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On the stability of vector bundles

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Abstract

In this work we present vector bundles and their properties. We will expose the construction of the moduli space of vector bundles on a smooth projective algebraic curve, following Gieseker´s construction, as a way of classifying them. We will talk about the notion of stability which let us make a construction for the semistable bundles and we will give some ideas about Geometric Invariant Theory, one-parameter subgroups, and their contributions to our problem. Afterwards, we give the Harder-Narashiman filtration for unstable vector bundles. Finally, we consider the Kempf's one-parameter subgroup, as the best way of destabilizing for an unstable point and we apply it to the case of vector bundles to obtain a destabilizing filtration. The main result of the work is to conclude that the Kempf filtration and the Harder-Narashiman filtration are really the same.

Mathematical Subject Classification (2010): primary 14H60; secondary 14D20.

Key words: moduli space, vector bundles, algebraic curves, Geometric Invariant Theory, Harder-Narasimhan filtration, Kempf one-parametric subgroup

Resumen

En este trabajo presentamos los fibrados vectoriales y sus propiedades. Expondremos la construcción del espacio de móduli de fibrados vectoriales sobre una curva algebraica proyectiva no singular, siguiendo la construcción de Gieseker, como forma de clasificarlos. Hablaremos acerca de la noción de estabilidad que nos permite realizar una construcción para los fibrados semiestables y daremos algunas ideas acerca de la Teoría de Invariantes Geométricos, los subgrupos uniparamétricos, y sus contribuciones a nuestro problema. Después, damos la filtración de Harder-Narasimhan para los fibrados vectoriales inestables. Finalmente, consideramos el subgrupo uniparamétrico de Kempf, como la mejor forma de desestabilizar un punto inestable y lo aplicamos al caso de fibrados vectoriales para obtener una filtración desestabilizante. El principal resultado del trabajo es concluir que la filtración de Kempf y la filtración de Harder-Narasimhan son realmente la misma.

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Palabras clave: espacio de moduli, fibrados vectoriales, curvas algebraicas, Teoría de Invariantes Geométricos, filtración de Harder-Narasimhan, subgrupo uniparamétrico de Kempf

Contents

Introduction

Vector bundles are an important object of study. They make precise the idea of a family of vector spaces parametrized by another space, which can be a topological space, a differential manifold, an algebraic variety, etc. In algebraic geometry, they are used to provide functions over algebraic varieties. A complex projective algebraic variety is a compact topological space, thus its global functions to the complex field are only the constants. Then, vector bundles provide us global sections -meromorphic functions- over algebraic varieties, apart from the global functions. Knowing whether a vector bundle has global sections or not and, in the affirmative case, computing the dimension of the space of global sections, is precisely knowing the dimension of the vector space of global functions the vector bundle defines over our variety.

In fact, vector bundles are a particular case of a more general construction, principal bundles (where the transition functions map in a group G, in general different from the general linear group). Principal bundles have important applications in topology, differential geometry and gauge theories in physics. They provide a unifying framework for the theory of fiber bundles in the sense that all fiber bundles with structure group G determine a unique principal G-bundle from which the original bundle can be reconstructed.

There are two main purposes in this report. The first one is to study vector bundles. The second one is to pave the way for the study of tensors, which are vector bundles with additional structure.

Dealing with vector bundles, beyond the background of the framework given in Chapter 1, we encounter the idea of moduli. In mathematics, to study something is, in one way or another, to classify the objects we are working with. And this is the idea of what a moduli space is: a good way of parametrizing a family of objects, meaning that the moduli space has a good structure (it will be an algebraic variety) and that isomorphism classes of objects correspond with points of it. We will construct a moduli space for vector bundles over a given curve, with their algebraic invariants fixed, degree, rank and genus.

The first construction was due to Mumford, who gave the moduli space for stable sheaves over algebraic curves. The next was the construction of Narashiman and Seshadri for semistable sheaves over curves, with which we get the compactness of the moduli space. Gieseker and Maruyama extended it to the case of an algebraic variety higher dimension and later, Simpson gave another method to do it. The last result of Adrian Langer obtains a moduli space for semistable sheaves over an algebraic variety of dimension n, in characteristic p. The difficulty in higher dimension is to prove the boundedness of semistable sheaves.

Here we are going to follow the construction of the moduli space of semistable vector bundles over curves due to David Gieseker. To complete, in some way, the classification, we use the filtration given by Harder and Narasimhan. This filtration characterizes the unstable vector bundles and it is the best way to destabilize them.

In order to study the semistable vector bundles, as well as the unstable ones, we use the powerful machinery of the Geometric Invariant Theory, due to Mumford. This framework appears in the heart of many problems in geometry. In our case, it gives numerical criteria to determine the type of the bundle through one-parameter subgroups, which can destabilize or not.

In the unstable case some one-parameter subgroups turn out to be better than others. And this was the inspiration of George Kempf: the fact that there exists a way to destabilize a unstable object which is the best one. The main purpose of this work is to prove that the Harder-Narasimhan filtration and the Kempf filtration (the filtration provided by the one-parameter subgroup defined in Kempf's article) are the same. Both of them are the best way to destabilize a unstable vector bundle.

In the case of tensors there have been several attempts at giving a similar destabilizing filtration. The stability for tensors depends on a parameter which relates the vector bundle and the additional information (a morphism). We cannot obtain a maximum destabilizing object because it can destabilize the vector bundle or the morphism but not both at the same time. The aim is to use the Kempf's one-parameter subgroup to give the best way of destabilizing for tensors.

There are very important theorems of Mehta and Ramanathan on the restriction of a semistable vector bundle from a variety to a subvariety. However, they are proved only in the case of sheaves. Being able to give a Kempf filtration in the case of tensors would definitely help in order to extend these results.

Chapter 1

Background

In this section we present the concepts and properties that we are going to need in this work. We want to emphasize that the theory and most of the general results are stated in the framework of schemes. To deal with schemes is more difficult because they represent the natural generalization of algebraic varieties but with a very technical language. Given that the main argument can be followed substituting scheme for algebraic variety we have preferred to take the easier way. Similarly we have omitted several considerations about sheaf cohomology and Chern classes. For a complete treatment of schemes and cohomology of sheaves, see [Ha].

We give the basic definitions about sheaves over algebraic curves. We present vector bundles and their connection with classes of Cech cohomology, and we discuss the definition of the degree of a vector bundle. For all of this, we suppose the reader to be familiarized with algebraic varieties.

In this work X will be a smooth projective algebraic variety of dimension 1 over $\mathbb{C}$, which is the same that a Riemann surface. We call it algebraic curve or just curve.

1.1 Sheaves

Definition 1.1.1. *A* presheaf $\mathcal{F}$ *of abelian groups (resp. rings, modules...) over a topological space* X *consists of the data*

- *for every open subset* $U \subset X$*, an abelian group* $\mathcal{F}(U)$
- *for every inclusion* $V \subset U$ *of open subsets of* X*, a morphism of abelian groups* $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$*, subject to the conditions*
  1. $\mathcal{F}(\emptyset) = 0$*, where* $\emptyset$ *is the empty set*
  2. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ *is the identity map*
  3. *if* $W \subset V \subset U$ *are three open subsets, then* $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

We define a presheaf of rings, modules, etc..., by replacing the words "abelian group" in the definition by "ring", "module", etc.

If $\mathcal{F}$ is a presheaf on X, we refer to $s \in \mathcal{F}(U)$ as the sections of the presheaf $\mathcal{F}$ over the open set U, and we use the notation $\Gamma(U, \mathcal{F})$ to denote the group $\mathcal{F}(U)$. We call the maps ρ_{UV} restriction maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$, if $s \in \mathcal{F}(U)$.

Remark 1.1.2. *If we consider, for any topological space X, the category* $\mathfrak{Top}(X)$*, whose objects are the open subsets of X, and where the only morphisms are the inclusion maps, a presheaf is just a contravariant functor from the category* $\mathfrak{Top}(X)$ *to the category* $\mathfrak{Ab}$ *of abelian groups.*

Definition 1.1.3. *A presheaf* $\mathcal{F}$ *over a topological space* X *is a* sheaf *if it satisfies the following supplementary conditions:*

4. *if* U *is an open set, if* $\{V_i\}$ *is an open covering of* U*, and if* $s \in \mathcal{F}(U)$ *is an element such that* $s|_{V_i} = 0$ *for all* i*, then* $s = 0$
5. *if* U *is an open set, if* $\{V_i\}$ *is an open covering of U, and if we have elements* $s_i \in \mathcal{F}(V_i)$ *for each* i*, with the property that for each* i, j*,* $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$*, then there is an element* $s \in \mathcal{F}(U)$ *such that* $s|_{V_i} = s_i$ *for each* i

(Note that condition (4.) implies that s is unique).

Definition 1.1.4. *If $\mathcal{F}$ is a presheaf over X, and if P is a point of X, we define the* stalk *$\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P, via the restriction maps ρ.*

Given a sheaf of rings $\mathcal{A}$ on X we can consider a sheaf of modules $\mathcal{F}$ over the sheaf of rings $\mathcal{A}$ in such a way that on each open set $U \subset X$, $\mathcal{F}(U)$ is a module over the ring $\mathcal{A}(U)$ and the restriction homomorphisms agree with the module structure. The sheaf $\mathcal{F}$ is said to be a sheaf of $\mathcal{A}$-modules.

Definition 1.1.5. *Let $\mathcal{A}$ be a sheaf of rings on X. A sheaf of $\mathcal{A}$-modules, $\mathcal{F}$, is* free *if $\mathcal{F} \simeq \mathcal{A}^r$, for any r. $\mathcal{F}$ is* locally free *if there exists a covering of $\{U_i\}$ of X such that $\mathcal{F}|_{U_i} \simeq \mathcal{A}^r|_{U_i}$, for any r.*

Definition 1.1.6. *Let $\mathcal{F}$ be a sheaf over X. The* support *of $\mathcal{F}$, denoted* $\operatorname{Supp}\mathcal{F}$*, is defined to be $\{P \in X : \mathcal{F}_P \neq 0\}$, where $\mathcal{F}_P$ is the stalk of $\mathcal{F}$ at P.*

Definition 1.1.7. *A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is a* torsion sheaf *if for each open set $U \subset X$ and for all $s \in \mathcal{F}(U)$, s is a torsion element ($m \neq 0$ is a torsion element of a module M over a ring A if there exists an element $a \in A$, $a \neq 0$ such that $a \cdot m = 0$).*

Definition 1.1.8. *A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is a* torsion-free sheaf *if for each open set $U \subset X$ the module $\mathcal{F}(U)$ has no torsion elements.*

Definition 1.1.9. *A sheaf $\mathcal{F}$ is* pure *if $\forall \mathcal{F}' \subset \mathcal{F}$, $\dim\operatorname{Supp}\mathcal{F}' = \dim\operatorname{Supp}\mathcal{F}$.*

The intuitive idea of a pure sheaf is a sheaf which is torsion-free if we restrict it to its support. Note that if $\mathcal{F}$ is torsion-free, then $\mathcal{F}$ is pure.

Remark 1.1.10. *Note that to be locally free implies to be torsion free. Indeed, if a sheaf $\mathcal{F}$ is locally free, in a neighborhood of each point is isomorphic to certain $\mathcal{A}^r$, thus locally has no torsion elements. If our variety X is a smooth curve, the converse is also true because the locus of singularities of a torsion free sheaf over X has at least codimension 2, so it is locally free. Another way to see this is to consider the local algebra problem. We can see a torsion-free sheaf of modules over the structure sheaf of rings of a curve, locally, as a torsion-free, finitely generated module over a* *local regular ring of dimension 1, which is, in particular, a principal ideal domain. Then, we have a structure theorem for finitely generated modules over a principal ideal domain, so if the module is torsion-free, hence is free. Therefore, our sheaf will be locally free. If the curve X is non-smooth, it is not true: see the local ring of a node $\frac{(X,Y)}{(XY)}$, which is not a principal ideal domain (the ideal of the origin, $\frac{(X,Y)}{(XY)}$ is not a principal ideal, therefore $\widetilde{\frac{(X,Y)}{(XY)}}$ is not locally free as a sheaf of modules).*

However, the converse is not true in general. See the example of $\widetilde{(X,Y)}$ as a sheaf of $\mathbb{C}[X,Y]$-modules over Spec $\mathbb{C}[X,Y]$. The stalk at the origin has dimension 2. Otherwise it has dimension 1. Therefore $\mathcal{F}$ is a torsion-free sheaf because, locally, the modules have no torsion elements but it is no a locally free sheaf because we cannot give a trivialization in a neighborhood of the origin.

