M-theory and cohomotopy

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I. From 11d sugra to M-theory

II. Where do fields live?

III. (Twisted) Cohomotopy vs. (twisted) cohomology description of the M-theory fields
   1. Rationally.
   2. Integrally.
   3. Differentially.

IV. Further applications: branes and gauge theory

Joint with: Urs Schreiber, Domenico Fiorenza, Dan Grady, Vincent Braunack-Mayer

[BMSS] = Braunack-Mayer-S.-Schreiber
[FSS] = Fiorenza-S.-Schreiber
[GS] = Grady-S.
[S] = S.
[SS] = S.-Schreiber
Richness of M-theory

- Differential cohomology
- Higher geometry
- Supergeometry
- Generalized cohomology
- Twisted cohomology
- Equivariant homotopy
- Parametrized homotopy
I. From 11d supergravity to M-theory
Bosonic 11D supergravity

- **Bosonic Lagrangian**: given by the eleven-form [Cremmer-Julia-Scherk]

\[
\mathcal{L}^{\text{bos}}_{11} = R \ast 1 - \frac{1}{2} G_4 \wedge \ast G_4 - \frac{1}{6} G_4 \wedge G_4 \wedge C_3
\]

- **Equations of motion**: The variation \( \frac{\delta L^{(11), \text{bos}}_{11}}{\delta C_3} = 0 \) for \( C_3 \) gives the corresponding equation of motion

\[
d \ast G_4 + \frac{1}{2} G_4 \wedge G_4 = 0 .
\]

- **Bianchi identity**:

\[d G_4 = 0 .\]

- The second order equation (1) can be written in a first order form, by first writing \( d (\ast G_4 + \frac{1}{2} C_3 \wedge G_4) = 0 \) so that

\[
\ast G_4 = G_7 := d C_6 - \frac{1}{2} C_3 \wedge G_4 ,
\]

where \( C_6 \) is the potential of \( G_7 \), the Hodge dual field strength to \( G_4 \) in 11 dimensions.
The effect of the fermions

- The femionic field $\psi \in \Gamma(S \otimes TM)$ (the gravitino) satisfies the generalized Dirac equation, the Rarita-Schwinger equation

$$D_{RS}\psi = 0, \quad \psi \in \Gamma(S \otimes T^*M).$$

(involves mixing of terms).

- The fields themselves are in fact combinations of bosonic and fermionic fields. Physics literature usually writes:

$$G^\text{super}_4 = G_4 + \bar{\psi}\Gamma_2\psi$$

$\sim$ topology/geometry $\sim$ topology/geometry

- Similarly for the connections

$$\omega^\text{super} = \omega + \text{fermion-bilinears}$$

[See Duff-Nilsson-Pope]

Strategy: Extract topology/higher geometry from bosons and fermions separately.
II. Where do fields live?
Nontrivial physical entities, such as fields, charges, etc. generically take values in cohomology.

- **Generalized**: Capture essential topological and bundles aspects.
- **Twisting**: Account for symmetries via automorphisms.
- **Differentially refined**: Include geometric data, such as connections, Chern character form, smooth structure, smooth representatives of maps ...
Differential refinement

- Introduce geometric data via differential forms (connections, Chern forms, \(\cdots\)), i.e., retain differential form representatives of cohomology classes.

\[
\begin{align*}
\Omega^\bullet(M) & \quad \text{adjoin} \quad \text{adjoin} \quad \text{adjoin} \\
H^\bullet(M; \mathbb{Q}) & \quad \text{refinement} \quad \text{refinement} \\
\hat{H}^\bullet(M; \mathbb{Q}) & \quad \text{refinement} \\
H^\bullet(M; \mathbb{Z}) & \quad \text{refinement} \\
\hat{H}^\bullet(M; \mathbb{Z}) & \\
E^\bullet(M) & \quad \text{refinement} \\
\hat{E}^\bullet(M)
\end{align*}
\]

- Amalgam of an underlying (topological) cohomology theory and the data of differential forms:

  \[
  \text{Differential gen. cohomology} \quad \rightarrow \quad \text{Forms}
  \]

  \[
  \text{Gen. cohomology} \quad \rightarrow \quad \text{de Rham cohomology}
  \]

- That is, we have a fiber product or twisted product

  “Differential cohomology = Cohomology \times_{\text{de Rham}} \text{Forms}”
Differential generalized cohomology

- Start with a generalized cohomology theory $h$
- $\Omega(X, h_*):= \Omega(X) \otimes_{\mathbb{Z}} h_*$ Smooth differential forms with coefficients in $h_* := h(*)$
- $\Omega_{\text{cl}}(X, h_*) \subseteq \Omega(X, h_*)$ closed forms
- $H_{dR}(X, h_*)$ cohomology of the complex $(\Omega(X, h_*), d)$

Definition

A smooth extension of $h$ is a contravariant functor

\[ \tilde{h}: \text{Compact Smooth Manifolds} \rightarrow \text{Graded Abelian Grps} \]

\[ \begin{array}{ccc}
\Omega_{\text{cl}}(X, h_*) & \rightarrow & H_{dR}(X, h_*) \\
\downarrow & & \uparrow \\
\tilde{h}(X) & \rightarrow & h(X) \\
\downarrow & & \\
h(X) & & \downarrow \\
\end{array} \]
Examples ([GS])

1. Type I (II) RR fields live in twisted differential KO-theory $\widehat{KO}_{\tau}$ (K-theory $\widehat{K}_{\tau}$).
2. Differential refinements of various twisted cohomology theories.

