Six-Functor Formalisms

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Preface

These are lectures notes for a course in Winter 2022/23.

The goal of the lectures is to present some recent developments around six-functor formalisms, in particular

(1) the abstract theory of 6-functor formalisms;
(2) the 2-category of cohomological correspondences, and resulting simplifications in the proofs of Poincaré–Verdier duality results;
(3) the relation between 6-functor formalisms and "geometric rings";
(4) many examples of 6-functor formalisms, both old and new.

Many thanks to all the participants for very helpful discussions!

October 2022

1. Lecture I: Introduction

The first main goal of these lectures is to answer the question: What is a 6-functor formalism?

For a long time, the answer was "you know it when you see it": The first such formalism was established during the development of étale cohomology of schemes, and it was soon realized that very similar formalisms also exist in other contexts, such as for (usual) cohomology of topological spaces, or for D-modules on algebraic varieties in characteristic 0. But as I will discuss below, it is a highly nontrivial task to formalize all the structure implicit in a 6-functor formalism, and the abstract notion was only formalized recently.

To get started, we will recall the 6-functor formalism on (nice) topological spaces. For this lecture, let us work with the category C of finite-dimensional locally compact Hausdorff spaces X, say (for simplicity) those that can be written as locally closed subsets of some $\mathbb{R}^n$. We are interested in studying the cohomology of X, say $H^i(X,\mathbb{Z})$. In this generality, where X might be a Cantor set, the good definition of cohomology is not singular cohomology, but instead the sheaf cohomology. To define it, one starts with the abelian category $\mathrm{Ab}(X)$ of abelian sheaves on X and the global sections functor

$$H^0(X,-):\mathrm{Ab}(X)\to\mathrm{Ab}.$$

This is a left exact functor, and one can define its right derived functors

$$H^i(X,-):\mathrm{Ab}(X)\to\mathrm{Ab}.$$

Applied to the constant sheaf $\mathbb{Z}$ on X, this defines the cohomology groups $H^i(X,\mathbb{Z})$ as desired.

Reflecting on this definition, one is led to also contemplate the full derived functor

$$R\Gamma(X,-):D(\mathrm{Ab}(X))\to D(\mathrm{Ab}).$$

In particular, one is led to contemplate the derived category $D(X,\mathbb{Z}):=D(\mathrm{Ab}(X))$ of abelian sheaves on X, for any such X. As a categorically minded person, one is also immediately led to contemplate its functoriality in X. Namely, for any $f:X\to Y$, there is an exact pullback functor $f^*:\mathrm{Ab}(Y)\to\mathrm{Ab}(X)$ inducing a functor

$$f^*:D(Y,\mathbb{Z})\to D(X,\mathbb{Z})$$

(our first functor!) which has a right adjoint

$$f_*:D(X,\mathbb{Z})\to D(Y,\mathbb{Z})$$

(our second functor!). In the case where $Y=*$ is a point, the functor f^* is the "constant sheaf functor", while $f_*=R\Gamma(X,-)$ is the functor introduced above. In particular, the cohomology of X,

$$R\Gamma(X,\mathbb{Z})=f_*\mathbb{Z}=f_*f^*\mathbb{Z}\in D(*,\mathbb{Z})=D(\mathbb{Z}),$$

has a simple description in terms of these functors.

What is the functor f_* in general? It is a relative version of cohomology. Ideally, one would like to say that it interpolates the cohomology of all the fibres. While this is not true in general, it is true when f is proper (i.e. the preimage of any compact subset is compact):

Theorem 1.1 (Proper Base Change). *Let $f:X\to Y$ be a proper map in C and let $y\in Y$ with fibre $X_y=X\times_Y\{y\}$, with inclusions $i_X:X_y\to X$ and $i:\{y\}\to Y$. Take any $A\in D(X,\mathbb{Z})$. Then the natural map*

$$(f_*A)_y\to R\Gamma(X_y,i^*A)$$

is an isomorphism, where $(f_*A)_y := i^*f_*A \in D(\{y\},\mathbb{Z}) = D(\mathbb{Z})$ *is the stalk of* f_*A *at* y.

More generally, for any other map $g : Y' \to Y$ *with base change* $f' : X' = X \times_Y Y' \to Y'$ *(and* $g' : X' \to X$*), the natural base change transformation*

$$g^*f_* \to f'_*g'^*$$

of functors $D(X,\mathbb{Z}) \to D(Y',\mathbb{Z})$ *is an isomorphism.*

Here, $g^*f_* \to f'_*g'^*$ is adjoint to

$$f'^*g^*f_* = g'^*f^*f_* \to g'^*$$

where the latter map comes from the counit map $f^*f_* \to \mathrm{id}$.

At this point, we have defined the cohomology $R\Gamma(X,\mathbb{Z})$ of any $X \in C$. In fact, we have also encoded the pullback maps $R\Gamma(Y,\mathbb{Z}) \to R\Gamma(X,\mathbb{Z})$ for $f : X \to Y$: There is a unit map $\mathbb{Z} \to f_*f^*\mathbb{Z} = f_*\mathbb{Z}$, and taking $R\Gamma(Y,-)$ produces the desired map as $R\Gamma(Y, f_*\mathbb{Z}) = R\Gamma(X,\mathbb{Z})$ as the lower-shriek functors compose.

But there are further important structures on cohomology. For example:

Theorem 1.2 (Künneth Formula). *For proper* X *and* Y*, there is a natural isomorphism*

$$R\Gamma(X,\mathbb{Z}) \otimes R\Gamma(Y,\mathbb{Z}) \cong R\Gamma(X \times Y,\mathbb{Z}).$$

Here, $\otimes : D(\mathbb{Z}) \times D(\mathbb{Z}) \to D(\mathbb{Z})$ is the tensor product on $D(\mathbb{Z})$, i.e. the derived tensor product.

Even the formulation of this theorem requires us to contemplate the tensor product on $D(\mathbb{Z})$; and of course one is then led to also contemplate the tensor product on $D(X,\mathbb{Z})$ for any X. (This arises from the usual tensor product on $\mathrm{Ab}(X)$ by deriving.) Thus, we consider

$$- \otimes - : D(X,\mathbb{Z}) \times D(X,\mathbb{Z}) \to D(X,\mathbb{Z})$$

(our third functor!), which again has a (partial) right adjoint

$$\underline{\mathrm{Hom}}(-,-) : D(X,\mathbb{Z})^{\mathrm{op}} \times D(X,\mathbb{Z}) \to D(X,\mathbb{Z})$$

(our fourth functor!) characterized by the adjunction

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B,C)) \cong \mathrm{Hom}(A \otimes B, C).$$

How do these new functors interact with the previous ones? The pullback functors f^* are symmetric monoidal, i.e. commute with the tensor product, so in particular

$$f^*(A \otimes B) \cong f^*A \otimes f^*B.$$

Note that these isomorphisms here are really extra data, that ought to be subject to further compatibilities. Fortunately, there is a theory of symmetric monoidal categories and symmetric monoidal functors that makes it possible to express these compatibilities.

There is also a compatibility between tensor products and pushforward, again in the proper case.

Theorem 1.3 (Projection Formula). *Let* $f : X \to Y$ *be proper and* $A \in D(X,\mathbb{Z})$, $B \in D(Y,\mathbb{Z})$. *Then the natural map*

$$f_*A \otimes B \to f_*(A \otimes f^*B)$$

is an isomorphism.

The map is adjoint to

$$f^*(f_*A \otimes B) \cong f^*f_*A \otimes f^*B \to A \otimes f^*B.$$

This is enough to prove the Künneth Formula. In fact, if X and Y are proper and $A \in D(X, \mathbb{Z})$ and $B \in D(Y, \mathbb{Z})$, we claim that there is a natural isomorphism

$$R\Gamma(X, A) \otimes R\Gamma(Y, B) \cong R\Gamma(X \times Y, A \boxtimes B)$$

where $A \boxtimes B := p_1^*A \otimes p_2^*B \in D(X \times Y, \mathbb{Z})$. Here, we use the maps

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & \overset{p}{\searrow} & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{p_X} & *. \end{array}$$

Indeed,

$$\begin{aligned} R\Gamma(X \times Y, A \boxtimes B) &= p_*(p_1^*A \otimes p_2^*B) \\ &= p_{X*}(p_{1*}(p_1^*A \otimes p_2^*B)) \\ &\cong p_{X*}(A \otimes p_{1*}p_2^*B) \\ &\cong p_{X*}(A \otimes p_X^*p_{Y*}B) \\ &\cong p_{X*}A \otimes p_{Y*}B = R\Gamma(X, A) \otimes R\Gamma(Y, B). \end{aligned}$$

Finally, there is one more important structural feature of cohomology: Duality.

Theorem 1.4 (Poincaré Duality). *Assume that X is a compact oriented manifold of dimension d. Then there is a natural isomorphism*

$$R\Gamma(X, \mathbb{Z})[d] \cong R\Gamma(X, \mathbb{Z})^\vee$$

where $-^\vee = \underline{\mathrm{Hom}}(-, \mathbb{Z})$ is the duality functor on $D(\mathbb{Z})$.

In other words, up to shift, cohomology is self-dual. In fact, one can state a more precise version applying to any sheaf $A \in D(X, \mathbb{Z})$ with dual $A^\vee = \underline{\mathrm{Hom}}(A, \mathbb{Z})$:

$$R\Gamma(X, A^\vee)[d] \cong R\Gamma(X, A)^\vee.$$

In fact, one can even replace the $\mathbb{Z}$ in the dual $\underline{\mathrm{Hom}}(-, \mathbb{Z})$ by any $B \in D(\mathbb{Z})$, thus leading to

$$R\Gamma(X, \underline{\mathrm{Hom}}(A, f^*B))[d] \cong \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma(X, A), B).$$

In categorical language, this means precisely that the functor $B \mapsto f^*B[d]$ is right adjoint to $f_* = R\Gamma(X, -) : D(X, \mathbb{Z}) \to D(\mathbb{Z})$. Even more generally:

Theorem 1.5 (Verdier duality). *Let $f : X \to Y$ be a proper map that is a "manifold bundle" (i.e. locally on X and Y of the form $Y \times \mathrm{ball} \to Y$) of relative dimension d. Then the functor $f_* : D(X, \mathbb{Z}) \to D(Y, \mathbb{Z})$ admits a right adjoint which is given by*

$$f^* \otimes \omega_{X/Y}$$

for some sheaf $\omega_{X/Y} \in D(X, \mathbb{Z})$ that is locally isomorphic to $\mathbb{Z}[d]$.

The "oriented" assumption in Poincaré duality ensures that $\omega_{X/*} \cong \mathbb{Z}[d]$ globally on X.

