

# Manifolds, Higher Categories and Topological Field Theories

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- 1 "Topological (Quantum) Field Theories", i.e. tensor functors

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- 1 "Topological (Quantum) Field Theories", i.e. tensor functors

$$Z : \mathit{Bord}_n \rightarrow \mathcal{C}$$

- 2 Sheaves on  $\mathit{Man}_n$ , the category of  $n$ -dimensional manifolds (with morphisms open embeddings)

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For a symmetric monoidal  $(\infty, n)$ -category  $\mathcal{C}$ , an  $n$ -dimensional  $\mathcal{C}$ -valued TQFT is a symmetric monoidal functor

$$Z : Bord_n \rightarrow \mathcal{C}.$$

## Theorem ("Cobordism Hypothesis" (Lurie))

*Let  $\mathcal{C}$  be a symmetric monoidal  $(\infty, n)$ -category. Then there is an equivalence*

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## Remark

- 1 This theorem indicates an intimate connection between the theory of manifolds and higher categories. It was inspirational in our work, but we don't make use of it.
- 2 In practice, it is difficult to produce examples of TQFT's using this theorem.

Let  $Man_n$  be the (quasi-)category of framed  $n$ -dimensional manifolds and framed embeddings. It has a natural Grothendieck topology. Thus, we can consider local invariants of  $n$ -manifolds as sheaves on  $Man_n$ .

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Idea: we can improve matters by considering additional structure on manifolds; namely, the notion of transversality.

## Definition

Let  $n\text{Man}^{\text{th}}$  denote the  $(\infty, 1)$ -category whose objects are pairs

$$(M, S \subset M)$$

where  $M$  is a framed  $n$ -dimensional manifold and  $S$  is a compact subcomplex of  $M$  stratified by submanifolds, such that  $M$  admits a cover by basic opens, each of which has a non-empty intersection with  $S$ .

A morphism  $(M, S) \rightarrow (N, T)$  is a framed embedding  $f : M \hookrightarrow N$  together with a path  $\gamma$  from  $f(S)$  to  $T$  in the space of stratified submanifolds of  $N$ .

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Let  $n\text{Man}_0^{\text{th}} \subset n\text{Man}^{\text{th}}$  be the subcategory consisting of morphisms such that the corresponding path of stratified submanifolds does not decrease the number of components of strata.

## Definition

*A sheaf with transversality on  $n\text{Man}$  is a functor*

$$\Psi : n\text{Man}^{\text{th}} \rightarrow \text{Spaces}$$

*which restricts to a sheaf on  $n\text{Man}_0^{\text{th}}$*

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## Remark

*Gromov's h-principle gives a geometric way of producing sheaves with transversality. Namely, a sheaf valued in the (ordinary) category of topological spaces*

$$\Phi : n\text{Man} \rightarrow \text{Top}$$

*together with a notion of transversality which satisfies the h-principle produces a sheaf with transversality.*

*The requirement for objects on  $n\text{Man}^{\text{th}}$  to have covers by basics intersecting the subcomplex is related to the failure of a sheaf satisfying the h-principle to be a homotopy sheaf.*

There is a natural functor

$$\text{oblv} : \underline{n\text{Man}}^{\text{th}} \rightarrow n\text{Man}^{\text{th}}$$

Given a sheaf with transversality on  $n\text{Man}$

$$\Psi : n\text{Man}^{\text{th}} \rightarrow \text{Spaces},$$

we obtain a sheaf with transversality on  $\underline{n\text{Man}}$ , by pulling back along  $\text{oblv}$ .

By the theorem David described, this gives a functor

$$\rho : \{\text{th-sheaves on } n\text{Man}\} \rightarrow \{(\infty, n)\text{-categories}\}.$$