- Fields in $M$-theory are proposed to live in a theory of this type [S06]. Which one?
III. (Twisted) Cohomotopy vs. (twisted) cohomology
description of the M-theory fields
Cohomotopy versus cohomology

- **Cohomology** of $Y$ with $R$-coefficients: $[Y, K(R, n)] \cong H^n(Y; R)$.  
  old

- **Cohomotopy** of $Y$ with $R$-coefficients: $[Y, S^n_R] \cong \pi^n_R(Y)$.  
  new

Compare cohomotopy to cohomology of various flavors:

1. **Rational**: $S^4_Q$ vs. $H^4(-; \mathbb{Q})$.
2. **Integral**: $S^4_Z$ vs. $H^4(-; \mathbb{Z})$.
3. **Differential**: $\hat{S}^4$ vs. $\hat{H}^4(-)$. 
1. Rationally
Connection to rational homotopy theory

Definition

The field equations of (a limit) of M-theory on an 11-dimensional manifold \( Y^{11} \) are

\[
\begin{align*}
    d * G_4 & = \frac{1}{2} G_4 \wedge G_4 \\
    dG_4 & = 0
\end{align*}
\]

Q. What topological & geometric information can the above system provide us?

- **Rational structures**: Differential forms, rational cohomology, rational homotopy theory ...
- **More refined structures**: (twisted) 2-gerbes, (twisted) String structures, orientations ...

A priori, \( G_4 \) should be described by a map \( f : Y \rightarrow K(\mathbb{Z}, 4) \sim H^4(Y; \mathbb{Z}) \)

Differential refinement \( \hat{G}_4 \) corresponds to \( Y \rightarrow B^3 U(1)_\nabla \sim \hat{H}^4(Y) \)

Product structure on Eilenberg-MacLane spaces is *cup product*, with no a priori information about *trivialization*.

- Need \((G_4, G_7)\) satisfying above \(\Leftrightarrow \ Y \rightarrow ?\).
- Need \((\hat{G}_4, \hat{G}_7)\) satisfying above \(\Leftrightarrow \ Y \rightarrow \hat{?}\).
Rational degree four twists [S]

- Consider a 3-form $C_3$ with $G_4 = dC_3$. We can build a differential with $G_4$ as $d_{G_4} = d + v_3^{-1}G_4 \wedge$

Observation

The de Rham complex can be twisted by a differential of the form $d + v_{2i-1}^{-1}G_{2i} \wedge$ provided that $G_{2i}$ is closed and $v_{2i-1}$ is Grassmann algebra-valued.

- Form a duality-symmetric graded uniform degree form $G = v_3^{-1}G_4 + v_6^{-1}G_7$. This expression can now be used to twist the de Rham differential, leading to $d_G = d + G \wedge = d + v_3^{-1}G_4 \wedge + v_6^{-1}G_7 \wedge$.

Observation

The de Rham complex can be twisted by the differential $d_G$ provided

- $\{v_3, v_3\} = v_6$
- $dG_7 = \frac{1}{2} G_4 \wedge G_4$.

The first condition is the M-theory gauge algebra and the second is the equation of motion.
Observation (The **Sullivan model** as the equations of motion \([S]\))

The above equations correspond to the Sullivan DGCA model of the 4-sphere \(S^4\)

\[
\mathcal{M}(S^4) = (\wedge(y_4, y_7); \ dy_7 = y_4^2, \ dy_4 = 0)
\]

What about the factor of \(\frac{1}{2}\)?
- **Whitehead bracket** \([\iota_4, \iota_4]_W : S^7 \to S^4\) generates \(\mathbb{Z} (\mathbb{Q})\)-summand in \(\pi_7(S^4)\).
- There is an extra symmetry as we are in the dimension of a Hopf fibration, i.e. \(\sigma\) the \(\mathbb{H}\)-Hopf map and so the generator is \(\sigma = \frac{1}{2}[\iota_4, \iota_4]_W\).

Observation (**Quillen model** as the M-theory gauge algebra \([FSS]\))

The Sullivan model for \(S^{2n}\) is given by the DGCA

\[
\mathcal{M}(S^{2n}) = (\wedge(x_{2n}, x_{4n-1}); \ dx_{2n} = 0, \ dx_{4n-1} = x_{2n}^2) .
\]

Imposing the Maurer-Cartan equation on the degree 1 element \(x_{2n}\xi_{1-2n} + x_{4n-1}\xi_{2-4n}\) we find the Lie bracket dual to the differential is given by

\[
[\xi_{1-2n}, \xi_{1-2n}] = 2\xi_{2-4n}
\]

with all the other brackets zero.

Example \((n = 2)\)

The graded Lie algebra \(\mathbb{R}\xi_{-3} \oplus \mathbb{R}\xi_{-6}\) with bracket \([\xi_{-3}, \xi_{-3}] = 2\xi_{-6}\) (Quillen model) can be identified with the M-theory gauge Lie algebra.
Proposal ([S])

Higher gauge fields in M-theory are cocycles in (rational) cohomotopy.

Developed via *Rational Homotopy Theory (RHT)* in [FSS]: \( X \overset{(G_4, G_7)}{\longrightarrow} S^4_R \).

- \([Y, S^4_Q] = \pi^4_Q(Y)\) rational cohomotopy.
- Ultimately interested in full \( \text{Map}(Y, S^4) \ni f \).
- Geometry + physics \( \Rightarrow \) differential cohomotopy [FSS]
- Formulate in stacks/chain complexes.

**RHT.** Generalized Chern character maps are examples of *rationalization*

\[
\begin{array}{ccc}
\text{Rational homotopy theory} & \overset{\text{rationalization}}{\Rightarrow} & \text{Full homotopy theory} \\
\downarrow & & \downarrow \\
\text{Sullivan model construction} & \overset{\text{Chern character}}{\rightarrow} & \text{L}_\infty\text{-valued differential forms} \\
\end{array}
\]

RHT amenable to computations due to *Sullivan models*: differential graded-commutative algebras (dgc-algebras) on a finite number of generating elements (spanning the rational homotopy groups) subject to differential relations (enforcing the intended rational cohomology groups). In Sugra: “FDA”s.
### Examples

