Principal String 2-Group Bundles and M-Branes

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Based on:

- arXiv:1602.03441 with Getachew Alemu Demessie
- arXiv:1604.01639 with Brano Jurčo and Martin Wolf
- work in progress with Lennart Schmidt

Motivation

Future progress in string theory seems to depend on more mathematical input.

String-/M-theory as it used to be

- Every 10 years a "string revolution"
- Every 2-3 years one new big fashionable topic to work on

This changed: No more revolutions or really big fashionable topics.

My explanation

We need more input from maths, in particular category theory:

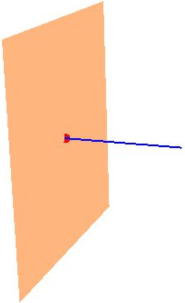
- 2-form gauge potential B-field: Gerbes or principal 2-bundles
- String Field Theory: L_∞-algebras or semistrict Lie ∞-algebras
- Double Field Theory: Courant algebroids and beyond or
  symplectic Lie 2-algebroids
- (2,0)-theory: parallel transport of string-like objects
  full non-abelian higher gauge theory

We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.
G. Moore and N. Seiberg, 1989

What does categorification mean?
One of Jeff Harvey's questions to identify the "generation PhD>1999" at Strings 2013.

Motivation: The Dynamics of Multiple M5-Branes

To understand M-theory, an effective description of M5-branes would be very useful.

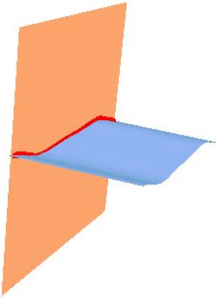


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory
- Study string theory via gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- Holy grail: (2,0)-theory (conjectured 1995)



What We Know

Multiple M5-branes are described by a $\mathcal{N} = (2,0)$ superconformal field theory.

What we know:

- String theory considerations: conformal fixed point in 6d
  Witten, Strominger 1995
- Field content: $\mathcal{N} = (2,0)$ supermultiplet in 6d:
  - a self-dual 3-form field strength
  - five (Goldstone) scalars
  - fermionic partners
- A theory of essentially tensionless light strings
- Supergravity decouples, so study string dynamics separately
- Observables: Wilson surfaces, i.e. parallel transport of strings
- No Lagrangian description known
- As important as $\mathcal{N} = 4$ super Yang-Mills for string theory
- Huge interest in string theory: AGT, AdS_7-CFT_6, S-duality, ...
- Mathematics: Geom. Langlands, Khovanov Homology, ...

Parallel Transport of Strings is Problematic

The lack of surface ordering renders a parallel transport of strings problematic.

Parallel transport of particles in representation of gauge group G:

- holonomy functor $\mathsf{hol} : \text{path } \gamma \mapsto \mathsf{hol}(\gamma) \in \mathsf{G}$
- $\mathsf{hol}(\gamma) = P\exp(\int_\gamma A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

- map $\mathsf{hol} : \text{surface } \sigma \mapsto \mathsf{hol}(\sigma) \in \mathsf{U}(1)$
- $\mathsf{hol}(\sigma) = \exp(\int_\sigma B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering

Imagine parallel transport of string with gauge degrees in Lie(G):

$$\begin{tikzcd}[column sep=huge]
\bullet & {} \arrow[l, bend right=60, ""{name=U1, below}] \arrow[l, ""{name=Ma, above}, ""{name=Mb, below}] \arrow[l, bend left=60, ""{name=D1, above}] & \bullet \arrow[l, bend right=60, ""{name=U2, below}] \arrow[l, ""{name=Na, above}, ""{name=Nb, below}] \arrow[l, bend left=60, ""{name=D2, above}]
\arrow[Rightarrow, from=U1, to=Ma, "g_1"] \arrow[Rightarrow, from=Mb, to=D1, "g_1'"] \arrow[Rightarrow, from=U2, to=Na, "g_2"] \arrow[Rightarrow, from=Nb, to=D2, "g_2'"]
\end{tikzcd}$$

Consistency of parallel transport requires:

$$(g_1' g_2')(g_1 g_2) = (g_1' g_1)(g_2' g_2)$$

This renders group G abelian.