Definition 1.1.11. *If $\mathcal{F}$ is a torsion-free sheaf, $\mathcal{F}' \subset \mathcal{F}$ is* saturated *if $\mathcal{F}/\mathcal{F}'$ is a torsion-free sheaf. If $\mathcal{F}$ is a pure sheaf, $\mathcal{F}' \subset \mathcal{F}$ is* saturated *if $\mathcal{F}/\mathcal{F}'$ is a pure sheaf.*

1.2 Vector bundles

The main objects about which we will discuss are vector bundles over a curve X. Here we give the definition:

Definition 1.2.1. *A (complex)* vector bundle of rank r over X *is a smooth manifold E together with a smooth morphism $\pi : E \longrightarrow X$ with the following properties. There is an open covering $\{U_i\}$ of X such that for every U_i there is a biholomorphism h_i making the following diagram commutative*

$$\begin{array}{ccc} E|_{U_i} & \xrightarrow{h_i} & U_i \times \mathbb{C}^r \\ {\scriptstyle \pi|_{U_i}} \downarrow & \swarrow {\scriptstyle p_1} & \\ U_i & & \end{array}$$

where p_1 is projection to the first factor (in other words, π is locally trivial*). And for every pair (i,j), the composition $h_i \circ h_j^{-1}$ is linear on the fibers, i.e.*

$$h_i \circ h_j^{-1}(x,v) = (x, g_{ij}(v)),$$

where $g_{ij} : U_i \cap U_j \longrightarrow GL(r, \mathbb{C})$ *is a holomorphic morphism. The morphisms* g_{ji} *are called the* transition functions.

Given two vector bundles over X, namely (E, π) and (E', π'), an isomorphism between them is an isomorphism $\varphi : E \longrightarrow E'$ which is compatible with the linear structure. That is, $\pi = \pi' \circ \varphi$ and the covering $\{U_i\} \bigcup \{U'_i\}$ together with the isomorphisms h_i, $h'_i \circ \varphi$ is a linear structure on E as before.

Given two vector bundles E and F, there is a vector bundle $E \otimes F$ whose fiber $E \otimes F|_x$ over $x \in X$ is canonically isomorphic to the tensor product of the fibers $E|_x \otimes F|_x$. If g_{ij} and h_{ij} are the transition functions of E and F, then the transition functions describing $E \otimes F$ are of the form $g_{ji} \otimes h_{ij}$. Analogously, the usual constructions with vector spaces can be defined for vector bundles: the dual $E^\vee$, symmetric and skew-symmetric products $Sym^m E$, $\bigwedge^m E$, etc...

Remark 1.2.2. *The set of isomorphism classes of vector bundles of rank* r *on* X *is canonically bijective to the Cech cohomology set* $\check{H}^1(X, \underline{GL(r, \mathbb{C})})$. *Indeed, we choose a trivialization of the vector bundle in open sets* $\{U_i\}_{i \in I}$ *and we consider the chain complex where*

$$\begin{array}{l} C^0 = \{f_i : U_i \longrightarrow GL(r, \mathbb{C})\} \\ C^1 = \{g_{ij} : U_{ij} = U_i \cap U_j \longrightarrow GL(r, \mathbb{C})\} \\ C^2 = \{h_{ijk} : U_{ijk} = U_i \cap U_j \cap U_k \longrightarrow GL(r, \mathbb{C})\} \end{array}$$

and the coboundary map is

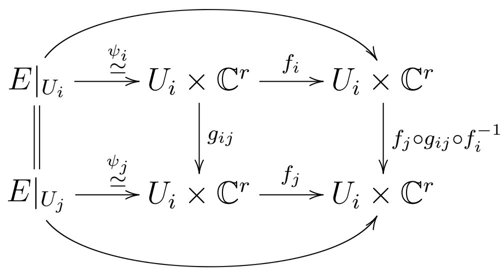
$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2$$

$$f_i \longmapsto f_i \circ f_j^{-1}$$

$$g_{ij} \longmapsto g_{ij} \circ g_{kj}^{-1} \circ g_{ki}$$

We can identify each vector bundle E *of rank* r *with an element of* $\check{H}^1(X, \underline{GL(r, \mathbb{C})}) = \dfrac{\operatorname{Ker} d^1}{\operatorname{Im} d^0}$, *associating the element of* C^2 *given by the transition functions* g_{ij} *in each intersection of the trivialization. This element is in* $\operatorname{Ker} d^1$ *because the functions* g_{ij}

satisfy the cocycle condition. Moreover, two elements associated to the same vector bundle differ by an element of $\operatorname{Im} d^0$*, as the following diagram shows:*



A local section of a vector bundle E is a morphism $s : U_i \longrightarrow E$ such that $\pi|_{U_i} \circ s = id|_{U_i}$.

Given a vector bundle E over X we define the locally free sheaf $\mathcal{E}$ of its sections, which assigns to each open set U the set of sections on $U \subset X$, $\mathcal{E}(U) = \Gamma(U, \pi^{-1}(U))$. This provides an equivalence of categories between the categories of vector bundles and that of locally free sheaves, (see [We], Thm. 1.13, p. 40). Therefore, if no confusion seems likely to arise, we will use the words "vector bundle" and "locally free sheaf" interchangeably.

1.3 Degree of a vector bundle

We define a *divisor* D in a curve X as a formal sum of points $D = \sum_i n_i P_i$ locally finite (hence finite), where the coefficients n_i are integers. Given a divisor D and given a point $x \in X$, there exists a meromorphic function $h = \frac{f_1 \cdot \ldots \cdot f_l}{g_1 \cdot \ldots \cdot g_s}$ in a neighborhood U of x which defines the divisor D, i.e. the points of $D \cap U$ with positive coefficient are exactly the zeroes of the $f's$ and those with negative coefficient are exactly the zeroes of the $g's$. Then, for every divisor D there exists a covering $\{U_i\}$ of X where, for each open set U_i, D has meromorphic equation h_i. In each $U_{ij} = U_i \cap U_j$ the meromorphic function $g_{ij} = \frac{h_i}{h_j}$ is an unitary element of $\mathcal{O}^*(U_{ij})$ since it has no zeroes or poles (both

h_i and h_j define the same divisor in U_{ij}). We call them transition functions

$$g_{ij} : U_{ij} \longrightarrow GL(1, \mathbb{C})$$

and we see they verify the cocycle condition. Therefore, given a divisor D, we can define a vector bundle of rank 1 as that with precisely the transition functions g_{ij} associated to the covering $\{U_i\}$.

We define the *degree* of a divisor $D = \sum_i n_i P_i$ over a curve X as the sum of the coefficients

$$\deg D = \sum_i n_i \ .$$

We want to give a definition of degree for vector bundles. With the correspondence between divisors and line bundles, we define the degree of a line bundle as the degree of its associated divisor. Roughly speaking, the degree of a line bundle will be the number of zeroes minus the numbres of poles of a rational section (counted with multiplicity).

If E is a vector bundle or a locally free sheaf over a smooth projective variety X of dimension n, we define the *determinant line bundle* as $\det E = \bigwedge^r E$. If E is torsion free, since X is smooth, we can still define its determinant as follows. The maximal open subset $U \subset X$ where E is locally free is *big* (with this we will mean that its complement has codimension at least two), because it is torsion free. Therefore, there is a line bundle $\det E|_U$ on U, and since U is *big* and X is smooth, this extends to a unique line bundle on X, which we call the determinant of E, denoted $\det E$. Then, we can define the degree of a vector bundle of rank r as the degree of its determinant line bundle.

The formal definition of the degree of a torsion free sheaf is given in terms of Chern classes.

Definition 1.3.1. *Let X be an algebraic curve embedded in a projective space, i.e. with an fixed ample line bundle $\mathcal{O}_X(1)$ corresponding to a divisor H. Let E be a torsion free sheaf on X. Its Chern classes are denoted $c_i(E) \in H^{2i}(X;\mathbb{Z})$. We define the* degree *of E*

$$deg_H E = \int c_1(E) \wedge c_1(H)^{n-1}$$

It can be proved that $\deg E = \deg(\det E)$.

Definition 1.3.2. *Given a sheaf $\mathcal{F}$ over X, we define its* Euler characteristic $\chi(E)$ *as*

$$\chi(E) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F})$$

Definition 1.3.3. *We define the* Hilbert polynomial *of a torsion-free sheaf E*

$$P_E(m) = \chi(E(m)) \ ,$$

where $E(m) = E \otimes \mathcal{O}_X(m)$ (called the twist *of E by m) and $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$.*

Chapter 2

The moduli space of vector bundles

Here we are going to give a rather good answer to the problem of classifying vector bundles over a given algebraic curve X, fixing the numerical invariants as data. This is the idea of constructing the moduli space of vector bundles, meaning an algebraic variety which parametrizes the bundles. We will follow the construction of a moduli space for semistable torsion free sheaves on a smooth projective curve, given by David Gieseker in 1977 (see [Gi1]). There is a more recent construction, due to Simpson (see [Si]), which we will comment in Apendix A.

The construction of the moduli space will require of Mumford's Geometric Invariant Theory (see [Mu]) about which we will give the main ideas.

2.1 The concept of moduli

Moduli spaces were born to give a solution to classification problems, specifically in algebraic geometry. A general classification problem consist of a collection of objects A and an equivalence relation $\sim$ on A; the problem is to describe the set of equivalence classes $A/\sim$. Sometimes there are discrete invariants which partition $A/\sim$ into a countable number of subsets, but in algebraic geometry this very rarely gives a complete solution of the problem. Almost always there exist "continuous families" of objects of A, and we would like to give $A/\sim$ some algebro-geometric structure to reflect this fact. This is the object of the theory of moduli.

More precisely, the existence of non-trivial automorphisms of the objects being classified makes it difficult to have a moduli space as the set of equivalence classes. However, it is often possible to consider a modified moduli problem of classifying the original objects together with additional data, chosen in such a way that the identity is the only automorphism respecting also the additional data. With a suitable choice of the rigidifying data, the modified moduli problem will have a moduli space.

The modern formulation of moduli problems and definition of moduli spaces dates back to Alexander Grothendieck, (1960/1961), *"Techniques de construction en géométrie analytique. I. Description axiomatique de l'espace de Teichmüller et de ses variantes."*, in which he described the general framework, approaches and main problems using Teichmüller spaces in complex analytical geometry as an example. The talks in particular describe the general method of constructing moduli spaces, fixing data in the moduli problem.

Geometric Invariant Theory (GIT), developed by David Mumford, shows that under suitable conditions we can give a solution to the problem.

Another general approach is primarily associated with Michael Artin. Here the idea is to start with any object of the kind to be classified and study its deformation theory.

Example 2.1.1. *The complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is a moduli space. It is the space of lines in $\mathbb{C}^{n+1}$ which pass through the origin. More generally, the Grassmannian $\mathcal{GR}(k,n)$ is the moduli space of all k-dimensional linear subspaces of $\mathbb{C}^n$.*

Example 2.1.2. *We consider the collection A of all the non-singular complex cubics. By a change of coordinates we can think them of the form $y^2 = x(x-1)(x-\lambda)$, where $\lambda \in \mathbb{C}$. Then we define $j = j(\lambda) = 2^8 \dfrac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$, called the j-invariant of a curve. We see that all the non-singular complex cubics are parametrized by the affine complex line (an algebraic variety), by the j-invariant of a curve. Hence, to classify cubics is the same that to give a 1-dimensional variety where each point corresponds to a cubic.*

Remark 2.1.3. *The word "moduli" was coined by Riemann, in his celebrated paper of 1857 on abelian functions.*

2.2 Gieseker construction of the moduli space of vector bundles

Once we have an idea of what a moduli space is in mathematics, let us go to construct a moduli space for vector bundles over a given curve X. We will construct a moduli space for a certain class of vector bundles, the semistable ones. Here we give a definition of stability which will arise naturally from the construction at the end.