<table>
<thead>
<tr>
<th></th>
<th>Rational super space</th>
<th>Loop super $L_\infty$-algebra</th>
<th>Chevalley-Eilenberg super dgc-algebras (&quot;Sullivan models&quot;, &quot;FDA&quot;s)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General</strong></td>
<td>$X$</td>
<td>$\mathbb{I}X$</td>
<td>$\text{CE}(\mathbb{I}X)$</td>
</tr>
<tr>
<td><strong>Super spacetime</strong></td>
<td>$T^d,1</td>
<td>N$</td>
<td>$\mathbb{R}^d,1</td>
</tr>
<tr>
<td><strong>Eilenberg-MacLane</strong></td>
<td>$K(\mathbb{R}, p + 2)$</td>
<td>$\mathbb{R}[p + 1]$</td>
<td>$\mathbb{R}[c_{p+2}]/(d c_{p+2} = 0)$</td>
</tr>
<tr>
<td><strong>Odd-dimensional sphere</strong></td>
<td>$S^{2k+1}\mathbb{R}$</td>
<td>$\mathbb{I}(S^{2k+1})$</td>
<td>$\mathbb{R}[\omega_{2k+1}]/(d \omega_{2k+1} = 0)$</td>
</tr>
<tr>
<td><strong>Even-dimensional sphere</strong></td>
<td>$S^{2k}\mathbb{R}$</td>
<td>$\mathbb{I}(S^{2k})$</td>
<td>$\mathbb{R}[\omega_{2k},\omega_{4k-1}]/\left(\begin{array}{l}d \omega_{2k} = 0 \ d \omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k}\end{array}\right)$</td>
</tr>
<tr>
<td><strong>M2-extended super spacetime</strong></td>
<td>$\widetilde{T^{10,1}</td>
<td>32}$</td>
<td>$\mathfrak{m}2\breve{\text{brane}}$</td>
</tr>
</tbody>
</table>
Reduction via a circle bundle $\Rightarrow$ new functors formalizing dimensional reduction via loop (and mapping) spaces with rich structure retained (topological, geometric, gauge).

The rational data of $S^4$ on the total space $Y^{11}$ of a circle bundle $S^1 \to Y^{11} \to X^{10}$ leads exactly to rational data of twisted K-theory on base $X^{10}$. [see Vincent’s talk]

Even if we take flat + rational we can still see a lot of structure: Study of cocycles in Super-Minkowski space recovers cocycles in rational twisted K-theory.

Furthermore, T-duality can be derived at the level of supercocycles.
Branes from supercocycles

- **Superspace formulation of 11d supergravity** [D’Auria-Fre]: fully controlled by an iterated pair of invariant super-cocycles $\mu_{M2}$ and $\mu_{M5}$ on $D = 11, N = 1$ super Minkowski spacetime.

- In the super homotopy-theoretic formulation [FSS]:

  $$
  \begin{array}{ccc}
  K(\mathbb{R}, 3) & \xrightarrow{\mu_{M5}} & K(\mathbb{R}, 7) \\
  \downarrow & & \downarrow \\
  \overline{T}^{10,1|32} & \xrightarrow{\mu_{M2}} & \overline{T}^{10,1|32} \\
  \text{fib}(\mu_{M2}) & & \\
  \end{array}
  $$

  $\mu_{M5} = \frac{1}{5!} (\overline{\psi} \Gamma_{a_1 \cdots a_5} \psi) e^{a_1} \wedge \cdots \wedge e^{a_5} + h_3 \wedge \mu_{M2}$

  $\mu_{M2} = \frac{i}{2} (\overline{\psi} \Gamma_{a_1 a_2} \psi) e^{a_1} \wedge e^{a_2}$

  which are the super-flux forms to which the M2-brane and M5-brane couple, in their incarnation as Green-Schwarz-type sigma models [FSS].

  - $\overline{T}^{10,1|32} = \text{m2brane}$ arises as the homotopy fiber of $\mu_{M2}$ and is the extended super Minkowski spacetime or the M2-brane super Lie 3-algebra.

  - $\mu_{M2} = \text{super-form component of magnetic flux}$ sourced by charged M5-branes.

  - $\mu_{M5} = \text{super-form component of electric flux}$ source by charged M2-branes.

So these cocycles are avatars of M-brane charge/flux at the level of super RHT.
Twisted K-theory in type II from M-theory

1 Type IIA. [BMSS] The double dimensional reduction of rational M-brane supercocycles \((\mu_{M2}, \mu_{M5})\) is indeed the tuple of F1/Dp-brane supercocycles \((\mu_{F1}, \mu_{D0}, \mu_{D2}, \mu_{D4}, \mu_{D6}, \mu_{D8})\) in rational twisted K-theory, which the literature demands to be the rational image of a cocycle in actual twisted K-theory.

<table>
<thead>
<tr>
<th>Objects</th>
<th>Cohomology theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-branes</td>
<td>twisted Cohomotopy</td>
</tr>
<tr>
<td>D-branes</td>
<td>twisted K-theory</td>
</tr>
</tbody>
</table>

[see talk by Vincent]

double dimensional reduction/oxidation

2 Type IIB. Characterization of T-duality for circle and sphere bundles using RHT [FSS].

- **Novel effect**: T-duality in super-exceptional spacetimes in 11d M-theory [FSS][SS].
2. Integrally
Rationally and stably $S^4_\mathbb{Q}$ is just the Eilenberg-MacLane space $K(\mathbb{Q}, 4)$, and

$$H^4(Y^{11}; \mathbb{Q}) \cong \pi^4_s(Y^{11}) \otimes \mathbb{Q}.$$ 

In the unstable case, schematically, we have

Rational cohomotopy = Rational cohomology + trivialization of the cup square

**Integrally** and **stably** we do see new effects.

- In between full non-abelian cohomotopy and abelian ordinary cohomology sits **stable cohomotopy**, represented not by actual spheres, but by their stabilization to the sphere spectrum.
- There is a description of the C-field in each one of these flavors [FSS][BMSS].

