EL_∞-algebras, Generalized Geometry, and Tensor Hierarchies

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Based on:

- arXiv:2106.00108 with Leron Borsten and Hyungrok Kim
- arXiv:1908.08086 with Lennart Schmidt

Motivation: Five Questions

1) What is the algebraic structure underlying Courant algebroids?
2) What is alg. structure underlying multisymplectic manifolds?
3) What are "good" curvatures for non-abelian gauge potentials?
4) What do Leibniz algebras integrate to?
5) What is a small cofibrant replacement for the operad $\mathcal{L}ie$?

First question

What is the algebraic structure underlying Courant algebroids?

Answers in the literature:

- Roytenberg (2002):
  An (exact) Courant algebroid is the symplectic dg-manifold
  $$\mathcal{V}_2 = T^*[2]T[1]M~, \quad \omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta^\mu~,$$
  $$Q = \{S, -\}~, \quad S = \xi^\mu p_\mu + \tfrac{1}{3!}\varpi_{\mu\nu\kappa}\xi^\mu\xi^\nu\xi^\kappa$$
  for M some manifold. Dorfman and Courant brackets:
  $$[X,Y]_D = \{QX, Y\}~, \quad [X,Y]_C = \tfrac{1}{2}\big(\{QX,Y\} - \{QY,X\}\big)$$
- These brackets fit into two structures:
  - $[-,-]_C$ part of L_∞-algebra, Getzler (2009), Zambon (2010)
  - $[-,-]_D$ part of dg-Leibniz algebra cf. Rogers (2011).
- Is there more to it?

Relevance of first question

This may be seem as a niche question, but:

- Courant algebroids underlie Hitchin's Generalized Geometry
- Application in supergravity: Generalized tangent bundle
- All generalized tangent bundles are symplectic L_∞-algebroids
- Dorfman bracket structure relevant in tensor hierarchies
- Currently relevant: Double and Exceptional Field Theory.

In order to further understand the above:
understand symplectic L_∞-algebroids!

How to construct "good" curvatures for non-abelian gauge potentials in presence of B-field?

Answers in the literature:

- Use Chern-Simons terms:

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]\ , \quad H = \mathrm{d}B + (A, \mathrm{d}A) + \tfrac{1}{3}(A,[A,A])$$

Bergshoeff et al. (1982), Chapline et al. (1983)

- This is at odds with the "conventional" non-abelian gerbes:

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]\ , \quad H = \mathrm{d}B - \tfrac{1}{3}(A,[A,A])$$

Breen/Messing (2001), Aschieri, Cantini, Jurco (2003)

- Sati, Schreiber (2009): adjust definition of curvatures

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]\ , \quad H = \mathrm{d}B + (A,F) - \tfrac{1}{3}(A,[A,A])$$

- Where does $(-,-)$ come from?

We need reasonable higher principal bundles with connections.

Physics:

- Heterotic supergravity
- Tensor hierarchies of gauged supergravity and EFT
- 6d superconformal field theories

Mathematics:

- Higher geometry would be much less beautiful.

Summary

All these questions:

1) Algebraic structure underlying symplectic L_∞-algebroids?
2) Algebraic structure underlying multisymplectic manifolds?
3) Algebraic structure underlying higher curvature forms?
4) Cofibrant replacement of $\mathcal{L}ie$?
5) How do you integrate Leibniz algebras?

have a simple, unifying answer:

EL_∞-algebras

Initial observations

Generalized Geometry:

- The Dorfman bracket is part of a hemistrict Lie 2-algebra.

Coupling B-field to non-abelian gauge potential:

- Additional algebraic structure is an alternator of Lie 2-algebra.

Gauged supergravity:

- Embedding tensor yields (weak) Lie 2-algebra

Mathematics literature:

- Roytenberg (2007): weak Lie 2-alg. or 2-term EL_∞-algebras
- Dehling (2017): weak Lie 3-alg. or 3-term EL_∞-algebras

Conclusions

We are looking for a weak generalization of L_∞-algebras, generalizing the 2- and 3-term EL_∞-algebras in the literature.

Key ingredients: Operads + Koszul duality

Sketch:

- Won't need much more than an intuitive understanding
- Useful framework for describing algebras and their relations
- Symmetric operad $\mathcal{O}$:
  - Abstract operations with n inputs and 1 output:

  - Composition prescription: equalities between "trees"
  - Also: unit and symmetric group action on inputs
- Algebras over $\mathcal{O}$: ops are multilinear maps on vector spaces
- Examples: $\mathcal{L}ie$, $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}eib$
- "Homotopy $\mathcal{O}$-algebras or $\mathcal{O}_\infty$-algebra is an algebra over the Koszul resolution of the Koszul-dual cooperad."