Definition 2.2.1. *A torsion free sheaf E over an algebraic variety X is called* (semi)stable *if for all proper subsheaves $F \subset E$,*

$$\frac{P_F}{\operatorname{rk} F} \underset{(<)}{\leq} \frac{P_E}{\operatorname{rk} E} .$$

A torsion free sheaf is called unstable if it is not semistable. Sometimes this is referred to as Gieseker (or Maruyama) stability.

Definition 2.2.2. *A torsion free sheaf E over X is* slope-(semi)stable *if for all proper subsheaves $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$,*

$$\frac{\deg F}{\operatorname{rk} F} \underset{(<)}{\leq} \frac{\deg E}{\operatorname{rk} E} .$$

The number $\deg E/\operatorname{rk} E$ is called the slope of E. A torsion free sheaf E is called slope-unstable if it is not slope-semistable. Sometimes this is referred to as Mumford (or Takemoto) stability.

Using Riemann-Roch theorem, we find

$$P_E(m) = \operatorname{rk} E \frac{m^n}{n!} + (\deg E - \operatorname{rk} E \frac{\deg K}{2}) \frac{m^{n-1}}{(n-1)!} + \cdots$$

where K is the canonical divisor. From this it follows that

$$\text{slope-stable} \Longrightarrow \text{stable} \Longrightarrow \text{semistable} \Longrightarrow \text{slope-semistable}$$

Note that, if $n = 1$, Gieseker and Mumford (semi)stability coincide, because the Hilbert polynomial has degree 1.

We suppose our curve X embedded in a projective space with a very ample line bundle $H = \mathcal{O}_X(1)$ (called a polarization of X). The concept of stability depends on the Hilbert polynomial of E which depends on $\deg(E)$, and the degree of E depends at first on the polarization of X by definition in terms of Chern classes:

$$\deg_H E = \int c_1(E) \wedge c_1(H)^{n-1} ,$$

where n is the dimension of X. See that, for 1-dimensional varieties ($n = 1$) the stability of E does not depend on the polarization.

To begin with the construction we have to remark the idea of boundedness and the crucial *Vanishing Theorem of Serre.*

Definition 2.2.3. *Let m be an integer. A vector bundle E over a variety X is said to be m-*regular *if*

$$H^i(E(m-i)) = 0 ,$$

for all $i > 0$.

As a consequence of the definition, if E is m-regular, then $E(m)$ is generated by global sections (the evaluation map is surjective), and E is again m'-regular for all integers $m' \geq m$.

Theorem 2.2.4 (Serre's Vanishing Theorem). *Given a vector bundle E over a variety X, there exists an $m \in \mathbb{Z}$ big enough such that E is m-regular, i.e., $h^i(E(m-i)) = 0, i > 0$ and $E(m)$ is generated by global sections.*

Definition 2.2.5. *A set $\mathcal{E} = \{E\}$ of vector bundles is* bounded *if there exists a family $\mathcal{E} \longrightarrow X \times T$ parametrized by scheme of finite type T.*

In our context, a scheme of finite type T means a finite union of finite dimension varieties.

For the proof of the two following propositions we refer to [H-L1].

Proposition 2.2.6. *Let E be a vector bundle over X. If $\{E'\}$ is a family of vector subbundles, $E' \subset E$, with their numerical invariants bounded, then $\{E'\}$ is a bounded family. Similarly, if $\{E''\}$ is a family of quotients, $E \to E''$, with their numerical invariants bounded, then $\{E''\}$ is a bounded family.*

Here, the numerical invariants are the coefficients of the Hilbert polynomial.

Proposition 2.2.7. *Given a bounded family $\{E\}$, there exists an $m \in \mathbb{Z}$ such that E is m-regular $\forall E \in \{E\}$.*

Maruyama proved that all the semistable vector bundles E over an n-dimensional algebraic variety X are a bounded family. Thus, we can choose the same m for all the vector bundles in the result of Serre. Then, we twist our semistable vector bundle E with $\mathcal{O}_X(m)$ in such a way that $H^i(E(m)) = 0, i > 0$.

So we have all of ingredients we need to begin the construction of the moduli space. In the following, let X be an algebraic curve with a polarization H. We fix a line bundle $L \in \mathrm{Pic}(X)$, and let E be a semistable vector bundle of rank r and degree d over X such that $\det(E) = \bigwedge^r(E) \simeq L$. Let $m \in \mathbb{Z}$ an integer working for all the semistable vector bundles E in the Vanishing Theorem of Serre. Thus, we can compute the dimension of the space of global sections with the Riemann-Roch theorem:

$$h^0(E(m)) = \chi(E(m)) = \deg(E(m)) + r(1-g) = d + rm + r(1-g) = N \ ,$$

where $\deg(E(m)) = d + rm$ is the well-known formula for the degree.

Serre's theorem says also that $E(m)$ is generated by global sections, so there exists a surjective map between sheaves, the evaluation morphism:

$$\begin{array}{ccc} H^0(E(m)) \otimes \mathcal{O}_X(U) & \stackrel{\mathrm{ev}}{\twoheadrightarrow} & E(m)(U) \\ s \otimes 1|_U & \longmapsto & s|_U \end{array} ,$$

for each open set U in X.

Remark 2.2.8. *The morphism to be surjective means that*

$$(1, ..., 1) \in \mathcal{O}_X \oplus ... \oplus \mathcal{O}_X \longmapsto s_1(x) + ... + s_N(x) \in E(m)(x) \ ,$$

generate all the r-dimensional vector space $E(m)(x)$ for each $x \in X$.

We want to define the previous map in $\mathbb{C}^N \otimes \mathcal{O}_X$, $N = \dim H^0(E(m))$. For this, we have to give an isomorphism α

$$H^0(E(m)) \otimes \mathcal{O}_X \overset{\alpha}{\simeq} \mathbb{C}^N \otimes \mathcal{O}_X = (\mathcal{O}_X)^N = \mathcal{O}_X \oplus ... \oplus \mathcal{O}_X$$

and then, we define the new surjective map:

$$\begin{array}{ccc} \mathbb{C}^N \otimes \mathcal{O}_X & & \\ {\scriptstyle \alpha}\downarrow & \searrow & \\ H^0(E(m)) \otimes \mathcal{O}_X & \xrightarrow[\text{ev}]{} & E(m) \end{array}$$

We will denote $\mathcal{U} \overset{\alpha}{\simeq} H^0(E(m))$, so we have $\mathcal{U} \otimes \mathcal{O}_X \twoheadrightarrow E(m)$. Taking cohomology we obtain

$$\mathcal{U} = H^0(\mathcal{U} \otimes \mathcal{O}_X) \longrightarrow H^0(E(m)) \ ,$$

a homomorphism between vector spaces.

Now we take the r-exterior power to obtain

$$\wedge^r \mathcal{U} \longrightarrow \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r(E(m))) \ ,$$

where the second morphism is given by taking a wedge product of r sections of $E(m)$ to get a section of $\wedge^r(E(m))$. Besides, we have $H^0(\wedge^r(E(m))) = H^0(\wedge^r E \otimes \mathcal{O}_X(rm)) \simeq H^0(L \otimes \mathcal{O}_X(rm)) = \mathcal{W}$, where we take an isomorphism $det(E) = \bigwedge^r(E) \overset{\beta}{\simeq} L$. Thus, we have homomorphisms of vector spaces

$$\wedge^r \mathcal{U} \longrightarrow \mathcal{W} \, .$$

Therefore, a pair (E, α) given by a semistable vector bundle E over X, of rank r, degree d and determinant line bundle isomorphic to L, and a choice of basis

$H^0(E(m)) \overset{\alpha}{\simeq} \mathbb{C}^N$ corresponds to a point of $\mathrm{Hom}(\wedge^r \mathcal{U}, \mathcal{W})$. We consider the projectivization $\mathbb{P}(\mathrm{Hom}(\wedge^r \mathcal{U}, \mathcal{W}))$ and we see (E, α) as a point in a projective space. The group $SL(N, \mathbb{C})$ represents the changes of basis in $H^0(E(m))$, so it gives an action on $\mathcal{U}$ and induces an action on $\mathbb{P}(\mathrm{Hom}(\wedge^r \mathcal{U}, \mathcal{W}))$.

Note that two isomorphisms $\bigwedge^r(E) \overset{\beta}{\simeq} L$ and $\bigwedge^r(E) \overset{\beta'}{\simeq} L$ only differ by multiplication by a scalar. Then, the points (E, α) and (E, α') map to the same point in the projective space.

Remark 2.2.9. *In fact, the changes of basis in $H^0(E(m))$ are represented by the action of $GL(N, \mathbb{C})$, but when we consider the projective space, two changes differing by multiplication by an scalar go to the same point. Similarly,* $\det(E) = \bigwedge^r(E) \simeq L$ *is fixed apart from a scalar so, in the projective space, all the isomorphisms are the same, hence we do not have to make an additional quotient due to that election.*

Now we give the definition of the *Quot*-scheme, due to Grothendieck.

Definition 2.2.10. *Let X an algebraic projective variety and let $\mathcal{F}$ a sheaf over X. We fix a polynomial P. We define the* Quot-scheme

$$Quot_{\mathcal{F},X,P} = \{\mathcal{F} \twoheadrightarrow E : \ P_E = P\} ,$$

the set of all the quotients where E is another sheaf over X with Hilbert polynomial P. An isomorphism of quotients is an isomorphism $\alpha : E \to E'$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F} & \overset{q}{\twoheadrightarrow} & E \\ \| & & \downarrow \alpha \, \simeq \\ \mathcal{F} & \overset{q'}{\twoheadrightarrow} & E' \end{array}$$

In our case, we can consider $\mathcal{F} = \mathcal{U} \otimes \mathcal{O}_X(-m)$ and the quotients are

$$\mathcal{F} = \mathcal{U} \otimes \mathcal{O}_X(-m) \twoheadrightarrow E ,$$

twisting by $-m$. Note that all our vector bundles E have the same Hilbert Polynomial, namely $P_E = d + r(1 - g)$. Then, we can see the pairs (E, α) as points of the *Quot*-scheme:

$$Z = \{(E \ , \ \alpha : \mathcal{U} \simeq H^0(E(m))\} \subset Quot_{\mathcal{U} \otimes \mathcal{O}_X(-m), X, P = d + r(1-g)}$$

Finally, to obtain the moduli space of semistable vector bundles we have to quotient our subset $Z \subset Quot$ by the action of $SL(N, \mathbb{C})$, the group representing the changes of basis in $H^0(E(m))$.

2.3 Geometric Invariant Theory

Let G be an algebraic group. Recall that a right action on a variety X is a morphism $\sigma : X \times G \to X$, which we will usually denote $\sigma(x, g) = x \cdot g$, such that $x \cdot (gh) = (x \cdot g) \cdot h$ and $x \cdot e = x$, where e is the identity element of G. A left action is analogously defined, with the associative condition $(hg) \cdot x = h \cdot (g \cdot x)$.

The orbit of a point $x \in X$ is the image $x \cdot G$. A morphism $f : X \to Y$ between two varieties endowed with G-actions is called *G-equivariant* if it commutes with the actions, that is $f(x) \cdot g = f(x \cdot g)$. If the action on Y is trivial (i.e. $y \cdot g = y$ for all $g \in G$ and $y \in Y$), then we also say that f is *G-invariant*.

