$E L_{\infty}$-algebras, Generalized Geometry, and Tensor Hierarchies

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Based on:
- arXiv:2106.00108 with Leron Borsten and Hyungrook Kim
- arXiv:1908.08086 with Lennart Schmidt
Motivation: Five Questions

1) What is the algebraic structure underlying Courant algebroids?
2) What is alg. structure underlying multisymplectic manifolds?
3) What are “good” curvatures for non-abelian gauge potentials?
4) What do Leibniz algebras integrate to?
5) What is a small cofibrant replacement for the operad $\mathcal{L}ie$?
What is the algebraic structure underlying **Courant algebroids**?

Answers in the literature:

- **Roytenberg (2002):**
  An (exact) **Courant algebroid** is the symplectic dg-manifold
  \[ \mathcal{V}_2 = T^*[2]T[1]M, \quad \omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta^\mu, \]
  \[ Q = \{S, -\}, \quad S = \xi^\mu p_\mu + \frac{1}{3!} \omega_{\mu \nu \kappa} \xi^\mu \xi^\nu \xi^\kappa \]
  for \( M \) some manifold. **Dorfman** and **Courant** brackets:
  \[ [X, Y]_D = \{QX, Y\}, \quad [X, Y]_C = \frac{1}{2} (\{QX, Y\} - \{QY, X\}) \]

- These brackets fit into two structures:
  - \([-,-]_C\) part of **\( L_\infty \)-algebra**, **Getzler (2009)**, **Zambon (2010)**
  - \([-,-]_D\) part of dg-Leibniz algebra cf. **Rogers (2011)**.

- **Is there more to it?**
This may be seem as a niche question, but:

- Courant algebroids underlie Hitchin’s Generalized Geometry
- Application in supergravity: Generalized tangent bundle
- All generalized tangent bundles are symplectic $L_\infty$-algebroids
- Dorfman bracket structure relevant in tensor hierarchies
- Currently relevant: Double and Exceptional Field Theory.

In order to further understand the above: understand symplectic $L_\infty$-algebroids!
How to construct “good” curvatures for non-abelian gauge potentials in presence of $B$-field?

Answers in the literature:

- Use Chern-Simons terms:
  \[ F = dA + \frac{1}{2}[A, A] \, , \quad H = dB + (A, dA) + \frac{1}{3}(A, [A, A]) \]
  Bergshoeff et al. (1982), Chapline et al. (1983)

- This is at odds with the “conventional” non-abelian gerbes:
  \[ F = dA + \frac{1}{2}[A, A] \, , \quad H = dB - \frac{1}{3}(A, [A, A]) \]

- Sati, Schreiber (2009): adjust definition of curvatures
  \[ F = dA + \frac{1}{2}[A, A] \, , \quad H = dB + (A, F) - \frac{1}{3}(A, [A, A]) \]

- Where does $(-, -)$ come from?
We need reasonable higher principal bundles with connections.

Physics:
- Heterotic supergravity
- Tensor hierarchies of gauged supergravity and EFT
- 6d superconformal field theories

Mathematics:
- Higher geometry would be much less beautiful.
All these questions:

1) Algebraic structure underlying symplectic $L_\infty$-algebroids?
2) Algebraic structure underlying multisymplectic manifolds?
3) Algebraic structure underlying higher curvature forms?
4) Cofibrant replacement of $\mathcal{L}ie$?
5) How do you integrate Leibniz algebras?

have a simple, unifying answer:

$EL_\infty$-algebras
Generalized Geometry:
- The Dorfman bracket is part of a hemistrict Lie 2-algebra.

Coupling $B$-field to non-abelian gauge potential:
- Additional algebraic structure is an alternator of Lie 2-algebra.

Gauged supergravity:
- Embedding tensor yields (weak) Lie 2-algebra

Mathematics literature:
- Roytenberg (2007): weak Lie 2-alg. or 2-term $EL_\infty$-algebras
- Dehling (2017): weak Lie 3-alg. or 3-term $EL_\infty$-algebras

Conclusions
We are looking for a weak generalization of $L_\infty$-algebras, generalizing the 2- and 3-term $EL_\infty$-algebras in the literature.
Key ingredients: Operads + Koszul duality

Sketch:

- Won’t need much more than an intuitive understanding
- Useful framework for describing algebras and their relations
- Symmetric operad $\mathcal{O}$:
  - Abstract operations with $n$ inputs and 1 output:

$$\ldots,$$  

  - Composition prescription: equalities between “trees”
  - Also: unit and symmetric group action on inputs
- Algebras over $\mathcal{O}$: ops are multilinear maps on vector spaces
- Examples: Lie, Ass, Com, Leib
- “Homotopy $\mathcal{O}$-algebras or $\mathcal{O}_\infty$-algebra is an algebra over the Koszul resolution of the Koszul-dual cooperad.”
Recall: **Chevalley–Eilenberg algebra** of a Lie algebra \( \mathfrak{g} \)

- **Graded vector space** \( V = \mathfrak{g}[1]^{*} \), coords. \( \xi^{\alpha} \), \( |\xi^{\alpha}| = 1 \).
- **Vector field** or differential on polynomial functions:
  \[
  Q = -\frac{1}{2} f_{\beta \gamma}^{\alpha} \xi^{\beta} \xi^{\gamma} \frac{\partial}{\partial \xi^{\alpha}} , \quad Q^2 = 0 , \quad |Q| = 1
  \]
- **Lie bracket** \( [\tau_{\alpha}, \tau_{\beta}] = f^{\gamma}_{\alpha \beta} \tau_{\gamma} , \quad Q^2 = 0 \Leftrightarrow \text{Jacobi identity} \)

**Generalize to Chevalley–Eilenberg algebra of** \( L_{\infty} \)-**algebra**:

- \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{i} \)
- \( Q \) most general with \( Q^2 = 0 \) and \( |Q| = 1 \)
- **Structure constants in** \( Q \): \( \mu_{i} : \mathfrak{g}^{\wedge i} \to \mathfrak{g} \), \( |\mu_{i}| = 2 - i \)
- \( Q^2 = 0 \Leftrightarrow \text{homotopy Jacobi identities} \)
- For \( \mathfrak{g} = \bigoplus_{i \leq 0} \mathfrak{g}_{i} \): categorified Lie algebras
Operadic perspective:

- $\mathcal{L}ie$ has Koszul-dual $\mathcal{L}ie^\dagger = \mathcal{C}om$
- Therefore:
  \[
  \begin{align*}
  L_\infty\text{-algebras} & \leftrightarrow \text{dg-com algebras} \\
  \mu_i & \leftrightarrow Q 
  \end{align*}
  \]
- Semifree dg-$\mathcal{C}om$-algebra give homotopy $\mathcal{L}ie$-algebra.
- Similarly for
  - $\mathcal{A}ss^\dagger = \mathcal{A}ss$: produces $A_\infty$-algebras
  - $\mathcal{L}eib^\dagger = \mathcal{Z}inb$: produces homotopy Leibniz algebras
  - ...

**Question:** which operad produces weak $L_\infty$-/EL$_\infty$-algebras?
Hemistrict Lie 2-algebras: differential graded algebras $\mathcal{L}$ with

$$\varepsilon_2 : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, \quad |\varepsilon_2| = 0, \quad \text{alt} : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, \quad |\text{alt}| = -1$$

Generalize, preserving differential compatibility: $h\text{Lie}$-algebras

**$h\text{Lie}$-algebras**

Graded vector space $\mathcal{L}$ with

$$\varepsilon_1 : \mathcal{L} \to \mathcal{L}, \quad |\varepsilon_1| = 1, \quad \varepsilon^i_2 : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, \quad |\varepsilon^i_2| = -i$$

such that

$$
\varepsilon_1(\varepsilon_1(x_1)) = 0,
\varepsilon_1(\varepsilon^i_2(x_1, x_2)) = \pm \varepsilon^i_2(\varepsilon_1(x_1), x_2) \pm \varepsilon^i_2(x_1, \varepsilon_1(x_2)) + \varepsilon^{i-1}_2(x_1, x_2) \mp \varepsilon^{i-1}_2(x_2, x_1)
\varepsilon^i_2(\varepsilon^i_2(x_1, x_2), x_3) = \pm \varepsilon^i_2(x_1, \varepsilon^i_2(x_2, x_3)) \mp \varepsilon^i_2(x_2, \varepsilon^i_2(x_1, x_3)) \pm \varepsilon^{i+1}_2(x_2, \varepsilon^{i-1}_2(x_3, x_1))
\varepsilon^{i+1}_2(\varepsilon^i_2(x_1, x_2), x_3) = \pm \varepsilon^{i+1}_2(x_2, \varepsilon^{i-1}_2(x_3, x_1))
\varepsilon^i_2(\varepsilon^{j-1}_2(x_1, x_2), x_3) = \pm \varepsilon^j_2(\varepsilon^i_2(x_1, x_2), x_3) \mp \varepsilon^j_2(x_2, \varepsilon^i_2(x_1, x_3)) \pm \varepsilon^{i+1}_2(x_3, \varepsilon^{j-1}_2(x_1, x_2))
\varepsilon^{i+1}_2(\varepsilon^{j-1}_2(x_1, x_2), x_3) = \pm \varepsilon^{i+1}_2(x_3, \varepsilon^{j-1}_2(x_1, x_2))
$$

