty]{0}$, which is a reduced C^∞-ring. So $I \subset A$ is C^∞-radical iff A/I is reduced. Note that if $\varphi: A \to B$ is a homomorphism of C^∞-rings and $J \subset B$ is a C^∞-radical ideal, then $\varphi^{-1}(J)$ is also C^∞-radical. (For we have $\varphi: A/\varphi^{-1}(J) \rightarrowtail B/J$, so if B/J is reduced then so is $A/\varphi^{-1}(J)$.)

It will be useful to have a description of reduced C^∞-rings in terms of "generators and relations." If E is any set, and $I \subset C^\infty(\mathbb{R}^E)$ is an ideal, there is a filter of zerosets in $\mathbb{R}^E$,

$$\hat{I} = \{Z(f) \mid f \in I\}$$

(which is proper if I is), and conversely, for any filter Φ of zerosets in $\mathbb{R}^E$ there is an ideal

$$\Phi' = \{f \in C^\infty(\mathbb{R}^E) \mid Z(f) \in \Phi\}$$

(which is proper if Φ is). For an ideal $I \subset C^\infty(\mathbb{R}^E)$, we call the ideal $(\hat{I})'$ the C^∞-*radical* of I, and again denote it by $\sqrt[\infty]{I}$. This is consistent with our earlier terminology, since

2.2. LEMMA. *Let $A \cong C^\infty(\mathbb{R}^E)/I$ be an arbitrary C^∞-ring. Then A is reduced iff $I = \sqrt[\infty]{I}$ in the sense that*

$$Z(g) = Z(f) \quad \text{and} \quad f \in I \Rightarrow g \in I.$$

Proof. It suffices to observe that for $f \in C^\infty(\mathbb{R}^E)$, $A\{1/f\}$ is trivial iff $f \in (\hat{I})'$. But $A\{1/f\}$ is trivial iff $\exists g \in I\ Z(g) \subseteq Z(f)$ (by the explicit description of $A\{1/f\}$ we gave in Sect. 1), iff $f \in (\hat{I})'$. ∎

Note that from Lemma 2.2 it follows that a finitely presented C^∞-ring, i.e., a ring of the form $C^\infty(\mathbb{R}^n)/(f_i,..., f_k)$, is reduced iff it is point-determined, as defined, e.g., in Kock [7].

2.3. LEMMA. *Let A be a C^∞-ring, and $I \subset A$ a prime ideal. Then $\sqrt[\infty]{I}$ is also prime. Or in more algebraic terms, if A is a C^∞-domain, then so is A_{red}.*

Proof. We may restrict ourselves to the case where $A = C^\infty(\mathbb{R}^E)$. Clearly, if Φ is a prime filter of zerosets in $\mathbb{R}^E$ (i.e., a filter with $F \cup G \in \Phi \Rightarrow F \in \Phi$ or $G \in \Phi$) then Φ' is a prime ideal. So we need to show that if $I \subset C^\infty(\mathbb{R}^E)$ is a prime ideal, then $\hat{I}$ is a prime filter. Suppose F and G are zerosets in $\mathbb{R}^E$ with $F \cup G = Z(\varphi)$ and $\varphi \in I$. We may assume that $\varphi \geqslant 0$ (replace φ by φ^2). Let $D \subset E$ be a finite subset containing the coordinates involved in F, G, and φ; i.e., there are closed sets $\tilde{F}, \tilde{G} \subset \mathbb{R}^D$ and a smooth $\tilde{\varphi}: \mathbb{R}^D \to \mathbb{R}$ such that $\varphi = \tilde{\varphi} \circ \pi_D$, $F = \pi_D^{-1}(\tilde{F})$, $G = \pi_D^{-1}(\tilde{G})$. Choose smooth nonnegative functions $f, g: \mathbb{R}^D \to \mathbb{R}$ such that $Z(f) = \tilde{F}$, $Z(g) = \tilde{G}$, and let $\psi = f - g: \mathbb{R}^D \to \mathbb{R}$. Consider the closed sets $H = \{x \mid \psi(x) \leqslant 0\}$ and $K = \{x \mid \psi(x) \geqslant 0\}$. Since $\tilde{F} \cup \tilde{G} = Z(\tilde{\varphi})$, $H \cap Z(\tilde{\varphi}) = \tilde{F}$ and $K \cap Z(\tilde{\varphi}) = \tilde{G}$. So if we let h and k be nonnegative functions on $\mathbb{R}^D$ with $Z(h) = H$ and $Z(k) = K$, then $Z(\tilde{\varphi} + h) = \tilde{F}$ and $Z(\tilde{\varphi} + k) = \tilde{G}$. But $h \cdot k = 0$, so $(\tilde{\varphi} + h) \cdot (\tilde{\varphi} + k) \circ \pi_D = \varphi^2 + \varphi \cdot (h + k) \circ \pi_D \in I$. I is prime, so $(\tilde{\varphi} + h) \circ \pi_D \in I$ or $(\tilde{\varphi} + k) \circ \pi_D \in I$, i.e., either $F \in \hat{I}$ or $G \in \hat{I}$. Thus $\hat{I}$ is a prime filter. ∎

