M-theory and cohomotopy

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Outline

I. From 11d sugra to M-theory

II. Where do fields live?

III. (Twisted) Cohomotopy vs. (twisted) cohomology description of the M-theory fields

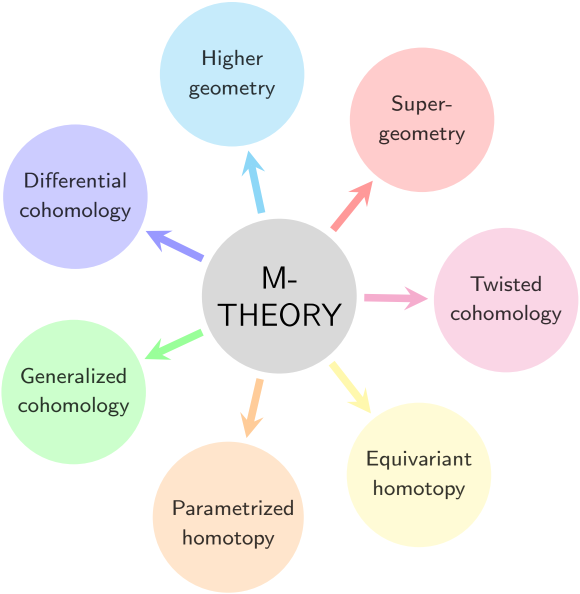
1. *Rationally.*
2. *Integrally.*
3. *Differentially.*

IV. Further applications: branes and gauge theory

Joint with: Urs Schreiber, Domenico Fiorenza, Dan Grady, Vincent Braunack-Mayer

`[BMSS]= Braunack-Mayer-S.-Schreiber`
`[FSS]= Fiorenza-S.-Schreiber`
`[GS]= Grady-S.`
`[S]= S.`
`[SS]= S.-Schreiber`

Richness of M-theory


M-THEORY

Higher geometry

Super-geometry

Twisted cohomology

Equivariant homotopy

Parametrized homotopy

Generalized cohomology

Differential cohomology

I. From 11d supergravity to M-theory

Bosonic 11D supergravity

- **Bosonic Lagrangian:** given by the eleven-form [Cremmer-Julia-Scherk]

$$\boxed{\mathcal{L}_{11}^{\text{bos}} = R * \mathbb{1} - \frac{1}{2} G_4 \wedge * G_4 - \frac{1}{6} G_4 \wedge G_4 \wedge C_3}$$

- **Equations of motion:** The variation $\frac{\delta L_{(\mathbf{11}),\text{bos}}}{\delta C_3} = 0$ for C_3 gives the corresponding equation of motion

$$\boxed{d * G_4 + \frac{1}{2} G_4 \wedge G_4 = 0} \,. \tag{1}$$

- **Bianchi identity:**

$$\boxed{dG_4 = 0} \,. \tag{2}$$

- The second order equation (1) can be written in a first order form, by first writing $d\left(*G_4 + \frac{1}{2} C_3 \wedge G_4\right) = 0$ so that

$$\boxed{*G_4 = G_7 := dC_6 - \frac{1}{2} C_3 \wedge G_4} \,, \tag{3}$$

where C_6 is the potential of G_7, the Hodge dual field strength to G_4 in 11 dimensions.

The effect of the fermions

- The femionic field $\psi \in \Gamma(S \otimes TM)$ (the gravitino) satisfies the generalized Dirac equation, the Rarita-Schwinger equation

$$\boxed{D_{RS}\psi = 0 \,, \qquad \psi \in \Gamma(S \otimes T^*M)} \,.$$

  (involves mixing of terms).

- The fields themselves are in fact combinations of bosonic and fermionic fields. Physics literature usually writes:

$$G_4^{\text{super}} = \underbrace{G_4}_{\rightsquigarrow\textbf{topology}/\textbf{geometry}} + \underbrace{\overline{\psi}\Gamma_2\psi}_{\rightsquigarrow\textbf{topology}/\textbf{geometry}}$$

- Similarly for the connections

$$\omega^{\text{super}} = \omega + \text{fermion-bilinears}$$

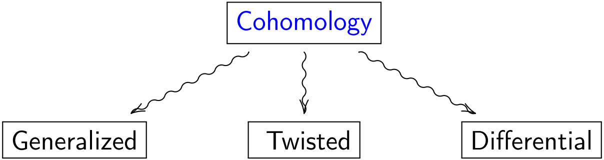
[See Duff-Nilsson-Pope]

Strategy: Extract topology/higher geometry from bosons and fermions separately.

II. Where do fields live?

Generalities on what physics wants

Nontrivial physical entities, such as fields, charges, etc. generically take values in cohomology.



Cohomology

Generalized

Twisted

Differential

I. **Generalized**: Capture essential topological and bundles aspects.

II. **Twisting**: Account for symmetries via automorphisms.

III. **Differentially refined**: Include geometric data, such as connections, Chern character form, smooth structure, smooth representatives of maps ...

Differential refinement

- Introduce geometric data via differential forms (connections, Chern forms, $\cdots$), i.e., retain differential form representatives of cohomology classes.

$$\begin{tikzcd}
 & \Omega^\bullet(M) \arrow[dl, squiggly, "adjoin"'] \arrow[d, squiggly, "adjoin"] \arrow[dr, squiggly, "adjoin"] & \\
H^\bullet(M;\mathbb{Q}) \arrow[d, squiggly, "\text{refinement}"] & H^\bullet(M;\mathbb{Z}) \arrow[d, squiggly, "\text{refinement}"] & E^\bullet(M) \arrow[d, squiggly, "\text{refinement}"] \\
\widehat{H}^\bullet(M;\mathbb{Q}) & \widehat{H}^\bullet(M;\mathbb{Z}) & \widehat{E}^\bullet(M)
\end{tikzcd}$$

- Amalgam of an underlying (topological) cohomology theory and the data of differential forms:

$$\begin{tikzcd}
\boxed{\text{Differential gen. cohomology}} \arrow[r] \arrow[d] & \text{Forms} \arrow[d] \\
\text{Gen. cohomology} \arrow[r] & \text{de Rham cohomology}
\end{tikzcd}$$

- That is, we have a fiber product or twisted product

$$\text{“Differential cohomology} = \text{Cohomology} \times_{\text{de Rham}} \text{Forms”}$$

Differential generalized cohomology

- Start with a generalized cohomology theory h
- $\Omega(X, h_*) := \Omega(X) \otimes_{\mathbb{Z}} h_*$ Smooth differential forms with coefficients in $h_* := h(*)$
- $\Omega_{\mathrm{cl}}(X, h_*) \subseteq \Omega(X, h_*)$ closed forms
- $H_{\mathrm{dR}}(X, h_*)$ cohomology of the complex $\big(\Omega(X, h_*), d\big)$

Definition

A smooth extension of h is a contravariant functor

$\widehat{h}$: **Compact Smooth Manifolds** $\longrightarrow$ **Graded Abelian Grps**

$$\begin{tikzcd}
 & \Omega_{\mathrm{cl}}(X, h_*) \arrow[d] \\
\widehat{h}(X) \arrow[ur, "R"] \arrow[dr, "I"] & H_{\mathrm{dR}}(X, h_*) \\
 & h(X) \arrow[u]
\end{tikzcd}$$

Full structure

Twisted ∩ Differential ∩ Generalized

$$\begin{tikzcd}
 & \Omega^\bullet(M) \arrow[dl, rightsquigarrow, "\text{adjoin}"'] \arrow[d, rightsquigarrow] \arrow[dr, rightsquigarrow, "\text{adjoin}"] & & \textbf{Geometric} \\
H^\bullet(M;\mathbb{Q}) \arrow[loop above, "\mathit{twist}_H"] \arrow[d, rightsquigarrow, "\text{refinement}"] & H^\bullet(M;\mathbb{Z}) \arrow[loop above, "\mathit{twist}_H"] \arrow[d, rightsquigarrow, "\text{refinement}"] & E^\bullet(M) \arrow[loop above, "\mathit{twist}_E"] \arrow[d, rightsquigarrow, "\text{refinement}"] & \textbf{Topological} \\
\widehat{H^\bullet}(M;\mathbb{Q}) \arrow[loop below, "\mathit{twist}_{\widehat{H}}"] & \widehat{H^\bullet}(M;\mathbb{Z}) \arrow[loop below, "\mathit{twist}_{\widehat{H}}"] & \widehat{E^\bullet}(M) \arrow[loop below, "\mathit{twist}_{\widehat{E}}"] & \textbf{Combined}
\end{tikzcd}$$

Examples ([GS])

1. Type I (II) RR fields live in twisted differential KO-theory $\widehat{KO}_{\hat{\tau}}$ (K-theory $\widehat{K}_{\hat{\tau}}$).
2. Differential refinements of various twisted cohomology theories.