If G acts on a projective variety X, $\psi : G \times X \to X$, a *linearization* of the action on an ample line bundle $\mathcal{O}_X(1)$ consists of giving an action on the total space L of the line bundle $\mathcal{O}_X(1)$, $\sigma : G \times L \longrightarrow L$, such that for every $g \in G$ and $x \in X$ there exists a isomorphism which takes a fiber onto another $L_x \longrightarrow L_{g \cdot x}$ (σ is linear along the fibers and the projection $L \to X$ is equivariant). It is the same thing as giving, for each $g \in G$, an isomorphism of line bundles $\widetilde{g} : \mathcal{O}_X(1) \longrightarrow \varphi_g^* \mathcal{O}_X(1)$, ($\varphi_g = \psi(g, \cdot)$) which also satisfies the previous associative property. We say also that $\sigma = \widetilde{\psi}$ is a lifting to L of the action ψ:

$$\begin{array}{ccc} G \times L & \xrightarrow{\sigma = \widetilde{\psi}} & L \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow{\psi} & X \end{array}$$

If $\mathcal{O}_X(1)$ is very ample, then a linearization is the same thing as a representation of G on the vector space $H^0(\mathcal{O}_X(1))$ such that the natural embedding

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))^\vee)$$

is equivariant.

At this point, we have a group $G = SL(N, \mathbb{C})$ acting on $Z \subset Quot \hookrightarrow \mathbb{P}(\mathrm{Hom}(\wedge^r \mathcal{U}, \mathcal{W}))$. We consider the set of orbits Z/G. When can we define Z/G as a variety $\mathcal{M}$, i.e., the points of Z/G correspond in a natural way to the points of $\mathcal{M}$?

Let us study the following example, due to David Gieseker, which clarifies the situation:

Example 2.3.1. *Given an integer N, let $V_N = \{\sum_{i+j=N} a_{ij} X_0^i X_1^j\}$ be the set of all homogeneous polynomials of degree N in two variables X_0 and X_1. Let $\mathbb{P}(V_N)$ be the set of lines in V_N. Then $G = SL(2, \mathbb{C})$ operates on V_N in the following way: if $g = (a_{ij}) \in G$ and $P \in V_N$, then*

$$P^g(X_0, X_1) = P(g^{-1} \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}) \ .$$

We have the following geometric interpretation of $f \in V_N$. The equation $\{f = 0\}$ defines a finite set of points with multiplicities in $\mathbb{P}^1$. The multiplicity of $p \in \mathbb{P}^1$ is just the order of vanishing or f at p. Thus $\{f = 0\}$ defines a divisor D_f on $\mathbb{P}^1$, and $\mathbb{P}(V_N)$ exactly corresponds to the space of divisors of degree N on $\mathbb{P}$. The corresponding action of G on the space of divisors is to move the divisors around by linear fractional transformations. We cannot form X/G as a variety, where $X = \mathbb{P}(V_N)$, since X/G is not Hausdorff in the classical topology. Indeed, consider $\overline{f} \in \mathbb{P}(V_N)$ and $f \in V_N$ a corresponding equation. We can find an element h in the orbit of f so that X_1^N occurs in $h(X_0, X_1)$ (i.e., $h(X_0, X_1) = a_0 X_0^N + a_1 X_0^{N-1} X_1 + \ldots + a_{N-1} X_0 X_1^{N-1} + X_1^N$). Set $h_t(X_0, X_1) = t^N h(tX_0, t^{-1}X_1)$ and note that $h_t(X_0, X_1)$ is in the orbit of f and h for every $t \neq 0$, since $h_t(X_0, X_1) = t^N \cdot h^{g_t}(X_0, X_1)$, with $g_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$ and all of them gives the same point in $X = \mathbb{P}(V_N)$. Then we cannot give X/G a Hausdorff structure so that $\phi : X \longrightarrow X/G$ is continuous. If ϕ were continuous, from

$$\lim_{t \to 0} h_t(X_0, X_1) = h_0(X_0, X_1) = X_1^N,$$

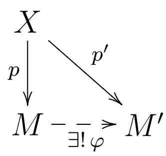
we would have

$$\phi(f) = \phi(h) = \lim_{t \to 0} \phi(h_t) = \phi(X_1^N).$$

So ϕ would have to map all orbits onto one element. The reason of this is that the polynomial X_1^N is not in the orbit of f and g, but is in its adherence. When we try to define a continuous quotient map, the adherent orbits have to go to the same point.

The example shows that, in order to obtain a "good" quotient (a quotient which is an algebraic variety), we have to make some considerations about the orbits of the action of our group G. Geometric Invariant Theory (GIT) will be a technique to construct such "good" quotients.

Definition 2.3.2. *Let X be a variety endowed with a G-action. A* categorical quotient *is a variety M with a G-invariant morphism $p : X \longrightarrow M$ such that for every other variety M', and G-invariant morphism p', there is a unique morphism φ with $p' = \varphi \circ p$*



Definition 2.3.3. *Let X be a variety endowed with a G-action. A* good quotient *is a variety M with a G-invariant morphism $p : X \longrightarrow M$ such that*

1. *p is surjective and affine*
2. *$p_*(\mathcal{O}_X^G) = \mathcal{O}_M$, where $\mathcal{O}_X^G$ is the sheaf of G-invariant functions on X.*
3. *If Z is a closed G-invariant subset of X, then $p(Z)$ is closed in M. Furthermore, if Z_1 and Z_2 are two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$.*

Definition 2.3.4. *A* geometric quotient *$p : R \to M$ is a good quotient such that $p(x_1) = p(x_2)$ if and only if the orbit of x_1 is equal to the orbit of x_2.*

Clearly, a geometric quotient is a good quotient, and a good quotient is a categorical quotient.

Let X be a projective variety, let G be a reductive algebraic group and an action $\sigma : G \times X \longrightarrow X$ of G on X. We call again σ a linearization of the action on an ample line bundle $\mathcal{O}_X(1)$. A closed point $x \in X$ is called *GIT-semistable* if, for some $m > 0$, there is a G-invariant section s of $\mathcal{O}_X(m)$, $s \in H^0(X, \mathcal{O}_X(m))$, such that $s(x) \neq 0$. If, moreover, the orbit of x is closed in the open set of all GIT-semistable points, it is called *GIT-polystable* and, if furthermore, this closed orbit has the same dimension as G (i.e. if x has finite stabilizer), then x is called a *GIT-stable* point. We say that a closed point of X is *GIT-unstable* if it is not GIT-semistable.

Using this definition, the stable points are precisely the polystable points with finite stabilizer.

Remark 2.3.5. *We consider X embedded in a projective space by the ample line bundle $\mathcal{O}_X(1)$*

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))^\vee) = \mathbb{P}(V)$$

We can see a section $s \in H^0(\mathcal{O}(m))$ as a polynomial homogeneous function of degree m in V. Then, the GIT-unstable points are those for which, for all $m > 0$, a G-invariant section $s \in H^0(\mathcal{O}(m))$ vanishes in the point. Therefore, as all homogeneous polynomials vanish at zero, the points which contains the zero in the closure of its orbit are GIT-unstable.

The core of Mumford's Geometric Invartiant Theory is the following theorem:

Theorem 2.3.6 (Mumford). *Let X^{ss} (respectively, X^s) be the subset of GIT-semistable points (respectively, GIT-stable). Both X^{ss} and X^s are open subsets. There is a good quotient $X^{ss} \longrightarrow X^{ss}/\!\!/G$ (where closed points are in one-to-one correspondence to the orbits of polystable points), the image $X^s/\!\!/G$ of X^s is open, $X/\!\!/G$ is projective, and the restriction $X^s \rightarrow X^s/\!\!/G$ is a geometric quotient (in the sense that it is an orbit space).*

A finite dimensional representation $\rho : G \longrightarrow GL(V)$ provides an action of G on $\mathbb{P}(V)$ and a linearization of this action in $\mathcal{O}_{\mathbb{P}(V)}(1)$, called again ρ. A point $[v] \in \mathbb{P}(V)$ represented by $v \in V$ is then semistable if and only if the closure of the orbit of v in V does not contain 0. It is polystable if, furthermore, its orbit is closed, and stable if the orbit of v in V is closed and the dimension of this orbit equals the dimension of G.

Finally, we define the moduli space of semistable vector bundles of rank r and degree d over an algebraic curve X. We had $Z = \{(E\ ,\ \alpha : \mathcal{U} \simeq H^0(E(m))\} \subset$

$Quot \hookrightarrow \mathbb{P}(\mathrm{Hom}(\wedge^r \mathcal{U}, \mathcal{W}))$. We consider the closure $\overline{Z}$ in *Quot* (in order to obtain a projective variety as a quotient) and we define the moduli space as the GIT-quotient

$$\mathcal{M}_X(r,d) = \overline{Z}/\!\!/SL(N,\mathbb{C}) \ .$$

Theorem (2.3.6) says that there is a good quotient $\overline{Z}^{ss} \longrightarrow \overline{Z}^{ss}/\!\!/G$. Gieseker and Maruyama proved that, in fact, the points of Z (the semistable vector bundles) are exactly the same that the GIT-semistable points of $\overline{Z}^{ss}$, so we have $Z = \overline{Z}^{ss}$ and the moduli space is

$$\mathcal{M}_X(r,d) = Z/\!\!/SL(N,\mathbb{C}) \ .$$

Therefore the unstable vector bundles which we had erased at the beginning were exactly the GIT-unstable points that we have had to erase now. Of course, very soon this notion of stability will be equivalent to the one given at the beginning, taking into account only the numerical invariants of the vector bundle.

2.4 One-parameter subgroups

Beyond the technical definitions, Geometric Invariant Theory gives us a numerical criterium, based on the use of one-parameter subgroups of G, to characterize GIT-semistable points.

Definition 2.4.1. *A* one-parameter subgroup *of* G*, denoted* $1-PS$*, is a non-trivial algebraic homomorphism* $\lambda : \mathbb{C}^* \longrightarrow G$*.*

In the following, we will consider the group $G = SL(N,\mathbb{C})$.

It follows from elementary representation theory that there is a basis $v_1, ..., v_N$ of $\mathbb{C}^N$ and $\lambda_i \in \mathbb{Z}$ such that

$$\lambda(t) = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_N} \end{pmatrix} ,$$

so we will refer it by giving the diagonal form. Note that $\sum \lambda_i = 0$ because $\lambda(t) \in SL(N, \mathbb{C})$.

Let X be the projective variety where the group G acts. Suppose that this action is linearized in a line bundle $\mathcal{O}_X(1)$ and call the linearization σ. Then, given a one-parameter subgroup λ of G and $x \in X$, we can define $\Phi : \mathbb{C}^* \longrightarrow X$ by $\Phi(t) = \lambda(t) \cdot x$. We say $\lim\limits_{t \to 0} \lambda(t) \cdot x = \infty$ if Φ cannot be extended to a map $\widetilde{\Phi} : \mathbb{C} \longrightarrow X$. If Φ can be extended, we write $\lim\limits_{t \to 0} \lambda(t) \cdot x = x_0$.

