Real, Complex, and Quarternionic Representations

JWR

10 March 2006

1 Group Representations

1. Throughout $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ denotes the complex numbers, $\mathbb{H}$ denotes the quaternions, and G denotes a compact group. A **real representation** of G is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{R}}(U)$ from G into the group of $\mathbb{R}$-linear automorphisms of a real vector space U. A **complex representation** of G is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$ from G into the group of $\mathbb{C}$-linear automorphisms of a complex vector space V. A **quaternionic representation** of G is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{H}}(W)$ from G into the group of $\mathbb{H}$-linear automorphisms of a quaternionic vector space W.

2. We define some functors. The **dual space** of a complex vector space V is the complex vector space

$$V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

The **complexification** of a real vector space U is the complex vector space

$$\mathbb{C} \otimes_{\mathbb{R}} U, \quad i(z \otimes u) = (iz) \otimes u.$$

The **quarternionification** of a complex vector space V is the quarternionic vector space

$$\mathbb{H} \otimes_{\mathbb{C}} V, \quad p(q \otimes v) = (qp) \otimes v, \qquad p, q \in \mathbb{H}, \quad v \in V$$

where $\mathbb{C}$ acts on $\mathbb{H}$ from the right in the tensor product so that $z(q \otimes v) = q \otimes zv$ for $z \in \mathbb{C}$. For a complex vector space V we denote by

$$V_{\mathbb{R}}$$

the real vector space obtained by restricting the scalars to $\mathbb{R} \subset \mathbb{C}$. For a quarternionic vector space W we denote by

$$W_{\mathbb{C}}$$

the complex vector space obtained by restricting the scalars to $\mathbb{C} \subset \mathbb{H}$. Each of these operations is functorial. and hence yields new representations from old. For example, when $G \to \mathrm{Aut}_{\mathbb{C}}(V) : g \mapsto g_V$ is a complex representation, the dual representation is $G \to \mathrm{Aut}_{\mathbb{C}}(V^*) : g \mapsto (g_V^*)^{-1}$.

Theorem 3. *Let $G \to \mathrm{Aut}_{\mathbb{C}}(V)$ be a complex representation. Then the following are equivalent.*

(1) *$V = \mathbb{C} \otimes U$ is the complexification of a real representation $G \to \mathrm{Aut}_{\mathbb{R}}(U)$.*

(2) *V admits an equivariant real structure. (A* **real structure** *on a complex vector space V is an anti linear map $S : V \to V$ such that $S^2(v) = v$.)*

(3) *There is an equivariant isomorphism $B : V \to V^*$ such that $B^* = B$.*

Also the following are equivalent.

(4) *$V = W_{\mathbb{C}}$ is obtained from a quaterionic representation $G \to \mathrm{Aut}_{\mathbb{H}}(W)$ by restriction of the scalars.*

(5) *V admits an equivariant quaternionic structure. (A* **quaternionic structure** *on a complex vector space V is an anti linear map $S : V \to V$ such that $S^2(v) = -v$.)*

(6) *There is an equivariant isomorphism $B : V \to V^*$ such that $B^* = -B$.*

Proof. First we prove (1) $\iff$ (2) and (4) $\iff$ (5).

(1) $\Longrightarrow$ (2). The complexification $V = \mathbb{C} \otimes_{\mathbb{R}} U$ of a real vector space U has the real structure $S_U(z \otimes u) = \bar{z} \otimes u$.

(2) $\Longrightarrow$ (1). A real structure S on a complex vector space V determines a real vector space $U = \{u \in V : S(u) = u\}$ and an isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} U \to V : z \otimes u \mapsto zu.$$

(4) $\Longrightarrow$ (5). The complex vector space $W_{\mathbb{C}}$ obtained from a quaternionic vector space W by restricting the scalars to $\mathbb{C} \subset \mathbb{H}$ has a quaternionic structure S_W defined by $S_W(w) = jw$ for $w \in W$.

(5) $\Longrightarrow$ (4). A complex vector space V with a quaternionic structure S determines a quaternionic vector space W via $W = V$ and

$$qv = (q_0 + q_1 i)v + (q_2 + q_3 i)S(v), \qquad q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$$

where $q_0, q_1, q_2,_3 \in \mathbb{R}$.