- Fields in *M-theory* are proposed to live in a theory of this type [S06]. Which one?

III. (Twisted) Cohomotopy vs. (twisted) cohomology description of the M-theory fields

Cohomotopy versus cohomology

- Cohomo**logy** of Y with R-coefficients: $[Y, K(R,n)] \cong H^n(Y;R)$. old
- Cohomo**topy** of Y with R-coefficients: $[Y, S^n_R] \cong \pi^n_R(Y)$. new

Compare cohomotopy to cohomology of various flavors:

1. Rational: $S^4_{\mathbb{Q}}$ vs. $H^4(-;\mathbb{Q})$.
2. Integral: $S^4_{\mathbb{Z}}$ vs. $H^4(-;\mathbb{Z})$.
3. Differential: $\widehat{S}^4$ vs. $\widehat{H}^4(-)$.

1. Rationally

Connection to rational homotopy theory

Definition

The field equations of (a limit) of M-theory on an 11-dimensional manifold Y^{11} are

$$\begin{aligned} d * G_4 &= \tfrac{1}{2} G_4 \wedge G_4 \\ dG_4 &= 0 \end{aligned}$$

- **Q. What topological & geometric information can the above system provide us?**
  - Rational structures: Differential forms, rational cohomology, rational homotopy theory ...
  - More refined structures: (twisted) 2-gerbes, (twisted) String structures, orientations ...

- A priori, G_4 should be described by a map $f : Y \to K(\mathbb{Z}, 4) \rightsquigarrow H^4(Y; \mathbb{Z})$
- Differential refinement $\widehat{G}_4$ corresponds to $Y \to B^3U(1)_\nabla \quad \rightsquigarrow \widehat{H}^4(Y)$
- Product structure on Eilenberg-MacLane spaces is *cup product*, with no a priori information about *trivialization*.
- Need (G_4, G_7) satisfying above $\Leftrightarrow \quad Y \to ?$.
- Need $(\widehat{G}_4, \widehat{G}_7)$ satisfying above $\Leftrightarrow \quad Y \to \widehat{?}$.

Rational degree four twists [S]

- Consider a 3-form C_3 with $G_4 = dC_3$. We can build a differential with G_4 as $d_{G_4} = d + v_3^{-1} G_4 \wedge$

Observation

The de Rham complex can be twisted by a differential of the form $d + v_{2i-1}^{-1} G_{2i} \wedge$ provided that G_{2i} is closed and v_{2i-1} is Grassmann algebra-valued.

- Form a duality-symmetric graded uniform degree form $G = v_3^{-1} G_4 + v_6^{-1} G_7$. This expression can now be used to twist the de Rham differential, leading to

$$d_G = d + G\wedge = d + v_3^{-1} G_4 \wedge + v_6^{-1} G_7 \wedge \; .$$

Observation

The de Rham complex can be twisted by the differential d_G provided

- $\{v_3, v_3\} = v_6$
- $dG_7 = \frac{1}{2} G_4 \wedge G_4$.

The first condition is the M-theory gauge algebra and the second is the equation of motion.

Observation (The **Sullivan model** as the equations of motion [S])

The above equations correspond to the Sullivan DGCA model of the 4-sphere S^4

$$\mathcal{M}(S^4) = \big(\wedge(y_4, y_7);\ dy_7 = y_4^2,\ dy_4 = 0\big)$$

What about the factor of $\frac{1}{2}$?

- Whitehead bracket $[\iota_4, \iota_4]_W : S^7 \to S^4$ generates $\mathbb{Z}$ $(\mathbb{Q})$-summand in $\pi_7(S^4)$.
- There is an extra symmetry as we are in the dimension of a Hopf fibration, i.e. σ the $\mathbb{H}$-Hopf map and so the generator is $\sigma = \frac{1}{2}[\iota_4, \iota_4]_W$.

Observation (**Quillen model** as the M-theory gauge algebra [FSS])

The Sullivan model for S^{2n} is given by the DGCA

$$\mathcal{M}(S^{2n}) = \big(\wedge(x_{2n}, x_{4n-1});\ dx_{2n} = 0,\ dx_{4n-1} = {x_{2n}}^2\big)\ .$$

Imposing the Maurer-Cartan equation on the degree 1 element $x_{2n}\xi_{1-2n} + x_{4n-1}\xi_{2-4n}$ we find the Lie bracket dual to the differential is given by

$$[\xi_{1-2n}, \xi_{1-2n}] = 2\xi_{2-4n}$$

with all the other brackets zero.

Example ($n = 2$)

The graded Lie algebra $\mathbb{R}\xi_{-3} \oplus \mathbb{R}\xi_{-6}$ with bracket $[\xi_{-3}, \xi_{-3}] = 2\xi_{-6}$ (Quillen model) can be identified with the M-theory gauge Lie algebra.

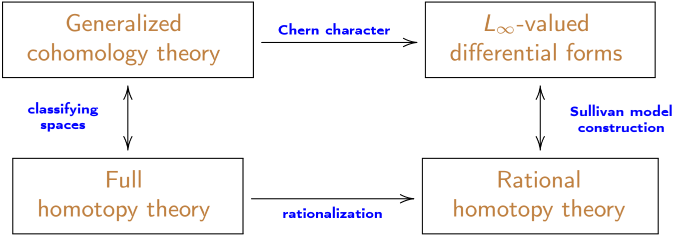
Proposal ([S])

Higher gauge fields in M-theory are cocycles in (rational) cohomotopy.

Developed via *Rational Homotopy Theory (RHT)* in [FSS]: $X \xrightarrow{(G_4, G_7)} S^4_{\mathbb{R}}$.

- $[Y, S^4_{\mathbb{Q}}] = \pi^4_{\mathbb{Q}}(Y)$ rational cohomotopy.
- Ultimately interested in full $\mathrm{Map}(Y, S^4) \ni f$.
- Geometry + physics ⇒ differential cohomotopy [FSS]
- Formulate in stacks/chain complexes.

RHT. Generalized Chern character maps are examples of *rationalization*



RHT amenable to computations due to *Sullivan models*: differential graded-commutative algebras (dgc-algebras) on a finite number of generating elements (spanning the rational homotopy groups) subject to differential relations (enforcing the intended rational cohomology groups). In Sugra: "FDA"s.