Then, the numerical criterium is the folowing:

Theorem 2.4.2. *With the previous notations:*

- x *is* semistable *if for all* 1*-PS* λ*,* $\lim\limits_{t \to 0} \lambda(t) \cdot x \neq 0$ *or* $\lim\limits_{t \to 0} \lambda(t) \cdot x = \infty$*.*
- x *is* polystable *if it is semistable and for all* 1*-PS* λ *such that* $\lim\limits_{t \to 0} \lambda(t) \cdot x = x_0$*,* $\exists g \in G$ *with* $x_0 = g \cdot x$
- x *is* stable *if for all* 1*-PS* λ*,* $\lim\limits_{t \to 0} \lambda(t) \cdot x = \infty$ *(then the stabilizer of* x *is finite)*
- x *is* unstable *if there exists a* 1*-PS* λ *such that* $\lim\limits_{t \to 0} \lambda(t) \cdot x = 0$

The point x_0 is clearly a fix point for the $\mathbb{C}^*$ on X induced by λ. Thus, $\mathbb{C}^*$ acts on the fiber of $L = \mathcal{O}_X(1)$ over x_0, say, with weight γ. One defines

$$\mu_\sigma(\lambda, x) := \gamma \ .$$

Note that this γ is the minimum exponent of the diagonal of the one-parameter subgroup which acts on a non-zero component of the point x, $\lambda(t) \cdot (x_1, ..., x_N) = (t^{\lambda_1} \cdot x_1, \dots, t^{\lambda_N} \cdot x_N)$. We will call this γ the *minimum relevant exponent.*

With the definition of γ we can give the Hilbert-Mumford criterion of GIT-stability:

Theorem 2.4.3 (Hilbert-Mumford criterion). *With the previous notations:*

- *x is* semistable *if for all 1-PS λ, $\mu_\sigma(\lambda, x) \leq 0$*
- *x is* polystable *if x is semistable and for all 1-PS λ with $\mu_\sigma(\lambda, x) = 0$, $\exists g \in G$ with $x_0 = g \cdot x$*
- *x is* stable *if for all 1-PS λ, $\mu_\sigma(\lambda, x) < 0$*
- *x is* unstable *if there exists a 1-PS λ such that $\mu_\sigma(\lambda, x) > 0$*

Example 2.4.4. *Returning to the example of Gieseker (2.3.1), we can apply this criterion. We had $G = SL(2, \mathbb{C})$ and $V_N = \{\sum_{i+j=N} a_{ij} X_0^i X_1^j\}$. Let us consider the one-parameter subgroup of G*

$$\lambda(t) = \begin{pmatrix} t^{-r} & 0 \\ 0 & t^r \end{pmatrix}, r > 0 \ .$$

Let P be a polynomial in V_N and we ask when $\lim_{t \to 0} P^{\lambda(t)} = \lim_{t \to 0} \lambda(t) \cdot P = 0$. If $P(X_0, X_1) = \sum a_{ij} X_0^i X_1^j$, then $P^{\lambda(t)}(X_0, X_1) = \sum a_{ij} X_0^i X_1^j t^{r(i-j)}$. So $\lim_{t \to 0} P^{\lambda(t)} = 0$ means that $a_{ij} = 0$ if $j \geq i$. That is, P has a factor of X_0^k with $k > \frac{N}{2}$, hence, in that case, P is unstable. For a general monoparametric, we can see that P is semistable if and only if P has no linear factors of degree $> \frac{N}{2}$.

Remark 2.4.5. *A theorem of GIT says that, if $G \cdot v$ is the orbit of a point $v \in V$, in its closure $\overline{G \cdot v}$ there is an unique orbit $Y \subset \overline{G \cdot v}$ such that Y is closed in $\overline{G \cdot v}$, so it is closed also in the whole space V. The GIT-polystable points are in correspondance with these closed orbits. Two orbits, $G \cdot v$ and $G \cdot w$, with the same closed orbit Y in their closures $\overline{G \cdot v}$, $\overline{G \cdot w}$, are called S-equivalent.*

Remark 2.4.6. *Geometric Invariant Theory says that we can go to every point in the closure of an orbit through one-parameter subgroups. Then the GIT-stability measures if 0 is an adherent point or not, and if there exist another adherent orbits to one given or not. The points of the moduli space are in correspondence with the distinguished closed orbits in (2.4.5), so we are classifying polystable points modulo S-equivalence.*

2.5 Application of GIT to the construction of the moduli space

Recall that we have the set of all the isomorphism classes of semistable vector bundles E of rank r, degree d and determinant line bundle L fixed, over an algebraic curve X. We have given a correspondence between each one of this E with a certain choice of basis, and a point of $Z \subset Quot$, a subvariety of a projective space. Thanks to GIT we can make a good quotient for the GIT-semistable points of $\overline{Z}$ (in order to erase the previous choice of basis, seen as a group acting on $Quot$), where the space obtained is an algebraic variety

$$\mathcal{M}_X(r,d) = \overline{Z}^{ss} /\!\!/ SL(N,\mathbb{C}) \ .$$

The main result here, due to Gieseker and Maruyama, is that every semistable vector bundle E with Hilbert Polynomial fixed (rank and degree fixed) and $\det(E) = L$, with L line bundle fixed, gives a GIT-semistable point of $\overline{Z}$ for any choice of basis (for any choice of isomorphism $\mathbb{C}^N \overset{\alpha}{\simeq} H^0(E(m))$). The proof of the result is rather long and technical. Thus, $Z = \overline{Z}^{ss}$ and

$$\mathcal{M}_X(r,d) = Z /\!\!/ SL(N,\mathbb{C}) \ .$$

Now, we apply the Hilbert-Mumford criteria in our case. We have a monoparametric $\lambda : \mathbb{C}^* \longrightarrow SL(N,\mathbb{C})$, where N is the dimension of $H^0(E(m)) \overset{\alpha}{\simeq} \mathbb{C}^N$, $N = \chi(E(m))$. Let us call $V = H^0(E(m))$ and V^i each one of the eigenspaces of the diagonalization of λ, where $V = \bigoplus_i V^i$ and $V_i = \bigoplus_{j=1}^i V^j$. Let $E_i(m)$ be the sheaf generated by the sections of V_i, $\operatorname{rk} E_i(m) = r_i$, and let us denote $E^i(m) = E_i(m)/E_{i-1}(m)$, $\operatorname{rk} E^i(m) = r^i$. Then, we can compute the weight of the action in the limit point, which is the same that the minimum relevant exponent, denoted $\mu(\lambda, E)$. It is given by

$$\mu(\lambda,E) = \dim V \cdot \sum_{n\in\mathbb{Z}} n \cdot r^n = \sum_{n\in\mathbb{Z}} n \cdot (r^n \dim V - r \cdot \dim V^n) = -\sum_{n\in\mathbb{Z}} (r_n \cdot \dim V - r \cdot \dim V_n) \ ,$$

and, in terms of the exponents λ_i of the diagonal of λ

$$\mu(\lambda,E) = \sum_{i=1}^N \lambda_i(-\dim V^i \cdot r + \dim V \cdot r^i) \ .$$

Note that, although the first sum is taken over all the integers, it will have only a finite number of nonzero terms.

The criteria concludes that, E is semistable (resp. stable), if and only if,

$$\dim(V') \cdot r \underset{(<)}{\leq} \dim V \cdot r' ,$$

for all V' a non-trivial proper subspace of V, $E'(m)$ the subsheaf generated by the global sections of V' and r' its rank.

See that $\dim V = \chi(E(m))$ and $\dim V' < h^0(E'(m))$. Then, if E is a unstable vector bundle, there exists V' strictly contained in V such that

$$\frac{\chi(E'(m))}{r} = \frac{h^0(E'(m))}{r'} > \frac{\dim V'}{r'} > \frac{\dim V}{r} = \frac{\chi(E(m))}{r} .$$

Since we have

$$\chi(E(m)) = h^0(E(m)) = \frac{d + rm + r(1-g)}{r} = \frac{d + r(1-g)}{r} + m =$$
$$= h^0(E) + m = \chi(E) + m ,$$

(and same for $E'(m)$) the characterization of the stability through minimum relevant exponents is the same that the definition of the Giesecker-stability, given in (2.2.1):

a torsion free sheaf E over an algebraic variety X is called *unstable* if for all proper subsheaves $F \subset E$,

$$\frac{P_F}{\operatorname{rk} F} > \frac{P_E}{\operatorname{rk} E} .$$

In the case of curves, it is the same that the slope-stability, given in (2.2.2):

a torsion free sheaf E over X is *unstable* if for all proper subsheaves $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$,

$$\frac{\deg F}{\operatorname{rk} F} > \frac{\deg E}{\operatorname{rk} E} .$$

Chapter 3

The Harder-Narasimhan filtration

In the previous chapter we have constructed an algebraic variety which parametrizes the semistable vector bundles of rank r and degree d over an algebraic curve X. This left out of the classification of the vector bundles over X the unstable ones. To study them, there is a powerful instrument, namely the Harder-Narasimhan filtration, which will exhibit unstable vector bundles as extensions of semistable ones. Hence, in some way, the classification problem will be completed.

Definition 3.0.1. *Let E be a torsion-free sheaf of rank r. A Harder-Narasimhan filtration for E is an increasing sequence*

$$\{0\} = E_0 \subsetneqq E_1 \subsetneqq E_2 \subsetneqq ... \subsetneqq E_{l-1} \subsetneqq E_l = E$$

verifying that the quotients $E^i = E_i/E_{i-1}\ \forall i \in \{1, ..., l\}$ are semistable sheaves of rank r, with slopes $\mu^i = \mu_{E^i} = \frac{d_i - d_{i-1}}{r_i - r_{i-1}}$ satisfying $\mu^1 > \mu^2 > \mu^3 > ... > \mu^{l-1} > \mu^l = \mu$.

Let E be a vector bundle of rank r and degree d over an algebraic curve X. We supose that E is unstable and let $\mu = \frac{d}{r}$ be its slope. By definition of stability there are subbundles E' of rank $r' < r$ and degree d, $\{0\} \subsetneqq E' \subsetneqq E$ such that $\mu' = \frac{d'}{r'} > \mu = \frac{d}{r}$. We choose E_1 with $\mu_1 > \mu$ to be maximal and of maximal rank between those of maximal slope (i.e. if $\exists E_1'$ with $\mu_1' = \mu_1$, then $E_1' \subseteq E_1$). We will call it *the maximal destabilizing of E*. Now we consider the subbundle $F = E/E_1$. If it is semistable we have finished and the Harder-Narasimhan filtration is $\{0\} \subsetneqq E_1 \subsetneqq E$. If not, in analogy to the previous case there exists $\{0\} \subsetneqq F_1 \subsetneqq F$, of maximal slope and of maximal rank between those of maximal slope, hence we have

$$\begin{array}{ccccccc} \{0\} & \subsetneqq & E_1 & \subsetneqq & E_2 & \subsetneqq & E \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \{0\} & \subsetneqq & E_2/E_1 = F_1 & \subsetneqq & E/E_1 = F \end{array}$$

verifying two properties:

- E_2/E_1 is semistable. Indeed, if $E_2/E_1 = F_1$ was not semistable, there would exists $\{0\} \subsetneqq F_2 \subsetneqq F_1$ with $\mu_{F_2} > \mu_{F_1}$, contradicting the choice of F_1.
- $\mu^1 = \mu_{E_1/\{0\}} > \mu^2 = \mu_{E_2/E_1}$, because if we had $\mu^1 \leq \mu^2 \iff \frac{d_1}{r_1} \leq \frac{d_2-d_1}{r_2-r_1} \iff d_1r_2 - d_1r_1 \leq d_2r_1 - d_1r_1 \iff \frac{d_1}{r_1} \leq \frac{d_2}{r_2} \iff \mu_1 = \mu_{E_1} \leq \mu_2 = \mu_{E_2}$, and we have chosen E_1 of maximal slope between the subbundles of E and $E_1 \subsetneqq E_2$.

Repeating the process, if the quotient $G = E/E_2$ is not semistable, we choose $\{0\} \subsetneqq G_1 \subsetneqq G$ with maximal slope and rank and we will have

$$\begin{array}{ccccccccc} \{0\} & \subsetneqq & E_1 & \subsetneqq & E_2 & \subsetneqq & E_3 & \subsetneqq & E \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \{0\} & \subsetneqq & E_2/E_1 = F_1 & \subsetneqq & E_3/E_1 = F_2 & \subsetneqq & E/E_1 = F \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & \{0\} = F_1/E_2 & \subsetneqq & E_3/E_2 = G_1 & \subsetneqq & E/E_2 = G \end{array} .$$

By analogy we will have that $F_2/F_1 = \frac{E_3/E_1}{E_2/E_1} \approx E_3/E_2$ is semistable and that

$$\mu_{F_1=E_2/E_1} > \mu_{G_1=E_3/E_2} \iff \mu^2 > \mu^3 .$$

Iterating until we get a semistable quotient E/E_{l-1}, we obtain the Harder-Narasimhan filtration:

$$\{0\} \subsetneqq E_1 \subsetneqq E_2 \subsetneqq \ldots \subsetneqq E_{l-1} \subsetneqq E_l = E$$

which verifies:

- $E^i = E_i/E_{i-1}$ is semistable, $\forall i \in \{1, ..., l\}$ $(E_0 = \{0\})$
- $\mu^1 > \mu^2 > \mu^3 > ... > \mu^{l-1} > \mu^l = \mu$ $(\mu^i = \mu_{E^i} = \frac{d_i - d_{i-1}}{r_i - r_{i-1}})$.