<table>
<thead>
<tr>
<th>Cohomology theory</th>
<th>Rational cohomology</th>
<th>Integral cohomology</th>
<th>Stable cohomotopy</th>
<th>Non-abelian cohomotopy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cocycle</td>
<td>$G_4$</td>
<td>$\tilde{G}_4$</td>
<td>$\Sigma^\infty c$</td>
<td>$c$</td>
</tr>
</tbody>
</table>

**Hypothesis H.** The C-field is charge-quantized in cohomotopy theory, even non-rationally.

Cancellation of main anomalies of M-theory follows naturally from cohomotopy:

1. C-field charge quantization in twisted cohomotopy implies various fundamental anomaly cancellation and quantization conditions [FSS].
2. Similar effects for D-branes and orientifolds [SS].
Lifting rational $S^4$ to integral $S^4$

- If we start with the rational 4-sphere $S^4_{\mathbb{Q}}$, then how can we lift it to an “integral” space?
- The actual 4-sphere $S^4$ stands out as not only the most natural but the finite-dimensional one.

\[ S^4_{\mathbb{Q}} \quad \Downarrow \quad \text{Integral, torsion} \quad \Downarrow \quad S^4_{\mathbb{Q}} \]

\[ Y \quad \text{Rational, non-torsion} \quad \Downarrow \quad S^4 \]

- Start with integral cohomology as describing the (shifted/twisted) C-field and then transition to a description in terms of cohomotopy. By representability, this amounts to lifting

\[ S^4 \quad \Downarrow \quad \text{Nonlinear prequantum} \quad \Downarrow \quad S^4 \]

\[ Y \quad \text{Linear quantum} \quad \Downarrow \quad K(\mathbb{Z}, 4) \quad . \]

The map $\iota$ assembles, upon taking homotopy classes, into the integral cohomology $H^4(S^4; \mathbb{Z})$ generated by a fundamental class.

- Description:

\[
\text{C-field in } \pi^4(Y^{11}) \iff \text{C-field in } H^4(Y^{11}; \mathbb{Z}) + \text{nontrivial conditions.}
\]
Proposition (Integral Postnikov tower for $S^4$ [GS])

Note that at the top level the three conditions vanish necessarily on $Y^{11}$, for dimension reasons.
Cohomotopy in deg 4 \sim Integral 4-cohomology + four sets of obstructions.

Pulling back to spacetime $Y$, where the fundamental class $\iota_4$ pulls back to the field

$$G_4 - \frac{1}{2} \lambda =: \tilde{G}_4 = f^* \iota_4$$

where $\lambda = \frac{1}{2} p_1$ is the first Spin characteristic class of $TY$.

**(i) First obstruction.**

$$\text{Sq}^2 \tilde{G}_4 \equiv 0 \in H^6(Y; \mathbb{Z}_2).$$

This follows from anomaly cancellation in M-theory [FSS].

**(ii) Second obstruction.**

$$f^*(\alpha_7) \equiv 0 \in H^7(Y; \mathbb{Z}_2)$$

where $\alpha_7$ is a secondary operation, restricting fiberwise to $\text{Sq}^2 \iota_5$.
No candidate degree 5 classes.

**(iii) Third obstructions.**

$$f^*("\text{Sq}^4 \iota_4" ) \equiv 0 \in H^8(Y; \mathbb{Z}_8)$$

- Note that by construction, this implies also that (upon mod 2 reduction)

  $$f^*(\text{Sq}^4 \iota_4) = \text{Sq}^4 f^* (\iota_4) = \text{Sq}^4 \tilde{G}_4 = \tilde{G}_4 \cup \tilde{G}_4 = 0 \in H^8(Y; \mathbb{Z}_2).$$

- Recall that rationally we have the EOM $d \ast G_4^{\text{form}} = \frac{1}{2} G_4^{\text{form}} \wedge G_4^{\text{form}}$.
- Coefficients being $\mathbb{Z}_8$ rather than $\mathbb{Z}_2$: Fields reduced modulo 4:

  $\frac{1}{2} \lambda \sim$ modding out $p_1$ by 4. (Pontrjagin square operation).
We also have $P_3^1 \iota_4 = 0$.

- Mod 3 reductions are shown to play a prominent role in topological considerations in M-theory [S], where similar conditions, including $P_3^1 \rho_3 G_4 = 0$, have been highlighted in the context of Spin K-theory.

**Fourth obstruction.**

\[
f^*(P_{11}) = 0
\]

where $P_{11}$ is a class which fiberwise restricts to $Sq^4 \iota_7$.

- Reminiscent of $G_4 \wedge G_7$.
- The universal coefficient theorem gives detectable effect for M-theory on orientable spacetimes.

**Fifth obstructions.**

\[
\text{"Sq}^8 \iota_4 = 0, \quad \iota_4^3 = 0, \quad P_5^1 \iota_4 = 0
\]

These obstructions necessarily vanish on $Y^{11}$. However on a 12-manifold $Z^{12}$, for analyzing the congruences of the Chern-Simons term in the M-theory action, the three conditions are nontrivial (but natural to have).
Proposition (Cohomotopy vs. cohomology for the C-field)

Consider the M-theory (shifted) C-field $\tilde{G}_4$ as an integral cohomology class in degree four. Then if $\tilde{G}_4$ lifts to a cohomotopy class $G_4 \in \pi^4(Y^{11})$ the following obstructions necessarily vanish

1. $Sq^2\tilde{G}_4 = 0 \in H^6(Y^{11}; \mathbb{Z}_2)$.
2. $P^1_3(\tilde{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_3)$.
3. $Sq^4\tilde{G}_4 = \tilde{G}_4 \cup \tilde{G}_4 = 0 \in H^8(Y^{11}; \mathbb{Z}_2)$.
4. If $G_4 = 0$ and $dC_3 = 0$ can be lifted to an integral class $\tilde{C}_3$, then we also have $Sq^3Sq^1\tilde{C}_3 = 0 \in H^7(Y^{11}; \mathbb{Z}_2)$.
5. If $dG_7 = G_4 \wedge G_4 = 0$ and $G_7$ can be lifted to an integral class $\tilde{G}_7$, then we also have the condition $Sq^4\tilde{G}_7 = 0 \in H^{11}(Y^{11}; \mathbb{Z}_2)$. 
**Consequences:**

1. **Congruences for the action** The Chern-Simons term in the action

\[ \frac{1}{6} \int_{Y^{11}} C_3 \wedge G_4 \wedge G_4. \]

Since \( C_3 \) may not be globally defined in general, one may consider \( Y^{11} \) as the boundary of a 12-manifold \( Z^{12} \) and analyzes the globally well defined term

\[ \frac{1}{6} \int_{Z^{12}} G_4 \wedge G_4 \wedge G_4 \tag{7} \]

[Witten]: usual quantization law of \( G_4 \) does not give rise to a well defined Chern-Simons action, as (7) might fail to be integral by a factor of 6.