Examples

	Rational super space	Loop super L_∞-algebra	Chevalley-Eilenberg super dgc-algebras (“Sullivan models”, “FDA”s)
General	X	$\mathfrak{l}X$	$\mathrm{CE}(\mathfrak{l}X)$
Super spacetime	$\mathbb{T}^{d,1\vert\mathbf{N}}$	$\mathbb{R}^{d,1\vert\mathbf{N}}$	$\mathbb{R}[\{\psi^\alpha\}_{\alpha=1}^{N}, \{e^a\}_{a=0}^{d}] \Big/ \left(\begin{array}{lcl} d\,\psi^\alpha & = & 0 \\ d\,e^a & = & \overline{\psi}\,\Gamma^a\,\psi \end{array} \right)$
Eilenberg-MacLane space	$K(\mathbb{R}, p+2)$ $\simeq_{\mathbb{R}}\ B^{p+1}S^1$	$\mathbb{R}[p+1]$	$\mathbb{R}[c_{p+2}] \,/\, (\, d\,c_{p+2} \; = \; 0\,)$
Odd-dimensional sphere	$S^{2k+1}_{\mathbb{R}}$	$\mathfrak{l}(S^{2k+1})$	$\mathbb{R}[\omega_{2k+1}] \,/\, (\, d\,\omega_{2k+1} \; = \; 0\,)$
Even-dimensional sphere	$S^{2k}_{\mathbb{R}}$	$\mathfrak{l}(S^{2k})$	$\mathbb{R}[\omega_{2k}, \omega_{4k-1}] \Big/ \left(\begin{array}{lcl} d\,\omega_{2k} & = & 0 \\ d\,\omega_{4k-1} & = & -\omega_{2k} \wedge \omega_{2k} \end{array} \right)$
M2-extended super spacetime	$\widehat{\mathbb{T}^{10,1\vert 32}}$	$\mathfrak{m2brane}$	$\mathbb{R}[\{\psi^\alpha\}_{\alpha=1}^{32}, \{e^a\}_{a=0}^{10}, h_3] \Big/ \left(\begin{array}{lcl} d\,\psi^\alpha & = & 0 \\ d\,e^a & = & \overline{\psi}\,\Gamma^a\,\psi \\ d\,h_3 & = & \frac{i}{2}(\overline{\psi}\Gamma_{ab}\psi) \wedge e^a \wedge e^b \end{array} \right)$

Consequences [FSS][BMSS]

1. Reduction via a circle bundle ⇒ new functors formalizing dimensional reduction via loop (and mapping) spaces with rich structure retained (topological, geometric, gauge).
2. The rational data of S^4 on the total space Y^{11} of a circle bundle $S^1 \to Y^{11} \to X^{10}$ leads exactly to rational data of twisted K-theory on base X^{10}. → [see Vincent's talk]
3. Even if we take *flat + rational* we can still see a lot of structure: Study of cocycles in Super-Minkowski space recovers cocycles in rational twisted K-theory.
4. Furthermore, T-duality can be derived at the level of supercocycles.

Branes from supercocycles

- **Superspace formulation of 11d supergravity** [D'Auria-Fre]: fully controlled by an iterated pair of invariant super-cocycles μ_{M2} and μ_{M5} on $D = 11$, $N = 1$ super Minkowski spacetime.
- In the super homotopy-theoretic formulation [FSS]:

$$\begin{tikzcd}
K(\mathbb{R},3) \arrow[d] & & K(\mathbb{R},3) \arrow[d] \\
\boxed{\widehat{\mathbb{T}^{10,1|\mathbf{32}}}} \arrow[rr, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{fib}(\mu_{\mathrm{M2}})"'] & & K(\mathbb{R},7) \arrow[d] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{\mathrm{M2}}"] & & K(\mathbb{R},4)
\end{tikzcd}$$

$\mathbb{R}-$quaternionic Hopf fibration

$$\mu_{\mathrm{M5}} = \frac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right)e^{a_1}\wedge\cdots\wedge e^{a_5} + h_3 \wedge \mu_{\mathrm{M2}}$$

$$\mu_{\mathrm{M2}} = \frac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right)e^{a_1}\wedge e^{a_2}$$

which are the super-flux forms to which the M2-brane and M5-brane couple, in (4) their incarnation as Green-Schwarz-type sigma models [FSS].

- $\widehat{\mathbb{T}^{10,1|\mathbf{32}}} = \mathfrak{m2brane}$ arises as the homotopy fiber of μ_{M2} and is the extended super Minkowski spacetime or the M2-brane super Lie 3-algebra.
- $\mu_{\mathrm{M2}} =$ super-form component of *magnetic flux* sourced by charged M5-branes.
- $\mu_{\mathrm{M5}} =$ super-form component of *electric flux* source by charged M2-branes.

So these cocycles are avatars of M-brane charge/flux at the level of super RHT.

Twisted K-theory in type II from M-theory

1. **Type IIA.** [BMSS] The double dimensional reduction of rational M-brane supercocycles $(\mu_{\mathrm{M2}}, \mu_{\mathrm{M5}})$ is indeed the tuple of F1/D*p*-brane supercocycles $(\mu_{\mathrm{F1}}\mu_{\mathrm{D0}}, \mu_{\mathrm{D2}}, \mu_{\mathrm{D4}}, \mu_{\mathrm{D6}}, \mu_{\mathrm{D8}})$ in rational twisted K-theory, which the literature demands to be the rational image of a cocycle in actual twisted K-theory.

Objects	**Cohomology theory**
M-branes	twisted Cohomotopy
D-branes	twisted K-theory

double dimensional reduction/oxidation

[see talk by Vincent]

2. **Type IIB.** Characterization of T-duality for circle and sphere bundles using RHT [FSS].

- **Novel effect**: T-duality in super-exceptional spacetimes in 11d M-theory [FSS][SS].

2. Integrally

1. Rationally and stably $S^4_{\mathbb{Q}}$ is just the Eilenberg-MacLane space $K(\mathbb{Q}, 4)$, and

$$H^4(Y^{11}; \mathbb{Q}) \cong \pi^4_s(Y^{11}) \otimes \mathbb{Q} \,.$$

2. In the unstable case, schematically, we have

Rational cohomotopy = Rational cohomology + trivialization of the cup square

Integrally and *stably* we do see new effects.

- In between full non-abelian cohomotopy and abelian ordinary cohomology sits stable cohomotopy, represented not by actual spheres, but by their stabilization to the sphere spectrum.
- There is a description of the C-field in each one of these flavors [FSS][BMSS].

Cohomology theory	Rational cohomology	Integral cohomology	Stable cohomotopy	Non-abelian cohomotopy
Cocycle	G_4	$\widetilde{G}_4$	$\Sigma^\infty c$	c

Hypothesis H. *The C-field is charge-quantized in cohomotopy theory, even non-rationally.*

Cancellation of main anomalies of M-theory follows naturally from cohomotopy:

1. C-field charge quantization in twisted cohomotopy implies various fundamental anomaly cancellation and quantization conditions [FSS].
2. Similar effects for D-branes and orientifolds [SS].

Lifting rational S^4 to integral S^4

- If we start with the rational 4-sphere $S^4_{\mathbb{Q}}$, then how can we lift it to an "integral" space?
- The actual 4-sphere S^4 stands out as not only the most natural but the finite-dimensional one.

$$\begin{tikzcd}
 & S^4 \arrow[d] \\
Y \arrow[ur, dotted, "\textbf{Integral, torsion}"] \arrow[r, "\textbf{Rational, non-torsion}"] & S^4_{\mathbb{Q}}
\end{tikzcd} \tag{5}$$

- Start with integral cohomology as describing the (shifted/twisted) C-field and then transition to a description in terms of cohomotopy. By representability, this amounts to lifting

$$\begin{tikzcd}
 & S^4 \arrow[d, "\iota"] \\
Y \arrow[ur, dotted, "\textbf{Nonlinear prequantum}"] \arrow[r, "\textbf{Linear quantum}"] & K(\mathbb{Z}, 4)\ .
\end{tikzcd} \tag{6}$$

  The map ι assembles, upon taking homotopy classes, into the integral cohomology $H^4(S^4; \mathbb{Z})$ generated by a fundamental class.