Although here we show the construction of the filtration in the case of vector bundles, the proof given works for torsion-free sheaves. The result, where the existence and the uniqueness of a maximal destabilizing sheaf in each step are proved, is the following:

Theorem 3.0.2. *Every torsion-free sheaf has a unique Harder-Narasimhan filtration.*
Proof. *See [H-L1], Thm. 1.3.4, p. 16.* ■

Finally, as we announced at the beginning of this section, we can exhibit unstable vector bundles as extensions of semistable ones in this way. Given an unstable vector bundle we have its Harder-Narasimhan filtration

$$0 \subset E_1 \subset E_2 \subset ... \subset E_{l-1} \subset E_l = E \subset E/E_l \subset 0 \; .$$

This breaks into short exact sequences

$$0 \longrightarrow \underset{semistable}{E_1} \longrightarrow E_2 \longrightarrow \underset{semistable}{E_2/E_1} \longrightarrow 0 \; ,$$

$$0 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \underset{semistable}{E_3/E_2} \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow E_{l-1} \longrightarrow E \longrightarrow \underset{semistable}{E/E_{l-1}} \longrightarrow 0$$

where the vector bundle on the right is semistable.

Using the Harder-Narasimhan filtration we can think of semistable sheaves as building blocks for torsion-free sheaves.

We will show that we can give an easy expression for the Harder-Narasimhan filtration in the case $X = \mathbb{P}^1_{\mathbb{C}}$.

Example 3.0.3. *Suppose that the curve X is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. We know, by an important theorem of Grothendieck, (see [H-L1], Thm. 1.3.1, p. 14) that a vector bundle E over $\mathbb{P}^1_{\mathbb{C}}$ splits in line bundles*

$$E = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_3) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_s)$$

with $a_1 > a_2 > ... > a_s$, and we call b_i the number of times each line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_i)$ appears. Thus the slope is the average of a_i :

$$\mu = \frac{\deg E}{rk\ E} = \frac{a_1 b_1 + ... + a_s b_s}{b_1 + ... + b_s}$$

It is clear that

$$E_1 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1)$$

with

$$\mu^1 = \frac{a_1 + \overbrace{\cdots}^{b_1} + a_1}{b_1} = a_1 > \mu$$

and

$$F = E/E_1 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_3) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_r) .$$

Then also

$$E^2 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2)$$

which lifts to

$$E_2 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) .$$

It is clear that each E^i is semistable and $\mu^1 > \mu^2 > \mu^3 > ... > \mu^{s-1} > \mu^s = \mu$.

Chapter 4

The Kempf filtration

We said in Remark 2.4.6 that Geometric Invariant Theory measured, through one-parameter subgroups, if zero was in the closure of the orbit of a given point or not, that is, if the point was unstable or not. Understanding one-parameter subgroups as paths which give a way of approaching a limit point, we can think that there are 1-PS better than others in the sense that they go to the limit point faster than others.

With this idea, we can say that the Harder-Narasimhan filtration of an unstable vector bundle represents, in some way, the best one-parameter subgroup. This means the following: we know that, if E is unstable, there is any 1-PS that destabilizes E, and the Harder-Narasimhan filtration is the best way to destabilize E, so maybe it has to correspond to the best one-parameter subgroup, with this point of view.

George R. Kempf, in [Ke], explores this idea generalizing the main results of Mumford in GIT from an algebraic closed field to a perfect field. He find an *"special almost unique"* one-parameter subgroup with the best properties, *to move most rapidly toward the origin.*

Suppose that E is a GIT-unstable point. Then, there exists a 1-PS λ such that $\mu(\lambda, E) > 0$, its minimum relevant exponent. The article of Kempf says that, if there exists a 1-PS with these conditions, the function $\dfrac{\mu(\lambda, E)}{\|\lambda\|}$ has maximum value (as a function of λ) and the 1-PS that achieves this maximum is unique, where $\|\lambda\| = \sqrt{\sum_i \lambda_i^2}$ is the norm of the 1-PS and $\mu(\lambda, E)$ the minimum relevant exponent.

We will call this 1-PS, the Kempf one-parameter subgroup. It will be the fastest 1-PS which destabilizes our vector bundle E and we will prove that it corresponds to the Harder-Narasimhan filtration, hence both ideas are really the same.

4.1 Convex cones

Here we are going to describe a pure linear geometry technique which will be very useful in the developing of the results.

Endow $\mathbb{R}^t$ with an inner product $(\cdot,\cdot)$ defined by a matrix

$$\begin{pmatrix} b^1 & & \\ & \ddots & \\ & & b^t \end{pmatrix}$$

where b^i are positive integers. Let

$$\mathcal{C} = \left\{ x \in \mathbb{R}^t : x_1 < x_2 < \cdots < x_t \right\}$$

Let $v = (v_1, \cdots, v_t) \in \mathbb{R}^t$. Define

$$\mu_v(\lambda) = \frac{(\lambda, v)}{\|\lambda\|} \tag{4.1}$$

We want to find a vector $\lambda \in \overline{\mathcal{C}}$ which maximizes this function (assuming that there exists $\lambda \in \overline{\mathcal{C}}$ with $\mu_v(\lambda) > 0$).

Let $w^i = -b^i v_i$, $w_0 = 0$, $w_i = w^1 + \cdots + w^i$, $b_0 = 0$, and $b_i = b^1 + \cdots + b^i$. We draw a graph joining the points with coordinates (b_i, w_i). Note that this graph has t segments, each segment has slope $-v_i$ and width b^i. This is the graph drawn with a continuous line in the figure. Now draw the convex envelope of this graph (discontinuous line in the figure), whose coordinates we denote by $(b_i, \widetilde{w}_i)$, and let $\lambda_i = -\widetilde{w}^i / b^i$. In other words, the quantities $-\lambda_i$ are the slopes of the convex envelope graph. Note that the vector $\lambda = (\lambda_1, \cdots, \lambda_t)$ belongs to $\overline{\mathcal{C}}$ by construction.

Remark 4.1.1. *If $\widetilde{w}_i > w_i$, then $\lambda_{i+1} = \lambda_i$.*

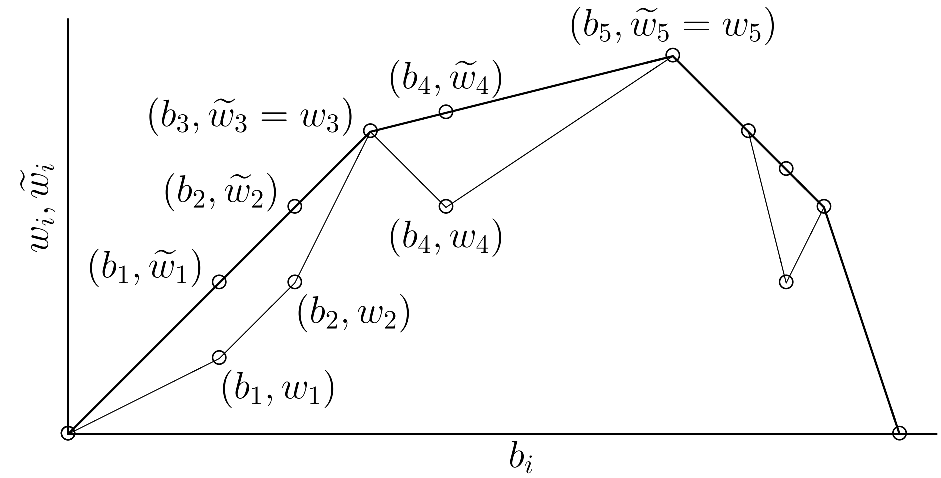


Figure 4.1: Graph

Theorem 4.1.2. *The vector λ defined in this way gives a maximum for the function μ_v.*

Before proving the theorem we need some lemmas.

Lemma 4.1.3. *Let λ be the point in $\overline{\mathcal{C}}$ which is closest to v. Then λ achieves the maximum of μ_v.*

Proof. For any $\alpha \in \mathbb{R}^{>0}$, the vector $\alpha\lambda$ is also in $\overline{\mathcal{C}}$, so in particular λ is the closest point in the line $\alpha\lambda$ to v. This point is the orthogonal projection of v into the line $\alpha\lambda$, and the distance is

$$||v|| \sin \beta(v, \lambda) \tag{4.2}$$

where $\beta(v, \lambda)$ is the angle between v and λ. But a vector $\lambda \in \overline{\mathcal{C}}$ minimizes (4.2) if and only if it maximizes

$$||v|| \cos \beta(v, \lambda) = \frac{(v, \lambda)}{\|\lambda\|}$$

so the lemma is proved. ■

We say that an affine hyperplane in $\mathbb{R}^t$ separates a point v from $\overline{\mathcal{C}}$ if v is on one side of the hyperplane and all the points of $\overline{\mathcal{C}}$ are on the other side or on the hyperplane.

Lemma 4.1.4. *A point $\lambda \in \overline{\mathcal{C}}$ gives minimum distance to v if and only if the hyperplane $\lambda + (v-\lambda)^\perp$ separates v from $\overline{\mathcal{C}}$.*

Proof. $\Rightarrow$) Assume that there is a point $w \in \overline{\mathcal{C}}$ on the same side of the hyperplane as v. The segment going from λ to w is in $\overline{\mathcal{C}}$ (by convexity), but there are points in this segment (near λ), which are closer to v than λ.

$\Leftarrow$) Let λ be a point in $\overline{\mathcal{C}}$ such that $\lambda + (v-\lambda)^\perp$ separates v from $\overline{\mathcal{C}}$. Let $w \in \overline{\mathcal{C}}$ be another point. Let w' be the intersection of the hyperplane and the segment which goes from w to v. Since the hyperplane separates $\overline{\mathcal{C}}$ from v, either $w' = w$ or w' is in the interior of the segment. Therefore

$$d(w,v) \geq d(w',v) \geq d(\lambda, v) \ ,$$

where the last inequality follows from the fact that λ is the orthogonal projection of v to the hyperplane. ■

Proof of the theorem. Let λ be the vector in the hypothesis of the theorem. Using lemmas 4.1.3 and 4.1.4, it is enough to check that the hyperplane $\lambda + (v-\lambda)^\perp$ separates v from $\overline{\mathcal{C}}$.

Let $\lambda + \epsilon \in \overline{\mathcal{C}}$. The condition that $\lambda + \epsilon$ belongs to $\overline{\mathcal{C}}$ means that

$$\epsilon_i - \epsilon_{i+1} \leq \lambda_{i+1} - \lambda_i \tag{4.3}$$

The hyperplane separates v from $\overline{\mathcal{C}}$ if and only if $(v-\lambda, \epsilon) \leq 0$ for all such ϵ. Therefore we calculate (using the convention $\widetilde{w}_0 = 0$, $w_0 = 0$ and $\epsilon_{t+1} = 0$)

$$(v-\lambda, \epsilon) = \sum_{i=1}^{t} b^i (v_i - \lambda_i)\epsilon_i = \sum_{i=1}^{t} (-w^i + \widetilde{w}^i)\epsilon_i =$$

$$= \sum_{i=1}^{t} \big((\widetilde{w}_i - \widetilde{w}_{i-1}) - (w_i - w_{i-1})\big)\epsilon_i = \sum_{i=1}^{t} (\widetilde{w}_i - w_i)(\epsilon_i - \epsilon_{i+1})$$

If $\widetilde{w}_i = w_i$, then the corresponding summand is zero. On the other hand, if $\widetilde{w}_i > w_i$, then $\lambda_{i+1} = \lambda_i$ (Remark 4.1.1), and (4.3) implies $\epsilon_i - \epsilon_{i+1} \leq 0$. In any case, the summands are always non-positive, and hence $(v - \lambda, \epsilon) \leq 0$. ■

4.2 The m-Kempf filtration of an unstable bundle

Let E be a vector bundle of rank r and degree d over an algebraic curve X, and suppose that E is unstable.

Consider the following positive rational number

$$C = \max\{(\frac{d}{r} - \mu_{\max}(E))r(r-1) + gr(r-1) + 1 \ , \ 1\},$$

which we remark that is strictly greater than

$$(\frac{d}{r} - \mu_{\max}(E))r(r-1) + gr(r-1) \ ,$$

where $\mu_{\max}(E)$ is the maximum slope of the semistable factors of the Harder-Narashiman filtration of E. Recall that, for any proper subbundle $E' \subseteq E$, it is $\mu_{\max}(E') \leq \mu_{\max}(E)$.

