Mathematics. — *Concerning the homotopy groups of the components of the mapping space* Y^{S^p}. By SZE-TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

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1. Introduction.

Let Y be a connected compact absolute neighbourhood retract, [8, p. 58]. Let us denote by G^p the mapping space Y^{S^p}, which consists of the totality of the mappings of a p-sphere S^p into Y. Let $x_0 \in S^p$, $y_0 \in Y$ be given points, and denote by F^p the closed subset of G^p, which consists of the totality of the mappings $f \in G^p$ with $f(x_0) = y_0$. Let $\pi^p(Y)$ denote the p-th homotopy group of Y with x_0, y_0 as base points. Let F^p_α be the component of F^p which consists of the totality of the representatives of the element $\alpha \in \pi^p(Y)$. Since Y is arcwise connected, each component of G^p contains at least one component of F^p. Let G^p_α be the component of G^p which contains F^p_α.

The fundamental group of the component G^p_0 was first studied by M. ABE, [1]; the higher homotopy groups of G^p_0 were determined by the author in terms of those of Y, [6, § 10], during the early months of 1946. At that time, practically nothing was known concerning the homotopy properties of the component G^p_α, $\alpha \neq 0$. Most recently, it appears the work of G. W. WHITEHEAD, [10], in which an example has been given to show that G^p_0 and G^p_α are in general of the different homotopy types if $\alpha \neq 0$. In the present note, two isomorphisms will be given in § 2 regarding the structures of the homotopy groups of the component G^p_α, which indicate the close relation between WHITEHEAD products, [11], and the homotopy groups of G^p_α. They have been used to determine the homotopy groups of G^p_α in terms of those of Y for a certain number of special cases, of which the most interesting one is $Y = S^2$.

2. General theorems.

For each pair of elements $\alpha \in \pi^p(Y)$, $\beta \in \pi^q(Y)$, let us denote by $[\alpha, \beta] \in \pi^{p+q-1}(Y)$ the WHITEHEAD product of α and β, [11]. For $q > 1$ and a given $\alpha \in \pi^p(Y)$, the transformation $\beta \to [\alpha, \beta]$ is a homomorphism of $\pi^q(Y)$ into $\pi^{p+q-1}(Y)$, denoted by ϱ_α. Let K^q_α and J^{p+q-1}_α denote the kernel and the image of ϱ_α respectively. Choose $a \in F^p_\alpha$ as the base point for all the homotopy groups of F^p_α and G^p_α. There is a natural homomorphism μ: $\pi^q(F^p_\alpha) \to \pi^q(G^p_\alpha)$ induced by the injection mapping

$F_\alpha^p \to G_\alpha^p$. Let P_α^q be the image of $\pi^q(F_\alpha^p)$ under μ. For the higher homotopy groups of G_α^p, the following theorem is a direct consequence of the theorems of G. W. WHITEHEAD, [10, (2.4) and (3.2)].

Theorem 2.1. *For each $q > 1$ and $\alpha \in \pi^p(Y)$, we have*

(2.11) $$\pi^q(G_\alpha^p)/P_\alpha^q \approx K_\alpha^q,$$

(2.12) $$\pi^{p+q}(Y)/J_\alpha^{p+q} \approx P_\alpha^q.$$

According to S. EILENBERG, [2], the fundamental group $\pi^1(Y)$ is a group of operators for the group $\pi^p(Y)$ with the unit element of $\pi^1(Y)$ as unit operator. Let Q_α^1 denote the subgroup of $\pi^1(Y)$, wich consists of the totality of the elements $\omega \in \pi^1(Y)$ such that $\omega(\alpha) = \alpha$. For the fundamental group $\pi^1(G_\alpha^p)$, we shall prove the following theorem.

Theorem 2.2. *For each $\alpha \in \pi^p(Y)$ we have*

(2.21) $$\pi^1(G_\alpha^p)/P_\alpha^1 \approx Q_\alpha^1,$$

(2.22) $$\pi^{p+1}(Y)/J_\alpha^{p+1} \approx P_\alpha^1.$$

[Proof] Let τ denote the projection of G_α^p into Y, defined by $\tau(f) = f(x_0)$ for each $f \in G_\alpha^p$. Then τ is a fibre mapping, [10, (2.1)], and $\tau(F_\alpha^p) = y_0$. τ induces a homomorphism of $\pi^1(G_\alpha^p)$ into $\pi^1(Y)$ still denoted by τ.