The problem with this form of Verdier duality is that the assumptions on f combine a global assumption ("proper") with a local assumption ("manifold bundle"); this also makes a direct proof more complicated. Fortunately, there is a way to state a purely local form of Verdier duality.

Namely, for any map $f : X \to Y$ there is a further "proper pushforward" functor

$$f_! : D(X\mathbb{Z}) \to D(Y, \mathbb{Z})$$

(our fifth functor!). In this setting, this can be defined as the derived functor of the functor of sections with proper support. In particular, there is a natural transformation $f_! \to f_*$ that is an isomorphism when f is proper. Again, there is a right adjoint

$$f^! : D(Y, \mathbb{Z}) \to D(X, \mathbb{Z})$$

(the final sixth functor!) called the "exceptional inverse image" functor. Now we can state the local version:

THEOREM 1.6 (Verdier duality). *Let $f : X \to Y$ be a "manifold bundle" of relative dimension d. Then $f^!$ is isomorphic to $f^* \otimes \omega_{X/Y}$ where $\omega_{X/Y} = f^!\mathbb{Z}$ is locally isomorphic to $\mathbb{Z}[d]$.*

The advantage of this local statement is that it can be formally reduced to the case that $X = Y \times \text{ball}$ is the product of Y with an (open d-dimensional) ball; and by induction one can also assume that $d = 1$. Another advantage of introducing the functor $f_!$ is that it enables one to state more general versions of proper base change and the projection formula. In fact, if one replaces f_* by $f_!$ in proper base change and the projection formula, they are true without the hypothesis that f is proper! There is however a technical problem: In this more general version of proper base change and the projection formula, one does not have comparison maps a priori, and the isomorphisms are instead extra data that ought to be subject to further coherence isomorphisms.

Finally, we can say what a 6-functor formalism is, roughly speaking: There is some category C of geometric objects (topological spaces, schemes, stacks, analytic spaces, ...) and an association $X \mapsto D(X)$ from C to (triangulated/stable ∞-/...) categories, together with functors $(f^*, f_*, \otimes, \underline{\mathrm{Hom}}, f_!, f^!)$ satisfying several compatibilities. Notably, every second functor is the right adjoint of the previous one; pullback is symmetric monoidal; and $f_!$ satisfies general base change and projection formula. There are many examples: Etale cohomology (in various settings), but also D-modules, or mixed Hodge modules, or arithmetic D-modules, or...

However, it has been an open problem how to encode all the required compatibilities, especially if – as is the custom these days – one treats $D(X)$ as a stable ∞-category (which leads to an implicit infinite system of higher coherences). Moreover, establishing the foundational results of each formalism, especially Poincaré–Verdier duality, usually requires a heavy amount of work that is specific to each formalism.

Our goal in this course is to first define and study abstract 6-functor formalisms, following Mann. We will then set up some general machinery applying in this abstract generality that in particular reduces the proof of Poincaré–Verdier duality to a rather simple problem, building on work of Lu–Zheng. Afterwards, we will discuss many different examples of 6-functor formalisms. A curious phenomenon is that in most cases the association $X \mapsto D(X)$ can be factored as a composite

$$C \xrightarrow{F} \{\text{analytic stacks}\} \xrightarrow{D_{\mathrm{qc}}} \mathrm{Cat}$$

where the first functor F takes any $X \in C$ to some other kind of geometric object $F(X)$, sometimes a scheme but often rather a stack or even an analytic stack, and the second functor is the functor of taking the derived category of quasicoherent sheaves on an analytic stack. This gives a more geometric perspective on a 6-functor formalisms, as a functor F between different kinds of geometric objects. This perspective also leads to a relation between 6-functor formalisms and (analytic) “ring stacks” i.e. (commutative) ring objects in (analytic) stacks (this was first observed by Drinfeld in relation to prismatic cohomology).

A somewhat curious case of this phenomenon is in the context of this introductory lecture:

Exercise 1.7. For $X \in C$ as above a finite-dimensional locally compact Hausdorff space, consider the functor

$$\underline{X}^{\mathrm{sch}} : \mathrm{Schemes}^{\mathrm{op}} \to \mathrm{Sets}$$

taking any scheme S to the continuous maps $|S| \to X$. Show that $\underline{X}^{\mathrm{sch}}$ is a “pro-étale algebraic space”, i.e. admits a pro-étale surjection from a scheme. Moreover, show that there is a natural equivalence

$$D(X, \mathbb{Z}) \cong D_{\mathrm{qc}}(\underline{X}^{\mathrm{sch}}).$$

2. Lecture II: Six-Functor Formalisms

In the first part of this course, we want to develop some abstract theory of 6-functor formalisms; the second part will then discuss many examples.

For this first part, we will work in the following setup: A category C, thought of as the category of geometric objects, and a class of morphisms E of C, which will be the class of morphisms f of C for which the "exceptional" functors $f_!$ (and $f^!$) functors are defined. We will always assume the following hypotheses:

(1) The category C has all finite limits.
(2) The class of morphisms E contains all isomorphisms, and is stable under pullback and composition.

The naive idea of a 6-functor formalism is the following:[1]

(1) An association $X \mapsto D(X)$ from C to (∞-)categories $D(X)$.
(2) A symmetric monoidal structure $\otimes$ on $D(X)$ for each $X \in C$.
(3) For each $f : X \to Y$, a pullback functor $f^* : D(Y) \to D(X)$, compatible with the symmetric monoidal structure, and compatible with composition of maps in C.
(4) For each $f : X \to Y$ in E, a functor $f_! : D(X) \to D(Y)$, compatible with composition, and satisfying base change and projection formula isomorphisms.
(5) Moreover, there should right adjoints to $- \otimes A$ (i.e., internal Hom's), f^* (i.e. f_*), and $f_!$ (i.e. $f^!$) for $f \in E$.

It is rather easy to formalize items (1)–(3): Indeed, this is precisely encoded in a functor

$$C^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$$

where Cat_∞ denotes the (∞-)category of ∞-categories, and CMon denotes the commutative monoids in it (for the Cartesian product), in other words $\mathrm{CMon}(\mathrm{Cat}_\infty)$ is the (∞-)category of symmetric monoidal ∞-categories (with morphisms given by symmetric monoidal functors). Also, item (5) is just a condition. The whole difficulty in formalizing a 6-functor formalism lies in formalizing the data in (4). Let us list some of the data one would like to have:

After defining the functor $f_! : D(X) \to D(Y)$ for each $f \in E$, one has to supply in addition:

(1) For each $f : X \to Y$ and $g : Y \to Z$ in E, with composite $h : X \to Z$ thus also in E, an isomorphism $h_! \cong g_! f_!$ of functors $D(X) \to D(Z)$ (satisfying further associativity constraints for triple compositions).
(2) For each cartesian diagram

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD}$$

with f in E (and thus also $f' \in E$), a base change isomorphism

$$g^* f_! \cong f'_! g'^*$$

of functors $D(X) \to D(Y')$. Moreover, these isomorphisms should be compatible with composing base change isomorphisms (horizontally and vertically).

[1] In making the notions precise, we will be led to higher categorical notions, on which we will give a brief reminder below.

(3) For each $f : X \to Y$ in E, and $A \in D(X)$, $B \in D(Y)$, an isomorphism
$$f_! A \otimes B \cong f_!(A \otimes f^* B),$$
as functors
$$D(X) \times D(Y) \to D(X).$$
Moreover, this isomorphism should also be compatible with compositions in f in a suitable sense, as well as the base change isomorphisms. It should also be compatible with further writing B as a tensor product $B' \otimes B''$ and the symmetric monoidal structure of f^*, etc.pp.

Especially with the projection formula, which simultaneously uses all of $\otimes$, f^* and $f_!$, one sees that it quickly becomes tedious to explicitly write out all the compatibility isomorphisms (and associativity-type constraints) that have to be satisfied. This problem gets accentuated when $D(X)$ is actually an ∞-category, in which case even the associativity constraint is not just a condition, but a further datum that has to be subjected to a tower of higher associativity constraints (and similar remarks apply to the base change formula and projection formula).

In the context where $D(X)$ is an ∞-category, there have been at least two formalizations of the datum of a 6-functor formalism:

(1) By Liu–Zheng [**LZ12a**] in the context of étale cohomology of schemes. Their formalization is firmly rooted in Lurie's foundational works, but it relies heavily on the combinatorics of specific simplicial sets.
(2) By Gaitsgory–Rozenblyum [**GR17**] in the context of coherent cohomology of schemes. They propose a very nice formal structure, but their formalization makes use of the formalism of $(\infty, 2)$-categories which is much less developed (and in particular, they assumed certain statements about the formalism on faith).[2]

Very recently, Mann [**Man22b**, Appendix A.5] has found a definition that combines the best of both worlds: Like Liu–Zheng's work, it is firmly rooted in Lurie's formalism, while like Gaitsgory–Rozenblyum's work it is a nice definition.

The goal of this lecture is to state Mann's definition, but first we want to say a few words about higher categories.

Very roughly, a higher category is something like a category that besides objects and morphisms also allows 2-morphisms between morphisms, and possibly 3-morphisms between 2-morphisms, ad infinitum. There are basically two routes leading to higher categories:

(1) The collection of all categories is naturally a 2-category: The objects are categories, the morphisms are functors, and the 2-morphisms are natural transformations. Likewise, the collection of all 2-categories should naturally form a 3-category, etc.
(2) In homotopy theory, one studies the category of topological spaces up to homotopy. More precisely, the objects are topological spaces, while morphisms are homotopy classes of morphisms. However, passing to homotopy classes loses information – often, it is important to know how, and not just that, two maps are homotopic. This can be recorded if instead of passing to homotopy classes, one instead introduces homotopies as 2-morphisms. But then one is naturally led to introduce homotopies between homotopies as 3-morphisms, and so on.

[2]Moreover, their functors behave somewhat differently than in the outline above – they start with $\otimes$, f_* and $f^!$, and the latter two do not usually have left adjoints; and $f^!$ is symmetric monoidal which is usually not the case in our setup.

A property of the second example is that 2-morphisms (and all higher morphisms) are invertible (while this is not the case in the first example – there are natural transformations that are not isomorphisms). Somewhat curiously, it turns out that it is easier to allow higher morphisms if one insists that they are all invertible. This leads to the following very vague definition.

Slogan. For $\infty \geq n \geq m$, an (n, m)-category is a higher category having i-morphisms for $i \leq n$, which are invertible for $i > m$.