2.4. PROPOSITION. *Every C^∞-domain is local.*

Proof. It suffices to show this for finitely generated C^∞-rings. So let $A = C^\infty(\mathbb{R}^n)/I$ be a domain. I is a prime ideal, hence the corresponding filter $\hat{I}$ of closed sets is also prime (Lemma 2.3). Now let $f, g \in C^\infty(\mathbb{R}^n)$ with $f + g$ invertible in A. Then $\exists F \in \hat{I}$, $\forall x \in F$, $f(x) + g(x) \neq 0$. Let $U = \{x \mid f(x) \neq 0\}$, $V = \{x \mid g(x) \neq 0\}$. Then $F \subset U \cup V$, so by normality there are closed $G \subset U$ and $H \subset V$ with $F = G \cup H$. $\hat{I}$ is prime, so either $G \in \hat{I}$ or $H \in \hat{I}$, i.e., either f or g is invertible in A. ∎

The following proposition characterizes reduces C^∞-domains.

2.5. PROPOSITION. *A C^∞-ring is a reduced C^∞-domain* iff *it is C^∞-embeddable in a C^∞-field.*

Proof. $\Leftarrow$ is clear. For $\Rightarrow$ we wish to construct a C^∞-field in the usual way: let $A_0 = A$ be the reduced C^∞-domain under consideration, and let $A_{n+1} = A'_n$, where for a reduced C^∞-domain B, B' is the universal solution to inverting all nonzero elements in B. $F = \varinjlim_n A_n$ is a field. This proves $\Rightarrow$ provided we can show that each $A_n \to A_{n+1}$ is an injective C^∞-homomorphism. Arguing by induction, we prove that if B is a reduced C^∞-domain, then so is B' and $B \to B'$ is injective.

In fact this follows straightforwardly from the result of Muñoz and Ortega (Theorem 1.3). Indeed, write $B' = \varinjlim B\{\Delta^{-1}\}$, where the (filtered) colimit is taken over all finite subsets $\Delta \subset B - \{0\}$, and $B\{\Delta^{-1}\} = B\{b^{-1}\}$ of course, if b is the product of the elements of Δ.

Now suppose $a \in B$, and $a = 0$ in B'. Then there is a $b \in B - \{0\}$ such that $a = 0$ in $B\{b^{-1}\}$, so $B\{b^{-1}\}\{a^{-1}\} = B\{(a \cdot b)^{-1}\}$ is trivial. B is reduced, so $a \cdot b = 0$ in B; and B is a domain, so $a = 0$ in B. Thus $B \to B'$ is injective.