Let I denote the closed interval $(0, 1)$ of real numbers. Let $\xi \in \pi^1(G_\alpha^p)$ be represented by a mapping $\Phi: I \to G_\alpha^p$ such that $\Phi(0) = a = \Phi(1)$. Let $\psi = \tau\Phi$, then $\psi(0) = y_0 = \psi(1)$. ψ represents an element $\omega \in \pi^1(Y)$, and $\omega = \tau\xi$. Φ defines a homotopy $f_t: S^p \to Y$ by means of the relation $\Phi(t) = f_t$ for each $t \in I$. Since $f_0 = a = f_1$ and $f_t(x_0) = \psi(t)$ for each $t \in I$, it follows that $\omega(\alpha) = \alpha$. Hence $\omega \in Q_\alpha^1$. Conversely, suppose $\omega \in Q_\alpha^1$ be an arbitrary element, represented by a mapping $\psi: I \to Y$ with $\psi(0) = y_0 = \psi(1)$ From the Covering Homotopy Theorem, [7], it follows that there exists a mapping $\Phi: I \to G_\alpha^p$ such that $\tau\Phi = \psi$ and $\Phi(0) = a$. Φ defines a homotopy $f_t: S^p \to Y$ by the relation $\Phi(t) = f_t$ for each $t \in I$. Since $f_0 = a$ and $f_t(x_0) = \psi(t)$, f_1 represents the element $\omega(\alpha) \in \pi^p(Y)$. Hence $f_1 \in F_\alpha^p$, for $\omega \in Q_\alpha^1$. From the arcwise connectedness, it follows that there exists a homotopy $\Phi_t: I \to G_\alpha^p$, $0 \leqslant t \leqslant 1$, such that $\Phi_0 = \Phi$ $\Phi_t(0) = a$, $\Phi_t(1) \in F_\alpha^p$, and $\Phi_1(1) = a$. Let $\psi_t = \tau\Phi_t$; then $\psi_t: I \to Y$ $0 \leqslant t \leqslant 1$, is a homotopy such that $\psi_0 = \psi$ and $\psi_t(0) = y_0 = \psi_t(1)$ for each $t \in I$. Hence ψ_1 is also a representative of ω. Φ_1 represents an element $\xi \in \pi^1(G_\alpha^p)$ and $\tau\xi = \omega$. Hence, we have proved that the image of $\pi^1(G_\alpha^p)$ under the homomorphism τ is Q_α^1.

It is trivial that P_α^1 is contained in the kernel of τ. Conversely, suppose $\xi \in \pi^1(G_\alpha^p)$ be an arbitrary element of the kernel of τ, represented by a mapping $\Phi: S^1 \to G_\alpha^p$ with $\Phi(z_0) = a$, z_0 being a given point of S^1. Then the mapping $\psi=\tau\Phi$ represents the unit element of $\pi^1(Y)$; hence there exists a homotopy $\psi_t: S^1 \to Y$, $0 \leqslant t \leqslant 1$, such that $\psi_0=\psi$, $\psi_1(S^1)=y_0$, and $\psi_t(z_0)=y_0$ for each $0 \leqslant t \leqslant 1$. From the Covering Homotopy Theorem, it follows that there exists a homotopy $\Phi_t: S^1 \to G^p_\alpha$ such that $\Phi_0=\Phi$, $\tau\Phi_t=\psi_t$ and $\Phi_t(z_0)=a$ for each $0 \leqslant t \leqslant 1$. Since $\Phi(S^1) \subset F^p_\alpha$, we obtain $\xi \in P^1_\alpha$. Hence (2. 21) follows.

The isomorphism (2. 22) can be proved as (2. 12). Q. E. D.

For the use of the sequel, we mention the results of the author, [6, § 10], and M. ABE, [1], for the component G^p_0.

(2. 3) *If $q>1$, $\pi^q(G^p_0)$ is isomorphic with the direct sum of $\pi^{p+q}(Y)$ and $\pi^q(Y)$.*

(2. 4) *If Y is $(p+1)$-simple, [2], $\pi^1(G^p_0)$ is isomorphic with the direct product of $\pi^{p+1}(Y)$ and $\pi^1(Y)$.*

Theorem 2. 5. *If $\alpha+\beta=0$, then the components G^p_α and G^p_β are homeomorphic.*

[Proof] Let $\theta: S^p \to S^p$ be a homeomorphism of S^p which reverses orientation and has x_0 as a fixed point. Then, a homeomorphism h of G^p_α onto G^p_β is given by $h(f)=f\theta$ for each $f \in G^p_\alpha$. Q. E. D.