Eckmann and Hilton, 1962
Physicists 80'ies and 90'ies

Way out: 2-categories, Higher Gauge Theory.

Two operations $\circ$ and $\otimes$ satisfying Interchange Law:

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2) \ .$$

Objection to a classical (2,0)-theory

Without coupling constant, there shouldn't be classical effective descriptions in M-theory.

Standard objection beyond the previous no-go theorem:

- theory at conformal fixed points $\Rightarrow$ no dimensionful parameter
- fixed points are isolated $\Rightarrow$ no dimensionless parameter
- "No parameters $\Rightarrow$ no classical limit $\Rightarrow$ no Lagrangian."

Answers:

- Same arguments for M2-brane — Schwarz, 2004
- There, integer parameters arose from orbifold $\mathbb{R}^8/\mathbb{Z}_k$ — BLG, ABJM, 2008
- Same should happen for M5-branes
- Even if no Lagrangian, BPS-states may exist classically
- Even if not, study quantum features of related theories.

We want to categorify gauge theory

Need: suitable descriptions/definitions

Equivalent options:

1. finite descriptions via Wilson lines $\Rightarrow$ Parallel transport functor
2. infinitesimal description via connections $\Rightarrow$ Atiyah Algebroid

Encode gauge theory in parallel transport functor

Mackaay, Picken, 2000

- Every manifold comes with path groupoid $\mathcal{P}M = (PM \rightrightarrows M)$

$$x \xrightarrow{\gamma} y$$

- Gauge group gives rise to delooping groupoid $\mathsf{B}\mathsf{G} = (\mathsf{G} \rightrightarrows *)$
- parallel transport functor $\mathrm{hol} : \mathcal{P}M \to \mathsf{B}\mathsf{G}$:
  - assigns to each path a group element
  - composition of paths: multiplication of group elements
- Readily categorifies: Baez, Schreiber 2004
  - use path 2-groupoid with homotopies between paths
  - use delooping of categorified group
- Problem: Need to differentiate to get to cocycles

Recall: NQ-Manifolds

NQ-manifolds, known from BRST quantization, provide very useful language.

N-manifolds, NQ-manifold

- $\mathbb{N}$-graded manifold with coordinates of degree $0, 1, 2, \ldots$

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots$$

manifold   linear spaces

- Morphisms $\phi : M \to N$ are maps $\phi^* : C^\infty(N) \to C^\infty(M)$
- NQ-manifold: vector field Q of degree 1, $Q^2 = 0$
- Physicists: think ghost numbers, BRST charge, SFT

Examples:

- Tangent algebroid $T[1]M$, $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$, $Q = \mathrm{d}$
- Lie algebra $\mathfrak{g}[1]$, coordinates ξ^a of degree 1:

$$Q = -\tfrac{1}{2} f_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c}$$

Condition $Q^2 = 0$ is equivalent to Jacobi identity for f_{ab}^c

2. Atiyah Algebroid Sequence

A straightforward way to describe gauge theory is in terms of parallel transport functors.

(Flat) connection: splitting of Atiyah algebroid sequence

$$0 \longrightarrow P \times_{\mathsf{G}} \mathsf{Lie}(\mathsf{G}) \longrightarrow TP/\mathsf{G} \longrightarrow TM \longrightarrow 0$$

Atiyah, 1957

Related approach: Kotov, Strobl, Schreiber, ...

- Gauge potential from morphism of N-manifolds:

$$a : T[1]M \to \mathfrak{g}[1] \quad \longrightarrow \quad A^a_\mu \mathrm{d}x^\mu := a^*(\xi^a)$$

- Curvature: failure of a to be morphism of NQ-manifold:

$$F^a := (\mathrm{d} \circ a^* - a^* \circ Q)(\xi^a) = \mathrm{d}A^a + \tfrac{1}{2} f^a_{bc} A^b \wedge A^c$$

- Infinitesimal gauge transformations: flat homotopies
- Readily categorifies, but integration an issue

L_∞-Algebras, Lie 2-Algebras

NQ-manifolds provide an easy definition of L_∞-algebras.