To see that B' is reduced, choose $a \in B'$, $a \neq 0$, and suppose $B'\{a^{-1}\}$ is trivial. Then there is a $b \in B - \{0\}$ such that $a \in B\{b^{-1}\}$ and $B\{b^{-1}\}\{a^{-1}\}$ is trivial. By Theorem 1.4 there are $c, d \in B$ with c invertible in $B\{b^{-1}\}$ and $a \cdot c = d$ in $B\{b^{-1}\}$. So $B\{b^{-1}\}\{a^{-1}\} \simeq B\{b \cdot d^{-1}\}$, and hence $d = 0$ as before since B is a reduced domain, i.e., $a = 0$ in $B\{b^{-1}\}$.

The proof that B is a domain is similar: we need to show that if $a_1, a_2 \in B\{b^{-1}\}$ and $a_1 \cdot a_2 = 0$ then $a_1 = 0$ or $a_2 = 0$ in $B\{b^{-1}\}$. Take c_i, d_i with c_i invertible in $B\{b^{-1}\}$ and $a_i \cdot c_i = d_i$ in $B\{b^{-1}\}$. Then $B\{(bd_1d_2)^{-1}\} = B\{b^{-1}\}\{(a_1a_2)^{-1}\}$ is trivial, so $bd_1d_2 = 0$ in B. B is a domain, so $d_1 = 0$ or $d_2 = 0$. Thus $a_1 = 0$ or $a_2 = 0$ in $B\{b^{-1}\}$. ∎

Our next aim is to show that local C^∞-rings are Henselian. Recall that a local ring A with residue field k_A is *Henselian* if for every monic polymial $p(t)$ with coefficients in A, simple roots of p in k_A can be lifted to A, i.e.,

$$\forall \alpha \in k_A (p(\alpha) = 0 \neq p'(\alpha) \to \exists a \in A(p(a) = 0 \text{ in } A \;\&\; \pi(a) = \alpha)),$$

where π denotes the canonical map $A \to k_A$. (Such a lifting is necessarily unique.) For general information see Raynaud [13].

We will need the following version of the implicit function theorem.

2.6. LEMMA. (IFT for closed sets). *Let $f(x, t)$: $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be smooth, and let $F \subset \mathbb{R}^m$ be closed. Suppose γ: $\mathbb{R}^m \to \mathbb{R}$ is a smooth function such that*

$f(x, \gamma(x)) = 0 \neq (\partial f/\partial t)(x, \gamma(x))$ for all $x \in F$. Then there exists an open $U \supset F$ and a tube

$$B = \{(x, t) \in U \times \mathbb{R} \mid |t - \gamma(x)| < \rho(x)\}$$

with $\rho: U \to (0, \infty)$ smooth, such that for every $a \in U$ there is a unique $b \in \mathbb{R}$ such that $(a, b) \in B$ and $f(a, b) = 0$. Furthermore, the function $a \mapsto b$ is a smooth extension of $\gamma \mid F$.

Proof. This follows easily from the usual verion of the implicit function theorem. Indeed, by this usual version we find for each $x_0 \in F$ a neighbourhood U_{x_0} and *a* $\delta_{x_0} > 0$ such that for each $x \in U_{x_0}$, $f(x, -)$ has exactly one zero in $(\gamma(x_0) - \delta_{x_0}, \gamma(x_0) + \delta_{x_0})$. Let $V = \bigcup_{x_0 \in F} U_{x_0}$. By a partition of unity argument, we can find a smooth $\rho: V \to (0, \infty) \subset \mathbb{R}$ such that if $x \in V$ then $f(x, -)$ has at most one zero in $(\gamma(x) - \rho(x), \gamma(x) + \rho(x))$ (making V a little smaller if necessary). Let $U = \{x \in V \mid f(x, -)$ has exactly one zero in $(\gamma(x) - \rho(x), \gamma(x) + \rho(x))\}$. Then $U \supset F$, and by the implicit function theorem, U is open and the function on U which associates with $x \in U$ this unique zero is smooth. ∎