Any vector subbundle $E' \subseteq E$, of rank r' and degree d', such that $\mu(E') < \frac{d}{r} - C$, satisfies, for any natural number m, an estimate due to Le Poitier and Simpson (see [G-S1], [Sch] or [Si]):

$$h^0(E'(m)) \leq r'(\frac{r'-1}{r'}[\frac{d}{r} + (\frac{d}{r} - \mu_{\max}(E)) + m + 1]_+ + \frac{1}{r'}[\frac{d}{r} - C + m + 1]_+) \ ,$$

where $[x]_+ = \max\{0, x\}$. We denote

$$m_0 := \max\{-\frac{d}{r} - (\frac{d}{r} - \mu_{\max}(E)) - 1 \ , \ -\frac{d}{r} + C - 1\} \ ,$$

and we choose $m \geq \max\{m_0, m_G\}$, where m_G, depending on g, r, d, is the integer from which the construction of the moduli space of Gieseker is valid. Then we have the following bound of the space of global sections of $E'(m)$:

$$
\begin{aligned}
&h^0(E'(m)) \leq r'(\frac{r'-1}{r'}(\frac{d}{r}+(\frac{d}{r}-\mu_{\max}(E))+m+1)+\frac{1}{r'}(\frac{d}{r}-C+m+1))\\
&=r'(\frac{d}{r}+m+1+\frac{r'-1}{r'}(\frac{d}{r}-\mu_{\max}(E))-\frac{C}{r'})\\
&\leq r'(\frac{d}{r}+m+1+(\frac{d}{r}-\mu_{\max}(E))-\frac{C}{r'}),
\end{aligned}
$$

so that

$$
\begin{aligned}
&h^0(E'(m))r-\chi(E(m))r'\\
&\leq r'(\frac{d}{r}+m+1+(\frac{d}{r}-\mu_{\max}(E))-\frac{C}{r'})r-(d+rm+r(1-g))r'\\
&=r'(\frac{d}{r}-\mu_{\max}(E))r+r'r-r(1-g)r'-rC\\
&=r'(\frac{d}{r}-\mu_{\max}(E))r+grr'-rC\\
&\leq(\frac{d}{r}-\mu_{\max}(E))r(r-1)+gr(r-1)-C
\end{aligned}
$$

hence

$$h^0(E'(m))r-\chi(E(m))r'<0. \tag{4.4}$$

We fix $m > \max\{m_0, m_G\}$. Let V be a vector space of dimension $h^0(E(m))$. Fixing an isomorphism between V and $H^0(E(m))$, and using Gieseker's construction, we obtain a point in our orbit space, which is GIT-unstable with respect to the natural action of $SL(V)$.

Given a filtration $V_1 \subset \cdots \subset V_t = V$ and real numbers $\lambda_1 < \cdots < \lambda_t$, we define the function

$$\mu(V_\bullet,\lambda_\bullet)=\frac{\sum \dim V^i\lambda_i(r^i\frac{\dim V}{\dim V^i}-r)}{\sqrt{\sum \dim V^i\lambda_i^2}}$$

where $V^i = V_i/V_{i-1}$.

Theorem 4.2.1 (Kempf). *There is a unique filtration $V_i \subset V$ and weights $\lambda_1 < \cdots < \lambda_t$ for which Kempf's function $\mu(V_\bullet,\lambda_\bullet)$ achieves a maximum.*

Remark 4.2.2. *Note that this Kempf function is the same as the one given in the introduction of Chapter 4, where the numerator is the minimum relevant exponent and the denominator is the norm of the one-parameter subgroup, by associating a filtration of vector spaces with an* 1*-PS.*

Let $V_\bullet \subset V$ be the filtration given by Kempf's theorem. For each i, let $E_i^m \subset E$ be the subsheaf generated by $V_i \otimes \mathcal{O}_X(-m)$ under the evaluation map. We call this filtration, the m-Kempf filtration of E.

Proposition 4.2.3. *Every filter E_i^m in the m-Kempf filtration of E has slope $\mu(E_i^m) \geq \frac{d}{r} - C$.*

Proof. Following the notations in Section 4.1 we call $v_i = -r + \frac{\dim V \cdot r^i}{\dim V^i}$, $b^i = \dim V^i > 0$ (therefore $w^i = -b^i \cdot v_i = r \cdot \dim V^i - \dim V \cdot r^i$). Then the Kempf function $\mu(V_\bullet, \lambda_\bullet)$ is the same that the function $\mu_v(\lambda) = \frac{(\lambda, v)}{\|\lambda\|}$ in 4.1.2. See that $w_i = w^1 + \ldots + w^i = r \cdot \dim V_i - \dim V \cdot r_i$.

If it were $\mu(E_i^m) < \frac{d}{r} - C$ for any filter E_i^m, then,

$$w_i = r \cdot \dim V_i - \chi(E(m)) \cdot r_i \leq r \cdot h^0(E_i(m)) - \chi(E(m)) \cdot r_i < 0,$$

where the first inequality holds because the dimension of the space of global sections of E_i^m is, in general, greater than the dimension of V_i, the vector space of the sections which generate E_i^m. The second inequality follows from the bound given in (4.4).

Hence, for that index i, we would have $w_i < 0$. By 4.1.2 this graph must be the convex envelope as in Figure 4.1, so the slopes have to be decreasing. If for any index the height of the point in the graph, w_i, is negative, it will be $w_j < 0$, for every $j \geq i$, so $w_t < 0$, the last of them. However $w_t = r \dim V_t - r_t \dim V = 0$, which is a contradiction. ■

On the other hand, it is clear that $\mu(E_i^m) \leq \mu_{\max}(E)$. The important point is that, although E_i^m will in general depend on m, its slope is bounded above and below by numbers which do not depend on m, and furthermore it is a subsheaf of E. Hence, the set of possible isomorphism classes for E_i^m is bounded. So we can apply the Vanishing Theorem of Serre and there is an $m_1 \geq \max\{m_0, m_G\}$ such that, the sheaves E_i^m and $E^{m,i} = E_i^m / E_{i-1}^m$ are m_1-regular. In particular their higher cohomology groups vanish and they are generated by global sections. We also assume that m_1 is large enough so that the maximal destabilizing subsheaf of any $E^{m,i}$ is also m_1-regular. From now on, we will assume $m \geq m_1$.

Proposition 4.2.4. *With the previous notations we have that* $V_i = H^0(E_i^m)$*, so also* $\dim V_i = h^0(E_i^m)$.

Proof. Let $E_\bullet^m \subseteq E$ the m-Kempf filtration. We know that V_i generates the subsheaf E_i^m by definition, so we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \subset & V_1 & \subset & V_2 & \subset & \cdots & \subset & V_t = V \\ & & \cap & & \cap & & & & || \\ & & H^0(E_1^m) & \subset & H^0(E_2^m) & \subset & \cdots & \subset & H^0(E_t^m) = H^0(E) \end{array}$$

Suppose that there is any index i such that $V_i \neq H^0(E_i^m)$. Let i be such a subindex with $V_i \neq H^0(E_i^m)$ and $\forall j > i$ it is $V_j = H^0(E_j^m)$. Then we get the following diagram

$$\begin{array}{ccccccccc} V_i & \subset & V_{i+1} & \subset & V_{i+2} & \subset & \ldots & \subset & V \\ \cap & & || & & || & & & & || \\ H^0(E_i^m) & \subset & H^0(E_{i+1}^m) & & H^0(E_{i+2}^m) & & & & H^0(E) \end{array}$$

Hence, we have $V_i \subset H^0(E_i^m) \subset V_{i+1}$ and we consider a new filtration by addition of the filter $H^0(E_i^m)$:

$$0 \subset \cdots \subset V_i \subset H^0(E_i^m) \subset V_{i+1} \subset \cdots \subset V_t = V$$

Using again the notations of Section 4.1, let us call

$$(b_i, w_i) = (\dim V_i, \dim V_i \cdot r - r_i \cdot \dim V) ,$$

the graph of the m-Kempf filtration. Now we consider the graph of the new filtration, which has two new segments. The old segment linking the points $(b_i, w_i) =$

$(\dim V_i, \dim V_i \cdot r - r_i \cdot \dim V)$ and $(b_{i+1}, w_{i+1}) = (\dim V_{i+1}, \dim V_{i+1} \cdot r - r_{i+1} \cdot \dim V)$ has disappeared.

The new point corresponding to the new filter has coordinates

$$(h^0(E_i^m), h^0(E_i^m) \cdot r - r_i \cdot dimV) \ .$$

Hence, the first of our new segments has slope

$$\frac{(h^0(E_i^m) \cdot r - r_i \cdot \dim V) - (dimV_i \cdot r - r_i \cdot \dim V)}{h^0(E_i^m) - \dim V_i} = r$$

By the other hand, the first one of the slopes is

$$\frac{dimV_1 \cdot r - r_1 \cdot dimV}{dimV_1} = r - \frac{r_1 \cdot dimV}{\dim V_1} < r \ .$$

Then the Kempf function $\mu(V_\bullet, \lambda_\bullet)$ is the same that the function $\mu_v(\lambda) = \dfrac{(\lambda, v)}{\|\lambda\|}$ in 4.1.2, with $v_i = -r + \dfrac{\dim V \cdot r^i}{\dim V^i}$, $b^i = \dim V^i > 0$ (therefore $w^i = -b^i \cdot v_i = r \cdot \dim V^i - \dim V \cdot r^i$). Thus, by Theorem 4.1.2, the maximun for the function $\mu_v(\lambda)$ is achieved in the convex envelope graph, as in Figure 4.1. But, if a new segment has slope strictly bigger than another preceding one, the new point has to be in the convex envelope, so the first filtration could not be the convex envelope. This contradicts the fact that the first filtration was the maximal filtration.

Therefore $V_i = H^0(E_i^m)$, for all i. ■

4.3 Properties of the m-Kempf filtration of E

In the previous section we have seen that, for any $m \geq m_1$, all the filters E_i^m of the m-Kempf filtration of E are m_1-regular. Hence, $E_i^m(m_1)$ is generated by the subspace $H^0(E_i^m(m_1))$ of $H^0(E(m_1))$, and the filtration of sheaves

$$E_1^m \subset E_2^m \subset \cdots \subset E_{t-1}^m \subset E_t^m = E$$

is the filtration associated to the filtration of vector spaces

$$H^0(E_1^m(m_1)) \subset H^0(E_2^m(m_1)) \subset \cdots \subset H^0(E_{t-1}^m(m_1)) \subset H^0(E_t^m(m_1)) = H^0(E(m_1))$$

Therefore, for any $m \geq m_1$, the m-Kempf filtration of E has length $t \leq h^0(E(m_1)) =: N_1$, a bound which does not depend on m.

For each m, the ranks and degrees of the quotients $E^{m,i} = E_i^m/E_{i-1}^m$ make a list of pairs $((r^1,d^1),(r^2,d^2),\ldots)$ with length at most N_1. Since the ranks and degrees are bounded (with a bound independent of m), the number of lists which we get is finite.