Lie n-algebroid or n-term L_∞-algebroid:

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Lie n-algebra, n-term L_∞-algebra or Lie n-algebra:

$$* \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Example: Lie 2-algebra as 2-term L_∞-algebra

- NQ-manifold: $* \leftarrow W[1] \leftarrow V[2] \leftarrow * \leftarrow \ldots$, coords. w^a, v^i
- Homological vector field:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \tfrac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Structure constants: higher products μ_i on $W \leftarrow V[1]$

$$\mu_1(\tau_i) = m_i^a \tau_a\ , \quad \mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c\ , \quad \ldots\ , \quad \mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$$

- $Q^2 = 0$: Higher or homotopy Jacobi identity, e.g.

$$\mu_2(w_1, \mu_2(w_2, w_3)) + \text{cycl.} = \mu_1(\mu_3(w_1, w_2, w_3))$$

Higher gauge theory with Lie 2-algebra:

- Lie 2-algebra: $* \leftarrow W[1] \leftarrow V[2] \leftarrow * \leftarrow \dots$, coords. w^a, v^i
- Gauge potentials $T[1]M \to (W[1] \leftarrow V[2])$:

$$A^a_\mu \mathrm{d}x^\mu := a^*(w^a) \quad \text{and} \quad B^i_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = a^*(v^i)$$

- Curvature: failure of a to be morphism of NQ-manifold:

$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B)$$
$$\mathcal{H} := \mathrm{d}B + \mu_2(A,B) + \tfrac{1}{3!}\mu_3(A,A,A)$$

- Gauge transformations from flat homotopies ...

Local Higher Gauge Theories

The most interesting higher gauge theories for us live in 6 and 4 dimensions.

- "Fake curvature": $\mathcal{F} = \mathrm{d}A + \frac{1}{2}\mu_2(A, A) - \mu_1(B) = 0$
  Vanishing makes parallel transport reparam. invariant.
- 3-form curvature: $\mathcal{H} = \mathrm{d}B + \mu_2(A, B) + \frac{1}{3!}\mu_3(A, A, A)$

Gauge part of (2,0) theory

If (2,0) theory on $\mathbb{R}^{1,5}$ is a higher gauge theory, then gauge part is:

$$\mathcal{H} = *\mathcal{H}\ , \quad \mathcal{F} = 0\ .$$

Non-Abelian Self-Dual Strings

BPS equation for (2,0) theory on $\mathbb{R}^4$ ($\sim$ monopoles in 4d SYM)

$$\mathcal{H} = *\left(\mathrm{d}\Phi + \mu_2(A, \Phi)\right)\ , \quad \mathcal{F} = 0\ .$$

The Global Picture:

Principal 2-Bundles and Finite Gauge Transformations

Three steps:

1. Categorified notion of group, describing the symmetries
2. Categorified notion of principal bundle
3. Endow these bundles with categorified connections

Tool: Categorification

Categorification provides some guidelines in the construction of higher objects.

Category theory: excellent tool for deformations/generalizations.

Notions used: categorification, internalization and enrichment.

Idea: Mathematical objects are stuff, structures, structure eqns.

Translate as follows:

- stuff (sets) becomes categories
- structures (functions) become functors
- structure equations become structure isomorphisms

1. Categorified Groups

Categorifying a group, we arrive at the notion of a 2-group.

