AN INTRODUCTION TO THE DERIVED CATEGORY OF COHERENT SHEAVES

FRANCO ROTA

Abstract. In these notes we introduce the concept of derived category, with particular attention to the derived category of coherent sheaves of a smooth projective variety. These notes are meant to be a guide for someone approaching the subject for the first time, they try to provide examples and motivation to help intuition.

1. Introduction and motivation

Algebraic geometry in the 1950s and 1960s experienced a revolution, based mainly on the work of A. Grothendieck, which led to the modern approach to the subject. In fact, in those years the French school found a way to solve the foundational problems of the subject, which the Italian school of the earlier century wasn't able to address satisfactorily. Grothendieck himself presented the idea of a scheme over a field to generalize the notion of variety.

In those years, attention was drawn to the study of sheaves on algebraic varieties from a categorical point of view. The following result gives a taste of the categorical approach of those years:

Theorem 1.1 (Gabriel, [1]). *Let X and Y be smooth projective varieties, then an equivalence $\Phi : \textbf{\textit{Coh}}(X) \xrightarrow{\sim} \textbf{\textit{Coh}}(Y)$ induces an isomorphism $X \simeq Y$.*

The category of coherent sheaves on a variety proved to be an interesting object, and some of the issues present in the study of this category led to the construction of the derived category of coherent sheaves of a variety. Cohomology of sheaves was one of the most powerful instruments of investigation, but presented some peculiar features that weren't well understood. Some of the important functors in algebraic geometry are not exact, the construction of derived functors emphasized the importance of resolutions of a sheaf. In the study of resolutions, the passage to the category of complexes was pretty natural, as well as the following:

Definition 1.2. Let $\mathcal{A}$ be an abelian category, and $\mathrm{Kom}(\mathcal{A})$ be its category of complexes. Let $A^\bullet$,$B^\bullet$ be elements in $\mathrm{Kom}(\mathcal{A})$ and $f : A^\bullet \to B^\bullet$ be a morphism of complexes. It induces maps

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet),$$

we say that f is a *quasi-isomorphism* if all the maps $H^i(f)$ are isomorphisms.

Example 1.3. A resolution $F^\bullet \to A \to 0$ of an object A in an abelian category $\mathcal{A}$ induces a quasi-isomorphism of complexes:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & F_n & \longrightarrow & \dots & \xrightarrow{d} & F_0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \varepsilon} & & \big\downarrow{\scriptstyle 0} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Indeed, the only non vanishing cohomology object is in degree zero, and H^0 of the two complexes coincides: we have $H^0(F^\bullet) = F_0/\mathrm{Im}d = F_0/\mathrm{Ker}\varepsilon \simeq A = H^0(A)$ since a resolution is exact.

Example 1.4. A quasi-isomorphism in $\mathrm{Kom}(\mathcal{A})$ is in general not invertible. Consider as an example the category of complexes of abelian groups, and the free resolution of $\mathbb{Z}/2\mathbb{Z}$:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \varepsilon} & & \big\downarrow{\scriptstyle 0} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

The morphism ε is a quasi-isomorphism, but the only possible morphism of complexes in the opposite direction is the zero map. This map induces the zero map on cohomology, hence it's not an inverse to ε.

Bearing in mind these examples, let us proceed in the construction of the derived category of an abelian category.

2. Construction of the derived category of an abelian category

Let $\mathcal{A}$ be an abelian category (we will mainly be interested in the abelian category $\mathcal{A} = \mathbf{Coh}(X)$ of coherent sheaves on a smooth projective variety X). We build the derived category $D(\mathcal{A})$ by subsequent steps. We follow the construction as explained in [2]. An alternative construction, using localizations is presented in [3].

Step 1: The context in which the construction will take place is the category $\mathrm{Kom}(\mathcal{A})$ of complexes of objects in $\mathcal{A}$. Objects in this categories are complexes

$$\dots \to A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \to \dots$$

such that $d_A^i \circ d_A^{i-1} = 0$ for all $i \in \mathbb{Z}$ (these maps are called *differentials*). A morphism of complexes $f : A \to B$ is defined to be a collection of maps $f_i : A^i \to B^i$ that commute with the differentials. Cohomology is defined in this setting as

$$H^i(A) = \frac{\mathrm{Ker} d_A^{i+1}}{\mathrm{Im} d_A^i}.$$

In this category we also have a functor which will prove very important in the future:

Definition 2.1. For $A^\bullet \in \mathrm{Kom}(\mathcal{A})$, and f a morphism in $\mathrm{Kom}(\mathcal{A})$. Define $A^\bullet[1]$ by setting $(A^\bullet[1])^i = A^{i+1}$ and $d^i_{A^\bullet[1]} = (-d_A^{i+1})$, and $f[1]^i = f^{i+1}$. We call $T : A^\bullet \mapsto A^\bullet[1]$ the *shift functor*, it defines an equivalence of abelian categories (exercise).