Recall: Chevalley–Eilenberg algebra of a Lie algebra $\mathfrak{g}$

- Graded vector space $V = \mathfrak{g}[1]^*$, coords. ξ^α, $|\xi^\alpha| = 1$.
- Vector field or differential on polynomial functions:

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha}~, \quad Q^2 = 0~, \quad |Q| = 1$$

- Lie bracket $[\tau_\alpha, \tau_\beta] = f^\gamma_{\alpha\beta} \tau_\gamma$, $\quad Q^2 = 0 \;\Leftrightarrow$ Jacobi identity

Generalize to Chevalley–Eilenberg algebra of L_∞-algebra:

- $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$
- Q most general with $Q^2 = 0$ and $|Q| = 1$
- Structure constants in Q: $\mu_i : \mathfrak{g}^{\wedge i} \to \mathfrak{g}$, $|\mu_i| = 2 - i$
- $Q^2 = 0 \;\Leftrightarrow$ homotopy Jacobi identities
- For $\mathfrak{g} = \oplus_{i \leq 0} \mathfrak{g}_i$: categorified Lie algebras

Operadic perspective:

- $\mathcal{L}ie$ has Koszul-dual $\mathcal{L}ie^! = \mathcal{C}om$
- Therefore:

$$\begin{array}{ccc} L_\infty\text{-algebras} & \leftrightarrow & \text{dg-com algebras} \\ \mu_i & \leftrightarrow & Q \end{array}$$

- semifree dg-$\mathcal{C}om$-algebra give homotopy $\mathcal{L}ie$-algebra.
- Similarly for
  - $\mathcal{A}ss^! = \mathcal{A}ss$: produces A_∞-algebras
  - $\mathcal{L}eib^! = \mathcal{Z}inb$: produces homotopy Leibniz algebras

$$\vdots$$

- Question: which operad produces weak L_∞-/EL_∞-algebras?

Hemistrict Lie 2-algebras: differential graded algebras $\mathfrak{L}$ with

$$\varepsilon_2 : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_2| = 0 , \quad \mathsf{alt} : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L} , \quad |\mathsf{alt}| = -1$$

Generalize, preserving differential compatibility: $h\mathcal{L}ie$-algebras

$h\mathcal{L}ie$-algebras

Graded vector space $\mathfrak{L}$ with

$$\varepsilon_1 \ : \ \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_1| = 1 , \quad \varepsilon_2^i \ : \ \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_2^i| = -i$$

such that

$$\begin{aligned}
\varepsilon_1(\varepsilon_1(x_1)) &= 0 , \\
\varepsilon_1(\varepsilon_2^i(x_1,x_2)) &= \pm\varepsilon_2^i(\varepsilon_1(x_1),x_2) \pm \varepsilon_2^i(x_1,\varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1,x_2) \mp \varepsilon_2^{i-1}(x_2,x_1) \\
\varepsilon_2^i(\varepsilon_2^i(x_1,x_2),x_3) &= \pm\varepsilon_2^i(x_1,\varepsilon_2^i(x_2,x_3)) \mp \varepsilon_2^i(x_2,\varepsilon_2^i(x_1,x_3)) \mp \varepsilon_2^{i+1}(x_2,\varepsilon_2^{i-1}(x_3,x_1)) \\
\varepsilon_2^j(\varepsilon_2^i(x_1,x_2),x_3) &= \pm\varepsilon_2^{i+1}(x_2,\varepsilon_2^{j-1}(x_3,x_1)) \\
\varepsilon_2^i(\varepsilon_2^j(x_1,x_2),x_3) &= \pm\varepsilon_2^j(x_1,\varepsilon_2^i(x_2,x_3)) \mp \varepsilon_2^i(x_2,\varepsilon_2^j(x_1,x_3)) \pm \varepsilon_2^{i+1}(x_3,\varepsilon_2^{j-1}(x_1,x_2))
\end{aligned}$$

Generalizes hemistrict Lie 2-algs and specializes dg-Leibniz algs.

$$h\mathcal{L}ie \xrightarrow{\text{Koszul duality}} \mathcal{E}ilh$$

$\mathcal{E}ilh$-algebras

Graded vector space V, tensor products $\oslash_i$, $|\oslash_i| = i$, $i \in \mathbb{N}$,

$$a \oslash_i (b \oslash_j c) = \sum (\ldots \oslash_k \ldots) \oslash_l \ldots \ ,$$