Since conditions (2) and (5) and conditions (3) and (6) differ only in a sign we can treat (2) $\iff$ (3) and (5) $\iff$ (6) simultaneously. By averaging over G we get a real valued G-invariant $(\cdot, \cdot)$ Hermitean inner product on V. Let $\langle \cdot, \cdot \rangle$ denote its real part so

$$(v, w) = \langle v, w \rangle + i\langle iv, w \rangle, \qquad (zv, w) = \bar{z}(v, w), \qquad (v, w) = \overline{(w, v)}.$$

For $A \in \mathrm{End}_{\mathbb{R}}(V)$ denote by $A' \in \mathrm{End}_R(V)$ the unique map satisfying

$$\langle Av, w \rangle = \langle v, A'w \rangle.$$

(2),(5) $\implies$ (3),(6). By replacing (v,w) by $(v,w)+\overline{(Sv,Sw)}$ we may assume w.l.o.g. that

$$(Sv,Sw)=\overline{(v,w)}.$$

Define B by

$$B(v,w):=(Sv,w).$$

Then B is $\mathbb{C}$-biinear and

$$B(v,w)=\overline{(w,Sv)}=(Sw,S^2v)=\pm(Sw,v)=\pm B(w,v)$$

so $B^*=\pm B$ as required.

(3),(6) $\implies$ (2),(5). Given $B:V\to V^*$ define an equivariant $\mathbb{R}$-linear automorphism $A:V\to V$ by

$$\langle Av,w\rangle=\mathrm{Re}\ B(v,w)$$

where $\mathrm{Re}\, z\in\mathbb{R}$ denotes the real part of $z\in\mathbb{C}$. Form the polar decomposition

$$A=PS,\qquad P=\sqrt{AA'},\quad S=P^{-1}A$$

so $P=P'>0$ and $S'=S^{-1}$.

Step 1. A is anti linear. (Proof: $\langle Aiv,w\rangle=\mathrm{Re}\ B(iv,w)=\mathrm{Re}\ B(v,iw)=\langle Av,iw\rangle=-\langle iAv,i^2w\rangle=-\langle iAv,w\rangle$.)

Step 2. $A'=\pm A$. (Proof: $\langle Av,w\rangle=\mathrm{Re}\ B(v,w)=\pm\mathrm{Re}\ B(w,v)=\pm\langle Aw,v\rangle=\pm\langle w,A'v\rangle=\pm\langle A'v,w\rangle$.)

Step 3. AA' is $\mathbb{C}$-linear. (Proof: A and $A'=\pm A$ are both anti linear).

Step 4. P is $\mathbb{C}$-linear. (Proof: it is a power series in AA').

Step 5. S is anti linear. (Proof: P is $\mathbb{C}$-linear and A is anti linear).

Step 6. $S^2=\pm1$. (Proof: Since $A'=\pm A$ we have that A is normal in the sense that $AA'=A'A$. Hence $B=PS=SP$. But $P=P'>0$ and $S'=S^{-1}$ by the polar decomposition so $PS'=(SP)'=(PS)'=B'=\pm B=\pm PS$ so $S^{-1}=S'=\pm S$ as required.)

Steps 5 and 6 show that S is a structure map. It is equivariant as the construction is explicit. □

2 Irreducible Representations

A representation of a compact group has an invariant Hermitean inner product so the representation is **irreducible** (no invariant subspace) if and only if it is **indecomposable** (no invariant splitting).

Definition 4. A complex representation $G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is said to be **self dual** iff it is isomorphic to its complex dual V^* An irreducible complex representation is said to be of **real type** iff it satisfies the equivalent conditions (1-3) of Theorem 3, of **quaternionic type** iff it satisfies the equivalent conditions (4-6) of Theorem 3, and of **complex type** iff it is not self dual.