Cohomotopy implies the added condition that

\[ \tilde{G}_4^3 \equiv 0 \mod 3. \]

This, with \( \tilde{G}_4^2 = Sq^4(\tilde{G}_4) \equiv 0 \mod 2 \), gives result (without \( E_8 \)-gauge theory).

2. **The anomaly in the partition function** Quantization in cohomotopy yields the condition \( Sq^2(\tilde{G}_4) = 0 \) for some integral lift of \( G_4 \).

- Implies the vanishing of the DMW anomaly \( Sq^3(\tilde{G}_4) = 0 \) [FSS].
- Obstruction theory for \( S^4 \Rightarrow \) fields which contribute to the phase are just the field which lift to the first Postnikov stage in cohomotopy [GS].
Example (Flux compactification spaces)

Anti-de Sitter space $\text{AdS}_n \sim$ simply-connected cover $\tilde{\text{AdS}}_n$ of $\text{AdS}_n$.

1. $\tilde{\text{AdS}}_4 \times \mathbb{C}P^2 \times T^2$: Supersymmetry without supersymmetry [Duff-Lu-Pope] and T-duality [Bouwknegt-Evslin-Mathai]. $\pi^4(\mathbb{C}P^2) \cong \mathbb{Z}$ while $H^4(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$.

2. $\tilde{\text{AdS}}_7 \times \mathbb{R}P^4$: M-theory on an orientifold [Witten][Hori]. $\pi^4(\mathbb{R}P^4) \cong \mathbb{Z}_2$ while $H^4(\mathbb{R}P^4; \mathbb{Z}) = 0$, indeed shows that cohomotopy detects more.

3. $\tilde{\text{AdS}}_4 \times \mathbb{R}P^5 \times T^2$: $\pi^4(\mathbb{R}P^5)$ is cyclic or order 4, i.e. either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, while $H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2$.

4. $\tilde{\text{AdS}}_4 \times \mathbb{C}P^3 \times S^1$: $\pi^4(\mathbb{C}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ while $H^4(\mathbb{C}P^3; \mathbb{Z}) \cong \mathbb{Z}$, so that there is an extra contribution of $\mathbb{Z}_2$ present in cohomotopy.

5. For $\mathbb{H}P^2$: $\pi^4(\mathbb{H}P^2) \cong \mathbb{Z}$ while $H^4(\mathbb{H}P^2; \mathbb{Z}) \cong \mathbb{Z}$, and hence no new contribution.

6. For $\mathbb{O}P^2$: $\pi^4(\mathbb{O}P^2) \cong \mathbb{Z}$. while $H^4(\mathbb{O}P^2; \mathbb{Z}) = 0$, signaling a new effect. Important for bosonic M-theory ([Ramond][S]).

Interpretation and consequences? Work in progress (via Pontrjagin-Thom theory).
Twisted Cohomotopy theory [FSS]

In degree $d-1$ there is a canonical twisting on Riemannian $d$-manifolds, given by the unit sphere bundle in the orthogonal tangent bundle:

$$
\pi^{TX^d}(X^d) := \left\{ \begin{array}{l}
\text{continuous section = twisted cocycle} \\
S(TX^d) \longrightarrow S^{d-1}/O(d)
\end{array} \right\} / \sim \text{homotopy}
$$

Since the canonical morphism $O(d) \longrightarrow \text{Aut}(S^{d-1})$ is known as the $J$-homomorphism, we may call this $J$-twisted Cohomotopy theory, for short.
Hypothesis H: The C-field 4-flux & 7-flux forms in M-theory are subject to charge quantization in J-twisted Cohomotopy cohomology theory in that they are in the image of the non-abelian Chern character map from J-twisted Cohomotopy theory. 

⇒ Cancellation of main anomalies:

<table>
<thead>
<tr>
<th>Half-integral flux quantization</th>
<th>$\left[ G_4 + \frac{1}{4} p_1 \right] \in H^4(X, \mathbb{Z})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Background charge</td>
<td>$q(\tilde{G}_4) = \tilde{G}_4 \left( \tilde{G}_4 - \frac{1}{2} p_1 \right)$</td>
</tr>
<tr>
<td>DMW-anomaly cancellation</td>
<td>$W_7(TX) = 0$</td>
</tr>
<tr>
<td>Integral equation of motion</td>
<td>$Sq^3(\tilde{G}_4) = 0$</td>
</tr>
<tr>
<td>M5-brane anomaly cancellation</td>
<td>$I_{\text{ferm}}^{M5} + I_{\text{sd}}^{M5} + I_{\text{infl}}^{\text{bulk}} = 0$</td>
</tr>
<tr>
<td>M2-brane tadpole cancellation</td>
<td>$N_{M2} + q(\tilde{G}_4) = I_8$</td>
</tr>
</tbody>
</table>

Consequences for WZW model associated to M5-brane ⇒ [See talk by Domenico]
J-Twisted Cohomotopy and Topological $G$-Structure

For every topological coset space realization $G/H$ of an $n$-sphere, there is a canonical homotopy equivalence between the classifying spaces for $G$-twisted Cohomotopy and for topological $H$-structure (i.e., reduction of the structure group to $H$), as follows:

$$
\begin{array}{c}
\text{coset space structure} \\
\text{on topological $n$-sphere}
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\text{$G$-twisted Cohomotopy} \\
\text{topological $H$-structure}
\end{array}

S^n \xrightarrow{\text{homeo}} G/H \Rightarrow S^n \htpy G \simeq BH.

(One may think of this as “moving $G$ from numerator on the right to denominator on the left”.)

Existence of a $G$-structure is a non-trivial topological condition, so is the existence of $J$-twisted Cohomotopy cocycles.