Proposition 4.3.1. *There is an integer $m_2 \geq m_1$, such that, for $m \geq m_2$, the m-Kempf filtration of E has $r^i > 0$ for all i.*

Proof. Choose one of the lists of numerical data $((r^1,d^1),(r^2,d^2),\ldots)$. Let $m \geq m_1$ be an integer for which this list is realized. In other words, the m-Kempf filtration $E_i^m \subset E$ has $\mathrm{rk}(E^{m,i}) = r^i$ and $\deg(E^{m,i}) = d^i$. By Kempf's theorem, we know that this filtration achieves a maximum of the function

$$\sqrt{m}\,\mu(V_\bullet,\lambda_\bullet) = \frac{\sum \frac{\dim V^i}{m}\lambda_i m(r^i\frac{\dim V}{\dim V^i} - r)}{\sqrt{\sum \frac{\dim V^i}{m}\lambda_i^2}}$$

among all weighted filtrations (V_i,λ_i) with $\lambda_i < \lambda_{i+1}$ and $V_i \subset H^0(E(m))$. Note that this function coincides with $\mu_v(\lambda)$ defined in (4.1) if we set

$$b^i(m) = \frac{\dim V^i}{m} = r^i + \frac{d^i + r^i(1-g)}{m}$$

$$v_i(m) = m\big(r^i\frac{\dim V}{\dim V^i} - r\big) = \big(r^i d - r d^i\big)\frac{1}{r^i + \frac{d^i+r^i(1-g)}{m}}$$

Note that we can calculate $\dim V^i$ in terms of r^i and d^i thanks to Proposition 4.2.4. Therefore, using Theorem 4.1.2, the graph corresponding to the data b_i, v_i as in Figure 4.1 is convex, and $\lambda_i = v_i$.

Note that the numbers b^i and v_i only depend on r, d, r^i, d^i and m. Assume that, for some i, it is $r^i = 0$. For $m' \geq m$ large enough, the graph corresponding to these numbers is not convex (there is a segment with b^i too small, so it tends to a vertical segment). Therefore, the m'-Kempf filtration of E is not equal to E_i^m.

We can repeat this argument for each list in which there is some $r^i = 0$, and, since there is only a finite set of lists, we finally obtain a number $m_2 \geq m_1$ such that, if $m \geq m_2$, then the m-Kempf filtration of E has $r^i > 0$ for all i. ■

Given $m > m_2$, let E_i^m be the m-Kempf filtration of E. Let $m' > m_2$ be another integer. The graph associated to the filtration $H^0(E_i^m(m')) \subset H^0(E(m'))$ using the data

$$b^i(m') = \frac{\dim V^i}{m'} = r^i + \frac{d^i + r^i(1-g)}{m'}$$

$$v_i(m') = m'\big(r^i \frac{\dim V}{\dim V^i} - r\big) = \big(r^i d - r d^i\big) \frac{1}{r^i + \frac{d^i + r^i(1-g)}{m'}}$$

as in Figure 4.1 is called *the m'-graph associated to the m-Kempf filtration of E.* If we use the limiting data when m' tends to ∞

$$b^i(\infty) = r^i$$

$$v_i(\infty) = d - \frac{r}{r^i} d^i \ ,$$

the corresponding graph is called *the ∞-graph associated to the m-Kempf filtration of E.*

Proposition 4.3.2. *There is an integer $m_3 \geq m_2$, such that, for $m \geq m_3$, the ∞-graph and the m'-graph associated to the m-Kempf filtration of E is convex for all $m' > m$.*

Proof. Choose one of the lists of numerical data $((r^1, d^1), (r^2, d^2), \ldots)$. Let m be an integer for which this list is realized. In other words, the m-Kempf filtration $E_i^m \subset E$, has $r^i = \mathrm{rk}(E^{m,i})$ and $d^i = \deg(E^{m,i})$.

If the ∞-graph of the m-Kempf filtration of E is not convex, then it is also not convex for any m' after certain constant. This is because being convex is a closed condition, and the ∞-graph is the limit of the m'-graphs when m' tends to infinity.

We repeat this argument for each list, and, since there is only a finite set of lists, we obtain a constant m_3 which satisfies required conditions. ■

Let $m \geq m_3$, and consider the m-Kempf filtration of E. Its m-graph is convex, so the maximum of the function

$$(\sqrt{m}\,\mu_v(\lambda))^2 = \frac{(v,\lambda)^2}{(\lambda,\lambda)}$$

is achieved for $\lambda = v$, and the value is

$$m\mu_v(v)^2 = (v,v) = \sum \big(r^i + \frac{d^i + r^i(1-g)}{m}\big)\Big(\big(r^i d - r d^i\big)\frac{1}{r^i + \frac{d^i + r^i(1-g)}{m}}\Big)^2 .$$

Hence, for each list $((r^1,d^1),(r^2,d^2),\ldots)$ of numerical data we have a rational function on m. We say that $f_1 \prec f_2$ for two rational functions, if the inequality $f_1(m) < f_2(m)$ holds for $m \gg 0$. From the finite set of functions, let f be the maximal one with respect to this ordering. Note that this function is unique, because if $f_1(m) = f_2(m)$ for infinitely many values of m, then $f_1 = f_2$ as functions.

Since we have a finite set of functions, there is a value m_4, such that, for $m \geq m_4$, the values of f are strictly greater than the values of the other functions.

Proposition 4.3.3. *Let m' and m'' be integers with m', $m'' \geq m_4$. Then the m'-Kempf filtration of E is equal to the m''-Kempf filtration of E.*

Proof. By construction, the filtration

$$H^0(E_i^{m'}(m')) \subset H^0(E(m')) \tag{4.5}$$

is Kempf's filtration and, by maximality of f, the value of Kempf's function for this filtration is $f(m')$.

Now consider the filtration

$$H^0(E_i^{m''}(m')) \subset H^0(E(m')) \; . \tag{4.6}$$

Since $m'', m' \geq m_4$ (in particular $m' \geq m_1$, and then $E_i^{m''}$ is m'-regular) the value of Kempf's function for this filtration is also $f(m')$. Since this value coincides with the value for Kempf's filtration of $H^0(E(m'))$, by uniqueness the filtrations (4.5) and (4.6) coincide. Since m', $m'' \geq m_1$, this implies that the filtrations $E_i^{m'}$ and $E_i^{m''}$ coincide. ∎

If $m \geq m_4$, the m-Kempf filtration of E is called the Kempf filtration of E.

4.4 The Kempf filtration is the Harder-Narasimhan filtration

Finally we see that the Kempf filtration satisfies the properties of the Harder-Narasimhan filtration given in Chapter 3, so they are the same, by uniqueness of both filtrations.

Let $m \geq m_4$. Let

$$0 \subset V_1 \subset \cdots \subset V_t = V = H^0(E(m)) \tag{4.7}$$

be Kempf's filtration with weights $\lambda_1 < \cdots < \lambda_t$, and let

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_t = E \tag{4.8}$$

the associated filtration of sheaves, where E_i is the subsheaf generated by $V_i \otimes \mathcal{O}_X(-m)$.

By the previous work we know that E_i is m-regular for all i, and $V_i = H^0(E_i(m))$. Using an argument similar to Proposition 4.2.4, we also know that E_i is saturated in E, i.e., E/E_i is torsion free.

Theorem 4.4.1. *The filtration (4.8) coincides with the Harder-Narasimhan filtration of E.*

Proof. We will check that this filtration satisfies the properties of the Harder-Narasimhan filtration.

Step 1. $\mu(E^i) > \mu(E^{i+1})$

Kempf's theorem says that the weighted filtration (4.7) achieves the maximum of the function

$$\frac{\sum_{i=1}^t r^i \lambda_i(-\frac{\dim V^i}{r^i} r + \dim V)}{\sqrt{\sum_{i=1}^t r^i \lambda_i^2}}$$

If we set $b^i = r^i$, $v_i = -\frac{\dim V^i}{r^i} r + \dim V$, then this is the function $\mu_v(\lambda)$ in (4.1). Therefore, $\lambda_i = v_i$ by Theorem 4.1.2. Since $\lambda_i < \lambda_{i+1}$, we obtain

$$-\frac{\dim V^i}{r^i} r + \dim V < -\frac{\dim V^{i+1}}{r^{i+1}} r + \dim V$$

$$\frac{\dim V^i}{r^i} > \frac{\dim V^{i+1}}{r^{i+1}}$$

Applying Proposition 4.2.4 we get $\dim V^i = H^0(E^i(m)) = \chi(E^i(m))$ (the quotient of m–regular sheaves is m-regular), and using Riemann-Roch Theorem we obtain

$$\mu(E^i) > \mu(E^{i+1}) \ .$$

Step 2. E^i is semistable.

If it is not, let $\overline{F} \subset E^i$ be the maximal destabilizing subsheaf. It is of the form $\overline{F} = F/E_{i-1}$, with F a subsheaf of E_i. The sheaf F is an extension of m-regular sheaves, so it is also m-regular, and the sheaf $\overline{F}$ is the quotient of m-regular sheaves, so it is m-regular.

Consider the filtration

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{i-1} \subsetneq F \subsetneq E_i \subsetneq \cdots \subsetneq E_t = E$$

and the associated filtration of vector spaces

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq H^0(F(m)) \subsetneq V_i \subsetneq \cdots \subsetneq V_t = V \ .$$

By hypothesis, the convex envelope of the graph associated to this filtration is equal to the graph of the filtration (4.7). This means that the new dot, corresponding to the new term $H^0(F(m))$, must be on or below the segment joining the dots corresponding to V_{i-1} and V_i. Using the fact that the sheaves involved are m-regular, this implies

$$\mu(\overline{F}) \le \mu(E^i)$$

but this contradicts the assumption that $\overline{F}$ is destabilizing. ■

Appendix A

Simpson construction of the moduli space

Here we give briefly the construction of the moduli space of vector bundles following the method of Carlos Simpson (see [Si]).

Let E be vector bundle over an algebraic curve X, of rank r and degree d, and we suppose E to be slope-semistable. All the vector bundles with these properties are a bounded family, so we can choose an integer m such that we can apply the Vanishing Theorem of Serre, in order to have that $h^1(E(m)) = 0$ and $E(m)$ is generated by global sections. Thus, we can compute their Hilbert Polynomials with the Riemann-Roch theorem:

$$\chi(E(m)) = h^0(E(m)) = \deg(E(m)) + r(1-g) = d + rm + r(1-g) = N \ .$$

In a similar way to the Gieseker method, we have the evaluation morphism

$$\begin{array}{ccc} H^0(E(m)) \otimes \mathcal{O}_X(U) & \stackrel{\text{ev}}{\twoheadrightarrow} & E(m)(U) \\ s \otimes 1|_U & \longmapsto & s|_U \end{array} ,$$

for each open set U in C, and we make a choice of basis to define the surjective morphism in $\mathbb{C}^N \otimes \mathcal{O}_X$:

$$\begin{array}{ccc} \mathbb{C}^N \otimes \mathcal{O}_X & & \\ \downarrow{\scriptstyle \alpha} & \searrow & \\ H^0(E(m)) \otimes \mathcal{O}_X & \xrightarrow[\text{ev}]{} & E(m) \end{array}$$

.

We take the kernel of each morphism and we obtain the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_X^N \longrightarrow E(m) \longrightarrow 0 \ ,$$

where $\mathbb{C}^N \otimes \mathcal{O}_X = \mathcal{O}_X^N$. Now, we twist the sequence with an integer $l - m$, such that $l \gg m \gg 0$, to get $H^1(\mathcal{K}(l-m)) = 0$ (by the same reasons there exists an l for all the bounded family). Then,

$$0 \longrightarrow \mathcal{K}(l-m) \longrightarrow \mathcal{O}_X^N(l-m) \longrightarrow E(l) \longrightarrow 0 \ ,$$

and, taking cohomology,

$$0 \longrightarrow H^0(\mathcal{K}(l-m)) \longrightarrow H^0(\mathcal{O}_X^N(l-m)) \longrightarrow H^0(E(l)) \longrightarrow 0 \ ,$$

where the zero in the right corresponds to $H^1(\mathcal{K}(l-m))$.

Therefore, pairs $\{E, \alpha\}$ correspond to quotients $H^0(\mathcal{O}_X^N(l-m)) \twoheadrightarrow H^0(E(l))$, which can be seen as vector subspaces of dimension $h^0(E(l))$ in a vector space of dimension $h^0(\mathcal{O}_X^N(l-m))$. Thus, we can see our Quot-scheme inside a projective variety, a Grassmannian:

$$\mathit{Quot} \subseteq \mathcal{GR}(h^0(\mathcal{O}_X^N(l-m)), r(l) + (d + r(1-g))) \ .$$

From here, the rest of the process is similar, applying Geometric Invariant Theory and one-parameter subgroups to our parameter space, to obtain a GIT quotient. The conclusion is that, the slope-semistable vector bundles we had chosen at the beginning, are the GIT-semistable points at the end.

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