- Description:

$$\boxed{\text{C-field in } \ \pi^4(Y^{11}) \iff \text{C-field in } \ H^4(Y^{11}; \mathbb{Z}) + \text{nontrivial conditions.}}$$

Proposition (Integral Postnikov tower for S^4 [GS])

$$\begin{tikzcd}
 & & S^4 & \\
 & & \vdots & \\
 & K(\mathbb{Z}_2, 10) \arrow[r] & (S^4)_4 \arrow[r, "\text{“}\mathrm{Sq}^8 \iota_4\text{”}, \iota_4^3, \mathcal{P}_5^1 \iota_4 = 0"] \arrow[r, phantom, shift right=3ex, "\text{holds}"] \arrow[d] & K(\mathbb{Z}_{240}, 12) = K(\mathbb{Z}_{16}, 12) \times K(\mathbb{Z}_5, 12) \times K(\mathbb{Z}_3, 12) \\
 & K(\mathbb{Z}_{24}, 7) \arrow[r] & (S^4)_3 \arrow[r, "P_{11}"] \arrow[r, phantom, shift right=3ex, "\text{fourth obstruction}"] \arrow[d] & K(\mathbb{Z}_2, 11) \\
 & K(\mathbb{Z}_2, 6) \arrow[r] & (S^4)_2 \arrow[r, "(\text{“}\mathrm{Sq}^4 \iota_4\text{”}, \mathcal{P}_3^1 \iota_4)"] \arrow[r, phantom, shift right=3ex, "\text{third obstruction}"] \arrow[d] & K(\mathbb{Z}_{24}, 8) = K(\mathbb{Z}_8, 8) \times K(\mathbb{Z}_3, 8) \\
 & K(\mathbb{Z}_2, 5) \arrow[r] & (S^4)_1 \arrow[r, "\alpha_7"] \arrow[r, phantom, shift right=3ex, "\text{second obstruction}"] \arrow[d] & K(\mathbb{Z}_2, 7) \\
Y \arrow[rr, "\text{integral Cohomology}"] \arrow[uurr, dotted, "\text{first lifting}"] \arrow[uuurr, dotted, "\text{second lifting}"] \arrow[uuuurr, dotted, "\text{third lifting}"] \arrow[uuuuurr, dotted, bend left, "\text{fourth lifting}"] \arrow[uuuuuurr, dotted, bend left, "\text{degree 4 cohomotopy}"] & & (S^4)_0 = K(\mathbb{Z}, 4) \arrow[r, "\mathrm{Sq}^2 \iota_4"] \arrow[r, phantom, shift right=3ex, "\text{first obstruction}"] & K(\mathbb{Z}_2, 6)
\end{tikzcd}$$

Note that at the top level the three conditions vanish necessarily on Y^{11}, for dimension reasons.

Cohomotopy in deg 4 ∼ Integral 4-cohomology + four sets of obstructions.

Pulling back to spacetime Y, where the fundamental class ι_4 pulls back to the field

$$G_4 - \tfrac{1}{2}\lambda =: \widetilde{G}_4 = f^*\iota_4$$

where $\lambda = \frac{1}{2}p_1$ is the first Spin characteristic class of TY.

(i) First obstruction. $\boxed{\mathrm{Sq}^2 \widetilde{G}_4 \overset{!}{=} 0 \in H^6(Y;\mathbb{Z}_2)}$.

This follows from anomaly cancellation in M-theory [FSS].

(ii) Second obstruction. $\boxed{\mathrm{f}^*(\alpha_7) \overset{!}{=} 0 \in H^7(Y;\mathbb{Z}_2)}$

where α_7 is a secondary operation, restricting fiberwise to $\mathrm{Sq}^2\iota_5$.
No candidate degree 5 classes.

(iii) Third obstructions. $\boxed{\mathrm{f}^*(''\mathrm{Sq}^4\iota_4'') \overset{!}{=} 0 \in H^8(Y;\mathbb{Z}_8)}$

- Note that by construction, this implies also that (upon mod 2 reduction)

$$f^*(\mathrm{Sq}^4\iota_4) = \mathrm{Sq}^4 f^*(\iota_4) = \mathrm{Sq}^4 \widetilde{G}_4 = \widetilde{G}_4 \cup \widetilde{G}_4 = 0 \in H^8(Y;\mathbb{Z}_2)\ .$$

- Recall that rationally we have the EOM $d * G_4^{\mathrm{form}} = \frac{1}{2} G_4^{\mathrm{form}} \wedge G_4^{\mathrm{form}}$ ..
- Coefficients being $\mathbb{Z}_8$ rather than $\mathbb{Z}_2$: Fields reduced modulo 4: $\frac{1}{2}\lambda \rightsquigarrow$ modding out p_1 by 4. (Pontrjagin square operation).

We also have $\boxed{\mathcal{P}^1_3 \iota_4 = 0}$.

- Mod 3 reductions are shown to play a prominent role in topological considerations in M-theory [S], where similar conditions, including $\mathcal{P}^1_3 \rho_3 G_4 = 0$, have been highlighted in the context of *Spin K-theory*.

(iv) Fourth obstruction. $\boxed{\mathrm{f}^*(P_{11}) \overset{!}{=} 0}$

where P_{11} is a class which fiberwise restricts to $\mathrm{Sq}^4 \iota_7$.

- Reminiscent of $G_4 \wedge G_7$.
- The universal coefficient theorem gives detectable effect for M-theory on orientable spacetimes.

(v) Fifth obstructions. $\boxed{\text{``}\mathrm{Sq}^8 \iota_4 \overset{!}{=} 0, \quad \iota_4^3 \overset{!}{=} 0, \quad \mathcal{P}^1_5 \iota_4 \overset{!}{=} 0}$

These obstructions necessarily vanish on Y^{11}. However on a 12-manifold Z^{12}, for analyzing the congruences of the Chern-Simons term in the M-theory action, the three conditions are nontrivial (but natural to have).

Proposition (Cohomotopy vs. cohomology for the C-field)

Consider the M-theory (shifted) C-field $\widetilde{G}_4$ as an integral cohomology class in degree four. Then if $\widetilde{G}_4$ lifts to a cohomotopy class $\mathcal{G}_4 \in \pi^4(Y^{11})$ the following obstructions necessarily vanish

- **(i)** $\mathrm{Sq}^2 \widetilde{G}_4 = 0 \in H^6(Y^{11}; \mathbb{Z}_2)$.
- **(ii)** $\mathcal{P}^1_3(\widetilde{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_3)$.
- **(iii)** $\mathrm{Sq}^4 \widetilde{G}_4 = \widetilde{G}_4 \cup \widetilde{G}_4 = 0 \in H^8(Y^{11}; \mathbb{Z}_2)$.
- **(iv)** If $G_4 = 0$ and $dC_3 = 0$ can be lifted to an integral class $\widetilde{C}_3$, then we also have $\mathrm{Sq}^3\mathrm{Sq}^1 \widetilde{C}_3 = 0 \in H^7(Y^{11}; \mathbb{Z}_2)$.
- **(v)** If $dG_7 = G_4 \wedge G_4 = 0$ and G_7 can be lifted to an integral class $\widetilde{G}_7$, then we also have the condition $\mathrm{Sq}^4 \widetilde{G}_7 = 0 \in H^{11}(Y^{11}; \mathbb{Z}_2)$.