Notice that this is a special effect of twisted non-abelian generalized Cohomology: A non-twisted generalized cohomology theory (abelian or non-abelian) always admits at least one cocycle, namely the trivial or zero-cocycle. But here for non-abelian $J$-twisted Cohomotopy theory on 8-manifolds, the existence of 	extit{any} cocycle is a non-trivial topological condition.
Equivalence for Spin 8-manifolds

\[ S^7 \text{//} \text{Spin}(8) \simeq B \text{Spin}(7) \]

\[ S^7 \text{//} \text{Spin}(2) \cdot \text{Sp}(1) \simeq B \text{Sp}(1) \cdot \text{Sp}(1) \]
Stable vs. unstable

The quaternionic Hopf fibration.

\[ S^7 \xrightarrow{\cong} S(\mathbb{H}^2) \xrightarrow{(q_1,q_2) \mapsto [q_1:q_2]} \mathbb{H}P^1 \xrightarrow{\cong} S^4, \]

which represents a generator of the non-torsion subgroup in the 4-Cohomotopy of the 7-sphere, as shown on the left here:

\[ [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] \xrightarrow{\pi^4(S^7)} \Sigma^\infty S^4(S^7) \xrightarrow{\Sigma^\infty [S^7 \xrightarrow{h_{\mathbb{H}}} S^4]} \]

\[ (1,0) \in \mathbb{Z} \times \mathbb{Z}_{12} \xrightarrow{(n,a) \mapsto (n \text{ mod } 24)} \mathbb{Z}_{24} \ni 1 \]

- So composition with the quaternionic Hopf fibration can be viewed as a transformation that translates deg-7 to deg-4 Cohomotopy classes:

\[ X \xrightarrow{(h_{\mathbb{H}})_*(c)} S^4 \xrightarrow{h_{\mathbb{H}}} S^7 \xrightarrow{\pi^7(X)} \]

\[ \pi^7(X) \xrightarrow{(h_{\mathbb{H}})_*} \pi^4(X) \xrightarrow{(h_{\mathbb{H}})_*} \]

\[ \pi^4(X) \xrightarrow{(h_{\mathbb{H}})_*} \]
Proposition (Differential form data underlying twisted Cohomotopy)

Let $X$ be a simply connected smooth manifold and $\tau: X \to BO(n + 1)$ a twisting for Cohomotopy in degree $n$. Let $\nabla_\tau$ be any connection on the real vector bundle $V$ classified by $\tau$ with Euler form $\chi_{2k+2}(\nabla_\tau)$ (see [Mathai-Quillen]).

(i) If $n = 2k + 1$ is odd $n \geq 3$: a cocycle defining a class in the rational $\tau$-twisted Cohomotopy of $X$ is equivalently given by

$$
\pi^\tau_Q(X) \simeq \left\{ G_{2k+1} \mid d G_{2k+1} = \chi_{2k+2}(\nabla_\tau) \right\}/\sim.
$$

(ii) If $n = 2k$ is even, $n \geq 2$: a cocycle defining a class in the rational $\tau$-twisted Cohomotopy of $X$ is given by a pair of differential forms $G_{2k} \in \Omega^{2k}(X)$ and $G_{4k-1} \in \Omega^{4k-1}(X)$ such that

$$
\begin{align*}
d G_{2k} &= 0; \\
\pi^* G_{2k} &= \frac{1}{2} \chi_{2k}(\nabla_\hat{\tau})
\end{align*}
$$

$$
\begin{align*}
d G_{4k-1} &= -G_{2k} \wedge G_{2k} + \frac{1}{4} p_k(\nabla_\tau),
\end{align*}
$$

where $p_k(\nabla_\tau)$ is the $k$-th Pontrjagin form of $\nabla_\tau$, $\pi: E \to X$ is the unit sphere bundle over $X$ associated with $\tau$, $\hat{\tau}: E \to BO(n)$ classifies the vector bundle $\hat{V}$ on $E$ defined by the splitting $\pi^* V = \mathbb{R}_E \oplus \hat{V}$ associated with the tautological section of $\pi^* V$ over $E$, and $\nabla_\hat{\tau}$ is the induced connection on $\hat{V}$. That is,

$$
\pi^\tau_Q(X) \simeq \left\{ (G_{2k}, G_{4k-1}) \middle| \begin{array}{l}
d G_{2k} = 0, \\
\pi^* G_{2k} = \frac{1}{2} \chi_{2k}(\nabla_\hat{\tau})
\end{array} \right\}/\sim.
$$
3. Differentially
Differential refinement

- Refine the topological lift (5) to a geometric lift at the level of smooth stacks of the form

\[ \hat{S}^4 \]

\[ \Rightarrow \]

\[ \rightarrow B^3 U(1)_\nabla \]

(8)

where \( \hat{S}^4 \) is the differential refinement of the 4-sphere and \( B^3 U(1)_\nabla \) is the smooth stack of 3-bundles with connections

- This would require a differential refinement of the Postnikov tower which uses refinement of cohomology operations, primary (such as Steenrod operations) and secondary (such as Massey products) [GS].
Differential cohomotopy [Fiorenza-S.-Schreiber]

- **h-Hopf fibration:** \( S^3 \to S^7 \to S^4 \to \text{BSU}(2) \overset{c_2}{\to} K(\mathbb{Z}, 4) \).
- Rationalize: \( S^3_{\mathbb{Q}} \to S^7_{\mathbb{Q}} \to S^4_{\mathbb{Q}} \to (\text{BS}^3)_{\mathbb{Q}} \) which is equivalent to
  \[
  K(\mathbb{Q}, 7) \to S^4_{\mathbb{Q}} \to K(\mathbb{Q}, 4)
  \]