Differential:

$$Q(a \oslash_i b) = (-1)^i \big((Qa) \oslash_i b + (-1)^{|a|} a \oslash_i Qb \big)$$
$$+ (-1)^i (a \oslash_{i+1} b) - (-1)^{|a|\,|b|} (b \oslash_{i+1} a) \ .$$

Duality explicitly:

- $h\mathcal{L}ie$-algebra:

$$\varepsilon_1(\tau_\alpha) = m_\alpha^\beta \tau_\beta \ , \quad \varepsilon_2^i(\tau_\alpha, \tau_\beta) = m_{\alpha\beta}^{i,\gamma} \tau_\gamma$$

- $\mathcal{E}ilh$-algebra:

$$Qt^\alpha = \pm m_\beta^\alpha t^\beta \pm m_{\beta\gamma}^{i,\alpha} t^\beta \oslash_i t^\gamma$$

EL_∞-algebras

EL_∞-algebra are homotopy $h\mathcal{L}ie$-algebras.
That is, graded vector space $\mathfrak{L}$ with higher products

$$\begin{aligned}
\varepsilon_1 &: \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_1| = 1\ ,\\
\varepsilon_2^i &: \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_2^i| = -i\\
\varepsilon_3^{ij} &: \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_3^{ij}| = -i-j\ ,\\
&\ \vdots \qquad\qquad \vdots
\end{aligned}$$

such that

$$\begin{aligned}
\varepsilon_1(\varepsilon_1(x_1)) &= 0\ ,\\
\varepsilon_1(\varepsilon_2^i(x_1,x_2)) &= \pm\varepsilon_2^i(\varepsilon_1(x_1),x_2) \pm \varepsilon_2^i(x_1,\varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1,x_2) \mp \varepsilon_2^{i-1}(x_2,x_1)\\
&\ \vdots \qquad\qquad \vdots
\end{aligned}$$

amounting to $Q^2=0$ in the corresponding dual $\mathcal{E}ilh$-algebra.

Note: if $\varepsilon_k^I = 0$ for $I \neq (0,0,\dots,0)$, then this is L_∞-algebra.

Properties of EL_∞-algebras

They generalize:

- (dg) Lie algebras
- L_∞-algebras
- Roytenberg's hemistrict and semistrict Lie 2-algebras
- Dehlings weak Lie 3-algebras

They specialize:

- Leibniz algebras
- homotopy Leibniz algebras

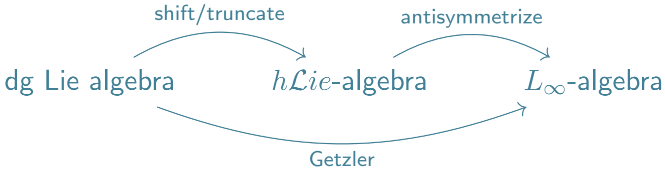
Properties:

- Modified homotopy transfer (modified tensor trick)
- Minimal model and strictification theorems
- EL_∞-algebras antisymmetrize to L_∞-algebras
- An L_∞-algebras in each quasi-isomorphism class
- $\Rightarrow$ They are weak Lie ∞-algebras

Answer to question 1:

What is the algebraic structure underlying Courant algebroids?

Derived bracket constructions



- Differential graded Lie algebras yield L_∞-algebras
  Fiorenza/Manetti (2006), Getzler (2009), ...
- Differential graded Lie algebras truncate to $h\mathcal{L}ie$-algebras
- $h\mathcal{L}ie$-algebras antisymmetrize to L_∞-algebras
- Note: $h\mathcal{L}ie$-algebras are much easier to handle!

Example: Courant algebroid

Differential graded Lie algebra Roytenberg (2002):

- Graded manifold $\mathcal{M} := T^*[2]T[1]M$, $\overset{0}{x^\mu}, \overset{1}{\xi^\mu}, \overset{1}{\zeta_\mu}, \overset{2}{p_\mu}$
- $\mathfrak{g} := C^\infty(T^*[2]T[1]M)$, degree is coordinate degree
- Lie bracket: Poisson bracket of $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta^\mu$
- differential: $Q = \{S, -\}$, $S = \xi^\mu p_\mu + \frac{1}{3!}\varpi_{\mu\nu\kappa}\xi^\mu\xi^\nu\xi^\kappa$

$h\mathcal{L}ie$-algebra:

- $\mathfrak{E} = \mathfrak{E}_{-1} \xrightarrow{\varepsilon_1} \mathfrak{E}_0 = C^\infty_0(\mathcal{M}) \xrightarrow{Q} C^\infty_1(\mathcal{M})$
- Higher products: $\varepsilon_1 = Q$, $\varepsilon^1_2 = \{-, -\}$, $\varepsilon^0_2 = \{Q-, -\}$

L_∞-algebra:

- $\mathfrak{L} = \mathfrak{E}$
- $\mu_2(x_1, x_2) = \frac{1}{2}(\varepsilon^0_2(x_1, x_2) \pm \varepsilon^0_2(x_2, x_1))$
- $\mu_3(x_1, x_2, x_3) = \frac{1}{3!}(\varepsilon^1_2(\varepsilon^0_2(x_1, x_2), x_3) \pm \ldots)$

Generalizes to all generalized tangent bundles

Answer to question 3

What are "good" curvatures for non-abelian gauge potentials?