Consequences:

1. **Congruences for the action** The Chern-Simons term in the action
$$\frac{1}{6}\int_{Y^{11}} C_3 \wedge G_4 \wedge G_4 \ .$$
Since C_3 may not be globally defined in general, one may consider Y^{11} as the boundary of a 12-manifold Z^{12} and analyzes the globally well defined term
$$\frac{1}{6}\int_{Z^{12}} G_4 \wedge G_4 \wedge G_4 \qquad (7)$$
[Witten]: usual quantization law of G_4 does not give rise to a well defined Chern-Simons action, as (7) might fail to be integral by a factor of 6. Cohomotopy implies the added condition that
$$\widetilde{G}_4^3 \equiv 0 \mod 3 \ .$$
This, with $\widetilde{G}_4^2 = \mathrm{Sq}^4(\widetilde{G}_4) \equiv 0$ mod 2, gives result (without E_8-gauge theory).
2. **The anomaly in the partition function** Quantization in cohomotopy yields the condition $\mathrm{Sq}^2(\widetilde{G}_4) = 0$ for some integral lift of G_4.
    - Implies the vanishing of the DMW anomaly $\mathrm{Sq}^3(\widetilde{G}_4) = 0$ [FSS].
    - Obstruction theory for $S^4 \Rightarrow$ fields which contribute to the phase are just the field which lift to the first Postnikov stage in cohomotopy [GS].

Example (Flux compactification spaces)

Anti-de Sitter space $\mathrm{AdS}_n \rightsquigarrow$ simply-connected cover $\widetilde{\mathrm{AdS}}_n$ of AdS_n.

1. $\underline{\widetilde{\mathrm{AdS}}_4 \times \mathbb{C}P^2 \times T^2}$: Supersymmetry without supersymmetry [Duff-Lu-Pope] and T-duality [Bouwknegt-Evslin-Mathai]. $\pi^4(\mathbb{C}P^2) \cong \mathbb{Z}$ while $H^4(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$.
2. $\underline{\widetilde{\mathrm{AdS}}_7 \times \mathbb{R}P^4}$: M-theory on an orientifold [Witten][Hori]. $\pi^4(\mathbb{R}P^4) \cong \mathbb{Z}_2$ while $H^4(\mathbb{R}P^4; \mathbb{Z}) = 0$, indeed shows that cohomotopy detects more.
3. $\underline{\widetilde{\mathrm{AdS}}_4 \times \mathbb{R}P^5 \times T^2}$: $\pi^4(\mathbb{R}P^5)$ is cyclic or order 4, i.e. either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, while $H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2$.
4. $\underline{\widetilde{\mathrm{AdS}}_4 \times \mathbb{C}P^3 \times S^1}$: $\pi^4(\mathbb{C}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ while $H^4(\mathbb{C}P^3; \mathbb{Z}) \cong \mathbb{Z}$, so that there is an extra contribution of $\mathbb{Z}_2$ present in cohomotopy.
5. $\underline{\text{For } \mathbb{H}P^2}$: $\pi^4(\mathbb{H}P^2) \cong \mathbb{Z}$ while $H^4(\mathbb{H}P^2; \mathbb{Z}) \cong \mathbb{Z}$, and hence no new contribution,
6. $\underline{\text{For } \mathbb{O}P^2}$: $\pi^4(\mathbb{O}P^2) \cong \mathbb{Z}$. while $H^4(\mathbb{O}P^2; \mathbb{Z}) = 0$, signaling a new effect. Important for bosonic M-theory ([Ramond][S]).

Interpretation and consequences? Work in progress (via Pontrjagin-Thom theory).

Twisted Cohomotopy theory [FSS]

In degree $d-1$ there is a canonical twisting on Riemannian d-manifolds, given by the unit sphere bundle in the orthogonal tangent bundle:

$$\underset{\text{Cohomotopy theory}}{\text{J-twisted}}\;\pi^{TX^d}(X^d) := \left\{ \begin{tikzcd} & \overset{\text{tangent}}{\overset{\text{unit sphere bundle}}{S(TX^d)}} \arrow[r] \arrow[d, "p"] & \overset{\text{universal tangent}}{\overset{\text{unit sphere bundle}}{S^{d-1}/\!\!/\mathrm{O}(d)}} \arrow[d] \\ X \arrow[ur, dashed, bend left, "\text{continuous section} \atop = \text{twisted cocycle}"] \arrow[r, equal] & X \arrow[r, "TX^d", "\text{classifying map of} \atop \text{tangent/frame bundle}"'] & B\mathrm{O}(d) \end{tikzcd} \right\} \Big/ \sim_{\frac{\text{homotopy}}{B\mathrm{O}(d)}}$$

$$\simeq \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\text{continuous function}"] \arrow[dr, "TX^d \atop \text{twist}"'] & {} \arrow[d, Rightarrow, "\text{homotopy}"] & S^{d-1}/\!\!/\mathrm{O}(d) \arrow[dl] \\ & B\mathrm{O}(d) & \end{tikzcd} \right\} \Big/ \sim_{\frac{\text{homotopy}}{B\mathrm{O}(d)}}$$

Since the canonical morphism $\mathrm{O}(d) \longrightarrow \mathrm{Aut}(S^{d-1})$ is known as the *J-homomorphism*, we may call this *J-twisted Cohomotopy theory*, for short.

Twisted cohomotopy and anomalies [FSS]

Hypothesis H: *The C-field 4-flux & 7-flux forms in M-theory are subject to charge quantization in J-twisted Cohomotopy cohomology theory in that they are in the image of the non-abelian Chern character map from J-twisted Cohomotopy theory.*

⇒ Cancellation of main anomalies:

Half-integral flux quantization	$\Big[\, \underbrace{G_4 + \tfrac{1}{4} p_1}_{=: \widetilde{G}_4 \ \text{integral flux}} \,\Big] \in H^4(X, \mathbb{Z})$
Background charge	$\underbrace{q(\widetilde{G}_4)}_{\text{quadratic form}} = \widetilde{G}_4 \left(\widetilde{G}_4 - \underbrace{\tfrac{1}{2} p_1}_{=(\widetilde{G}_4)_0} \right)$
DMW-anomaly cancellation	$W_7(TX) = 0$
Integral equation of motion	$\underbrace{\mathrm{Sq}^3}_{=\beta \mathrm{Sq}^2} (\widetilde{G}_4) = 0$
M5-brane anomaly cancellation	$\underbrace{I^{\mathrm{M5}}_{\mathrm{ferm}}}_{\substack{\text{chiral} \\ \text{fermion}}} + \underbrace{I^{\mathrm{M5}}_{\mathrm{sd}}}_{\substack{\text{self-dual} \\ \text{3-flux}}} + \underbrace{I^{\mathrm{bulk}}_{\mathrm{infl}}}_{\substack{\text{bulk} \\ \text{inflow}}} = 0$
M2-brane tadpole cancellation	$\underbrace{N_{\mathrm{M2}}}_{\substack{\text{number of} \\ \text{M2-branes}}} + q(\widetilde{G}_4) = \underbrace{I_8}_{\substack{\text{One loop} \\ \text{polynomial}}}$

Consequences for WZW model associated to M5-brane ⇒ [See talk by Domenico]

J-Twisted Cohomotopy and Topological G-Structure

- For every topological coset space realization G/H of an n-sphere, there is a canonical homotopy equivalence between the classifying spaces for G-twisted Cohomotopy and for topological H-structure (i.e., reduction of the structure group to H), as follows:

$$\overset{\substack{\textbf{coset space structure} \\ \textbf{on topological } \boldsymbol{n}\textbf{-sphere}}}{S^n \underset{\textbf{homeo}}{\simeq} G/H} \qquad \Rightarrow \qquad \overset{\substack{\boldsymbol{G}\textbf{-twisted Cohomotopy /} \\ \textbf{topological } \boldsymbol{H}\textbf{-structure}}}{S^n /\!\!/ G \underset{\textbf{htpy}}{\simeq} BH} \;.$$

  (One may think of this as "moving G from numerator on the right to denominator on the left".)