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## Question

*How close is  $\rho$  to being an equivalence?*

Let  $\mathcal{C}$  be a 2-category. A morphism

$$f : x \rightarrow y$$

has a left adjoint  $f^L : y \rightarrow x$  if there are 2-morphisms

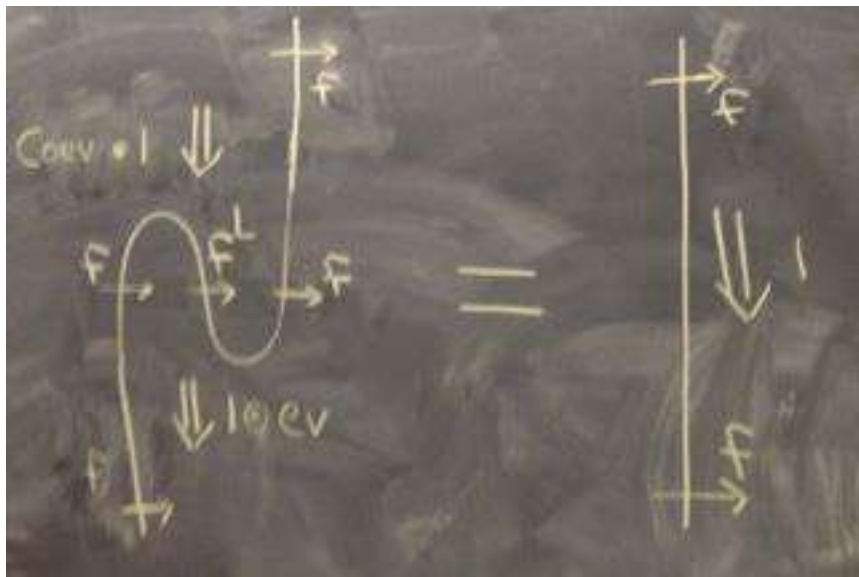
$$ev : f^L \circ f \rightarrow id_x \quad \text{and} \quad coev : id_y \rightarrow f \circ f^L$$

satisfying Zorro's lemma; i.e., the composite

$$f \xrightarrow{coev \otimes id} f \circ f^L \circ f \xrightarrow{id \otimes ev} f$$

is equal to the identity.

Similarly for right adjoints. In the same way, we have the notion of left and right adjoints for  $k$ -morphisms in an  $n$ -category (where  $0 < k < n$ ).



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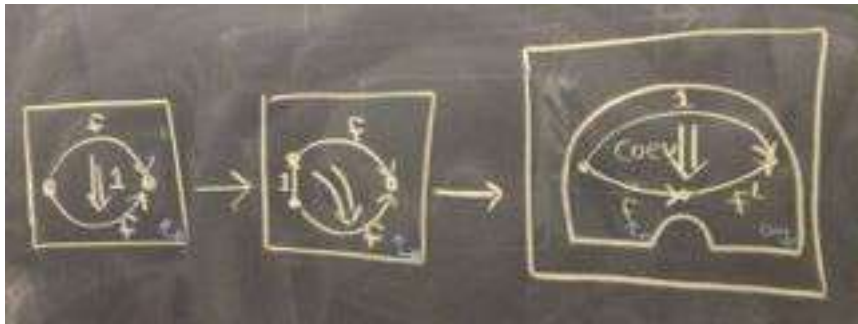
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## Theorem 80 (Ayala, R.)

*The functor*

$$\rho : \{\text{sheaves on } n\text{Man}\} \rightarrow \{(\infty, n)\text{-categories with adjoints}\}$$

*is an equivalence.*

There is a natural forgetful functor  $nMan^{\text{th}} \rightarrow nMan$   
Given a sheaf with transversality

$$\Psi : nMan^{\text{th}} \rightarrow Spaces,$$

we obtain the functor

$$\int \Psi : nMan \rightarrow Spaces$$

by left Kan extension along the forgetful functor. In this way, we obtain from  $\Psi$  invariants of manifolds.

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- When  $n = 1$ ,  $\int_{S^1} \Psi \simeq HH(\rho(\Psi))$ .
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- Should be related to blob homology defined by Morrison and Walker.

For a  $k$ -manifold  $M^k$ , let

$$\int_M^{\text{cat}} \Psi$$

be the sheaf with transversality on  $(n - k)\text{Man}$  given by

$$\int_M^{\text{cat}} \Psi(N, T) = \Psi(N \times M, T \times M).$$

## Proposition

*There is an isomorphism*

$$\int_M \Psi \simeq B(\rho(\int_M^{\text{cat}} \Psi)).$$

## Theorem (Ayala, R.)

The functors  $\int^{cat} \Psi$  give an  $n$ -dimensional TQFT

$$Z_\Psi : Bord_n \rightarrow Cat_{\infty,n}^{corr}.$$

$Cat_{\infty,n}^{corr}$  is the  $(\infty, n+1)$  category of correspondences of  $(\infty, n)$ -categories.

For  $n=1$ ,  $Cat_{\infty,1}^{corr}$  has objects  $(\infty, 1)$ -categories and

$$Cat_{\infty,1}^{corr}(\mathcal{C}, \mathcal{D}) := \text{Func}(\mathcal{C} \times \mathcal{D}^{op}, \text{Spaces}).$$

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## Remark

The  $n$ -dualizable objects of  $Cat_{\infty,n}^{corr}$  are exactly categories with adjoints.

# Picture of 1-Morphisms