2.7. THEOREM. *Let A and B be C^∞-rings, with B reduced, and let $\varphi: A \twoheadrightarrow B$ be a surjective C^∞-homomorphism which is local (i.e., $\varphi(a) \in U(B) \Rightarrow a \in U(A)$), and let $p(t)$ be a monic polynomial in $A[t]$. Then any simple root of $p(t)$ in B can be lifted to a root in A.*

Proof. Choosing a set E of generators for A, we can write $A = C^\infty(\mathbb{R}^E)/I$ and $B = C^\infty(\mathbb{R}^E)/J$, where $J \supset I$ and φ is the canonical quotient map. Let $\Phi = \{Z(f) \mid f \in I\}$ be the filter of zerosets corresponding to J, so $f \in J$ iff $Z(f) \in \Phi$ since B is reduced (Lemma 2.2). Suppose $p(t)$ is represented by

$$f(x, t) = t^n + f_1(x)\, t^{n-1} + \cdots + f_n(x),$$

with $f_i(x) \in C^\infty(\mathbb{R}^E)$, and let $r \in C^\infty(\mathbb{R}^E)$ be a simple root in B, that is,

$$f(x, r(x)) = 0 \qquad \text{on some } G \in \Phi,$$

$$\frac{\partial f}{\partial t}(x, r(x)) \neq 0 \qquad \text{on some } H \in \Phi.$$

Let $F = G \cap H$, and let $D \subseteq E$ be a finite set containing all the coordinates involved in the $f_i(x)$, $r(x)$, and F. So we can regard $f(x, t)$ as a function $\mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$, $r(x)$ as $\mathbb{R}^D \to \mathbb{R}$, and F as a closed subset of $\mathbb{R}^D$, and we have

$$f(x, r(x)) = 0 \neq \frac{\partial f}{\partial t}(x, r(x)), \qquad \forall x \in F.$$

By Lemma 2.6, there is an open $U \supset F$ and a smooth $s: U \to \mathbb{R}$ with $s|F = r|F$ (so $\varphi(s) = r$ since B is reduced), and $f(x, s(x)) = 0$ for all $x \in U$. We wish to conclude that $p(s) = 0$ in A, i.e., that $f(x), s(x)) \in I$. Let $g(x)$ be a characteristic function for U. Then $g(x) \cdot f(x, s(x)) = 0$ in $C^\infty(\mathbb{R}^E)$, hence in A. But g is invertible in B since $F \subset U$ and B is reduced, so g is invertible in A, and therefore $f(x, s(x)) = 0$ in A. ▮

Applying Theorem 2.7 to the special case where A is a local C^∞-ring and B is its residue field, we obtain

2.8. Corollary. *Every local C^∞-ring is Henselian.* ▮

It should be observed that in the proof of 2.7 we did not use that the function $f(x, t)$ representing $p(x)$ depended polynomially on t. So the argument remains valid if we assume $p \in A\{t\}$ rather than $p \in A[t]$. Rewriting the definition of Henselian local ring with $A\{t\}$ instead of $A[t]$ gives a notion which is more natural in the context of C^∞-rings, and which we call *C^∞-Henselian*. Thus, as a strengthening of 2.8 we have

2.8'. Corollary. *Every local C^∞-ring is C^∞-Henselian.* ▮

2.9. Corollary. *For every local C^∞-ring A we have (as rings) that $A \cong k_A \oplus m_A$, where k_A is the residue field and m_A is the maximal ideal.*

Proof. We show that the exact sequence $0 \to m_A \to A \to k_A \to 0$ is split-exact. Consider partial sections (K, s) of π, where K is a subfield ($\mathbb{R}$-algebra) of k_A. Let (K, s) be a maximal section (Zorn);

$$\begin{array}{ccc} & & A \\ & {\scriptstyle s}\nearrow & \downarrow{\scriptstyle \pi} \\ K & \subset & k_A \end{array}$$