All direct categorifications of gauge theory yield the following:

- Higher gauge Lie algebra
  - Two gauge Lie algebras: $\mathfrak{g}$ and $\mathfrak{h}$
  - Morphism $\mu_1 : \mathfrak{h} \to \mathfrak{g}$.
  - Action $\mu_2 : \mathfrak{g} \curvearrowright \mathfrak{h}$
- Higher non-abelian gauge potentials
  - $A \in \Omega^1(U, \mathfrak{g})$
  - $B \in \Omega^2(U, \mathfrak{h})$.
- Higher non-abelian curvature forms
  - Fake curvature $\mathrm{d}A + \frac{1}{2}\mu_2(A, A) + \mu_1(B)$
  - 3-form curvature $\mathrm{d}B + \mu_2(A, B)$

Many, many problems with this:

- BRST complex is not closed off-shell

  $\Rightarrow$ fake curvature 2-form needs to vanish

  $\Rightarrow$ but then everything becomes locally abelian...
- Principal bundles + ∇ are not trivially higher bundles
- Does not match mathematical or physical expectations

Adjusted kinematical data

Archetypal example: string Lie 2-algebra

$$\mathbb{R} \xrightarrow{0} \mathfrak{g}$$

Gauge potentials:

$$(A, B) \in \Omega^1(U) \otimes \mathfrak{g} \oplus \Omega^2(U)$$

Curvatures:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A]$$
$$H := \mathrm{d}B - \tfrac{1}{3!}\mu_3(A, A, A) + (A, F)$$
$$= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)}$$

Bianchi identities:

$$\mathrm{d}F + [A, F] = 0\ , \quad \mathrm{d}H - (F, F) = 0$$

Gauge transformations:

$$\delta A = \mathrm{d}\Lambda_0 + \mu_2(A, \Lambda_0) \qquad \delta F = -\mu_2(F, \Lambda_0)$$
$$\delta B = \mathrm{d}\Lambda_1 + (\Lambda_0, F) - \tfrac{1}{2}\mu_3(A, A, \Lambda_0) \qquad \delta H = 0$$

Evident question:

Where do the structure constants for adjustment come from?

Observation:
There is a family of quasi-isomorphic weak Lie 2-algebras

$$\mathfrak{string}^{\mathrm{wk},\alpha}_{\mathrm{sk}}(\mathfrak{g}) \;:=\; (\mathbb{R} \xrightarrow{0} \mathfrak{g}) \;,$$
$$\varepsilon_1(r) = 0 \;,$$
$$\varepsilon_2(x_1, x_2) = [x_1, x_2] \;, \quad \varepsilon_2(x_1, r) = 0 \;,$$
$$\varepsilon_3(x_1, x_2, x_3) = (1-\alpha)(x_1, [x_2, x_3]) \;,$$
$$\mathsf{alt}(x_1, x_2) = -2\alpha(x_1, x_2)$$

Conjecture:

Adjustment data from alternators in weak Lie n-algebras

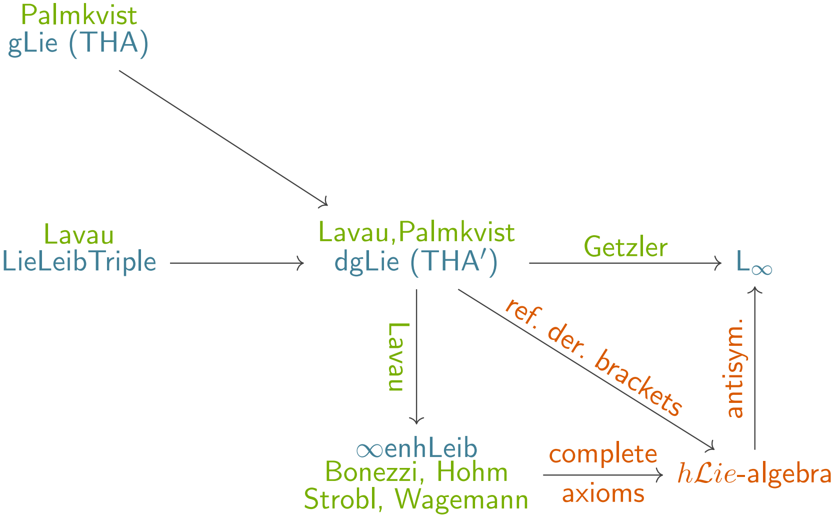
General adjustment

Theorem

Any EL_∞-algebra obtained by shift/truncation from a differential graded Lie algebra admits a mathematically natural adjustment of the definition of the resulting curvatures.