- Existence of a G-structure is a non-trivial topological condition, so is the existence of J-twisted Cohomotopy cocycles.
- Notice that this is a special effect of twisted non-abelian generalized Cohomology: A non-twisted generalized cohomology theory (abelian or non-abelian) always admits at least one cocycle, namely the trivial or zero-cocycle. But here for non-abelian J-twisted Cohomotopy theory on 8-manifolds, the existence of *any* cocycle is a non-trivial topological condition.

Equivalence for Spin 8-manifolds

$$
\begin{array}{c} S^7/\!\!/\mathrm{Spin}(8) \\ \simeq\ B\,\mathrm{Spin}(7) \end{array}
\quad\Longrightarrow\quad
\left\{
\begin{tikzcd}
\text{cocycle in J-twisted Cohomotopy} & \text{classifying space for J-twisted Cohomotopy theory} \\
 & S^7/\!\!/\mathrm{Spin}(8) \arrow[d] \\
X^8 \arrow[ur, dashed, bend left, "c"] \arrow[r, "TX^8"'] \arrow[ur, phantom, shift left=4, "\Downarrow\ \text{homotopy}"] & B\mathrm{Spin}(8) \\
\text{tangent spin structure} &
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\text{topological } \mathrm{Spin}(\mathbf{7}) \text{ structure} & \text{classifying space for topological } \mathrm{Spin}(\mathbf{7})\text{-structure} \\
 & B\mathrm{Spin}(7) \arrow[d, "Bi"] \\
X^8 \arrow[ur, dashed, bend left, "g"] \arrow[r, "TX^8"'] \arrow[ur, phantom, shift left=4, "\Downarrow\ \text{homotopy}"] & B\mathrm{Spin}(8) \\
\text{tangent spin structure} &
\end{tikzcd}
\right\}
$$

$$
\begin{array}{c} S^7/\!\!/\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \\ \simeq B\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \end{array}
\quad\Rightarrow\quad
\left\{
\begin{tikzcd}
\text{cocycle in } \mathrm{Sp}(\mathbf{2})\cdot\mathrm{Sp}(\mathbf{1})\text{-twisted Cohomotopy theory} & \text{classifying space for } \mathrm{Sp}(\mathbf{2})\cdot\mathrm{Sp}(\mathbf{1})\text{-twisted Cohomotopy theory} \\
 & S^7/\!\!/\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[d] \\
X^8 \arrow[ur, dashed, bend left, "c"] \arrow[r, "TX^8"'] \arrow[ur, phantom, shift left=4, "\Downarrow\ \text{homotopy}"] & B\mathrm{Spin}(8) \\
\text{tangent spin structure} &
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\text{topological } \mathrm{Sp}(\mathbf{1})\cdot\mathrm{Sp}(\mathbf{1}) \text{ structure} & \text{classifying space for topological } \mathrm{Sp}(\mathbf{1})\cdot\mathrm{Sp}(\mathbf{1})\text{-structure} \\
 & B\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \arrow[d, "Bi"] \\
X^8 \arrow[ur, dashed, bend left, "g"] \arrow[r, "TX^8"'] \arrow[ur, phantom, shift left=4, "\Downarrow\ \text{homotopy}"] & B\mathrm{Spin}(8) \\
\text{tangent spin structure} &
\end{tikzcd}
\right\}
$$

Stable vs. unstable

The quaternionic Hopf fibration.

quaternionic Hopf fibration

$$\begin{tikzcd}
S^7 \arrow[rrrr, bend left, "h_{\mathbb{H}}"] & \simeq & S(\mathbb{H}^2) \arrow[r, "(q_1, q_2) \mapsto {[q_1 : q_2]}"'] & \mathbb{H}P^1 & \simeq & S^4 ,
\end{tikzcd}$$

unit sphere in quaternionic 2-space

quaternionic projective 1-space

which represents a generator of the non-torsion subgroup in the 4-Cohomotopy of the 7-sphere, as shown on the left here:

non-abelian/unstable Cohomotopy group

stabilization

abelian/stable Cohomotopy group

quaternionic Hopf fibration

non-torsion generator

$$\begin{tikzcd}
{[S^7 \xrightarrow{h_{\mathbb{H}}} S^4]} & \pi^4(S^7) \arrow[r, "\Sigma^\infty"] & \mathbb{S}^4(S^7) & \Sigma^\infty [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] \\
(1,0) \in & \mathbb{Z} \times \mathbb{Z}_{12} \arrow[r, "(n,a) \mapsto (n \bmod 24)"'] & \mathbb{Z}_{24} & \ni 1
\end{tikzcd}$$

stabilized quaternionic Hopf fibration

torsion generator

- So composition with the quaternionic Hopf fibration can be viewed as a *transformation* that translates deg-7 to deg-4 Cohomotopy classes:

$$\begin{tikzcd}
 & S^7 \arrow[d, "h_{\mathbb{H}}"] \\
X \arrow[ur, dashed, "c"] \arrow[r, dashed, "(h_{\mathbb{H}})_*(c)"'] & S^4
\end{tikzcd}
\qquad
\begin{matrix} \text{7-Cohomotopy} \\ \\ \text{reflects into} \\ \\ \text{4-Cohomotopy} \end{matrix}
\qquad
\begin{tikzcd}
\pi^7(X) \arrow[d, "(h_{\mathbb{H}})_*"] \\
\pi^4(X)
\end{tikzcd}$$

Proposition (Differential form data underlying twisted Cohomotopy)

Let X be a simply connected smooth manifold and $\tau : X \to B\mathrm{O}(n+1)$ a twisting for Cohomotopy in degree n. Let ∇_τ be any connection on the real vector bundle V classified by τ with Euler form $\chi_{2k+2}(\nabla_\tau)$ (see [Mathai-Quillen]).

(i) If $n = 2k+1$ is odd $n \geq 3$: a cocycle defining a class in the rational τ-twisted Cohomotopy of X is equivalently given by

$$\boxed{\pi^\tau_{\mathbb{Q}}(X) \;\simeq\; \left\{ G_{2k+1} \,\middle|\, d\,G_{2k+1} = \chi_{2k+2}(\nabla_\tau) \right\}_{\big/\sim}}\,.$$

(ii) If $n = 2k$ is even, $n \geq 2$: a cocycle defining a class in the rational τ-twisted Cohomotopy of X is given by a pair of differential forms $G_{2k} \in \Omega^{2k}(X)$ and $G_{4k-1} \in \Omega^{4k-1}(X)$ such that

$$dG_{2k} = 0; \qquad \pi^* G_{2k} = \tfrac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}})$$
$$dG_{4k-1} = -G_{2k} \wedge G_{2k} + \tfrac{1}{4}p_{{}_k}(\nabla_\tau),$$

where $p_{{}_k}(\nabla_\tau)$ is the k-th Pontrjagin form of ∇_τ, $\pi\colon E \to X$ is the unit sphere bundle over X associated with τ, $\hat{\tau}\colon E \to B\mathrm{O}(n)$ classifies the vector bundle $\widehat{V}$ on E defined by the splitting $\pi^* V = \mathbb{R}_E \oplus \widehat{V}$ associated with the tautological section of $\pi^* V$ over E, and $\nabla_{\hat{\tau}}$ is the induced connection on $\widehat{V}$. That is,

$$\boxed{\pi^\tau_{\mathbb{Q}}(X) \;\simeq\; \left\{ \left(G_{2k}, G_{4k-1}\right) \,\middle|\, \begin{array}{c} d\,G_{2k} = 0\,, \qquad \pi^* G_{2k} = \tfrac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}}) \\ d\,G_{4k-1} = -G_{2k} \wedge G_{2k} + \tfrac{1}{4}p_{{}_k}(\nabla_\tau) \end{array} \right\}_{\Big/\sim}}\,.$$

3. Differentially

Differential refinement

- Refine the topological lift (5) to a geometric lift at the level of smooth stacks of the form

$$\begin{tikzcd}
 & \widehat{S^4} \arrow[d] \\
Y \arrow[ur, dotted, "\substack{\textbf{Differential cohomotopy,}\\ \textbf{prequantum and geometric}}"] \arrow[r, "\substack{\textbf{Differential cocycle,}\\ \textbf{quantum and geometric}}" description] & \mathbf{B}^3 U(1)_\nabla \, .
\end{tikzcd} \qquad (8)$$

  where $\widehat{S^4}$ is the differential refinement of the 4-sphere and $\mathbf{B}^3 U(1)_\nabla$ is the smooth stack of 3-bundles with connections

- This would require a differential refinement of the *Postnikov tower* which uses refinement of cohomology operations, primary (such as Steenrod operations) and secondary (such as Massey products) [GS].