Generalizes hemistrict Lie 2-algs and specializes dg-Leibniz algs.
\[ h\text{Lie} \xrightarrow{\text{Koszul duality}} \mathcal{E}ilh \]

**\( \mathcal{E}ilh\)-algebras**

Graded vector space \( V \), tensor products \( \otimes_i, |\otimes_i| = i, i \in \mathbb{N} \),

\[
a \otimes_i (b \otimes_j c) = \sum (\ldots \otimes_k \ldots) \otimes_l \ldots ,
\]

Differential:

\[
Q(a \otimes_i b) = (-1)^i ((Qa) \otimes_i b + (-1)^{|a|} a \otimes_i Qb) + (-1)^i (a \otimes_{i+1} b) - (-1)^{|a||b|} (b \otimes_{i+1} a) .
\]

Duality explicitly:

- **\( h\text{Lie}\)-algebra:**
  \[ \varepsilon_1(\tau_\alpha) = m_\alpha^\beta \tau_\beta , \quad \varepsilon_2^i(\tau_\alpha, \tau_\beta) = m_{\alpha\beta}^{i,\gamma} \tau_\gamma \]

- **\( \mathcal{E}ilh\)-algebra:**
  \[ Qt^\alpha = \pm m_\alpha^\beta t^\beta \pm m_{\beta\gamma}^{i,\alpha} t^\beta \otimes_i t^\gamma \]
**$EL_\infty$-algebras**

$EL_\infty$-algebra are **homotopy $h\mathcal{L}ie$-algebras**. That is, graded vector space $\mathcal{L}$ with higher products

$$
\varepsilon_1 : \mathcal{L} \to \mathcal{L}, \quad |\varepsilon_1| = 1,
$$

$$
\varepsilon^i_2 : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, \quad |\varepsilon^i_2| = -i
$$

$$
\varepsilon^{ij}_3 : \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, \quad |\varepsilon^{ij}_3| = -(i+j)
$$

such that

$$
\varepsilon_1(\varepsilon_1(x_1)) = 0,
$$

$$
\varepsilon_1(\varepsilon^i_2(x_1, x_2)) = \pm \varepsilon^i_2(\varepsilon_1(x_1), x_2) \pm \varepsilon^i_2(x_1, \varepsilon_1(x_2)) + \varepsilon^{i-1}_2(x_1, x_2) \mp \varepsilon^{i-1}_2(x_2, x_1)
$$

$$
\vdots \quad \vdots
$$

amounting to $Q^2 = 0$ in the corresponding dual $Eilh$-algebra.

**Note:** if $\varepsilon^I = 0$ for $I \neq (0, 0, \ldots, 0)$, then this is $L_\infty$-algebra.
Properties of $EL_\infty$-algebras

They generalize:
- (dg) Lie algebras
- $L_\infty$-algebras
- Roytenberg’s hemistrict and semistrict Lie 2-algebras
- Dehlings weak Lie 3-algebras

They specialize:
- Leibniz algebras
- homotopy Leibniz algebras

Properties:
- Modified homotopy transfer (modified tensor trick)
- Minimal model and strictification theorems
- $EL_\infty$-algebras antisymmetrize to $L_\infty$-algebras
- An $L_\infty$-algebras in each quasi-isomorphism class
- $\Rightarrow$ They are weak Lie $\infty$-algebras
Answer to question 1:

What is the algebraic structure underlying Courant algebroids?
Differential graded Lie algebras yield $L_\infty$-algebras
Fiorenza/Manetti (2006), Getzler (2009), ...