Group:

- Stuff: Underlying set G, unit $\mathbb{1}$
- Structure: Multiplication, inverse
- Structure equations: associativity, $g^{-1}g = \mathbb{1}$, $\mathbb{1}g = g\mathbb{1} = g$

2-Group:

- Stuff: A category $\mathscr{C}$, unit object $\mathbb{1}$
- Structure: Multiplication bifunctor $\otimes$, inverse functor inv
- Structure isomorphisms:
  - $\mathsf{a}_{x,y,z} : (x \otimes y) \otimes z \Rightarrow x \otimes (y \otimes z)$
  - $\mathsf{l}_x : x \otimes \mathbb{1} \Rightarrow x$, $\mathsf{r}_x : \mathbb{1} \otimes x \Rightarrow x$
  - $\mathsf{inv}(x) \otimes x \Rightarrow \mathbb{1} \Leftarrow x \otimes \mathsf{inv}(x)$

Example: Strict 2-Group $\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G}$,

- a, l, r all trivial, $\mathsf{inv}(x) \otimes x = \mathbb{1} = x \otimes \mathsf{inv}(x)$
- $\mathrm{id}(g) = (g, \mathbb{1}_{\mathsf{H}})$, $(g_1, h_1) \otimes (g_2, h_2) = (g_1 g_2, h_1(g_1 \rhd h_2))$, etc.

Čech groupoid of surjective submersion $Y \twoheadrightarrow M$, e.g. $Y = \sqcup_a U_a$:

$$\check{\mathscr{C}}(U) \ : \quad \bigsqcup_{a,b} U_{ab} \rightrightarrows \bigsqcup_a U_a \ , \quad U_{ab} \circ U_{bc} = U_{ac} \ .$$

Principal G-bundle

Transition functions are nothing but a functor $g : \check{\mathscr{C}}(U) \to (\mathsf{G} \rightrightarrows *)$

$$\begin{tikzcd} \sqcup U_{ab} \arrow[r, "g_{ab}"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{G} \arrow[d, shift left] \arrow[d, shift right] \\ \sqcup U_a \arrow[r, "*"] & * \end{tikzcd} \qquad g_{ab} g_{bc} = g_{ac}$$

Equivalence relations: natural isomorphisms.

Principal 2-bundle, structure Lie 2-group $\mathscr{G}$

Definition is clear: 2-functor $\check{\mathscr{C}}(U) \to (\mathscr{G} \rightrightarrows *)$.

Questions: Which notion of 2-category and which $\mathscr{G}$?

3. Towards Connections: Differentiation

There is a differentiation procedure of quasi-groupoids due to Ševera.

Recall: Connection on principal G-bundle: Lie(G)-valued 1-forms

We therefore need a way of differentiating Lie 2-groups.

Lie functor as suggested by Ševera, 2006

- Functors: supermanifolds to certain principal $\mathscr{G}$-bundles

$$X \mapsto \text{descent data subordinate to } X \times \mathbb{R}^{0|1} \twoheadrightarrow X$$

- Moduli: Lie($\mathscr{G}$) as an n-term complex of vector spaces
- Carries Hom($\mathbb{R}^{0|1}, \mathbb{R}^{0|1}$)-action $\to$ L_∞-algebra structure

Example: Differentiation of Lie group

There is a differentiation procedure of quasi-groupoids due to Ševera.

Ševera: want moduli of functor

$$X \mapsto \text{descent data subordinate to } X \times \mathbb{R}^{0|1} \twoheadrightarrow X$$

For a Lie group G:

$$g : X \times \mathbb{R}^{0|2} \to \mathsf{G}~, \quad g(\theta_0,\theta_1,x)g(\theta_1,\theta_2,x) = g(\theta_0,\theta_2,x)~.$$

This implies

$$g(0,\theta,x) = g(\theta,0,x)^{-1} \quad \text{and} \quad g(\theta_0,\theta_1,x) = g(\theta_0,0,x)(g(\theta_1,0,x))^{-1}$$

and we have a trivializing coboundary:

$$g(\theta_0,0,x) = \mathbb{1} + \alpha\theta_0~, \quad \alpha \in \mathsf{Lie}(\mathsf{G})[1]~.$$

We readily compute

$$g(\theta_0,\theta_1) = \mathbb{1} + \alpha(\theta_0 - \theta_1) + \tfrac{1}{2}[\alpha,\alpha]\theta_0\theta_1~.$$

With $Qg(\theta_0,\theta_1,x) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(\theta_0+\varepsilon,\theta_1+\varepsilon,x)$, we obtain the N$Q$-manifold description of $\mathsf{Lie}(\mathsf{G})$:

$$Q\alpha = -\tfrac{1}{2}[\alpha,\alpha]~.$$

Finite Gauge Transformations

The differentiation method can be extended to read off finite gauge transformations.