In particular through the works of Lurie (but building on previous work of Boardman–Vogt and Joyal), there is a well-developed notion of $(\infty, 1)$-categories. In the literature, these are now often simply called ∞-categories, and we will (reluctantly) do the same. But be warned that they do not generalize 2-categories (which are usually understood to mean $(2, 2)$-categories, not $(2, 1)$-categories)! Allowing non-invertible 2-morphisms, to get a theory of $(\infty, 2)$-categories, is part of the active area of higher category theory, and due to ignorancy we will not say anything about it. Roughly speaking, the situation seems to be that there are now many different models of $(\infty, 2)$-categories, and they have recently all been proved to be equivalent (in the relevant sense).

It turns out that the language of simplicial sets is a very convenient way to capture both the structure of categories, and the homotopy theory of topological spaces.

Definition 2.1.

(1) The simplex category Δ is the category of nonempty finite totally ordered sets, with weakly increasing morphisms. Equivalently, it has objects $\Delta^n \in \Delta$ for each $n \geq 0$ corresponding to the totally ordered set $\{0, 1, \ldots, n\}$, with $\mathrm{Hom}(\Delta^n, \Delta^m)$ given by the set of weakly increasing maps $\{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$.
(2) The category of simplicial sets sSet is the category freely generated under colimits by Δ. Equivalently, it is the functor category

$$\Delta^{\mathrm{op}} \to \mathrm{Set}$$

which contains $\Delta \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ via the Yoneda embedding.

We want to think of simplicial sets C as modelling some kind of categories. Thinking of C as a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$, we think of $C_n := C(\Delta^n)$ as the set of n-simplices in C, which we think of as functors from the category $\{0 \to 1 \to \ldots \to n\}$ to C. In other words:

(1) For $n = 0$, C_0 is the set of objects of C.
(2) For $n = 1$, C_1 is the set of morphisms of C, i.e. of pairs (X, Y) of objects of C together with a map $f : X \to Y$.
(3) For $n = 2$, C_2 is the set of commutative triangles in C, i.e. of triples (X, Y, Z) of C, maps $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$, as well as a witness that h is the composite of f and g.
(4) ...

In particular, this recipee defines a (fully faithful) functor from the (strict 1-)category of categories towards simplicial sets.

A critical property of a category should be that composites of morphisms are defined. This means that whenever we have objects X, Y, Z of C as well as maps $f : X \to Y$ and $g : Y \to Z$, there is a way to extend this to a whole 2-simplex of C (and in particular a composite $h : X \to Z$). Equivalently, denoting

$$\Lambda^2_1 = \Delta^1 \sqcup_{\Delta^0} \Delta^1 \subset \Delta^2$$

the inner horn, any map $\Lambda^2_1 \to C$ extends to $\Delta^2 \to C$. (More generally, for any $0 \leq i \leq n$, one defines the horn $\Lambda^n_i \subset \Delta^n$ obtained by removing the interior and the i-th face.) Moreover, composition should be unique, at least up to homotopy (which again, should be unique homotopy, ad infinitum). This condition of uniqueness up to all higher homotopies is that of being a trivial Kan fibration. This leads to the following definition.

Definition 2.2.

(1) A map $f : C \to D$ of simplicial sets is a trivial Kan fibration if for all $n \geq 0$ and any diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & C \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & D, \end{array}$$

there is an extension $\Delta^n \to C$ making both triangles commute (where $\partial\Delta^n \subset \Delta^n$ is obtained by removing the interior). Equivalently, the same condition holds with $\partial\Delta^n \hookrightarrow \Delta^n$ replaced by any injection of simplicial sets.

(2) An ∞-category is a simplicial set C such that

$$\mathrm{Map}(\Delta^2, C) \to \mathrm{Map}(\Lambda^2_1, C)$$

is a trivial Kan fibration.

Here, for simplicial sets C and D, $\mathrm{Map}(C, D)$ is the internal mapping object, with n-simplices given by $\mathrm{Hom}_{\mathrm{sSet}}(C \times \Delta^n, D)$. It turns out that C is an ∞-category if and only if for all $0 < i < n$, any map

$$\Lambda^n_i \to C$$

extends to the full n-simplex Δ^n.

Let us list some basic facts and definitions about ∞-categories.

(1) For any ∞-category C and any simplicial set D, the mapping $\mathrm{Map}(D, C)$ is an ∞-category; we write $\mathrm{Fun}(D, C)$ and call it the functor ∞-category (especially when D itself is an ∞-category).

(2) For an ∞-category C, one can defines its homotopy category $\mathrm{Ho}(C)$. Its objects are the same as the objects of C, while morphisms are homotopy classes of morphisms in C. Here, for objects X, Y of C and morphisms $f, g : X \to Y$, they are homotopic if there is a 2-simplex witnessing that $g : X \to Y$ is the composite of $f : X \to Y$ and the identity $Y \to Y$ (which is equivalent to the existence of a 2-simplex witnessing $g : X \to Y$ as the composite of the identity $X \to X$ and $f : X \to Y$).

(3) A map $f : X \to Y$ in C is defined to be an isomorphism[3] if it becomes an isomorphism in $\mathrm{Ho}(C)$. Equivalently, there is a morphism $g : Y \to X$ and 2-simplices witnessing the identity $X \to X$ as the composite $f : X \to Y$ and $g : Y \to X$, and the identity $Y \to Y$ as the composite $g : Y \to X$ and $f : X \to Y$.

(4) An ∞-groupoid is an ∞-category C such that all morphisms are invertible; equivalently, $\mathrm{Ho}(C)$ is a groupoid. This is equivalent to asking that C is a Kan complex, i.e. for all $0 \leq i \leq n$, $n > 0$, any horn $\Lambda^n_i \to C$ extends to $\Delta^n \to C$.

[3] sometimes called an equivalence

At this point, recall that Kan complexes form a model for the theory of homotopy types; for any topological space X, one can define the Kan complex whose n-simplices are the continuous maps from the topological n-simplex towards X, and this captures all homotopy groups of X. As discussed below, Kan complexes naturally form an ∞-category, and we will generally to refer to them as anima when they are thought of as objects of that ∞-category.

(5) For any ∞-category C, there is a maximal sub-∞-groupoid $C^{\simeq} \subset C$, consisting of all objects of C but restricting the morphisms to those that are equivalences in C.

(6) Given objects X, Y of C, one can form the pullback

$$\begin{array}{ccc} \mathrm{Map}_C(X,Y) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle (X,Y)} \\ \mathrm{Fun}(\Delta^1, C) & \xrightarrow{\mathrm{ev}_0,\mathrm{ev}_1} & C \times C; \end{array}$$

intuitively speaking, $\mathrm{Map}_C(X,Y)$ is the simplicial set of maps $X \to Y$ in C. Then $\mathrm{Map}_C(X,Y)$ is a Kan complex. Roughly speaking, this makes C into a "category enriched in anima". In fact, there is a natural way to treat any category enriched in anima as an ∞-category.

(7) The ∞-category Cat_∞ of ∞-categories has objects ∞-categories, and mapping anima $\mathrm{Fun}(C,D)^{\simeq} \subset \mathrm{Fun}(C,D)$ (i.e. one only allows invertible natural transformations as 2-morphisms). The ∞-category $\mathrm{An} \subset \mathrm{Cat}_\infty$ of anima is the full sub-∞-category on anima, i.e. ∞-groupoids.

Concerning the (non-set-theoretic) details, we note that everything in the above is precise, except for the implicit passage from categories enriched in Kan complexes to ∞-categories.[4] This latter procedure can be defined explicitly in terms of a homotopy-coherent nerve functor.

Another notion we will need is that of a symmetric monoidal structure on an ∞-category, and of (lax) symmetric monoidal functors; we will defer a more precise discussion of this to the next lecture. We only note here the following points:

(1) In a symmetric monoidal ∞-category $(C, \otimes)$, one has a notion of commutative monoid, which is roughly speaking an object $X \in C$ together with a unit map $1 \to X$ and a multiplication map $m : X \otimes X \to X$ together with (higher) unitality, commutativity, and associativity constraints.

(2) If C is an ∞-category admitting finite products, then C has the Cartesian symmetric monoidal structure, where $X \otimes Y = X \times Y$.

(3) The ∞-category Cat_∞ has finite products and thus the Cartesian symmetric monoidal structure. A symmetric monoidal ∞-category is a commutative monoid in $(\mathrm{Cat}_\infty, \times)$; we denote their ∞-category by $\mathrm{CMon}(\mathrm{Cat}_\infty)$.

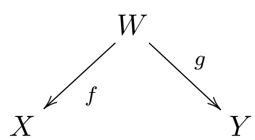
(4) If $(C, \otimes)$ and $(D, \otimes)$ are symmetric monoidal ∞-categories, a lax symmetric monoidal functor is a functor $F : C \to D$ together with (not necessarily invertible) maps $1_D \to F(1_C)$ and $F(X) \otimes F(Y) \to F(X \otimes Y)$ functorial in $X, Y \in C$, and compatible with the (higher) unitality, commutativity, and associativity constraints. A symmetric monoidal functor is one for which these natural transformations are equivalences.

[4] We note that with the given definitions, ∞-categories with mapping Kan complexes $\mathrm{Fun}^{\simeq}$ define a 1-category enriched in the 1-category of Kan complexes.

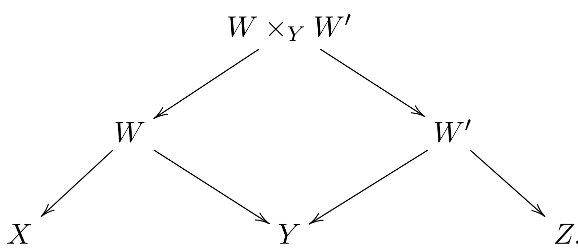
Finally, we can state Mann's definition of a 6-functor formalism. Recall that we start with a geometric setup (C, E), consisting of an (∞-)category C admitting finite limits, and a class of morphisms E stable under pullback and composition (and containing all isomorphisms). The following definition will be made more precise in the next lecture; we note that even if C is a 1-category (as is usually the case), $\mathrm{Corr}(C, E)$ will be a $(2, 1)$-category.

Definition 2.3. The symmetric monoidal ∞-category of correspondences $\mathrm{Corr}(C, E)$ is given as follows.

(1) The objects are the objects of C.
(2) The symmetric monoidal structure is the Cartesian symmetric monoidal structure of C.
(3) The morphisms are correspondences: $\mathrm{Hom}_{\mathrm{Corr}(C,E)}(X, Y)$ is given by the (∞-)groupoid of objects $W \in C$ together with maps



where $g \in E$.
(4) The composition of morphisms is given by the composition of correspondences, i.e. the composite of two correspondences $X \leftarrow W \to Y$ and $Y \leftarrow W' \to Z$ is given by the long triangle in the diagram



As in the naive discussion above, we will first formalize just three functors $\otimes$, f^* and $f_!$. In the following definition, Cat_∞ is endowed with the Cartesian symmetric monoidal structure.