3. Spaces with continuous multiplication.

Theorem 3. 1. *If Y admits a continuous multiplication with a two-sided identity e (e.g., if Y is a topological group), then G^p_0 and G^p_α are of the same homotopy type for each $\alpha \in \pi^p(Y)$.*

[Proof] According to G. W. WHITEHEAD, [10, p. 464], it remains to prove that there exists a mapping $\lambda: Y \to G^p_\alpha$ such that $\tau\lambda$ is the identity, where τ denotes the projection of G^p_α onto Y defined by $\tau(f)=f(x_0)$ for each $f \in G^p_\alpha$. For each $y \in Y$, let $\lambda_0 y \in G^p_0$ be the constant mapping of S^p into y. Hence λ_0 is a mapping of Y into G^p_0 such that $\tau\lambda_0$ is the identity. From the arcwise connectedness of Y, it follows that there exists a mapping $\Phi \in G^p_\alpha$ such that $\Phi(x_0)=e$. Define $\lambda: Y \to G^p$ by

$$\lambda y(x)=\Phi(x)\cdot\lambda_0 y(x), \qquad (y \in Y,\ x \in S^p).$$

Since $\lambda e=\Phi \in G^p_\alpha$ and Y is arcwise connected, it follows that $\lambda(Y) \subset G^p_\alpha$. Further, $\tau(\lambda y)=e\,.\,y=y$; hence $\tau\lambda$ is the identity. The proof has been completed. Q. E. D.

Corollary 3. 2. *If $Y=S^r$, $(r=1, 3, 7)$, then for each $\alpha \in \pi^p(S^r)$ and each $q \geqslant 1$ we have*

$$\pi^q(G^p_\alpha) \approx \pi^{p+q}(S^r)+\pi^q(S^r).$$

4. The sphere S^r.

In the present paragraph, let Y be the r-sphere S^r. Since the cases $r=1, 3, 7$ have been solved in (3. 2) and the case $r=2$ will be treated in § 5, we may suppose that $r>3$.

If $p<r$ or $p=r+2$, then the space G^p is connected and our proplem reduces to (2. 3) and (2. 4). It remains to investigate the case $p \geqq r$ and $p \neq r+2$.

Theorem 4. 1. *For each* $\alpha \in \pi^p(S^r)$, *we have*

(4. 11) $\pi^q(G^p_\alpha) \approx \pi^{p+q}(S^r), \quad (q<r-1);$

(4. 12) $\pi^{r-1}(G^p_\alpha) \approx \pi^{p+r-1}(S^r)/J^{p+r-1}_\alpha;$

(4. 13) $\pi^{r+1}(G^p_\alpha)$ *has a subgroup* $P^{p+r+1}_\alpha \approx \pi^{p+r+1}(S^r);$

(4. 14) $\pi^{r+2}(G^p_\alpha) \approx \pi^{p+r+2}(S^r)/J^{p+r+2}_\alpha.$

[Proof] These are immediate consequences of the general theorems (2. 1), (2. 2), and the facts $\pi^{r+2}(S^r)=0$ and $\pi^q(S^r)=0$ if $q<r$.

Lemma 4. 2. *If r is any positive even integer and α, β are arbitrary elements of $\pi^r(S^r)$ both different from zero, then $[\alpha, \beta] \in \pi^{2r-1}(S^r)$ is also different from zero.*

[Proof] Suppose, $[\alpha, \beta]=0$. Then by a corollary of G. W. WHITEHEAD, (10, p. 467), there exists a mapping $f: S^r \times S^r \to S^r$ of the type (α, β). It follows from a theorem of H. HOPF, [5, p. 431], that there exists an element of $\pi^{2r+1}(S^{r+1})$ with HOPF invariant ab, where a and b are the degrees of α and β. Since $r+1$ is odd, it follows that $ab=0$. Hence, at least one of the elements α, β must be zero. Q. E. D.

Following ALLEXANDROFF—HOPF, we shall denote by $\mathfrak{G}_0=\mathfrak{G}$ the infinite cyclic group, and by $\mathfrak{G}_m$ the finite cyclic group of the order m.

Theorem 4. 3. *If r is even and $\alpha \in \pi^r(S^r)$ is different from zero, then $K^r_\alpha=0$ and $J^{2r-1}_\alpha=\mathfrak{G}$.*

[Proof] This is a consequence of Lemma 4. 2.