Take $\alpha \in k_A - K$. If α is transcendental over K, we can extend s to a section on $K(\alpha) = K(x)$, contradicting maximality. And if α is algebraic over K, there is an irreducible monic polynomial f with $f(\alpha) = 0$, $f'(\alpha) \neq 0$. By Henselianness, α can be lifted to a root $\beta \in A$, $\pi(\beta) = \alpha$, an by sending α to β we obtain an extension of s to $K(\alpha)$, again contradicting maximality of s. So $K = k_A$. ▮

Recall from section 1 that every C^∞-ring has a canonical pre-order $<$. If $A = C^\infty(\mathbb{R}^E)/I$, then for $f \in C^\infty(\mathbb{R}^E)$ representing an element of A,

$$f > 0 \qquad \text{in } A \text{ iff } \exists g \in I, \forall x \in Z(g), f(x) > 0.$$

If A is a field, I is maximal (hence $I = \sqrt[\infty]{I}$), and the pre-order is a total order, i.e., $f \neq 0 \to f < 0$ or $f > 0$.

A totally ordered field is called *real closed* if it satisfies

(a) $x > 0 \Rightarrow \exists y \; x = y^2$,

(b) polynomials of odd degree have roots.

2.10. THEOREM. *Every C^∞-field is real closed.*

Proof. Let $K = C^\infty(\mathbb{R}^E)/I$ be a C^∞-field, and let Φ be the maximal filter of zerosets corresponding to the maximal ideal I. We have just remarked that K is totally ordered in a canonical way. Now condition (a) is trivial, and holds in fact in any C^∞-ring (use smooth Tietze, Sect. 1). To prove condition (b), let $p(t) \in K[t]$ be a polynomial of odd degree. We may assume that p is monic and irreducible, and hence that $(p, p') = (1)$ as ideals in $K[t]$. Or equivalently, the resultant determinant $\mathrm{Res}(p, p') \neq 0$ in K. Therefore, if $p(t)$ is represented by

$$f(x, t) = t^n + f_1(x)\, t^{n-1} + \cdots + f_n(x),$$

then for the function $R(x) = \mathrm{Res}(f(x, t), \partial f/\partial t(x, t))$, we have that $R(x) \neq 0$ for all x in some $F \in \Phi$.

As before, choose a finite $D \subset E$ containing all the coordinates involved in f and in F, and regard f as a function $\mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$ and F as a closed subset of $\mathbb{R}^D$. Let $U \subset \mathbb{R}^D$ be open, $F \subset U$, such that $R(x) \neq 0$ on U. $\mathbb{R}$ is real closed, so for each $x \in U$ the polynomial $f(x, t) \in \mathbb{R}[t]$ has a root. Let $r(x)$ be the first root. Then this root is simple since $R(x) \neq 0$, so $r \colon U \to \mathbb{R}$ is smooth by the implicit function theorem (the usual, not 2.6). r represents an element of K which is a root of p. ▮

A local ring A is called separably real closed if A is Henselian and k_A is real closed. So combining 2.8 and 2.10 we have

2.11. COROLLARY. *Every local C^∞-ring is separably real closed.* ▮

Although 2.11 has been proved for C^∞-rings in the topos of Sets, if follows that 2.11 is true for local C^∞-rings in any (Grothendieck) topos, since the notions involved are all coherent, (see Kock [6] and Joyal & Reyes [5]), so we can apply the completeness theorem of Makkai and Reyes [8].

As with Henselianness, we would like to define a "smooth" notion of real-closedness, using $K\{t\}$ instead of $K[t]$, *which implies the usual algebraic notion.* Analysing the proof of 2.10 we see that we need is to replace condition (b) that odd polynomials have roots by a "transversal intermediate value" condition:

$\forall f \in K\{t\}\ ((f(0) \cdot f(1) < 0$ in K & $(f, f') = (1)$ as ideals in $K\{t\} \Rightarrow \exists \alpha \in K,\ 0 < \alpha < 1,$ & $f(\alpha) = 0).$