Definition 2.4. A 3-functor formalism is a lax symmetric monoidal functor

$$D : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty.$$

The reader should take a moment to appreciate how concise this definition is! The claim is that this encodes the 3 functors $\otimes$, f^*, $f_!$, and (all of) their relations:

(1) On objects, D defines an association $X \mapsto D(X)$.
(2) The lax symmetric monoidal structure defines a natural "exterior tensor product" functor $D(X) \otimes D(X) \to D(X \times X)$. Together with the (diagonal) pullback defined just below, this defines the tensor product $\otimes$ on $D(X)$.
(3) For any map $f : X \to Y$, the correspondence $Y \xleftarrow{f} X = X$ defines the pullback functor $f^* : D(Y) \to D(X)$.
(4) For any map $f : X \to Y$ in E, the correspondence $X = X \xrightarrow{g} Y$ defines the functor $f_! : D(X) \to D(Y)$.

In particular, a correspondence $X \xleftarrow{f} W \xrightarrow{g} Y$ gets sent to the functor $g_! f^* : D(X) \to D(Y)$. The compatibility of this with composition amounts to the base change formula. Next time, we will also discuss how to find the projection formula.

Finally, as promised we define a 6-functor formalism.

Definition 2.5. A 6-functor formalism is a 3-functor formalism

$$D : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty$$

for which the functors $- \otimes A$, f^* and $f_!$ admit right adjoints.

We note that no further coherences are necessary here: Adjoints automatically acquire all relevant coherences.

3. Lecture III: Symmetric monoidal ∞-categories

Let us recall the definition of 3-functor formalisms from the last lecture. Given a geometric setting, given by an ∞-category C admitting finite limits, together with some class E of morphisms of C that is stable under pullback and composition (and contains all isomorphisms), one defines a symmetric monoidal ∞-category $\mathrm{Corr}(C, E)$ of correspondences.

Definition 3.1. A 3-functor formalism is a lax symmetric monoidal functor

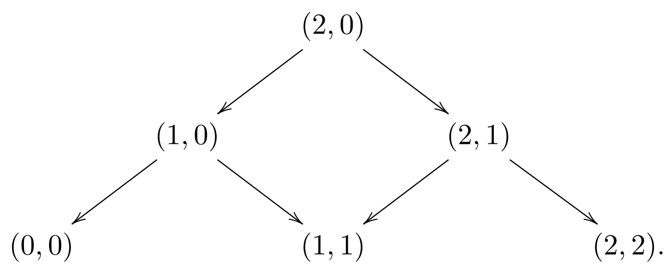
$$D : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty.$$

The goal of this lecture is to unpack this definition, and in particular give some background on symmetric monoidal ∞-categories and (lax) symmetric monoidal functors.

But first, let us give an honest definition of $\mathrm{Corr}(C, E)$ as an ∞-category. For any $n \geq 0$, let

$$(\Delta^n)^2_+ \subset (\Delta^n)^{\mathrm{op}} \times \Delta^n$$

be the subset spanned by those simplices $(i, j) \in \{0, 1, \ldots, n\}^2$ with $i \geq j$. For example, for $n = 2$, this corresponds to the category

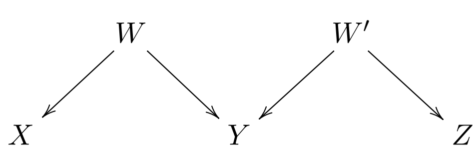


Varying n, this defines a cosimplicial category $(\Delta^\bullet)^2_+$.

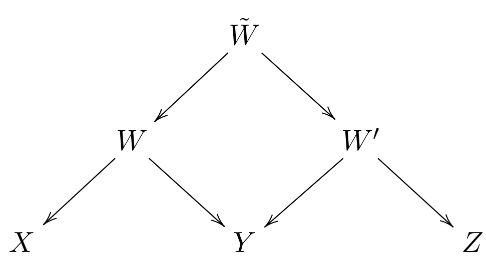
Definition/Proposition 3.2. *The correspondence ∞-category $\mathrm{Corr}(C, E)$ is the simplicial set whose n-simplices are maps from $(\Delta^n)^2_+$ to C with the following two properties:*

(1) *All arrows going down-right are in E.*
(2) *All small squares are Cartesian.*

One has to prove that this is, indeed, an ∞-category, i.e. that all inner horns $\Lambda^n_i \subset \Delta^n$ can be filled, $0 < i < n$. For $i = 1$, $n = 2$, this amounts to completing a diagram



in C, with $W \to Y$ and $W' \to Z$ in E, to a diagram



such that the square is cartesian, and also $\tilde{W} \to W'$ is in E. But this can be filled in, as C has pullbacks, and E is stable under pullbacks. The general assertion is [**LZ12a**, Lemma 6.1.2].

Now we want to endow $\mathrm{Corr}(C, E)$ with a symmetric monoidal structure. This requires a digression on the general notion of symmetric monoidal ∞-categories.

Recall that classically, a symmetric monoidal category is a category C together with:

(1) A unit object $1_C \in C$;
(2) A "tensor product" $- \otimes - : C \times C \to C$;
(3) A "unitality constraint" $u_X : 1 \otimes X \cong X$ functorially in X;
(4) A "commutativity constraint" $c_{X,Y} : X \otimes Y \cong Y \otimes X$ functorially in X, Y;
(5) An "associativity constraint" $a_{X,Y,Z} : (X \otimes Y) \cong Z \cong X \otimes (Y \otimes Z)$ functorially in X, Y, Z,

subject to a number of commutative diagrams, like the pentagon axiom and the hexagon axiom. What these axioms express is that for all finite sets I, and all objects $X_i \in C$ enumerated by $i \in I$, there is a well-defined

$$\bigotimes_{i \in I} X_i \in C.$$

Of course, one way to obtain this is to order $I \cong \{1, \ldots, n\}$ and then inductively define the tensor product. Using the commutativity and associativity constraint, any two such choices can be related by an isomorphism, and the extra axioms ensure that if one goes around in a circle using these constraints, one will always end up with the identity. For example, the pentagon axioms says that, when filled in with the evident associativity constraints, the diagram

$$\begin{array}{ccccc} ((X \otimes Y) \otimes Z) \otimes W & \longrightarrow & (X \otimes (Y \otimes Z)) \otimes W & \longrightarrow & X \otimes ((Y \otimes Z) \otimes W) \\ \downarrow & & & & \downarrow \\ (X \otimes Y) \otimes (Z \otimes W) & & \longrightarrow & & X \otimes (Y \otimes (Z \otimes W)) \end{array}$$

commutes.

Clearly, it would quickly become tedious to adapt this definition to higher categories, and a more conceptual approach is required. But the idea should just be that for any finite set I and any collection of objects $X_i \in C$ for $i \in I$, one can unambiguously define $\bigotimes_{i \in I} X_i$.

Let us first take a step back, and use this perspective to define commutative monoids in ∞-categories. Again, in a commutative monoid X, addition should give well-defined "sum all elements" maps from X^I to X for any finite set I.

Definition 3.3. A commutative monoid in an ∞-category C is a functor

$$X : \mathrm{Fin}^{\mathrm{part}} \to C$$

from the category of finite sets with partially defined maps, such that for all finite sets I the map

$$X(I) \to \prod_{i \in I} X(\{i\}) = X(*)^I$$

is an isomorphism (where the map is induced by the partially defined maps $I \to \{i\}$ sending i to i, and the rest nowhere).

Note that on objects, X is determined by $X(*)$; indeed, $X(I) \cong X(*)^I$ by the condition. Thus, a commutative monoid X is given by extra structure on $X(*)$, and we will sometimes abuse notation and write $X = X(*)$. The map $\emptyset \to *$ in $\mathrm{Fin}^{\mathrm{part}}$ defines a unit map $* = X(\emptyset) \to X(*)$; while for any finite set I, the projection map $I \to *$ defines a map $X(*)^I \cong X(I) \to X(*)$ "summing all elements". For a general partially defined map $f : I \dashrightarrow J$, the induced map $X^I = X(I) \to X^J = X(J)$ is on the j-th coordinate given by summing over $\prod_{i \in f^{-1}(j)} X_i$.

This definition can, in particular, be applied to $C = \mathrm{Cat}_\infty$, leading to the first definition of a symmetric monoidal ∞-category.

Definition 3.4. A symmetric monoidal ∞-category is a commutative monoid in Cat_∞.

Thus, giving a symmetric monoidal ∞-category requires us to write down functors

$$\mathrm{Fin}^{\mathrm{part}} \to \mathrm{Cat}_\infty.$$

It turns out that it is, in general, hard to write down functors towards Cat_∞ directly. This is an issue that already occurs in classical category theory, where it was solved by Grothendieck. A standard situation in algebraic geometry is to consider the functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$$

taking any scheme S to the category $\mathrm{QCoh}(S)$ of quasicoherent sheaves on S (or in the context of the first lecture, the functor

$$\mathrm{Top}^{\mathrm{op}} \to \mathrm{Cat}$$

taking any topological space X to the category $\mathrm{Ab}(X)$ of abelian sheaves on X). Any map $f : S' \to S$ defines a pullback functor

$$f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S').$$

But when one composes two morphisms, $(fg)^*$ is not literally identical to $g^* f^*$. Instead, one only has a natural isomorphism between the two, and of course this needs to be subject to a coherence axiom.

Grothendieck's solution was to write down certain categorical fibrations instead. Namely, one can define the category

$$\mathrm{QCoh}_{\mathrm{Sch}} = \{(S \in \mathrm{Sch}, \mathcal{M} \in \mathrm{QCoh}(S))\} \to \mathrm{Sch}$$

consisting of pairs $(S, \mathcal{M})$ of a scheme S and a quasicoherent sheaf $\mathcal{M}$ on S; and where morphisms $(S', \mathcal{M}') \to (S, \mathcal{M})$ are pairs of a morphism $f : S' \to S$ and a morphism $f^*\mathcal{M} \to \mathcal{M}'$. It is straightforward to define this category, and a functor to Sch which is a Cartesian fibration (for whose definition see below). But then a theorem says that Cartesian fibrations over a category C (e.g. $C = \mathrm{Sch}$) are equivalent to functors $C^{\mathrm{op}} \to \mathrm{Cat}$, giving the desired functor.

The following discussion can be found in [**Lur09**, Chapter 2, 3], see in particular [**Lur09**, Proposition 2.4.2.8] for the equivalence of the following definition with other definitions.