Step 2: Cohomology is still well behaved if we focus our attention to the homotopy category $K(\mathcal{A})$. This category has the same objects as $\mathrm{Kom}(\mathcal{A})$, and the morphisms are equivalence classes of morphisms in $Kom(\mathcal{A})$:

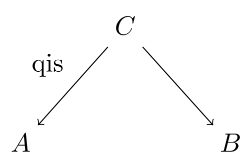
$$\mathrm{Hom}_{K(\mathcal{A})}(A, B) = \mathrm{Hom}_{Kom(\mathcal{A})}(A, B)/_{\sim}$$

where two morphisms of complexes $f, g : A \to B$ are said to be equivalent if there exists a homotopy P, i.e. a collection of maps $P^i : A^i \to B^{i-1}$, such that $f^i - g^i = P^{i+1}d^i_A + d^{i-1}_B P^i$ for all i.

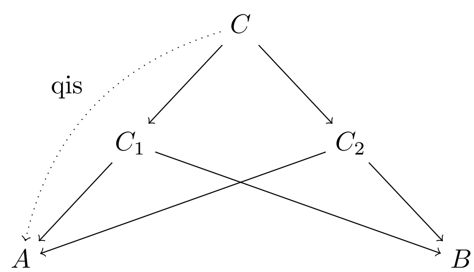
Step 3: Now we introduce the technical feature which allows to identify quasi-isomorphic objects. The derived category of $\mathcal{A}$ is the category whose objects are complexes of elements in $\mathcal{A}$,

$$\mathrm{ob}(D(\mathcal{A})) = \mathrm{ob}(K(\mathcal{A})) = \mathrm{ob}(\mathrm{Kom}(\mathcal{A})),$$

and morphisms are equivalence classes of *roofs*, i.e. diagrams of the form



(note that roofs have to be morphisms, if we want to allow quasi-isomorphisms to be considered invertible). Two such roofs are said to be equivalent if a diagram



exists and commutes in $K(\mathcal{A})$. It is noteworthy that commutativity is only required up to homotopy of complexes, the more restricting requirement of commutativity in $\mathrm{Kom}(\mathcal{A})$ would give trouble in developing the theory. It is possible to give a well behaved composition of roofs, the details are presented in [2, Chp. 2].

Remark 2.2. In the derived category, quasi-isomorphic objects are identified. This means that two objects are isomorphic if their cohomologies are isomorphi via maps induced by a map of complexes. Then, it is not enough that two complexes have the same cohomologies for them to be isomorphic.

We omit the necessary verifications related to the construction above (especially the fact that a roof is an acceptable notion of morphism), but it's useful to mention an object that comes into play during these verifications and plays a central role when we endow the derived category with a triangulated structure.

Definition 2.3. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Its *mapping cone* $C(f)$ is the complex

$$C(f)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f)} = \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

Sinthetically, we write $C(f) = A[1] \oplus B$.

Remark 2.4. By construction, the mapping cone of $f : A^\bullet \to B^\bullet$ admits natural morphisms of complexes $B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]$, where τ is an inclusion in the second summand, and π is projection on the first.

3. Triangulated categories

The main result in this section is that the derived category of an abelian category has the structure of a triangulated category.

Definition 3.1. Let $\mathcal{D}$ be an additive category. The structure of a *triangulated category* on $\mathcal{D}$ is given by an additive equivalence $T : \mathcal{D} \to \mathcal{D}$ called the *shift functor*, and a set of *distinguished triangles*

$$A \to B \to C \to T(A)$$

respecting the axioms TR1-TR4 below.

Before stating the axioms, we introduce some notation. For any integer n we will denote by $A[n]$ the object $T^n(A)$ for $A \in \mathcal{D}$, and if $f : A \to B$ is a morphism we denote by $f[n]$ the morphism $T^n(f) : A[n] \to B[n]$. A morphism between two triangles consists of a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

and it is an isomorphism when f, g and h are. Here come the axioms:
TR1.