Differential cohomotopy [Fiorenza-S.-Schreiber]

- $\mathbb{H}$-Hopf fibration: $S^3 \longrightarrow S^7 \longrightarrow S^4 \longrightarrow B\mathrm{SU}(2) \xrightarrow{c_2} K(\mathbb{Z}, 4)$.
- Rationalize: $S^3_{\mathbb{Q}} \longrightarrow S^7_{\mathbb{Q}} \longrightarrow S^4_{\mathbb{Q}} \longrightarrow (BS^3)_{\mathbb{Q}}$ which is equivalent to

$$K(\mathbb{Q}, 7) \longrightarrow S^4_{\mathbb{Q}} \longrightarrow K(\mathbb{Q}, 4)$$

- Rational homotopy of spaces can be modelled using L_∞-algebras.
- The Eilenberg-MacLane spaces $K(\mathbb{Q}, n) = B^n\mathbb{Q}$ can be modelled using algebras via chain complexes: $b^n\mathbb{Q} = \mathbb{Q}[n]$.
- Lie 7- algebra $\mathfrak{s}^4$ is defined by $\mathrm{CE}(\mathfrak{s}^4) = \mathbb{R}[g_4, g_7]$ with g_k in degree k and with the differential defined by $dg_4 = 0, dg_7 = g_4 \wedge g_4$.
- Has a natural structure of infinitesimal $\mathbb{R}[2]$-quotient of $\mathbb{R}[6]$, i.e., there exists a natural homotopy fiber sequence of L_∞-algebras

$$\begin{tikzcd} \mathbb{R}[6] \arrow[r] \arrow[d] & \mathfrak{s}^4 \arrow[d, "p"] \\ 0 \arrow[r] & \mathbb{R}[3] . \end{tikzcd} \tag{9}$$

Theorem (FSS)

The system $(\widehat{G}_4, \widehat{G}_7)$ forms a cocycle in differential cohomotopy.

Differential refinements: $\mathbf{B}^3U(1)_\nabla$ vs. $\widehat{S}^4$

- Let $\mathfrak{s}^4$ be the Lie 7-algebra whose corresponding Chevellay-Eilenberg algebra is the exterior algebra on generators g_4 and g_7 with relations

$$dg_4 = 0 \ , \qquad dg_7 = g_4 \wedge g_4 \ .$$

- As a de Rham model for flat 1-forms with values in S^4 we take the sheaf on the site of Cartesian spaces given by the assignment

$$\Omega^1_{\mathrm{fl}}(-;\mathfrak{s}^4) : U \longmapsto \mathrm{hom}_{\mathrm{dgcAlg}}(\mathrm{CE}(\mathfrak{s}^4), \Omega^*(U)) \ ,$$

  for each Cartesian space $U \cong \mathbb{R}^n$. (The homotopy type of $\Omega^1_{\mathrm{fl}}(-;\mathfrak{s}^4)$ can be computed via the Sullivan construction as the $\mathbb{R}$-local 4-sphere $S^4_{\mathbb{R}}$).
- Then pulling back along the canonical map $S^4 \to S^4_{\mathbb{R}}$, we get a smooth stack

$$\begin{tikzcd} \widehat{S}^4 \arrow[r] \arrow[d] & \Omega^1_{\mathrm{fl}}(-;\mathfrak{s}^4) \arrow[d] \\ S^4 \arrow[r] & S^4_{\mathbb{R}} \ . \end{tikzcd}$$

Definition (Differential unstable cohomotopy)

For a smooth manifold X, let $i(X)$ denote its embedding as a smooth stacks via its sheaf of smooth plots. Then the differential cohomotopy of X in degree 4 is defined as the pointed set $\widehat{\pi}^4_u(X) := \pi_0 \mathrm{Map}\big(i(X), \widehat{S}^4\big)$ where the maps on the right are those of smooth stacks.

Differential cohomotopy: stably

- Stably, S^4 has only torsion groups in higher degrees and hence the canonical map $S^4 \to K(\mathbb{R}, 4)$ is a stable $\mathbb{R}$-local equivalence.
- Geometrically, the realification if modeled by closed 4-forms $\Omega^4_{\mathrm{cl}}(-)$.
- Stable differential cohomotopy in degree 4 fits into a pullback square

$$\begin{tikzcd}
\widehat{\Sigma^\infty S^4} \arrow[r] \arrow[d] & H\Big(\tau^{\leq 0}\Omega^{4+*}(-)\Big) \arrow[d] \\
\Sigma^\infty S^4 \arrow[r] & \Sigma^4 H\mathbb{R} \, .
\end{tikzcd}$$

where $\Omega^{4+*}(-)$ denotes the de Rham complex, shifted so that Ω^4 is in degree zero, and $\tau^{\leq 0}$ truncates the complex in degree zero so that the complex is concentrated in negative degrees. The functor H denotes the Eilenberg-MacLane functor which turns a chain complex into a spectrum.

Definition (Differential stable cohomotopy)

Let X be a smooth manifold with $i(X)$ its associated smooth stack. The *stable* differential cohomotopy group of X is defined as

$$\widehat{\pi}^4_s(X) := \pi_0 \mathrm{Map}\big(i(X); (\widehat{\Sigma^\infty S^4})_0\big).$$

where the subscript 0 denotes the deg 0 component of the sheaf of spectra $\widehat{\Sigma^\infty S^4}$.

Geometric cycles

Definition (Geometric cohomotopy cocycles [GS])

If X is a smooth manifold, a morphism $\hat{c} : X \to \widehat{S}^4$ can be identified with a triple (c, h, ω) where

- **(i)** $c : X \to S^4$ is a cocycle in ordinary cohomotopy,
- **(ii)** $\omega : \mathrm{CE}(\mathfrak{s}^4) \to \Omega^*(X)$ is a DGA morphism, determined by specifying forms ω_4 and ω_7 on M satisfying $d\omega_7 = \omega_4^2$ and $d\omega_4 = 0$,
- **(iii)** and h is a homotopy interpolating between the rational cocycle represented by the form data and the rationalization of the classifying map $c : X \to S^4$. Thus, h exhibits a sort of *de Rham theorem* for cohomotopy.