- Rational homotopy of spaces can be modelled using \( L_\infty \)-algebras.
- The Eilenberg-MacLane spaces \( K(\mathbb{Q}, n) = B^n\mathbb{Q} \) can be modelled using algebras via chain complexes: \( b^n\mathbb{Q} = \mathbb{Q}[n] \).
- Lie 7- algebra \( s^4 \) is defined by \( CE(s^4) = \mathbb{R}[g_4, g_7] \) with \( g_k \) in degree \( k \) and with the differential defined by \( dg_4 = 0, dg_7 = g_4 \wedge g_4 \).
- Has a natural structure of infinitesimal \( \mathbb{R}[2] \)-quotient of \( \mathbb{R}[6] \), i.e., there exists a natural homotopy fiber sequence of \( L_\infty \)-algebras
  \[
  \begin{array}{ccc}
  \mathbb{R}[6] & \to & s^4 \\
  \downarrow & & \downarrow^p \\
  0 & \to & \mathbb{R}[3].
  \end{array}
  \]

**Theorem (FSS)**

*The system \((\hat{G}_4, \hat{G}_7)\) forms a cocycle in differential cohomotopy.*
Differential refinements: $B^3 U(1)_\nabla$ vs. $\hat{S}^4$

- Let $s^4$ be the Lie 7-algebra whose corresponding Chevellay-Eilenberg algebra is the exterior algebra on generators $g_4$ and $g_7$ with relations
  \[ dg_4 = 0 \, , \quad dg_7 = g_4 \wedge g_4 . \]

- As a de Rham model for flat 1-forms with values in $S^4$ we take the sheaf on the site of Cartesian spaces given by the assignment
  \[ \Omega^1_\fil(-; s^4) : U \mapsto \text{hom}_{dgcAlg}(\text{CE}(s^4), \Omega^*(U)) , \]
  for each Cartesian space $U \cong \mathbb{R}^n$. (The homotopy type of $\Omega^1_\fil(-; s^4)$ can be computed via the Sullivan construction as the $\mathbb{R}$-local 4-sphere $S^4_{\mathbb{R}}$).

- Then pulling back along the canonical map $S^4 \to S^4_{\mathbb{R}}$, we get a smooth stack
  \[ \begin{array}{ccc}
  \hat{S}^4 & \to & \Omega^1_\fil(-; s^4) \\
  \downarrow & & \downarrow \\
  S^4 & \to & S^4_{\mathbb{R}}.
  \end{array} \]

**Definition (Differential unstable cohomotopy)**

For a smooth manifold $X$, let $i(X)$ denote its embedding as a smooth stacks via its sheaf of smooth plots. Then the differential cohomotopy of $X$ in degree 4 is defined as the pointed set $\hat{\pi}^4_u(X) := \pi_0 \text{Map}(i(X), \hat{S}^4)$ where the maps on the right are those of smooth stacks.
Differential cohomotopy: stably

- Stably, $S^4$ has only torsion groups in higher degrees and hence the canonical map $S^4 \to K(\mathbb{R}, 4)$ is a stable $\mathbb{R}$-local equivalence.
- Geometrically, the realification if modeled by closed 4-forms $\Omega^4_{\text{cl}}(-)$.
- Stable differential cohomotopy in degree 4 fits into a pullback square

$$
\begin{array}{ccc}
\Sigma^\infty S^4 & \to & H\left(\tau^{\leq 0} \Omega^4 +^* (-)\right) \\
\downarrow & & \downarrow \\
\Sigma^\infty S^4 & \to & \Sigma^4 H\mathbb{R}.
\end{array}
$$

where $\Omega^4 +^* (-)$ denotes the de Rham complex, shifted so that $\Omega^4$ is in degree zero, and $\tau^{\leq 0}$ truncates the complex in degree zero so that the complex is concentrated in negative degrees. The functor $H$ denotes the Eilenberg-MacLane functor which turns a chain complex into a spectrum.

Definition (Differential stable cohomotopy)

Let $X$ be a smooth manifold with $i(X)$ its associated smooth stack. The stable differential cohomotopy group of $X$ is defined as

$$
\hat{\pi}^4_s(X) := \pi_0 \text{Map}(i(X); (\Sigma^\infty S^4)_0).
$$

where the subscript 0 denotes the deg 0 component of the sheaf of spectra $\Sigma^\infty S^4$. 
Geometric cycles

Definition (Geometric cohomotopy cocycles [GS])

If $X$ is a smooth manifold, a morphism $\hat{c} : X \to \hat{S}^4$ can be identified with a triple $(c, h, \omega)$ where

(i) $c : X \to S^4$ is a cocycle in ordinary cohomotopy,

(ii) $\omega : CE(S^4) \to \Omega^*(X)$ is a DGA morphism, determined by specifying forms $\omega_4$ and $\omega_7$ on $M$ satisfying $d\omega_7 = \omega_4^2$ and $d\omega_4 = 0$,

(iii) and $h$ is a homotopy interpolating between the rational cocycle represented by the form data and the rationalization of the classifying map $c : X \to S^4$. Thus, $h$ exhibits a sort of de Rham theorem for cohomotopy.
Proposition (Differential refinement of Postnikov tower of the sphere)

\[ K(\mathbb{Z}_{15}, 11) \rightarrow (\hat{S}^4)_{7} \]
\[ K(\mathbb{Z}_{24} \times \mathbb{Z}_{3}, 10) \rightarrow (\hat{S}^4)_{6} \rightarrow K(\mathbb{Z}_{15}, 12) \]
\[ K(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, 9) \rightarrow (\hat{S}^4)_{5} \rightarrow K(\mathbb{Z}_{24} \times \mathbb{Z}_{3}, 11) \]
\[ K(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, 8) \rightarrow (\hat{S}^4)_{4} \rightarrow K(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, 10) \]
\[ K(\mathbb{Z}_{12}, 7) \times K(\mathbb{Z}, 7) \rightarrow (\hat{S}^4)_{3} \rightarrow K(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, 9) \]
\[ K(\mathbb{Z}_{2}, 6) \rightarrow (\hat{S}^4)_{2} \xrightarrow{(?, \iota^2_4)} K(\mathbb{Z}_{12}, 8) \times B^7 U(1) \]
\[ K(\mathbb{Z}_{2}, 5) \rightarrow (\hat{S}^4)_{1} \xrightarrow{\alpha_7^l} K(\mathbb{Z}_{2}, 7) \]
\[ (\hat{S}^4)_0 = B^3 U(1) \xrightarrow{Sq^2 \rho_2^l} K(\mathbb{Z}/2, 6) \]

where we have identified the first few obstructions.
Consider the differentially refined M-theory (shifted) C-field \( \widehat{G}_4 \) as an integral cohomology class in degree four. Then if \( \widehat{G}_4 \) lifts to a cohomotopy class \( G_4 \in \widehat{\pi}^4(Y^{11}) \) the following obstructions necessarily vanish