Corollary 5. *A self dual irreducible complex representation is either of real type or of quaternionic type and not both.*

Proof. Suppose that $B : V \to V^*$ is an isomorphism. Then at least one of the two maps $B + B^*$ and $B - B^*$ is non zero and so an isomorphism by Schur's Lemma. Theorem 3 says that V is of real type in the former case and of quaternionic type in the latter case. Also by Schur's Lemma any two isomorphisms are non zero multiples of each other so there cannot be both a symmetric and a skew symmetric isomorphism between V and V^*. □

Corollary 6. *Abbreviate $r(V) = V_{\mathbb{R}}$, $f(W) = W_{\mathbb{C}}$, $d(V) = V^*$, $c(U) = \mathbb{C} \otimes U$, and $h(V) = \mathbb{H} \otimes_{\mathbb{C}} V$, There is a list $\{U_m\}_m$ of real representations, a list $\{V_n\}_n$ of complex representations, and a list $\{W_p\}_p$ of quaternionic representations such that*

(I) *The list $\{c(U)_m\}_m \cup \{V_n\}_n \cup \{V_n^*\}_n \cup \{f(W_p)\}_p$ contains exactly one representative of every irreducible complex representation.*

(II) *The list $\{U_m\}_m \cup \{r(V_n)\}_n \cup \{rf(W_p)\}_p$ contains exactly one representative of every irreducible real representation.*

(III) *The list $\{hc(U_m)\}_m \cup \{h(V_n)\}_n \cup \{W_p\}_p$ contains exactly one representative of every irreducible quaternionic representation.*

Lemma 7. *For representations of G these functors satisfy the following identities:*

$$rc = 2, \quad cr = 1 + d, \quad dc = c, \quad dr = r, \quad rd = r,$$

$$d^2 = 1, \quad hf = 2, \quad hd = h, \quad fh = 1 + d.$$

Here = means equivariant isomorphic and + means direct sum.

Proof. See [1] Proposition 6.1 page 95. □

Corollary 8. *Say that a complex representation has an* **irreducible real form** *iff it is the complexification of an irreducible real representation. Then an irreducible complex representation V is of*

(I) *real type if and only if V has irreducible real form,*

(II) *complex type if and only if $V \oplus V^*$ has irreducible real form,*

(III) *quaternionic type if and only if $V \oplus V$ has irreducible real form.*

Proof. In part (I) of Corollary 6 the representations in the list $\{c(U)_m\}_m$ are of real type, those in the list $\{V_n\}_n \cup \{V_n^*\}_n$ are of complex type and those in the list $\{f(W_p)\}_p$ are of quaternionic type. By Theorem 7) If $V = c(U_m)$ then V is the complexification of $U = U_m$. For $V = V_n$ or $V = V_n^*$ From $1 + d = cr$ in Theorem 7) we get $V_n \oplus V_n^* = (1 + d)V_n = cr(V_n)$ is the complexification of $U = r(V_n)$. we get that $V \oplus V^*$ is the complexification of $U = r(V)$. For $V = f(W_p)$ and $2 = cr$ in Theorem 7) we get that $V \oplus V$ is the complexification of $U = rf(W_p)$. In each case U is irreducible by part (II) of Theorem **??**. These values of U exhaust the real irreducible representations so this proves the converse as well. $\square$

Corollary 9. *In the notation of Corollary 6 the commutator algebras of the irredcucible real representations are given by*

$$\mathrm{End}_{\mathbb{R}}(U_m)^G = \mathbb{R}, \qquad \mathrm{End}_{\mathbb{R}}(r(V_n))^G = \mathbb{C}, \qquad \mathrm{End}_{\mathbb{R}}(rf(W_p))^G = \mathbb{H}.$$

Proof. See [1] Theorem 6.7 page 99. $\square$

Corollary 10. *Let $G \to \mathrm{Aut}_{\mathbb{C}}(V) : g \mapsto g_V$ be an irreducible real representation and $\chi_V : G \to \mathbb{C}$ be the character of V, i.e.*

$$\chi_V(g) := \mathrm{Trace}(g_V).$$

Then

$$\int_G \chi(g^2)\, dg = \begin{cases} 1 & \iff & V \text{ is of real type}, \\ 0 & \iff & V \text{ is of complex type}, \\ -1 & \iff & V \text{ is of quaternionic type}. \end{cases}$$

(The integral is with respect to Haar measure.)

Proof. See [1] Proposition 6.8 page 100. $\square$

3 Matrix Representations

We formulate the above in the language of matrices.