Theorem 4. 4. *If r is odd, then*

(4. 41) $K^r_\alpha=\mathfrak{G}$ *for each* $\alpha \in \pi^r(S^r)$;

(4. 42) $J^{2r-1}_\alpha=\mathfrak{G}_2$ *if* $\pi^{2r+1}(S^{r+1})$ *has no element with Hopf invariant 1 and α is of odd degree, and* $J^{2r-1}_\alpha=0$ *otherwise.*

[Proof] If $\pi^{2r+1}(S^{r+1})$ has an element of HOPF invariant 1, then by Theorem (3. 12) of G. W. WHITEHEAD, [10], we have $[\alpha, \beta]=0$ for each α, $\beta \in \pi^r(S^r)$; hence $K^r_\alpha=\mathfrak{G}$ and $J^{2r-1}_\alpha=0$. If $\pi^{2r+1}(S^{r+1})$ has no element with HOPF invariant 1, then $[\alpha, \beta]=0$ if and only if at least one of the elements α, β is of even degree. If α is of even degree, then $[\alpha, \beta]=0$ for each $\beta \in \pi^r(S^r)$; hence $K^r_\alpha = \mathfrak{G}$ and $J^{2r-1}_\alpha = 0$. If α is of odd degree, then K^r_α consists of the elements of even degree; hence $K^r_\alpha = \mathfrak{G}$, $J^{2r-1}_\alpha = \mathfrak{G}_2$. Q. E. D.

H. FREUDENTHAL, [4], announced without proof the existence of the elements of $\pi^{2r+1}(S^{r+1})$ with HOPF invariant 1 for every odd r. See also G. W. WHITEHAED, [9].

From (4. 3) and (4. 4), the following two theorems can be deduced easily.

Theorem 4. 5. *If r is even, then for each $\alpha \in \pi^r(S^r)$ different from zero, we have*

(4. 51) $\pi^{r-1}(G^r_\alpha) \approx \pi^{2r-1}(S^r)/J^{2r-1}_\alpha$, $J^{2r-1}_\alpha \approx \mathfrak{G}$;

(4. 52) $\pi^r(G^r_\alpha) \approx \pi^{2r}(S^r)/J^{2r}_\alpha$.

Theorem 4. 6. *If r is odd, then for each $\alpha \in \pi^r(S^r)$, we have*

(4. 61) $\pi^{r-1}(G^r_\alpha) \approx \pi^{2r-1}(S^r)/J^{2r-1}_\alpha$, *where* $J^{2r-1}_\alpha = 0$ *or* $\mathfrak{G}_2$ *as described in* (4. 42);

(4. 62) $\pi^r(G^r_\alpha)/P^r_\alpha \approx \mathfrak{G}$.

5. The sphere S^2.

Throughout this paragraph, let Y be the 2-sphere S^2, and let $\iota \in \pi^2(S^2)$ denote the element represented by the identity of S^2.

Lemma 5. 1. *The generator γ of the group $\pi^3(S^2)$ can be so chosen that $[\iota, \iota]=2\gamma$.*

[Proof] Let γ^* be an arbitrary generator of $\pi^3(S^2)$; then we have $[\iota, \iota]=\delta m \gamma^*$, where $\delta = \pm 1$ and $m>0$. Let D be the subgroup of $\pi^3(S^2)$ generated by $[\iota, \iota]$, then we have $\pi^3(S^2)/D$ is isomorphic with $\mathfrak{G}_m$. On the other hand, let E denote the Einhängung operation of H. FREUDENTHAL, [3]; then E is a homomorphism of $\pi^3(S^2)$ onto $\pi^4(S^2)$. By a result of G. W. WHITEHEAD, [10, p. 470], the kernel of E is the subgroup D; hence $\pi^3(S^2)/D$ is isomorphic with $\mathfrak{G}_2$. Then it follows that $m=2$. Choosing $\gamma = \delta\gamma^*$ as new generator of $\pi^3(S^2)$, we have $[\iota, \iota]=2\gamma$. Q. E. D.

Lemma 5. 2. $[\gamma, \iota]=0$.

[Proof] This is contained in the second example of G. W. WHITEHEAD, [10, p. 474].

Theorem 5.3. *If* $Y=S^2$ *and* $\alpha\in\pi^2(S^2)$ *is different from zero, then*

(5.31) $\pi^1(G^2_\alpha)=\mathfrak{G}_{2m}$, *where* $m>0$ *is determined by* $\alpha=\pm m\iota$;

(5.32) $\pi^2(G^2_\alpha)=\mathfrak{G}_2$;

(5.33) $\pi^3(G^2_\alpha)=\mathfrak{G}$;

(5.34) $\pi^4(G^2_\alpha)/P^4_\alpha\approx\mathfrak{G}_2$, $P^4_\alpha\approx\pi^6(S^2)$;

(5.35) $\pi^5(G^2_\alpha)\approx\pi^7(S^2)/J^7_\alpha$.