- We have: Lie algebra element in terms of descent data g
- Perform a coboundary transformation to $\tilde{g}$
- Trivialize $\tilde{g}$, establish relation between moduli of g, $\tilde{g}$, e.g.

$$\tilde{\alpha} = p^{-1}\alpha p + p^{-1}Qp \ , \quad p \in C^\infty(X, \mathsf{G})$$

- Replacing Q by de Rham differential on patches yields finite gauge transformations B Jurco, CS, M Wolf, 1403.7185
- Can read off global patching of gauge potential forms.
- More elegant approach in B Jurco, CS, M Wolf, 1604.01639

Again: Everything readily categorifies.

All this is quite powerful...

We readily define Deligne cohomology for semistrict Lie 2-group bundles.

Example: principal $\mathscr{G}$-bundle with $\mathscr{G}$ semistrict Lie 2-group:

Cocycle data: $(m_{ab}, n_{abc}, A_a, \Lambda_{ab}, B_a)$. Cocycle relations:

$$n_{abc} : m_{ab} \otimes m_{bc} \Rightarrow m_{ac}$$

$$n_{acd} \circ (n_{abc} \otimes \mathrm{id}_{m_{cd}}) \circ \mathsf{a}^{-1}_{m_{ab},m_{bc},m_{cd}} = n_{abd} \circ (\mathrm{id}_{m_{ab}} \otimes n_{bcd})$$

$$\mathrm{d}A_a + A_a \otimes A_a + \mathsf{s}(B_a) = 0$$

$$\Lambda_{ab} : A_b \otimes m_{ab} \Rightarrow m_{ab} \otimes A_a - \mathrm{d}m_{ab}$$

$$\begin{aligned}
\Lambda_{cb} \circ (\mathrm{id}_{A_b} \otimes n_{bac}) \circ \mathsf{a}_{A_b,m_{ba},m_{ac}} &= \\
= (n_{bac} \otimes \mathrm{id}_{A_c} - \mathrm{d}n_{bac}) &\circ \big[\mathsf{a}^{-1}_{m_{ba},m_{ac},A_c} - \mathrm{id}_{\mathrm{d}(m_{ba}\otimes m_{ac})}\big] \circ \\
\circ (\mathrm{id}_{m_{ba}} \otimes \Lambda_{ca} - \mathrm{id}_{\mathrm{d}m_{ba}\otimes m_{ac}}) &\circ (\mathsf{a}_{m_{ba},A_c,m_{ac}} - \mathrm{id}_{\mathrm{d}m_{ba}\otimes m_{ac}}) \circ (\Lambda_{ab} \otimes \mathrm{id}_{m_{ac}})
\end{aligned}$$

$$\begin{aligned}
B_b \otimes \mathrm{id}_{m_{ab}} = \mu(A_b, A_b, m_{ab}) &+ \big[\mathrm{id}_{m_{ab}} \otimes B_a + \mu(m_{ab}, A_a, A_a)\big] \circ \\
&\circ \big[-\mathrm{d}\Lambda_{ab} - \Lambda_{ab} \otimes \mathrm{id}_{A_a} - \mu(A_b, m_{ab}, A_a)\big] \circ \\
&\circ \big[-\mathrm{id}_{\mathsf{s}(\mathrm{d}\Lambda_{ab})} - \mathrm{id}_{A_b} \otimes (\Lambda_{ab} + \mathrm{id}_{\mathrm{d}m_{ab}})\big]
\end{aligned}$$

B Jurco, CS, M Wolf, 1403.7185

We can now start to calculate and look for applications.