Definition 3.5. A functor $F : D \to C$ of ∞-categories is a coCartesian[5] fibration if it is an inner fibration (i.e. any inner horn can be lifted), and

(1) Any morphism of C admits a locally coCartesian lift;
(2) Composites of locally coCartesian lifts are locally coCartesian.

Here, a morphism $g : Y \to Y'$ in D is a locally coCartesian lift of a morphism $f : X \to X'$ if it is initial among all morphisms with source Y that lift f.

[5]This notion is dual to Cartesian fibrations, and classifies covariant functors to Cat_∞ as opposed to contravariant ones.

Theorem 3.6 (Lurie, "Straightening/Unstraightening"). *There is a natural equivalence between the ∞-categories of functors $C \to \mathrm{Cat}_\infty$ and the ∞-category of coCartesian fibrations over C.*

One has to be a bit careful here which morphisms of coCartesian fibrations one allows. Indeed, one can already observe that functors $C \to \mathrm{Cat}_\infty$ form most naturally an $(\infty, 2)$-category, and the above theorem restricts to the underlying $(\infty, 1)$-category here (allowing only invertible natural transformations). But coCartesian fibrations know about the non-invertible transformations. Namely, consider two coCartesian fibrations $F : D \to C$, $F' : D' \to C$ over C, and a functor $G : D \to D'$ over C. For each object X of C, this induces a functor $G_X : D_X \to D'_X$ between the fibres. Moreover, for any map $f : X \to X'$, one can look at the diagram

$$\begin{array}{ccc} D_X & \xrightarrow{D_f} & D_{X'} \\ {\scriptstyle G_X}\downarrow & & \downarrow{\scriptstyle G_{X'}} \\ D'_X & \xrightarrow{D'_f} & D'_{X'}, \end{array}$$

where D_f and D'_f are the induced functors (of locally coCartesian lifts). This diagram may not commute, but the initiality condition on locally coCartesian lifts gives a natural transformation

$$D'_f G_X \to G_{X'} D_f.$$

This map is an isomorphism if and only if G preserves locally coCartesian lifts.

In practice, all functors to Cat_∞ are constructed by constructing the corresponding coCartesian fibration instead. With this in mind, we redefine symmetric monoidal ∞-categories:

Definition 3.7 ([**Lur17**, Definition 2.0.0.7]). A symmetric monoidal ∞-category is a coCartesian fibration

$$C^\otimes \to \mathrm{Fin}^{\mathrm{part}}$$

such that, denoting $C = C^\otimes_*$ the fibre over the one-element set $* \in \mathrm{Fin}^{\mathrm{part}}$, for all finite sets I the functor

$$C^\otimes_I \to \prod_{i \in I} C,$$

induced by the partially defined maps $I \dashrightarrow \{i\}$ sending i to i (and the rest nowhere), is an equivalence.

Starting with a symmetric monoidal ∞-category $(C, \otimes)$ in the old sense, $C^\otimes$ is roughly given as follows. Objects of $C^\otimes$ are pairs $(I, (X_i)_{i \in I})$ of a finite set I and objects $X_i \in C$ for $i \in I$. A map

$$(I, (X_i)_{i \in I}) \to (J, (Y_j)_{j \in J})$$

in $C^\otimes$ is given by a partially defined map $f : I \dashrightarrow J$ together with maps $\bigotimes_{i \in f^{-1}(j)} X_i \to Y_j$ for all $j \in J$.

Finally, let us discuss (lax) symmetric monoidal functors. Recall that a lax symmetric monoidal functor $F : (C, \otimes) \to (D, \otimes)$ is, intuitively speaking, a functor $F : C \to D$ together with natural maps $\bigotimes_{i \in I} F(X_i) \to F(\bigotimes_{i \in I} X_i)$ for any finite I and objects X_i, $i \in I$. A symmetric monoidal functor is a lax symmetric monoidal functor for which these maps are all isomorphisms.

It turns out that lax symmetric monoidal functors are best defined using this perspective of coCartesian fibrations $C^\otimes \to \mathrm{Fin}^{\mathrm{part}}$.

Definition 3.8. Let $(C, \otimes)$ and $(D, \otimes)$ be symmetric monoidal ∞-categories, with corresponding coCartesian fibrations

$$C^{\otimes}, D^{\otimes} \to \mathrm{Fin}^{\mathrm{part}}.$$

A lax symmetric monoidal functor $(C, \otimes) \to (D, \otimes)$ is a functor $F^{\otimes} : C^{\otimes} \to D^{\otimes}$ over $\mathrm{Fin}^{\mathrm{part}}$ such that $F^{\otimes}$ preserves locally coCartesian lifts of the morphisms $I \dashrightarrow \{i\}$ sending $i \in I$ to i and the rest nowhere. The functor $F^{\otimes}$ is symmetric monoidal if it preserves all locally coCartesian lifts.

The condition on locally coCartesian lifts of the morphisms $I \dashrightarrow \{i\}$ is ensuring that $F^{\otimes}_I : C^{\otimes}_I \to D^{\otimes}_I$ is given by $F^I : C^I \to D^I$ under the equivalences $C^{\otimes}_I \cong C^I$, $D^{\otimes}_I \cong D^I$. Then the above remarks on maps between coCartesian fibrations is precisely ensuring that $F^{\otimes}$ induces maps $\bigotimes_{i \in I} F(X_i) \to F(\bigotimes_{i \in I} X_i)$.

Remark 3.9. Arguably, a drawback of the definition of symmetric monoidal ∞-categories in terms of coCartesian fibrations is that the definition is not evidently selfdual – it is not clear that if $(C, \otimes)$ is a symmetric monoidal ∞-category then so is $(C^{\mathrm{op}}, \otimes)$. There is a dual notion of colax symmetric monoidal functors, and this is nontrivial to define in this language. See however [**HHLN22**] for a detailed discussion of this duality.

If C is any ∞-category that admits finite products, it acquires a unique symmetric monoidal ∞-category structure whose operation is the cartesian product.[6] In particular, this applies to Cat_{∞}. The cartesian symmetric monoidal structure has a useful universal property.

Theorem 3.10 ([**Lur17**, Proposition 2.4.1.7]). *Let $(C, \otimes)$ be any symmetric monoidal ∞-category, and let D be an ∞-category admitting finite products, with its cartesian symmetric monoidal structure. Then lax symmetric monoidal functors $(C, \otimes) \to (D, \times)$ are equivalent to functors*

$$F : C^{\otimes} \to D$$

such that for all finite sets I and objects $X_i \in C$, $i \in I$, the map

$$F((I, (X_i)_{i \in I})) \to \prod_{i \in I} F(X_i)$$

is an equivalence (the map induced as usual by the forgetful maps $I \dashrightarrow \{i\}$ for $i \in I$).

To finish the discussion of the terms appearing in the definition of a 3-functor formalism, it remains to define $\mathrm{Corr}(C, E)$ as a symmetric monoidal ∞-category. Conveniently, $\mathrm{Corr}(C, E)^{\otimes}$ can be defined itself as a correspondence category in

$$C^{\times'} := ((C^{\mathrm{op}})^{\sqcup})^{\mathrm{op}}.$$

Here $(C^{\mathrm{op}})^{\sqcup}$ is the coCartesian fibration corresponding to the symmetric monoidal ∞-category C^{op} with coproducts (i.e., products in C). Concretely, $C^{\times'}$ as objects given by pairs $(I, (X_i)_{i \in I})$ as usual, but maps

$$(I, (X_i)_{i \in I}) \to (J, (Y_j)_{j \in J})$$

are given by partially defined maps $f : J \dashrightarrow I$ together with maps

$$X_i \to \prod_{j \in f^{-1}(i)} Y_j$$

[6]This is actually easier to see for coproducts.

for all $i \in I$ (instead of maps in the opposite direction). Thus, $C^{\times'}$ is not the coCartesian fibration over $\mathrm{Fin}^{\mathrm{part}}$ encoding the cartesian symmetric monoidal structure on C.

Let $E^\times$ be the class of morphisms of $C^{\times'}$ that lie over identity morphisms of $\mathrm{Fin}^{\mathrm{part}}$, and where all maps $Y_i \to X_i$ are in E.

Definition/Proposition 3.11 ([**LZ12a**, Proposition 6.1.3]). *Let* $\mathrm{Corr}(C,E)^\otimes = \mathrm{Corr}(C^{\times'}, E^\times)$. *This is naturally a coCartesian fibration over* $\mathrm{Fin}^{\mathrm{part}}$*, defining a symmetric monoidal structure on* $\mathrm{Corr}(C,E)$.

Objects of $\mathrm{Corr}(C,E)^\otimes$ are again pairs $(I,(X_i)_{i\in I})$. Morphisms $(I,(X_i)_{i\in I}) \to (J,(Y_j)_{j\in J})$ are given by partially defined maps $f : I \dashrightarrow J$ together with correspondences

$$\prod_{i\in f^{-1}(j)} X_i \leftarrow W_j \to Y_j$$

for all $j \in J$, where the morphism $W_j \to Y_j$ lies in E. We note that we needed the contravariant maps in $C^{\times'}$ to cancel the contravariance of the first map in the correspondences. We note in particular the following three kinds of morphisms of $\mathrm{Corr}(C,E)^\otimes$:

(1) For all $X \in C$, a map $(\{1,2\},(X,X)) \to (*,X)$ given by the projection $\{1,2\} \to *$ and the correspondence $X \times X \leftarrow X = X$ where the first map is the diagonal.
(2) For any map $f : X \to X'$ in C, a map $(*,X') \to (*,X)$ given by $* = *$ and the correspondence $X' \leftarrow X = X$.
(3) For any map $f : X \to X'$ in E, a map $(*,X) \to (*,X')$ given by $* = *$ and the correspondence $X = X \to X'$.

Finally, we can restate the definition of a 3-functor formalism.

Definition 3.12. A 3-functor formalism is a lax symmetric monoidal functor $\mathrm{Corr}(C,E) \to \mathrm{Cat}_\infty$. Equivalently, it is a functor

$$D : \mathrm{Corr}(C,E)^\otimes \to \mathrm{Cat}_\infty$$

such that for all finite sets I and $X_i \in C$, $i \in I$, the functor

$$D((I,(X_i)_{i\in I})) \to \prod_{i\in I} D(X_i)$$

is an equivalence.