- Any triangle of the form $A \xrightarrow{\text{id}} A \to 0 \to A[1]$ is distinguished;
- any triangle isomorphic to a distinguished one is distinguished;
- any morphism $A \xrightarrow{f} B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \to C \to A[1].$$

TR2. The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

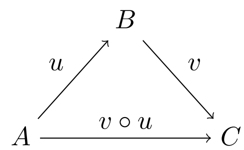
$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.
TR3. Suppose we have a diagram

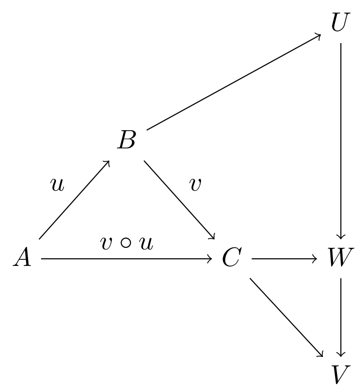
$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

where the leftmost square is commutative. Then the diagram can be completed to a morphism of triangles by a (not necessarily unique) $h : C \to C'$.

TR4. This is called the octahedral axiom, I found this formulation in some notes by Arend Bayer. For any composition of morphisms, i.e. a commutative diagram



then applying TR 1 to u,v,$v \circ u$ we get a diagram



the octahedral axiom requires the vertical line to be a distinguished triangle as well. Other representations of this diagram explain the reason for its name, see for example [3, IV, §1].

The derived category $D(\mathcal{A})$ of an abelian category can be endowed with the structure of a triangulated category ([3, IV, §2]). We have a natural choice for the functor T, given by the shift on $\mathrm{Kom}(\mathcal{A})$ (which descends to $D(\mathcal{A})$), and we can specify the class of distinguished triangles using cones:

Definition 3.2. A triangle

$$A_1 \to A_2 \to A_3 \to A_1[1]$$

in $D(\mathcal{A})$ is called *distinguished* if it is isomorphic in $D(\mathcal{A})$ to a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{\tau} C(f) \xrightarrow{\pi} A[1]$$

for a morphism of complexes f (cfr. remark 2.4).

Remark 3.3. The formulation of the octahedral axiom for the derived category of an abelian category becomes an axiom about cones of morphisms.

4. Derived functors

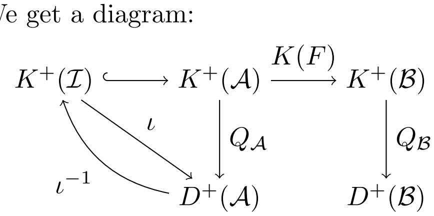
One of the main motivations for the development of the derived category is to get a better understanding of the behaviour of functors. A functor on the abelian category need not be exact. If we try to extend that naively to the derived category, we run into problems: the first tentative is to make F act on complexes componentwise. That seems to be a good choice since homotopic complexes are sent to homotopic complexes if we define an extension $K(F) : K(\mathcal{A}) \to K(B)$ in this way. However, if F is not an exact functor of abelian categories, it's hard to extend it with this procedure to the derived categries. In fact an acyclic complex $A^\bullet$ (i.e. an object which is zero in $D(\mathcal{A})$), may have an image $K(F)(A^\bullet)$ which is not acyclic. We will need to develop some techniques to extend functors that are not exact to the derived category. The result of this process will be called the derived functor RF of the functor F. To guarantee the existence of a derived functor, we need to assume some sort of exactness to start with. The construction is symmetric in the case of left or right exact functors, we will present in more detail the construction of the right derived functor of a left exact functor, since for this construction we need injective objects, and those are abundant in the category of coherent sheaves which is our ultimate goal.

Definition 4.1. An abelian category $\mathcal{A}$ is said to *have enough injectives* if for all objects A in $\mathcal{A}$ there exists an injection $A \to I$, where I is an injective object of $\mathcal{A}$ (recall that I being injective means that the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact).

If $\mathcal{A}$ contains enough injectives, every object in $\mathcal{A}$ admits an injective resolution. We have already remarked that this means that any object is quasi isomorphic to a complex of injective objects (cfr. rem. 1.3). More in general, denote by $K^+(\mathcal{A})$ the category of bounded from below complexes modulo homotopy, i.e. the full additive subcategory of $K(\mathcal{A})$ whose objects are complexes A^i with $A^i = 0$ for $i << 0$. Then we have:

Proposition 4.2. Let $\mathcal{I} \subset \mathcal{A}$ be the full subcategory of all injectives of an abelian category $\mathcal{A}$. Consider its homotopy category $K^+(\mathcal{I})$. If $\mathcal{A}$ has enough injectives, then for all objects $A^\bullet$ in $K^+(\mathcal{A})$ there exists $I^\bullet$ in $K^+(\mathcal{I})$, and a quasi-isomorphism $A^\bullet \to I^\bullet$. The composition of the inclusion $\mathcal{I} \subset \mathcal{A}$ with the natural functor $Q_{\mathcal{A}} : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ gives a natural exact functor $\iota : K^+(\mathcal{I}) \to D^+(\mathcal{A})$. Under these assumptions, this functor is an equivalence.