Applications to M-theory

An Application: $\mathcal{N} = (2,0)$ Theory from Twistors

Given a higher gauge group, one can readily construct a corresponding (2,0)-theory.

Theorem [Ward, 1977]

$$\begin{tikzcd} & \mathbb{C}^4 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ \mathbb{C}P^3_\circ & & \mathbb{C}^4 \end{tikzcd}$$

Holomorphic princ. bundles* over $\mathbb{C}P^3_\circ$
in 1:1 correspondence (mod gauge) with
Solutions to the 4d instanton equations

Idea: Put 2-bundles over twistor space for self-dual 3-forms in 6d

Theorem [CS & Wolf, 2012]

$$\begin{tikzcd} & \mathbb{C}^{6|16} \times \mathbb{C}P^3 \arrow[dl] \arrow[dr] & \\ Q^{6|4} & & \mathbb{C}^{6|16} \end{tikzcd}$$

Holomorphic princ. 2-bundles* over $Q^{6|4}$
in 1:1 correspondence (mod gauge) with
Solutions to $\mathcal{N} = (2,0)$ SCFT equations

$\Rightarrow$ Reduced search for $(2,0)$-theory to search for gauge structure.

Remaining Issue: Examples.

No truly non-abelian solutions of higher gauge equations are known

- In principle, we found a $(2,0)$-theory.
- We've generalized all this to ∞-groupoid bundles.
- But: How do we know we're not talking about the empty set?
- Popular claim: principal 2-bundles reduce to abelian gerbes.
- Problem: Find explicit, truly non-abelian configurations
- Input from M-theory: BPS $\Rightarrow$ self-dual strings in 4d
- Mathematics of gauge theory started with instanton sols.

Dirac Monopole:

- Extend Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ to $\mathbb{R}^3 \backslash \{0\}$
- $\mathfrak{u}(1)$-gauge potential A, solves $F_A = \mathrm{d}A = *\mathrm{d}\Phi$ for $\Phi \sim \frac{1}{2r}$

't Hooft-Polyakov monopole:

- Note: $S^3 \cong \mathsf{SU}(2)$
- Hopf fibration can be trivialized in $\mathsf{SU}(2) \times \mathbb{R}^3 \to \mathbb{R}^3$
- Obvious bundle map $S^3 \to S^2$ to $S^3 \times S^2 \to S^2$
- Interesting and non-singular monopole solution $F_A = *\mathrm{d}_A\Phi$

Abelian self-dual string:

- Tautological gerbe $\mathscr{G}$ over S^3 with DD-class 1 ($H = \mathrm{vol}_{S^3}$)
- solves $H := \mathrm{d}B = *\mathrm{d}\Phi$ over $\mathbb{R}^4 \backslash \{0\}$ for $\Phi \sim \frac{1}{2r^2}$

Truly non-Abelian self-dual string:

- $\mathscr{G}$ is a 2-group model for the string group
- Can trivialize $\mathscr{G}$ in trivial gerbe $\mathscr{G} \times S^3$
- Should yield very interesting and physically relevant solution
- Connective structure: work in progress with L. Schmidt

Summary:

- ✓ Clear physical and mathematical motivation to study HGT
- ✓ straightforward definition of higher gauge theory
- ✓ Can make cocycles etc. explicit to calculate with them
- ✓ Twistor constructions of (2,0) theory
- ✓ Relevant higher gauge group identified

Soon to come:

- ▷ Examples of 2-bundles with connections
- ▷ Link to higher quantization
- ▷ Localization for higher gauge theories

- Monopole/instanton solutions: gauge group from spin group $\mathsf{Spin}(3) \cong \mathsf{SU}(2)$, $\mathsf{Spin}(4) \cong \mathsf{SU}(2) \times \mathsf{SU}(2)$
- Higher analogue of the spin group: String group
  Stolz, Teichner, Witten, ...
- Def. via Whitehead tower (iteratively delete homotopy groups)

$$\ldots \rightarrow \mathsf{String}(n) \rightarrow \mathsf{Spin}(n) \rightarrow \mathsf{Spin}(n) \rightarrow \mathsf{SO}(n) \rightarrow \mathsf{O}(n)$$