In particular, on objects this gives $X \mapsto D(X)$, and the three types of morphisms above induce the tensor product, the pullback f^*, and the exceptional pushforward $f_!$. In general, a map

$$(I,(X_i)_{i\in I}) \to (J,(Y_j)_{j\in J})$$

given by $f : I \dashrightarrow J$ and correspondences

$$\prod_{i\in f^{-1}(j)} X_i \xleftarrow{f_j} W_j \xrightarrow{g_j} Y_j$$

induces the functor

$$\prod_{i\in I} D(X_i) \to \prod_{j\in J} D(Y_j)$$

whose j-th component is given by $g_{j!}f_j^*(\boxtimes_{i\in f^{-1}(j)})$ where

$$\boxtimes_{i\in f^{-1}(j)} : \prod_{i\in f^{-1}(j)} D(X_i) \to D(\prod_{i\in f^{-1}(j)} X_i)$$

denotes the exterior tensor product (i.e., the tensor product of the pullbacks from each factor). Indeed, this easily follows by writing this correspondence as a composite of smaller correspondences. Now the compatibility with composites of correspondences is encoding all the desired compatibilities between tensor, pullback, and exceptional pushforward.

For example, we can get the projection formula

$$f_! A \otimes B \cong f_!(A \otimes f^* B)$$

along a morphism $f : X \to Y$ in E, by writing the correspondence

$$(\{1,2\},(X,Y)) \to (*,Y)$$

given by $X \times Y \leftarrow X \to Y$ as a composite in two ways. First, as the composite of

$$(\{1,2\},(X,Y)) \to (\{1,2\},(Y,Y))$$

given by the correspondences $X = X \to Y$ and $Y = Y = Y$, and

$$(\{1,2\},(Y,Y)) \to (*,Y)$$

given by the correspondence $Y \times Y \leftarrow Y = Y$; and as the composite of

$$(\{1,2\},(X,Y)) \to (*,X)$$

given by the correspondence $X \times Y \leftarrow X = X$, and

$$(*,X) \to (*,Y)$$

given by the correspondence $X = X \to Y$. Here, the first composite unravels to

$$(A,B) \mapsto (f_! A, B) \mapsto f_! A \otimes B,$$

and the second composite to

$$(A,B) \mapsto A \otimes f^* B \mapsto f_!(A \otimes f^* B).$$

Remark 3.13. In some other formulations of 6-functor formalisms, it is stressed that the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ should be $\mathcal{D}(Y)$-linear (where $\mathcal{D}(Y)$ acts on $\mathcal{D}(X)$ by tensoring with the pullback); this is a "homotopy coherent" version of the projection formula. This structure of $f_!$ as a $\mathcal{D}(Y)$-linear functor is in fact encoded in the lax symmetric monoidal functor $\mathcal{D} : \mathrm{Corr}(C,E) \to \mathrm{Cat}_\infty$. In fact, lax symmetric monoidal functors preserve algebras and modules, and the relevant structure can be defined already on $\mathrm{Corr}(C,E)$. Namely, any X defines a commutative algebra object in C^{op} (this is always true for the disjoint union symmetric monoidal structure, as there is a canonical map $X \sqcup X \to X$), and any map $f : X \to Y$ defines a map f^* of commutative algebras in C^{op}, and in particular makes X a module over Y. The symmetric monoidal functor $C^{\mathrm{op}} \to \mathrm{Corr}(C,E)$ induces the same structure in $\mathrm{Corr}(C,E)$. Now any map $f : X \to X'$ over Y in E induces a covariant map of the associated modules over Y in $\mathrm{Corr}(C,E)$. Thus, the same structure is present after applying $\mathcal{D}$, making $f_! : \mathcal{D}(X) \to \mathcal{D}(X')$ a map of $\mathcal{D}(Y)$-modules, i.e. $\mathcal{D}(Y)$-linear. In particular, taking $X' = Y$, the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ is naturally $\mathcal{D}(Y)$-linear.

In the next lecture, we will discuss a construction principle for 3-functor formalisms, showing how to get all these intricate coherences in practice.

4. Lecture IV: Construction of Six-Functor Formalisms

Let us again fix a geometric setting (C, E) as before. We are interested in constructing a lax symmetric monoidal functor

$$\mathcal{D} : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty.$$

Equivalently, this is described by a functor

$$\mathcal{D} : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Cat}_\infty$$

such that for any finite set I and $X_i \in C$ for $i \in I$, the functor

$$\mathcal{D}((I, (X_i)_i)) \to \prod_{i \in I} \mathcal{D}(X_i)$$

is an equivalence.

In practice, it is easy to construct a functor

$$\mathcal{D}_0 : C^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$$

to symmetric monoidal ∞-categories, encoding $\otimes$ and f^*. Note that $\mathcal{D}_0$ is equivalent to a lax symmetric monoidal functor

$$C^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

by [**Lur17**, Theorem 2.4.3.18]. The problem is now to extend this from $C^{\mathrm{op}} \cong \mathrm{Corr}(C, \mathrm{isom})$ to $\mathrm{Corr}(C, E)$.

In practice (at least in contexts of algebraic geometry), there are two special classes of morphisms $I, P \subset E$ of "open immersions" and "proper" maps such that for $f \in I$, the functor $f_!$ is the left adjoint of f^*, while for $f \in P$, the functor $f_!$ is the right adjoint of f^*. Moreover, any $f \in E$ admits a factorization $f = \overline{f} j$ with $j \in I$ and $\overline{f} \in P$, i.e. a compactification. As $f_!$ should be compatible with composition, we need to have $f_! = \overline{f}_! j_!$ where the latter two functors are defined (as the right resp. left adjoint of pullback). It is, however, far from clear that this is well-defined, especially as a functor of ∞-categories!

Example 4.1.

(1) If C is the category of locally compact Hausdorff topological spaces, one can take for I the open immersions, and for P the proper maps (i.e., preimages of compact subsets are compact). Any map $f : X \to Y$ admits a compactification $X \hookrightarrow \overline{X} \to Y$, for example $\overline{X} = \beta X \times_{\beta Y} Y$ using the Stone-Čech compactification.

(2) If C is the category of qcqs schemes and E is the class of separated morphisms of finite type, then one can again take for I the open immersions and for P the proper maps. Indeed, the Nagata compactifion theorem ensures that any separated map of finite type $f : X \to Y$ between qcqs schemes admits a compactification $X \hookrightarrow \overline{X} \to Y$.

One might expect that in order for $f_!$ to be defined uniquely as a functor of ∞-categories, one has to put some rather strong assumptions on C (and the classes E, I, P) – for example, the existence of some canonical compactification, or at least a canonical "cofinal" collection of such. Somewhat surprisingly, it turns out that minimal hypotheses ensure that $f_!$ is canonically defined. In fact, we will only make the following assumptions. (Only the assumptions on I and P are not obviously necessary, but they are very weak and satisfied in all practical situations.)

(1) Assumptions on I and P: The classes of morphisms I and P are stable under pullback and composition and contain all isomorphisms. If $f : X \to Y$ and $f' : X' \to Y$ are both in I (resp. in P) and $g : X \to X'$ is a map over Y, then $g \in I$ (resp. $g \in P$). If $f \in I \cap P$, then f is n-truncated for some n.[7] Any $f \in E$ is a composite $f = \overline{f} j$ with $j \in I$ and $\overline{f} \in P$.

(2) For any $f \in I$, the functor f^* admits a left adjoint $f_!$ which satisfies base change and the projection formula.

(3) For any $f \in P$, the functor f^* admits a right adjoint f_* which satisfies base change and the projection formula.

(4) For any Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{j} & Y \end{array}$$

with $j \in I$ (hence $j' \in I$) and $g \in P$ (hence $g' \in P$), the natural map $j_! g'_* \to g_* j'_!$ is an isomorphism.

REMARK 4.2. In the situation of (4), assume that $Y' \times_Y Y' = Y'$ and that there is some morphism $Z \to Y$ ("the complementary closed to $Y' \subset Y$") such that $Z \times_Y Y' = \emptyset$ (i.e., an initial object of C, and we assume $D(\emptyset) = 0$) and $D(Y) \to D(Y') \times D(Z)$ is conservative. Then the condition of (4) follows from the other conditions. Indeed, it becomes enough to check that the map is an isomorphism after pullback to Y' and to Z. But by (2) and (3), all functors in question commute with base change. After pullback to Z, Y' and X' become empty and there is nothing to check. After pullback to Y', the map j becomes an isomorphism and again there is nothing to check.

We note that in (2), (3) and (4), one is only asking that certain natural maps (defined by adjunctions) are isomorphisms. For example, the last map $j_! g'_* \to g_* j'_!$ is by definition adjoint to

$$g^* j_! g'_* \cong j'_! g'^* g'_* \to j'_!$$

using the base change isomorphism for $j_!$ from (2).

In fact, in (4) one would expect to have a similar isomorphism $j_! g'_* \cong g_* j'_!$ even if the diagram is not Cartesian. However, in that case one cannot a priori define a natural isomorphism. Fortunately, it turns out that this is a consequence of the axioms. To see this, we make the following construction.

CONSTRUCTION 4.3. For any $f \in I \cap P$, there is a natural isomorphism $f_! \cong f_*$ between the left and the right adjoint of f^*.

Indeed, we argue by induction on n such that f is n-truncated. If $n = -2$, then f is an isomorphism, and the claim is clear. In general, consider the cartesian diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

[7] This is automatic if C is a category (as opposed to an ∞-category). It means that for any $g : Z \to X$, the anima of lifts of g to Y is n-truncated, i.e. has $\pi_i = 0$ for $i > n$.

as well as $\Delta : X \to X \times_Y X$. Then $g \in I \cap P$ by pullback-stability, and hence $\Delta \in I \cap P$ as it is a map between X and $X \times_Y X$, which have compatible projections to X that are in $I \cap P$. Moreover, Δ is $n-1$-truncated, so by induction we have constructed an identification $\Delta_! \cong \Delta_*$. Now

$$f_! = f_! g_* \Delta_* \cong f_! g_* \Delta_! \cong f_* h_! \Delta_! = f_*$$

using respectively $g\Delta = \mathrm{id}_X$, the identification $\Delta_! \cong \Delta_*$, assumption (4), and $h\Delta = \mathrm{id}_X$.

Construction 4.4. For any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{j} & Y \end{array}$$

with $j, j' \in I$ and $g, g' \in P$, there is a natural isomorphism $j_! g'_* \cong g_* j_!$.

Indeed, consider the induced map $h : X' \to X \times_Y Y'$ with $j'' : X \times_Y Y' \to X$ and $g'' : X \times_Y Y' \to Y'$. Then $h \in I$ as it is a map between X' and $X \times_Y Y'$ both of whose projections to X are in I; and $h \in P$ as it is a map between X' and $X \times_Y Y'$ both of whose projections to Y' are in P. Thus, the previous construction gives us an isomorphism $h_! \cong h_*$. Now

$$j_! g'_* = j_! g''_* h_* \cong j_! g''_* h_! \cong g_* j''_! h_! = g_* j'_!$$

using respectively $g' = g''h$, the identification $g_! \cong g_*$, assumption (4), and $j''h = j'$.