Indeed, if $f(0) \cdot f(1) < 0$ and $(f, f') = (1)$, then in terms of a representing function $f(x, t)\colon \mathbb{R}^E \times \mathbb{R} \to \mathbb{R}$ this means that for some $F \in \Phi$ (Φ as in the proof of 2.10).

$$\forall x \in F\left(f(x, 0) \cdot f(x, 1) < 0 \;\&\; \forall t \in \mathbb{R}\left(f(x, t) = 0 \Rightarrow \frac{\partial f}{\partial t}(x, t) \neq 0\right)\right).$$

Now choose a finite $D \subset E$ as in the proof of 2.10, so as to be able to regard f as a function $\mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$ and F as a closed subset of $\mathbb{R}^D$, and use compactness of $[0, 1]$ to find an open $U \supset F$ such that

$$\forall x \in U\left(f(x, 0) \cdot f(x, 1) < 0 \;\&\; \forall t \in [0, 1]\left(f(x, t) \neq 0 \text{ or } \frac{\partial f}{\partial t}(x, t) \neq 0\right)\right).$$

Now the usual implicit function theorem implies that the function $r\colon U \to \mathbb{R}$, $r(x) =$ the first zero of $f(x, -)$ in $[0, 1]$, is smooth.

To sum up this discussion, let us give a stronger formulation of theorem 2.10.

2.10′. THEOREM. *Every C^∞-field is C^∞-real closed in the sense that it satisfies*

$$\forall f \in K\{t\}((f(0) \cdot f(1) < 0 \;\&\; 1 \in (f, f') \subset K\{t\}) \Rightarrow \exists \alpha \in (0, 1) \subset K\colon f(\alpha) = 0).$$

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REFERENCES

1. M. DEMAZURE AND P. GABRIEL, "Groupes Algébriques," Tome 1, Masson, Paris 1970.
2. E. J. DUBUC, Sur les modèles de la Géometrie Différentielle Synthétique, *Cahiers Topologie Géom. Différentielle* **XX** (1979).

3. E. J. DUBUC, C^∞-schemes, *Amer. J. Math.* **103** (1981).
4. M. HAKIM, Topos Annelés et Schémas Relatifs, Ergebnisse der Mathematik, Vol. 64, Springer, Berlin, 1972.
5. A. JOYAL AND G. E. REYES, Separably real closed local rings, Sydney category seminar reports 1982, *in* "Proceedings of the 5th Symposium of Logic in Latin America," Bogotá 1981, to appear.
6. A. KOCK, Formally real local rings and infinitesimal stability, *in* "Aarhus Var. Publ. Series," Vol. 30, 1979.
7. A. KOCK, "Synthetic Differential Geometry," Cambridge Univ. Press, London, 1981.
8. M. MAKKAI AND G. E. REYES, "First Order Categorical Logic," Lecture notes in Math., Vol. 611, Springer, Berlin, 1977.
9. B. MALGRANGE, "Ideal of Differentiable Functions," Oxford Univ. Press, London, 1966.
10. J. MUÑOZ AND J. M. ORTEGA, Sobre las álgebras localmente convexas, Collectanae Mathematica **XX** (1969), 2.
11. I. MOERDIJK AND G. E. REYES, "A Smooth Version of the Zariski Topos," Amsterdam report, 1983; *Advan. in Math.*, in press.
12. I. MOERDIJK AND G. E. REYES, "C^∞-rings" (Chapt. I of a monograph in preparation on models of SDG, Rapports de Recherches, Dépt. Math., Université de Montréal, 1983.
13. M. RAYNAUD, "Anneaux Locaux Henséliens," Lecture Notes in Math., Vol. 169, Springer, Berlin, 1970.
14. G. E. REYES AND G. E. WRAITH, A note on tangent bundles in a category with a ring object, *Math. Scand.* **42** (1978).
15. J.-C. TOUGERON, "Idéaux de Fonctions Différentiables," Ergebnisse der Mathematik, Vol. 71, Springer, Berlin, 1972.