- Definition only up to homotopy, as a group: ∞-dimensional
- 2-group models:
  - ∞-dimensional strict 2-group — Baez et al., Nikolaus et al.
  - finite-dimensional quasi 2-group — Schommer-Pries
- Higher gauge theory 1602.03441, G A Demessie and CS
- Conjecture: Gauge 2-group for M5-branes is $\mathsf{String}(E_8)$

Review: The 't Hooft-Polyakov Monopole

The 't Hooft-Polyakov Monopole is a non-singular solution with charge 1.

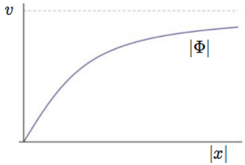
Recall 't Hooft-Polyakov monopole (e_i generate $\mathfrak{su}(2)$, $\xi = v|x|$):

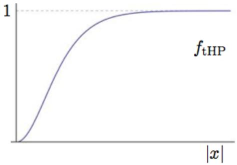
$$\Phi = \frac{e_i x^i}{|x|^2}\left(\xi \coth(\xi) - 1\right) , \quad A = \varepsilon_{ijk}\frac{e_i x^j}{|x|^2}\left(1 - \frac{\xi}{\sinh(\xi)}\right) \, \mathrm{d}x^k$$

- At S^2_∞: $\Phi \sim g(\theta) e_3 g(\theta)^{-1}$.
  $g(\theta) : S^2_\infty \to \mathrm{SU}(2)/\mathrm{U}(1)$: winding 1
- Charge $q = 1$ with

$$2\pi q = \frac{1}{2}\int_{S^2_\infty} \frac{\mathrm{tr}\left(F^\dagger \Phi\right)}{||\Phi||} \quad \text{with} \quad ||\Phi|| := \sqrt{\frac{1}{2}\,\mathrm{tr}\left(\Phi^\dagger \Phi\right)}$$

- Higgs field non-singular:





Elementary Solutions: A Non-Abelian Self-Dual String

We can write down a non-abelian self-dual string with winding number 1.

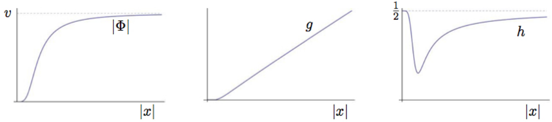
Self-Dual String (Lie 2-algebra $\mathfrak{su}(2)\times\mathfrak{su}(2)\xleftarrow{\mu_1}\mathbb{R}^4$, $\xi=v|x|^2$):

$$\Phi=\frac{e_\mu x^\mu}{|x|^3}\,f(\xi)\ ,\quad B_{\mu\nu}=\varepsilon_{\mu\nu\kappa\lambda}\frac{e_\kappa x^\lambda}{|x|^3}\,g(\xi)\ ,\quad A_\mu=\varepsilon_{\mu\nu\kappa\lambda}D(e_\nu,e_\kappa)\,\frac{x^\lambda}{|x|^2}\,h(\xi)$$

- Solves indeed $H=\star\nabla\Phi$ for right $f(\xi)$, $g(\xi)$, $h(\xi)$
- At S^∞_3: $\Phi\sim g(\theta)\rhd e_4$. $g(\theta):S^3_\infty\to\mathsf{SU}(2)$ has winding 1.
- Charge $q=1$:

$$(2\pi)^3q=\tfrac{1}{2}\int_{S^3_\infty}\frac{(H,\Phi)}{||\Phi||}\quad\text{with}\quad ||\Phi||:=\sqrt{\tfrac{1}{2}(\Phi,\Phi)}\ ,$$

- Higgs field non-singular:



Principal String 2-Group Bundles and M-Branes

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