Now, consider a left exact functor F between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Assume that $\mathcal{A}$ has enough injectives. By ι^{-1} we denote a quasi-inverse of the equivalence ι given above, ι^{-1} consists in choosing a complex of injective objects quasi-isomorphic to a given complex. We get a diagram:



where $K(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ denotes the extension of F to homotopy categories.

Definition 4.3. Let F be a left exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, and assume that $\mathcal{A}$ has enough injectives. The *right derived functor* of F is the functor:

$$RF = Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1} : D^+(\mathcal{A}) \to D^+(\mathcal{B}). \tag{4.1}$$

Then for any complex $A^\bullet \in D^+(\mathcal{A})$ we define

$$R^iF(A^\bullet) = H^i(RF(A^\bullet)).$$

The induced additive functors $R^iF : \mathcal{A} \to \mathcal{B}$ are the *higher derived functors* of F.

An example: The derived functor of Hom. Consider an object $A \in \mathcal{A}$ and the left exact functor $\mathrm{Hom}\,(A, \ )$ from $\mathcal{A}$ to the category of abelian groups. If $\mathcal{A}$ contains enough injectives, set

$$\mathrm{Ext}^i\,(A, \ ) = H^i \circ R\mathrm{Hom}\,(A, \ )\,.$$

As an example, we will compute these objects and prove the following:

Proposition 4.4. In the above setting, suppose $B \in \mathcal{A}$. Then we have natural isomorphisms

$$\mathrm{Ext}^i\,(A, B) \simeq \mathrm{Hom}_{D(\mathcal{A})}\,(A, B[i])$$

(regard A and B as complexes concentrated in degree zero).

Proof. First, compute $R\mathrm{Hom}\,(A, B)$. Using the definition, we choose an injective resolution for B, sai $\mathcal{I}^\bullet$. This complex lives in $K^+(\mathcal{I})$, which is naturally embedded in $K^+(\mathcal{A})$. Then we apply the functor $\mathrm{Hom}\,(A, \ )$ to $\mathcal{I}^\bullet$ and get a complex $(\mathrm{Hom}\,(A, I^i))_{i\in\mathbb{Z}}$. The object $R\mathrm{Hom}\,(A, B)$ is isomorphic to this complex in $D(\mathcal{A})$, therefore $\mathrm{Ext}^i\,(A, B)$ is computed by the cohomology of $(\mathrm{Hom}\,(A, I^i))_{i\in\mathbb{Z}}$. Consider the diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle f} & & & & \\
\dots & \longrightarrow & I^{i-1} & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow & \dots
\end{array}$$

Then f defines a morphism of complexes precisely if it is a cycle, i.e. it lies in the kernel of $\mathrm{Hom}\,(A, I^i) \to \mathrm{Hom}\,(A, I^{i+1})$. Moreover, f lies in the image of $\mathrm{Hom}\,(A, I^{i-1}) \to \mathrm{Hom}\,(A, I^i)$ if and only if f is nullhomotopic via the homotopy g. This shows that $\mathrm{Ext}^i\,(A, B) \simeq \mathrm{Hom}_{K(\mathcal{A})}\,(A, \mathcal{I}^\bullet[i])$. Since $\mathcal{I}^\bullet$ is a complex of injectives, we have (see [2, Lemma 2.39]) $\mathrm{Hom}_{K(\mathcal{A})}\,(A, \mathcal{I}^\bullet[i]) \simeq \mathrm{Hom}_{D(\mathcal{A})}\,(A, \mathcal{I}^\bullet[i])$. But $B[i]$ and $\mathcal{I}^\bullet[i]$ are isomorphic in $D(\mathcal{A})$, so we proved the proposition. □

5. Examples

In the next examples we will illustrate the proposition above.