Now we can also see that if $f \in E$ has two different compactifications, then the two induced possible definitions of $f_!$ can be identified.

Construction 4.5. Let $f : X \to Y$ in E be written in two ways as $f = \overline{f} j$ and $\overline{f}' j'$, with $j, j' \in I$ and $\overline{f}, \overline{f}' \in P$. Then there is a natural isomorphism

$$\overline{f}_* j_! \cong \overline{f}'_* j'_!.$$

Indeed, let $\overline{X}$ and $\overline{X}'$ be the implicit compactifications of X over Y, and consider $Z = \overline{X} \times_Y \overline{X}'$. The projection to Y lies in P, and the diagonal map $X \to X \times_Y X \to \overline{X} \times_Y \overline{X}'$ is a morphism in E (one can check that morphisms of E also satisfy the 2-out-of-3 property). Thus, it can be factored as $X \to \overline{X}'' \to \overline{X} \times_Y \overline{X}'$ where $\overline{f}'' : \overline{X}'' \to Y$ still lies in P, and $j'' : X \to \overline{X}''$ lies in I. It now suffices to construct isomorphisms

$$\overline{f}_* j_! \cong \overline{f}''_* j''_! \cong \overline{f}'_* j'_!.$$

Restricting attention to one half, we can (renaming $\overline{X}''$ as $\overline{X}'$) assume there is a map $g : \overline{X}' \to \overline{X}$ over Y. But then $g_* j'_! \cong j_!$ by the previous construction (in the degenerate case where one of the maps in P is the identity), and post-composing with $\overline{f}_*$ gives the desired isomorphism.

The previous constructions merely serve as an indication that the statement of the following theorem (which is really a construction) is sensible.

Theorem 4.6 (Mann [**Man22b**, Proposition A.5.10], Liu–Zheng). *Under the above assumptions, there is a canonical extension of $\mathcal{D}_0$ to a lax symmetric monoidal functor*

$$\mathcal{D} : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty$$

such that for $f \in I$, $f_!$ is the left adjoint of f^*, while for $f \in P$, $f_!$ is the right adjoint of f^*.

Sketch of Construction. In this sketch, we will ignore the lax symmetric monoidal structure; this can be taken care of by thinking about the functor $\mathrm{Corr}(C,E)^{\otimes} \to \mathrm{Cat}_\infty$ instead, where the source is itself a correspondence ∞-category to which the constructions below apply. (Indeed, one only has to replace C below by $C^{\times\prime}$.)

An important component of the construction are multisimplicial sets; more precisely, bi- and trisimplicial sets. Note that $\mathrm{Corr}(C,E)$ can be constructed in two steps. First, one can define a bisimplicial set $\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ whose (n,m)-simplices are maps

$$(\Delta^n)^{\mathrm{op}} \times \Delta^m \to C$$

satisfying an analogue of the condition defining $\mathrm{Corr}(C,E)$; i.e., sending morphisms of the second variable to morphisms in E, and making all squares cartesian. Now for any bisimplicial set X, one can define a simplicial set $\delta_+^* X$ whose n-simplices are given by maps $\Delta_+^{(n,n)} \to X$ where $\Delta^{(n,n)}$ is the bisimplicial set given by an (n,n)-simplex, and $\Delta_+^{(n,n)} \subset \Delta^{(n,n)}$ denotes the sub-bisimplicial set spanned by vertices (i,j) with $i \geq j$. Unraveling the definitions, we have

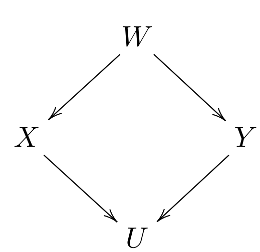
$$\mathrm{Corr}(C,E) = \delta_+^* \mathrm{Corr}(C,E)_{\mathrm{contra,co}}.$$

There is a more standard way to pass from a bisimplicial set X to a simplicial set $\delta^* X$, via taking as n-simplices the maps $\Delta^{(n,n)} \to X$. The inclusion $\Delta_+^{(n,n)} \subset \Delta^{(n,n)}$ induces a comparison map

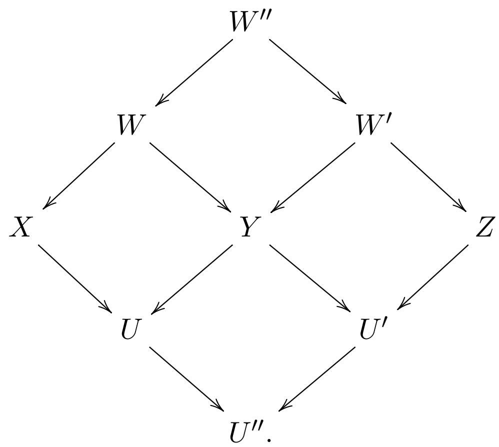
$$\delta^* X \to \delta_+^* X.$$

Note that in general neither $\delta^* X$ nor $\delta_+^* X$ is an ∞-category; and in our case, $\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ is not an ∞-category. Thus, we are here really using the interpretation of ∞-categories as simplicial sets, and ambient simplicial sets that are not ∞-categories.

Remark 4.7. Vertices of $\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ are objects of C, while edges (from X to Y) are given by cartesian squares



where the right-vertical arrows are in E. But given two such composable edges, one cannot in general compose them to a 2-simplex, which would be a diagram



Indeed, it is in general hard to find U''.

Theorem 4.8 ([**LZ12b**, Theorem 4.27]). *For any bisimplicial set X, the map $\delta^* X \to \delta^*_+ X$ is a categorical equivalence. In other words, for any ∞-category $\mathcal{E}$, the functor*

$$\mathrm{Fun}(\delta^*_+ X, \mathcal{E}) \to \mathrm{Fun}(\delta^* X, \mathcal{E})$$

of ∞-categories is an equivalence.

Remark 4.9. One might wonder whether $\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ has enough morphisms for this to be true, as few correspondences $X \leftarrow W \to Y$ extend to a cartesian square. But any correspondence for which one of $W \to X$ and $W \to Y$ is an isomorphism naturally lifts to an edge of $\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ (by taking $U = X$ or $U = Y$), and any correspondence can be canonically written as the composite of two such. As composites of morphisms are unique (up to contractible choice) in an ∞-category, any morphism in $\mathrm{Corr}(C,E)$ lifts canonically (up to contractible choice) to any fibrant replacement of $\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$ (i.e. any ∞-category equipped with a categorical equivalence from this simplicial set).

Using the theorem, it is thus sufficient to construct a map of simplicial sets

$$\delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}} \to \mathrm{Cat}_\infty.$$

At this point, tri-simplicial sets enter the picture. Namely, we consider the trisimplicial set

$$\mathrm{Corr}(C,I,P)_{\mathrm{contra,co,co}}$$

whose (n,m,k)-simplices are maps

$$(\Delta^n)^{\mathrm{op}} \times \Delta^m \times \Delta^k \to C$$

sending morphisms of the second variable to morphisms in I, morphisms of the third variable to morphisms in P, and making all squares and cubes cartesian. We can turn this into a bisimplicial set by contracting the last two factors, getting a natural map to $\mathrm{Corr}(C,E)_{\mathrm{contra,co}}$; and then we can turn both into simplicial sets by further diagonal restriction. In summary, denoting by δ^* full diagonal restriction for both tri- and bi-simplicial sets, we get a map of simplicial sets

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,co,co}} \to \delta^*\mathrm{Corr}(C,E)_{\mathrm{contra,co}}.$$

Theorem 4.10 ([**LZ12b**, Theorem 5.4]). *This map is a categorical equivalence.*

Remark 4.11. This theorem guarantees, roughly speaking, that decomposing a map in E into maps in I and P is "unique up to contractible choice", and encompasses the explicit constructions above.

In other words, it is enough to construct a map of simplicial sets

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,co,co}} \to \mathrm{Cat}_\infty.$$

What is easy to construct is a map

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,contra,contra}} \to \mathrm{Cat}_\infty.$$

Indeed, there is a natural functor

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,contra,contra}} \to C^{\mathrm{op}}$$

recording the diagonal composition of morphisms; and we can compose this with the given functor $\mathcal{D}_0$ to Cat_∞.

The idea now is to pass to adjoints in the second and third variable. In general, for a trisimplicial set X, giving a map $\delta^*X \to \mathrm{Cat}_\infty$ means to give for each (n,m,k)-simplex $\Delta^{(n,m,k)}$ of X a map

$$\Delta^n \times \Delta^m \times \Delta^k \to \mathrm{Cat}_\infty,$$

functorially in the simplex of X. This is equivalent to

$$\Delta^n \times \Delta^m \to \mathrm{Fun}(\Delta^k, \mathrm{Cat}_\infty)$$

where the latter is (up to equivalence) given by ∞-categories $\mathcal{C}_0,\ldots,\mathcal{C}_k$ with functors $F_i : \mathcal{C}_i \to \mathcal{C}_{i+1}$ for $i = 0,\ldots,k-1$. One can define a sub-∞-category $\mathrm{Fun}^L(\Delta^k, \mathrm{Cat}_\infty)$ whose objects are those where each F_i is a left adjoint, i.e. admits a right adjoint G_i, and where the morphisms are those maps $\mathcal{C}_i \to \mathcal{C}_i'$ (commuting with the F_i) which also, after passing to adjoints, commute with the G_i. On this subcategory, we can pass to adjoints, giving a functor

$$\mathrm{Fun}^L(\Delta^k, \mathrm{Cat}_\infty) \to \mathrm{Fun}((\Delta^k)^{\mathrm{op}}, \mathrm{Cat}_\infty).$$

In the context of the trisimplicial set, this gives a map

$$\Delta^n \times \Delta^m \times (\Delta^k)^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

i.e. one has reversed the arrows in the third direction. Of course, the same discussion applies if one passes to left adjoints instead.

We apply these remarks to first pass to left adjoints in the second direction (which is made possible by condition (2) on base change (and projection formula when keeping track of the symmetric monoidal structure)) and then pass to right adjoints in the third variable (which is made possible by condition (3) on base change (and projection formula when keeping track of the symmetric monoidal structure), as well as condition (4)), to turn the functor

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,contra,contra}} \to \mathrm{Cat}_\infty$$

into a functor

$$\delta^*\mathrm{Corr}(C,I,P)_{\mathrm{contra,co,co}} \to \mathrm{Cat}_\infty$$

as desired. □

REMARK 4.12. The construction shows that if one formulates the requirement that $f_!$ is the left (resp. right) adjoint of f^* for $f \in I$ (resp. $f \in P$) in a suitably strong form, then the extension is in fact unique. Indeed, the only parts of the construction that were not of propositional nature were the parts about passing to left resp. right adjoints in the context of this tri-simplicial set, but doing so was essentially dictated by the desired outcome.