Note:

- Have explicit formulas for adjustment/curvatures
- String 2-algebra from dgLA $\Rightarrow$ adjusted higher gauge theory
- Tensor hierarchies from dgLA $\Rightarrow$ adjusted higher gauge theory


Palmkvist
gLie (THA)
Lavau
LieLeibTriple
Lavau,Palmkvist
dgLie (THA′)
Getzler
L∞
ref. der. brackets
antisym.
Lavau
∞enhLeib
Bonezzi, Hohm
Strobl, Wagemann
complete
axioms
hLie-algebra

Example: 5d max. supersymmetric Tensor Hierarchy

graded Lie algebra/tensor hierarchy algebras (reps. of $\mathfrak{e}_{6(6)}$)

$$\begin{array}{cccccccccccccccc} V_{\mathfrak{e}_{6(6)}} & = & V_{-5} & \oplus & V_{-4} & \oplus & V_{-3} & \oplus & V_{-2} & \oplus & V_{-1} & \oplus & V_0 & \oplus & V_1 \\ \rho_{(k)} & & \mathbf{27}\oplus\mathbf{1728} & & \mathbf{351}_c & & \mathbf{78} & & \mathbf{27} & & \mathbf{27}_c & & \mathbf{78} & & \mathbf{351} \end{array}$$

$h\mathcal{L}ie$-algebra:

$$\begin{array}{ccccccccccc} \mathfrak{E}_{\mathfrak{e}_{6(6)}} = & \mathfrak{E}_{-4} & \oplus & \mathfrak{E}_{-3} & \oplus & \mathfrak{E}_{-2} & \oplus & \mathfrak{E}_{-1} & \oplus & \mathfrak{E}_0 \\ & \mathbf{27}\oplus\mathbf{1728} & & \mathbf{351}_c & & \mathbf{78} & & \mathbf{27} & & \mathbf{27}_c \end{array}$$

Curvatures:

$$F^a = \mathrm{d}A^a + \tfrac{1}{2}X_{bc}{}^a A^b \wedge A^c + Z^{ab}B_b$$

$$H_a = \mathrm{d}B_a - \tfrac{1}{2}X_{ba}{}^c A^b \wedge B_c - \tfrac{1}{6}d_{abc}X_{de}{}^b A^c \wedge A^d \wedge A^e + d_{abc}A^b \wedge F^c + \Theta_a{}^\alpha C_\alpha$$

$$G_\alpha = \mathrm{d}C_\alpha - \tfrac{1}{2}X_{a\alpha}{}^\beta A^a \wedge C_\gamma + (\tfrac{1}{4}X_{a\alpha}{}^\beta t_{\beta b}{}^c + \tfrac{1}{3}t_{\alpha a}{}^d X_{(db)}{}^c)A^a \wedge A^b \wedge B_c$$
$$+ \tfrac{1}{2}t_{\alpha a}{}^b F^a \wedge B_b - \tfrac{1}{2}t_{\alpha a}{}^b H_b \wedge A^a - \tfrac{1}{6}t_{\alpha a}{}^b d_{bcd}A^a \wedge A^c \wedge F^d - Y_{a\alpha}{}^\beta D_\beta{}^a$$

Note:

- Adjustments are given by alternators of $h\mathcal{L}ie$-algebra
- Invisible at level of gauge L_∞-algebra
- L_∞-algebra + extra structure, cf. Palmer, CS (2013)

Summary

- Constructed hemistrict weak Lie ∞-algebras: $h\mathcal{L}ie$-algebras
- They have many applications:
  - arise from differential graded Lie algebras
  - generalized tangent bundles/symplectic L_∞-algebroids
  - adjusted higher gauge theories
  - in particular: tensor hierarchies
- Homotopy $h\mathcal{L}ie$-algebras are EL_∞-algebras
- These have a number of mathematical applications
- Physical applications of true/non-strict EL_∞-algebras?
- Lift above constructions to true/non-strict EL_∞-algebras?

Thank You!