Example 1. Let A and B be two objects in an abelian category $\mathcal{A}$ with enough injectives. Suppose furthermore that there exists a nontrivial extension $0 \to B \to C \to A \to 0$ of B by A. Regard A and B as complexes in $D(\mathcal{A})$ concentrated in degree zero. The only morphism of complexes between A and $B[1]$ is the zero morphism, the cone above that is $C(0) = A[1] \oplus B[1]$. Hence we get a triangle $A \xrightarrow{0} B[1] \to A[1] \oplus B[1]$ which, after an application of TR2, becomes the triangle

$$B \to A \oplus B \to A \xrightarrow{0} B[1]$$

which corresponds to the trivial extension of A by B (cfr. [2, Ex. 1.38]). On the other hand, Prop. 4.4 predicts the existence of a map $A \to B[1]$ for any nontrivial extension $0 \to A \to C \to B \to 0$. Indeed, denote by $E^\bullet$ the complex $\dots \to 0 \to B \to C \to 0 \to \dots$ where C is in position 0, then $E^\bullet$ is a resolution of A and there exists a quasi isomorphism $\varepsilon : E^\bullet \to A$ (cfr. Ex. 1.3). Moreover, there is a morphism of complexes $E^\bullet \xrightarrow{p} B[1]$ given by the identity map in position -1, and the zero map elsewhere. Together, these maps give a roof:

$$A \xleftarrow{\ \varepsilon\ } E^{\bullet} \xrightarrow{\ p\ } B[1]$$

which is a morphism in $\operatorname{Hom}_{D(\mathcal{A})}(A, B[1])$. The quasi-isomorphism ε does not have an inverse in $K(\mathcal{A})$: indeed, such an inverse would give an isomorphism $A \to C/B$ which would split the extension.

Example 2. Let's focus on a more geometric example, we will illutrate how the language of derived categories provides more information about the functor $\mathcal{H}om(\ \ ,\mathcal{O})$ than just computing the dual of a sheaf. Consider the point P in $\mathbb{P}^2$, and pick a free resolution of its ideal sheaf with two linear forms:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{I}_P \to 0 \tag{5.1}$$

corresponding to the local situation (say that $P = (0:0:1)$ and let $S = \mathbb{C}[x_0, x_1]$):

$$0 \to S \xrightarrow{(\cdot x_1, \cdot x_0)} S \oplus S \xrightarrow{\cdot x_0 - \cdot x_1} (x_0, x_1) \to 0$$

We are interested in dualizing the sequence (5.1). We can compute the duals of the objects using the ideal sheaf sequence:

$$0 \to \mathcal{I}_P \to \mathcal{O}_{\mathbb{P}^2} \to k(P) \to 0.$$

Dualizing this, we get

$$0 = \mathcal{H}om(k(P), \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{H}om(\mathcal{I}_P, \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{E}\mathrm{xt}^1(k(P), \mathcal{O}_{\mathbb{P}^2}) = 0$$

$$0 \to \mathcal{E}\mathrm{xt}^1(\mathcal{I}_P, \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{E}\mathrm{xt}^2(k(P), \mathcal{O}_{\mathbb{P}^2}) \to 0$$

where we obtain the vanishing of $\mathcal{E}\mathrm{xt}^1(k(P), \mathcal{O}_{\mathbb{P}^2})$ in the following way: it is supported at P, since its stalks are zero at every other point. Hence, it coincides with a skyscraper sheaf at P of the ring of its global sections $\Gamma(\mathcal{E}\mathrm{xt}^1(k(P), \mathcal{O}_{\mathbb{P}^2}))$. By [4, Prop. III 6.9] and since $k(P)$ is a skyscraper sheaf, these global sections are computed by the group $\operatorname{Ext}^1(k(P), \mathcal{O}_{\mathbb{P}^2}) \simeq H^1(\mathbb{P}^2, k(P)) = 0$ (by Grothendieck vanishing, for example). Similar arguments allow to conclude $\mathcal{H}om(k(P), \mathcal{O}_{\mathbb{P}^2}) = 0$ and $\mathcal{E}\mathrm{xt}^2(k(P), \mathcal{O}_{\mathbb{P}^2}) = k(P)$. This gives us $\mathcal{H}om(\mathcal{I}_P, \mathcal{O}_{\mathbb{P}^2}) \simeq \mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{E}\mathrm{xt}^1(\mathcal{I}_P, \mathcal{O}_{\mathbb{P}^2}) \simeq k(P)$. Then, the dual of the free resolution (5.1) is the sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(2) \to k(P) \to 0.$$

As in the previous example, this sequence defines a roof in $\operatorname{Hom}_{D(\mathcal{A})}(k(P), \mathcal{O}_{\mathbb{P}^2}[2]) \simeq \operatorname{Ext}^2(k(P), \mathcal{O}_{\mathbb{P}^2})$, it is not given by any morphism of complexes. It is also noteworthy that knowing this extension class bears more information than just knowing the dual of every object in the resolution, since we have a way to keep track of the maps in the resolution.

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Department of Mathematics, University of Utah, Salt Lake City, UT