Proposition (Differential refinement of Postnikov tower of the sphere)

$$\begin{tikzcd}
K(\mathbb{Z}_{15}, 11) \arrow[r] & (\widehat{S}^4)_7 \arrow[d] & \\
K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 10) \arrow[r] & (\widehat{S}^4)_6 \arrow[d] \arrow[r] & K(\mathbb{Z}_{15}, 12) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) \arrow[r] & (\widehat{S}^4)_5 \arrow[d] \arrow[r] & K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 11) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 8) \arrow[r] & (\widehat{S}^4)_4 \arrow[d] \arrow[r] & K(\mathbb{Z}_2 \times \mathbb{Z}_2, 10) \\
K(\mathbb{Z}_{12}, 7) \times K(\mathbb{Z}, 7) \arrow[r] & (\widehat{S}^4)_3 \arrow[d] \arrow[r] & K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) \\
K(\mathbb{Z}_2, 6) \arrow[r] & (\widehat{S}^4)_2 \arrow[d] \arrow[r, "(?, \hat{\iota}_{\mathbf{4}}^{\mathbf{2}})"] & K(\mathbb{Z}_{12}, 8) \times \mathbf{B}^7 U(1)_\nabla \\
K(\mathbb{Z}_2, 5) \arrow[r] & (\widehat{S}^4)_1 \arrow[d] \arrow[r, "\alpha_{\mathbf{7}} l"] & K(\mathbb{Z}_2, 7) \\
 & (\widehat{S}^4)_0 = \mathbf{B}^3 U(1)_\nabla \arrow[r, "\mathrm{Sq}^{\mathbf{2}} \rho_{\mathbf{2}} l"] & K(\mathbb{Z}/2, 6)
\end{tikzcd}$$

where we have identified the first few obstructions.

Proposition (Differential cohomotopy vs. cohomology for the C-field)

Consider the differentially refined M-theory (shifted) C-field $\widehat{G}_4$ as an integral cohomology class in degree four. Then if $\widehat{G}_4$ lifts to a cohomotopy class $\mathcal{G}_4 \in \widehat{\pi}^4(Y^{11})$ the following obstructions necessarily vanish

- **(i)** $\mathrm{Sq}^2 I(\widehat{G}_4) = 0 \in H^6(Y^{11}; \mathbb{Z}_2)$.
- **(ii)** $\mathcal{P}^1_3 I(\widehat{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_3)$.
- **(iii)** $\mathrm{Sq}^4 I(\widehat{G}_4) = I(\widehat{G}_4 \cup_{\mathrm{DB}} \widehat{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_2)$.
- **(iv)** If $\widehat{G}_4 = 0$ and C_3^{form} is quantized, with differential refinement $\widehat{C}_3$, then we also have $\mathrm{Sq}^3\mathrm{Sq}^1 I(\widehat{C}_3) = 0 \in H^7(Y^{11}; \mathbb{Z}_2)$.
- **(v)** If $dG_7^{\mathrm{form}} = G_4^{\mathrm{form}} \wedge G_4^{\mathrm{form}} = 0$ and G_7^{form} is quantized, with differential refinement $\widehat{G}_7$, then we also have the condition $\mathrm{Sq}^4 I(\widehat{G}_7) = 0 \in H^{11}(Y^{11}; \mathbb{Z}_2)$.

Remark (Obstruction in M-theory via higher bundles with connections)

Deligne-Beilinson cup product in M-theory $\widehat{G}_4 \cup_{\mathrm{DB}} \widehat{G}_4$ gives a 7-bundle with connection form locally given by $C_3^{\mathrm{form}} \wedge G_4^{\mathrm{form}}$ [FSS]. From the identification of the k-invariant at the second stage (the DB square): to lift past the 2nd stage in the Postnikov tower for $\widehat{S}^4$, this connection must be globally defined. In terms of differential cohomology, $a(C_3^{\mathrm{form}} \wedge G_4^{\mathrm{form}}) = \widehat{G}_4 \cup_{\mathrm{DB}} \widehat{G}_4$, where $a : \Omega^7(Y^{11}) \to \widehat{H}^8(Y^{11})$ is the canonical map

Example (Differential cohomotopy of flux compactification spaces)

LES in stable cohomotopy

$$\begin{tikzcd} \ldots \arrow[r] & \pi^3_s(X) \arrow[r, "\deg"] & \Omega^3(X) \arrow[r] & \widehat{\pi}^4_s(X) \arrow[r] & \pi^4_s(X) \arrow[r] & \ldots . \end{tikzcd}$$

allows to compute some examples.

(i) $\underline{\widetilde{\mathrm{AdS}_7 \times \mathbb{R}P^4}}$: $\widehat{\pi}^4_s\big(\widetilde{\mathrm{AdS}_7} \times \mathbb{R}P^4\big) \cong \widehat{H}^4\big(\widetilde{\mathrm{AdS}_7} \times \mathbb{R}P^4\big)$.

(ii) $\underline{\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2}$: $\widehat{\pi}^4_s\big(\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2\big) \cong \widehat{H}^4\big(\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2\big)$.

(iii) $\underline{\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2 \times T^2}$: $\widehat{\pi}^4_s\big(\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2 \times T^2\big) \cong \widehat{H}^4\big(\widetilde{\mathrm{AdS}_4} \times \mathbb{C}P^2 \times T^2\big)$.

(iv) $\underline{\widetilde{\mathrm{AdS}_4} \times \mathbb{R}P^5 \times T^2}$: $\pi^4(\mathbb{R}P^5)$ is order 4, either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, while $H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2$. Also $\pi^3(\mathbb{R}P^5)$ is finite. We therefore have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega^3(\mathbb{R}P^5) \arrow[r] & \widehat{\pi}^4(\mathbb{R}P^5) \arrow[r] & \pi^4(\mathbb{R}P^5) \arrow[r] & 0 . \end{tikzcd}$$

Since $\pi^4(\mathbb{R}P^5)$ is generated by $q_5\eta_4$, with $\eta_4 : S^5 \to S^4$ the two-fold suspension of the Hopf map, the induced map on H^4 necessarily vanishes. Hence, in this case, differential cohomotopy yields considerably different information than ordinary differential cohomology.

Back to  :

Cohomotopy $\Rightarrow$ branes and gauge theory

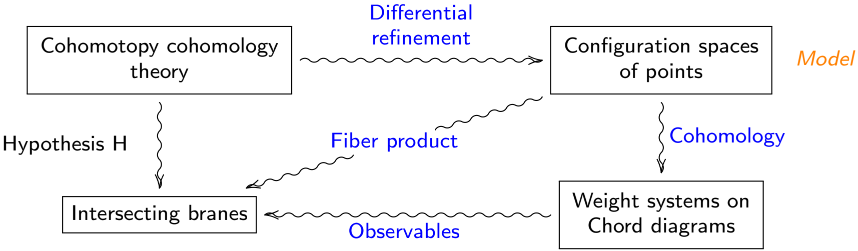
Differential cohomotopy and D-brane gauge theories

Zoom in beyond foundational/structural M-theoretic considerations [SS]:

(1) A differential refinement of Cohomotopy cohomology theory is given by *un-ordered configuration spaces of points*.

(2) The fiber product of such differentially refined Cohomotopy cocycle spaces describing $\mathrm{D6} \perp \mathrm{D8}$-brane intersections is homotopy-equivalent to the *ordered configuration space of points* in the transversal space.

(3) The higher observables on this moduli space are equivalently weight systems on horizontal chord diagrams.



Cohomotopy cohomology theory

Differential refinement

Configuration spaces of points

Model

Hypothesis H

Fiber product

Cohomology

Intersecting branes

Observables

Weight systems on Chord diagrams

Combining the above seemingly distinct mathematical areas reflect a multitude of effects expected on brane intersections in string theory. So aside from structural utility for M-theory, Hypothesis H implies:

- *M-theoretic observables on* $\mathrm{D6} \perp \mathrm{D8}$*-configurations* (cf. parametrized).
- *Chan-Paton observables.*
- *String topology operations.*
- *Multi-trace observables of BMN matrix model.*
- *Hanany-Witten states.*
- *BLG 3-Algebra observables.*
- *Bulk Wilson loop observables.* [See talk by Urs]
- *Single-trace observables*
- *of SYK & BMN model.*
- *Fuzzy funnel observables.*
- *Supersymmetric indices.*
- *'t Hooft string amplitudes.*

Top-down M-theory via Hypothesis H: knowledge about gauge field theory and perturbative string theory is not used in deriving the algebras of observables of M-theory, but only to interpret them.

Thank you!