Differential graded Lie algebras truncate to $h\mathcal{L}ie$-algebras
$h\mathcal{L}ie$-algebras antisymmetrize to $L_\infty$-algebras

Note: $h\mathcal{L}ie$-algebras are much easier to handle!
Example: Courant algebroid

Differential graded Lie algebra Roytenberg (2002):

- Graded manifold $\mathcal{M} := T^*[2]T[1]M$, $x^\mu, \xi^\mu, \zeta_\mu, p_\mu$
- $\mathfrak{g} := C^\infty(T^*[2]T[1]M)$, degree is coordinate degree
- Lie bracket: Poisson bracket of $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta^\mu$
- Differential: $Q = \{S, -\}$, $S = \xi^\mu p_\mu + \frac{1}{3!} \varpi_{\mu\nu\kappa} \xi^\mu \xi^\nu \xi^\kappa$

$h\mathcal{L}ie$-algebra:

- $\mathcal{E} = \mathcal{E}_{-1} \xrightarrow{\varepsilon_1} \mathcal{E}_0 = C^\infty_0(\mathcal{M}) \xrightarrow{Q} C^\infty_1(\mathcal{M})$
- Higher products: $\varepsilon_1 = Q$, $\varepsilon_2 = \{-, -\}$, $\varepsilon_2 = \{Q-, -\}$

$L_\infty$-algebra:

- $\mathcal{L} = \mathcal{E}$
- $\mu_2(x_1, x_2) = \frac{1}{2}(\varepsilon_2^0(x_1, x_2) \pm \varepsilon_2^0(x_2, x_1))$
- $\mu_3(x_1, x_2, x_3) = \frac{1}{3!}(\varepsilon_2^1(\varepsilon_2^0(x_1, x_2), x_3) \pm \ldots)$

Generalizes to all generalized tangent bundles
Answer to question 3

What are “good” curvatures for non-abelian gauge potentials?
All direct categorifications of gauge theory yield the following:

- **Higher gauge Lie algebra**
  - Two gauge Lie algebras: $\mathfrak{g}$ and $\mathfrak{h}$
  - Morphism $\mu_1 : \mathfrak{h} \to \mathfrak{g}$.
  - Action $\mu_2 : \mathfrak{g} \curvearrowright \mathfrak{h}$

- **Higher non-abelian gauge potentials**
  - $A \in \Omega^1(U, \mathfrak{g})$
  - $B \in \Omega^2(U, \mathfrak{h})$.

- **Higher non-abelian curvature forms**
  - Fake curvature $dA + \frac{1}{2} \mu_2(A, A) + \mu_1(B)$
  - $3$-form curvature $dB + \mu_2(A, B)$

Many, many problems with this:

- BRST complex is not closed off-shell
  - $\Rightarrow$ fake curvature $2$-form needs to vanish
    - $\Rightarrow$ but then everything becomes locally abelian...
- Principal bundles $+ \nabla$ are not trivially higher bundles
- Does not match mathematical or physical expectations
Adjusted kinematical data

Archetypal example: string Lie 2-algebra

\[ \mathbb{R} \xrightarrow{0} \mathfrak{g} \]

Gauge potentials:

\[ (A, B) \in \Omega^1(U) \otimes \mathfrak{g} \oplus \Omega^2(U) \]

Curvatures:

\[
F := dA + \frac{1}{2}[A, A] \\
H := dB - \frac{1}{3!}\mu_3(A, A, A) + (A, F) \\
= dB + (A, dA) + \frac{1}{3}(A, [A, A]) + \text{cs}(A)
\]

Bianchi identities:

\[ dF + [A, F] = 0 , \quad dH - (F, F) = 0 \]

Gauge transformations:

\[
\delta A = d\Lambda_0 + \mu_2(A, \Lambda_0) \\
\delta B = d\Lambda_1 + (\Lambda_0, F) - \frac{1}{2}\mu_3(A, A, \Lambda_0) \\
\delta F = -\mu_2(F, \Lambda_0) \\
\delta H = 0
\]
Observations

Evident question:

Where do the structure constants for adjustment come from?