11. The **classical algebraic groups** are the general linear group $\mathrm{GL}_n(\mathbb{F})$, the special linear group $\mathrm{SL}_n(\mathbb{F})$, the orthogonal group $\mathrm{O}_n(\mathbb{F})$, the special orthogonal group $\mathrm{GL}_n(\mathbb{F})$, and the symplectic group $\mathrm{Sp}_n(\mathbb{F})$ as follows.

$$\begin{aligned}
\mathrm{GL}_n(\mathbb{F}) &:= \{A \in \mathbb{F}^{n\times n} : \det(A) \neq 0\}, \\
\mathrm{SL}_n(\mathbb{F}) &:= \{A \in \mathbb{F}^{n\times n} : \det(A) = 1\}, \\
\mathrm{O}_n(\mathbb{F}) &:= \{A \in \mathrm{GL}_n(\mathbb{F}) : A^{-1} = {}^tA\}, \\
\mathrm{SO}_n(\mathbb{F}) &:= \{A \in \mathrm{SL}_n(\mathbb{F}) : A^{-1} = {}^tA\}, \\
\mathrm{Sp}_n(\mathbb{F}) &:= \{A \in \mathrm{GL}_n(\mathbb{F}) : A^{-1} = -J\,{}^tAJ\}.
\end{aligned}$$

In the definition of $\mathrm{Sp}_n(\mathbb{F})$, n is even and J is block diagonal with 2×2 blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the diagonal. In order that

$$\mathrm{GL}_n(\mathbb{F}) = \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}^n)$$

we identify the elements of $\mathbb{F}^n$ with column vectors where matrices act on the left and the scalars act on the right (so that $\mathbb{H}^n$ is a right vector space over $\mathbb{H}$). Then we have inclusions

$$\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{H})$$

corresponding to complexification and quaternionification and inclusions

$$\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R}), \qquad \mathrm{GL}_n(\mathbb{H}) \subset \mathrm{GL}_{2n}(\mathbb{C})$$

given by restriction of the scalars, i.e. replacing each entry $x + yi$ in a complex matrix by the 2×2 real matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and replacing each entry $u + vj$ in a quaternionic matrix by the 2×2 complex matrix $\begin{pmatrix} u & -v \\ \bar{v} & \bar{u} \end{pmatrix}$.

12. The **classical compact groups** are the unitary group $\mathrm{U}(n)$, the special unitary group $\mathrm{SU}(n)$, the real orthogonal group $\mathrm{O}(n)$, the real special orthogonal group $\mathrm{SO}(n)$, and the quaternionic unitary group $\mathrm{Sp}(n)$ as follows.

$$\begin{array}{lcl} \mathrm{U}(n) & := & \{A \in \mathrm{GL}_n(\mathbb{C}) : A^{-1} = {}^t\bar{A}\}, \\ \mathrm{SU}(n) & := & \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{U}(n), \\ \mathrm{O}(n) & := & \mathrm{O}_n(\mathbb{R}), \\ \mathrm{SO}(n) & := & \mathrm{SO}_n(\mathbb{R}), \\ \mathrm{Sp}(n) & := & \{A \in \mathrm{GL}_n(\mathbb{H}) : A^{-1} = {}^t\bar{A}\}. \end{array}$$

There is an isomorphism

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}_{2n}(\mathbb{C}).$$

(See [2] Exercise 7.4 page 99.) Every complex representation $G \to \mathrm{Aut}_{\mathbb{C}}(V)$ preserves a Hermitean form and hence is conjugate to a homomorphism

$$G \to \mathrm{U}(n).$$

A complex representation satisfies conditions (1-3) of Theorem 3 if and only if it can be represented by real matrices, i.e. if and only if it is conjugate to a homomorphism

$$G \to \mathrm{O}(n) \subset \mathrm{U}(n).$$

A complex representation satisfies conditions (4-6) of Theorem 3 if and only if can be represented by quaternionic matrices, i.e. if and only if it is conjugate to a homomorphism

$$G \to \mathrm{Sp}(n) \subset \mathrm{U}(2n).$$

References

[1] Theodor Bröcker & Tammo tom Dieck: *Representations of Compact on Lie Groups*, Springer GTM **98** 1985.

[2] William Fulton & Joe Harris: *Representation Theory, a first course*, Springer GTM **129** 1991.