(i) \( Sq^2 I(\widehat{G}_4) = 0 \in H^6(Y^{11}; \mathbb{Z}_2). \)

(ii) \( P_3^1 I(\widehat{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_3). \)

(iii) \( Sq^4 I(\widehat{G}_4) = I(\widehat{G}_4 \cup_{DB} \widehat{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_2). \)

If \( \widehat{G}_4 = 0 \) and \( \mathcal{C}_3^{\text{form}} \) is quantized, with differential refinement \( \widehat{C}_3 \), then we also have \( Sq^3 Sq^1 I(\widehat{C}_3) = 0 \in H^7(Y^{11}; \mathbb{Z}_2). \)

If \( dG_7^{\text{form}} = G_4^{\text{form}} \wedge G_4^{\text{form}} = 0 \) and \( G_7^{\text{form}} \) is quantized, with differential refinement \( \widehat{G}_7 \), then we also have the condition \( Sq^4 I(\widehat{G}_7) = 0 \in H^{11}(Y^{11}; \mathbb{Z}_2). \)

Remark (Obstruction in M-theory via higher bundles with connections)

Deligne-Beilinson cup product in M-theory \( \widehat{G}_4 \cup_{DB} \widehat{G}_4 \) gives a 7-bundle with connection form locally given by \( C_3^{\text{form}} \wedge G_4^{\text{form}} \) [FSS]. From identification of the \( k \)-invariant at 2nd stage (the DB square): to lift past the 2nd stage in the Postnikov tower for \( \widehat{S}^4 \), this connection must be globally defined. In terms of differential cohomology, \( a(C_3^{\text{form}} \wedge G_4^{\text{form}}) = \widehat{G}_4 \cup_{DB} \widehat{G}_4 \), where \( a : \Omega^7(Y^{11}) \to \widehat{H}^8(Y^{11}) \) is the canonical map.
Example (Differential cohomotopy of flux compactification spaces)

LES in stable cohomotopy

\[ \ldots \rightarrow \pi^3_s(X) \xrightarrow{\text{deg}} \Omega^3(X) \rightarrow \hat{\pi}^4_s(X) \rightarrow \pi^4_s(X) \rightarrow \ldots. \]

allows to compute some examples.

(i) \( \widetilde{\text{AdS}}_7 \times \mathbb{R}P^4 \): \( \hat{\pi}^4_s(\widetilde{\text{AdS}}_7 \times \mathbb{R}P^4) \cong \hat{H}^4(\widetilde{\text{AdS}}_7 \times \mathbb{R}P^4) \).

(ii) \( \widetilde{\text{AdS}}_4 \times \mathbb{C}P^2 \): \( \hat{\pi}^4_s(\widetilde{\text{AdS}}_4 \times \mathbb{C}P^2) \cong \hat{H}^4(\widetilde{\text{AdS}}_4 \times \mathbb{C}P^2) \).

(iii) \( \widetilde{\text{AdS}}_4 \times \mathbb{C}P^2 \times T^2 \): \( \hat{\pi}^4_s(\widetilde{\text{AdS}}_4 \times \mathbb{C}P^2 \times T^2) \cong \hat{H}^4(\widetilde{\text{AdS}}_4 \times \mathbb{C}P^2 \times T^2) \).

(iv) \( \widetilde{\text{AdS}}_4 \times \mathbb{R}P^5 \times T^2 \): \( \pi^4(\mathbb{R}P^5) \) is order 4, either \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), while \( H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2 \). Also \( \pi^3(\mathbb{R}P^5) \) is finite. We therefore have a short exact sequence

\[ 0 \rightarrow \Omega^3(\mathbb{R}P^5) \rightarrow \hat{\pi}^4(\mathbb{R}P^5) \rightarrow \pi^4(\mathbb{R}P^5) \rightarrow 0. \]

Since \( \pi^4(\mathbb{R}P^5) \) is generated by \( q_5 \eta_4 \), with \( \eta_4 : S^5 \rightarrow S^4 \) the two-fold suspension of the Hopf map, the induced map on \( H^4 \) necessarily vanishes. Hence, in this case, differential cohomotopy yields considerably different information than ordinary differential cohomology.
Cohomotopy $\Rightarrow$ branes and gauge theory
Differential cohomotopy and D-brane gauge theories

Zoom in beyond foundational/structural M-theoretic considerations [SS]:

1. A differential refinement of Cohomotopy cohomology theory is given by *un-ordered configuration spaces of points*.

2. The fiber product of such differentially refined Cohomotopy cocycle spaces describing $D6 \perp D8$-brane intersections is homotopy-equivalent to the *ordered configuration space of points* in the transversal space.

3. The higher observables on this moduli space are equivalently weight systems on horizontal chord diagrams.
Combining the above seemingly distinct mathematical areas reflect a multitude of effects expected on brane intersections in string theory. So aside from structural utility for M-theory, Hypothesis H implies:

- M-theoretic observables on $D6 \perp D8$-configurations (cf. parametrized).
- Chan-Paton observables.
- String topology operations.
- Multi-trace observables of BMN matrix model.
- Hanany-Witten states.
- BLG 3-Algebra observables.
- Bulk Wilson loop observables. [See talk by Urs]
- Single-trace observables of SYK & BMN model.
- Fuzzy funnel observables.
- Supersymmetric indices.
- ’t Hooft string amplitudes.

**Top-down M-theory via Hypothesis H**: knowledge about gauge field theory and perturbative string theory is not used in deriving the algebras of observables of M-theory, but only to interpret them.
Thank you!