Observation:

There is a family of quasi-isomorphic weak Lie 2-algebras

\[ \text{string}_{sk}^{\text{wk}, \alpha}(g) := (\mathbb{R} \xrightarrow{0} g), \]

\[ \varepsilon_1(r) = 0, \]

\[ \varepsilon_2(x_1, x_2) = [x_1, x_2], \quad \varepsilon_2(x_1, r) = 0, \]

\[ \varepsilon_3(x_1, x_2, x_3) = (1 - \alpha)(x_1, [x_2, x_3]), \]

\[ \text{alt}(x_1, x_2) = -2\alpha(x_1, x_2) \]

Conjecture:

Adjustment data from alternators in weak Lie \( n \)-algebras

Christian Saemann  \( EL_\infty \)-algebras, Generalized Geometry, and Tensor Hierarchies
Theorem

Any $EL_\infty$-algebra obtained by shift/truncation from a differential graded Lie algebra admits a \textbf{mathematically natural} adjustment of the definition of the resulting curvatures.

Note:

- Have explicit formulas for adjustment/curvatures
- String 2-algebra from dgLA $\Rightarrow$ adjusted higher gauge theory
- Tensor hierarchies from dgLA $\Rightarrow$ adjusted higher gauge theory
Example: Tensor hierarchies

- Palmkvist
  \[ g\text{Lie} \text{ (THA)} \]

- Lavaux
  \[ \text{LieLeibTriple} \]

- Lavaux, Palmkvist
  \[ \text{dgLie (THA')} \]

- Getzler
  \[ L_\infty \]

- Bonezzi, Hohm, Strobl, Wagemann
  \[ \infty\text{enhLeib} \]

- \[ \text{complete axioms} \]
  \[ h\text{Lie}-\text{algebra} \]

- Lavaux
  \[ \text{ref. der. brackets} \]
  \[ \text{antisym.} \]
graded Lie algebra/tensor hierarchy algebras (reps. of $\mathfrak{e}_{6(6)}$)

\[
V_{\mathfrak{e}_{6(6)}} = V_{-5} \oplus V_{-4} \oplus V_{-3} \oplus V_{-2} \oplus V_{-1} \oplus V_0 \oplus V_1
\]

\[
\rho_{(k)} = 27 \oplus 1728 \quad 351_c \quad 78 \quad 27 \quad 27_c \quad 78 \quad 351
\]

$h\mathcal{L}ie$-algebra:

\[
\mathfrak{e}_{\mathfrak{e}_{6(6)}} = \mathfrak{e}_{-4} \oplus \mathfrak{e}_{-3} \oplus \mathfrak{e}_{-2} \oplus \mathfrak{e}_{-1} \oplus \mathfrak{e}_0
\]

\[
27 \oplus 1728 \quad 351_c \quad 78 \quad 27 \quad 27_c
\]

Curvatures:

\[
F^a = dA^a + \frac{1}{2} X_{bc}^a A^b \wedge A^c + Z^{ab} B_b
\]

\[
H_a = dB_a - \frac{1}{2} X_{ba}^c A^b \wedge B_c - \frac{1}{6} d_{abc} X_{de}^b A^c \wedge A^d \wedge A^e + d_{abc} A^b \wedge F^c + \Theta^a_{\alpha} C_{\alpha}
\]

\[
G_\alpha = dC_\alpha - \frac{1}{2} X_{a\alpha}^\beta A^a \wedge C_\gamma + \left( \frac{1}{4} X_{a\alpha}^\beta t_{\beta b}^c + \frac{1}{3} t_{\alpha a}^d X_{(db)}^c \right) A^a \wedge A^b \wedge B_c + \frac{1}{2} t_{\alpha a}^b F^a \wedge B_b - \frac{1}{2} t_{\alpha a}^b H_b \wedge A^a - \frac{1}{6} t_{\alpha a}^b d_{bcd} A^a \wedge A^c \wedge F^d - Y_{a\alpha}^\beta D_{\beta}^a
\]

Note:

- **Adjustments** are given by alternators of $h\mathcal{L}ie$-algebra
- Invisible at level of gauge $L_{\infty}$-algebra
Constructed hemistrict weak Lie $\infty$-algebras: $h\mathcal{L}ie$-algebras

They have many applications:
- arise from differential graded Lie algebras
- generalized tangent bundles/symplectic $L_\infty$-algebroids
- adjusted higher gauge theories
- in particular: tensor hierarchies

Homotopy $h\mathcal{L}ie$-algebras are $E\mathcal{L}_\infty$-algebras

These have a number of mathematical applications

Physical applications of true/non-strict $E\mathcal{L}_\infty$-algebras?

Lift above constructions to true/non-strict $E\mathcal{L}_\infty$-algebras?
Thank You!