[Proof] (5.31) Since $Q^1_\alpha\subset\pi^1(S^2)=0$, it follows from (2.2) that $\pi^1(G^2_\alpha)\approx\pi^3(S^2)/J^3_\alpha$. Since $\alpha=\pm m\iota$, we have $[\alpha,\iota]=\pm 2m\gamma$; therefore, J^3_α is the subgroup of $\pi^3(S^2)$ generated by $2m\gamma$. Hence $\pi^3(S^2)/J^3_\alpha\approx\mathfrak{G}_{2m}$.

(5.32) $K^2_\alpha=0$, by (5.1); $J^4_\alpha=0$, by (5.2). Then, by (2.1),

$$\pi^2(G^2_\alpha)\approx\pi^4(S^2)\approx\mathfrak{G}_2.$$

(5.33) Since $J^4_\alpha=0$, we get $K^3_\alpha=\pi^3(S^2)$; since $\pi^5(S^2)=0$, we get $P^3_\alpha=0$. Hence $\pi^3(G^2_\alpha)\approx\pi^3(S^2)\approx\mathfrak{G}$.

(5.34) Since $\pi^5(S^2)=0$, we have $K^4_\alpha=\pi^4(S^2)$ and $J^6_\alpha=0$.

Then (5.34) follows from (2.1).

(5.35) follows from $K^5_\alpha\subset\pi^5(S^2)=0$. Q. E. D.

It would be worthwhile to mention here that

$\pi^1(G^2_0)=\mathfrak{G}$, $\pi^2(G^2_0)=\mathfrak{G}+\mathfrak{G}_2$, $\pi^3(G^2_0)=\mathfrak{G}$,

$\pi^4(G^2_0)\approx\pi^6(S^2)+\mathfrak{G}_2$, $\pi^5(G^2_0)\approx\pi^7(S^2)$.

(5.31) yields the complete solution of the classification of the homotopy types of the components of G^2 as stated in the

Theorem 5.4. *If* $Y=S^2$, *the two different components* G^2_α, G^2_β *of* G^2 *are of the same homotopy type, if and only if* α *and* β *are negative to each other.*

[Proof] If $\alpha+\beta=0$, then G^2_α and G^2_β are of the same homotopy type by (2.5). Conversely, if G^2_α and G^2_β are of the same homotopy type, then by (5.31), the absolute values of the degrees of α and β are equal. Hence $\alpha+\beta=0$, for they are supposed to different. Q. E. D.

The following two theorems can be proved by the similar methods as used in the proof of (5.3).

Theorem 5.5. *If* $Y=S^2$ *and* $\alpha\in\pi^3(S^2)$ *be arbitrary element then we have*

(5.51) $\pi^1(G^3_\alpha)=\mathfrak{G}_2$;

(5.52) $\pi^2(G^3_\alpha)=\mathfrak{G}$;

(5.53) $\pi^3(G^3_\alpha)/P^3_\alpha\approx\mathfrak{G}$;

(5.54) $\pi^4(G^3_\alpha)$ has a subgroup $P^4_\alpha\approx\pi^7(S^2)$;

(5.55) $\pi^5(G^3_\alpha)\approx\pi^8(S^2)/J^8_\alpha$.

Theorem 5.6. *If* $Y = S^2$ *and* $\alpha \in \pi^4(S^2)$ *be arbitrary element, then we have*

(5.61) $\pi^1(G^4_\alpha) = 0$;

(5.62) $\pi^2(G^4_\alpha)/P^2_\alpha \approx \mathfrak{G}$;

(5.63) $\pi^4(G^4_\alpha)$ *has a subgroup* $P^4_\alpha \approx \pi^8(S^2)$:

(5.64) $\pi^5(G^4_\alpha) \approx \pi^9(S^2)/J^9_\alpha$.

For $Y = S^2$, G^5 is connected; hence our problem has been solved in (2.3) and (2.4). For $p > 5$, I know no more than the general theorems in § 2.

6. Aspherical spaces.

Y is said to be aspherical, if $\pi^p(Y) = 0$ for each $p > 1$. Then Y is p-simple for each $p > 1$.

Since G^p, $p > 1$, is connected, we deduce from (2.3) and (2.4) the following theorem.

Theorem 6.1. *If* Y *is aspherical and* $p > 1$, *then* G^p *is also aspherical and* $\pi^1(G^p) \approx \pi^1(Y)$.

From (2.1), (2.2) and the 2-symplicity, we deduce the following

Theorem 6.2. *If* Y *is aspherical and* $\alpha \in \pi^1(Y)$, *then* G^1_α *is also aspherical and* $\pi^1(G^1_\alpha) \approx \pi^1(Y)$.

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