REMARK 4.13. To get a full 6-functor formalism, it remains to ask that all $\mathcal{D}(X)$ are closed symmetric monoidal ∞-categories; that all f^* admit right adjoints f_*; and that for $f \in P$, the functor f_* admits a further right adjoint $f^!$.

Appendix to Lecture IV: Passage to stacks

The main goal of the work of Liu–Zheng [**LZ12a**] was to extend the étale 6-functor formalism to stacks. Some of their work has been streamlined by Mann [**Man22b**, Appendix A.5]. Let us present a slightly different take on this passage to stacks. Our discussion here strongly relies on extended discussions of the author with Clausen, who in particular suggested to use the topology of universal $*$- and !-descent.

Start with any ∞-category C with all finite limits, and a class of morphisms E as usual. Let

$$D : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty$$

be a 6-functor formalism. We also assume that $D(X)$ is a presentable ∞-category for all $X \in C$, i.e. admits all colimits and for some cardinal κ it is generated under colimits by a set of κ-compact objects.

Our goal is to introduce a Grothendieck topology on C such that D extends to a 6-functor formalism on all sheaves (of anima) on C. Our principle will be to work with the finest possible topology, and try to find the largest possible class of morphisms for which the !-functors are defined. In the process, some of the conditions we impose may look a bit artificial; we will later discuss how to verify some of the conditions (like being a D-cover) in practice, see the second appendix to Lecture VI.

First, we extend D to all (small) presheaves of anima on C via right Kan extension, i.e. for any presheaf of anima $\tilde{X}$, we let

$$D(\tilde{X}) = \lim_{X \to \tilde{X}, X \in C} D(X).$$

This is still a symmetric monoidal presentable stable ∞-category. By [**Man22b**, Proposition A.5.16], one can moreover extend this to a 6-functor formalism with respect to the maps of presheaves of anima any of whose pullbacks to C are representable by a morphism in E.

Definition 4.14. Consider a family of maps $\{f_i : X_i \to Y\}$ in C.

(1) The maps f_i form a *cover of* Y *in the canonical topology* if for all objects $Z \in C$ and all pullbacks of $\{f_i : X_i \to Y\}$ along a map $Y' \to Y$ in C, the functor $\mathrm{Hom}(-, Z)$ satisfies descent along $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.
(2) The maps f_i *satisfy universal* $*$*-descent* if for all pullbacks of $\{f_i : X_i \to Y\}$ along a map $Y' \to Y$ in C, the functor $D(-)$ (with maps given by $*$-pullback) satisfies descent along $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.
(3) Assume that all f_i lie in E. The maps f_i *satisfy universal* !*-descent* if for all pullbacks of $\{f_i : X_i \to Y\}$ along a map $Y' \to Y$ from a presheaf of anima Y' on C, the functor $D(-)$ (with maps given by !-pullback) satisfies descent along $\{f_i \times_Y Y' : X_i \times_Y Y' \to Y'\}$.

The f_i *form a* D*-cover* if all $f_i \in E$ and they form a cover in the canonical topology, satisfy universal $*$-descent, and satisfy universal !-descent. The D*-topology on* C is the Grothendieck topology generated by the D-covers.

Remark 4.15. It seems artificial that one has to impose that the D-topology is subcanonical – one often has morphisms in C which are inverted by D, for example the morphism $X_{\mathrm{red}} \to X$ of schemes under the étale 6-functor formalism. In that case, one would expect that one can allow $X_{\mathrm{red}} \to X$ as a cover in the D-topology, but this is precluded by the requirement of being subcanonical. One way around is to restrict to reduced schemes from the start (or even absolutely weakly normal schemes, giving a canonical representative in each universal homeomorphism class).

Remark 4.16. There is an apparent asymmetry between the conditions (1) and (2) on the one hand, and condition (3) on the other: Namely (1) and (2) only ask descent after pullback to $Y' \in C$, while condition (3) asks it for all presheaves of anima. The reason is that in case (1) and (2) the given functor $\mathrm{Hom}(-, Z)$ resp. $D(-)$ is defined on all morphisms, and in those cases descent after pullback to any $Y' \in C$ implies descent after pullback to any presheaf of anima Y' (simply by writing Y' as a colimit of representables and using that the functors commute with limits). But in case (3), this automatic deduction does not work, and we need to ask for the stronger condition. It is not a priori clear how to verify these conditions in practice; for this, we refer to the second appendix to Lecture VI.

Now let $\tilde{C}$ be the ∞-category of sheaves of anima on C, for the D-topology. The functor $X \mapsto D(X)$ to symmetric monoidal presentable ∞-categories is a sheaf with respect to the D-topology, and thus gives a functor on $\tilde{C}$; for all $\tilde{X} \in \tilde{C}$, one still has

$$D(\tilde{X}) = \lim_{X \to \tilde{X}, X \in C} D(X).$$

Let $\tilde{E}_0$ be the class of maps $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that for any $Y \in C$ and $Y \to \tilde{Y}$, the pullback $f : X = \tilde{X} \times_{\tilde{Y}} Y \to Y$ satisfies $X \in C$ and $f \in E$.[8] The following result is an application of right Kan extension.

Proposition 4.17 ([**Man22b**, Proposition A.5.16]). *The 6-functor formalism D on (C, E) extends uniquely to a 6-functor formalism on $(\tilde{C}, \tilde{E}_0)$ such that the associated functor from $\tilde{C}^{\mathrm{op}}$ to symmetric monoidal presentable ∞-categories is given by the above recipe*

$$D(\tilde{X}) = \lim_{X \to \tilde{X}, X \in C} D(X).$$

We want to extend from $\tilde{E}_0$ to a larger class of morphisms $\tilde{E}$. Here are some desirable conditions.

Definition 4.18. Let $\tilde{E} \supset \tilde{E}_0$ be a class of morphisms of $\tilde{C}$ that is stable under pullback and composition.

(1) The class $\tilde{E}$ is *stable under disjoint unions* if whenever $\tilde{f}_i : \tilde{X}_i \to \tilde{Y}$ are morphisms in $\tilde{E}$, then also $\bigsqcup_i \tilde{f}_i : \bigsqcup_i \tilde{X}_i \to \tilde{Y}$ is in $\tilde{E}$.
(2) The class $\tilde{E}$ is *local on the target* if whenever $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism in $\tilde{C}$ such that for all $Y \in C$ with a map $Y \to \tilde{Y}$, the pullback $\tilde{X} \times_{\tilde{Y}} Y \to Y$ lies in $\tilde{E}$, then $\tilde{f} \in \tilde{E}$.
(3) Assume that D extends uniquely from $(\tilde{C}, \tilde{E}_0)$ to $(\tilde{C}, \tilde{E})$. The class $\tilde{E}$ is *local on the source* if whenever $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism in $\tilde{C}$ such that there is some map $\tilde{g} : \tilde{X}' \to \tilde{X}$ in $\tilde{E}$ that is of universal !-descent, and such that $\tilde{f}\tilde{g}$ lies in $\tilde{E}$, then $\tilde{f} \in \tilde{E}$.
(4) Assume that D extends uniquely from $(\tilde{C}, \tilde{E}_0)$ to $(\tilde{C}, \tilde{E})$. The class $\tilde{E}$ is *tame* if whenever $Y \in C$ and $\tilde{f} : \tilde{X} \to Y$ is a map in $\tilde{E}$, then there are morphisms $h_i : X_i \to Y$ in E and a morphism $\bigsqcup_i X_i \to \tilde{X}$ over Y that lies in $\tilde{E}$ and is of universal !-descent.

Remark 4.19. The condition of being local on the target may seem stricter than expected, as after pullback to $Y \in C$, no further localization in Y is allowed. But if $\tilde{E}$ is both local on the target and local on the source, then to check whether a morphism lies in $\tilde{E}$ it suffices to check it after pullback along an $\tilde{E}$-map $Y' \to Y$ that is of universal !-descent. Thus, for the formulation of the following theorem, one could also take this stronger condition of being "local on the target".

[8] For this condition to make sense, we need to make the D-topology subcanonical.

THEOREM 4.20. *There is a minimal collection of morphisms $\tilde{E} \supset \tilde{E}_0$ of $\tilde{C}$ such that D extends uniquely from $(\tilde{C}, \tilde{E}_0)$ to $(\tilde{C}, \tilde{E})$, and such that $\tilde{E}$ is stable under disjoint unions, local on the target, local on the source, and tame.*

PROOF. Consider the class A all possible collections of morphisms $\tilde{E}$ such that D extends uniquely to $(\tilde{C}, \tilde{E})$, and which are tame. Note that $\tilde{E}_0$ is an example (and minimal). We note that any filtered union of such collections is again such a collection. Given any such collection $\tilde{E}$, one can consider the collection $\tilde{E}'$ of all morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\tilde{X}$ can be written as a disjoint union of $\tilde{X}_i$ with all $\tilde{X}_i \to \tilde{Y}$ being in $\tilde{E}$. This class is stable under pullback and composition, it stays tame, and by [**Man22b**, Proposition A.5.12] the 6-functor formalism D extends uniquely to $(\tilde{C}, \tilde{E}')$. Thus, any class $\tilde{E} \in A$ can be minimally enlarged to be stable under disjoint unions.

Also, if $\tilde{E} \in A$, we can define a new class of morphisms $\tilde{E}'$ as being those morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that there is some $\tilde{E}$-map $\tilde{g} : \tilde{X}' \to \tilde{X}$ of universal !-descent for which $\tilde{f}\tilde{g}$ lies in $\tilde{E}$. This class is stable under pullback and composition, and by [**Man22b**, Proposition A.5.14] (an application of left Kan extension), the 6-functor formalism D extends uniquely to $(\tilde{C}, \tilde{E}')$; here, the special class of covers is taken to be those of universal !-descent. Moreover, $\tilde{E}'$ stays tame. Iterating this procedure countably many times, one can enlarge any $\tilde{E} \in A$ in a minimal way to make it local on the source. Combined with the procedure of making it stable under disjoint unions, we can also arrange both properties in a minimal way.

Now assume that $\tilde{E} \in A$ is stable under disjoint unions and local on the source. Consider the class $\tilde{E}'$ of morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that for all $Y \in C$ with a map $Y \to \tilde{Y}$, the pullback $\tilde{X} \times_{\tilde{Y}} Y \to Y$ lies in $\tilde{E}$. This class is clearly stable under pullback; we claim it is also stable under composition. For this, we show more generally that if $\tilde{Y}' \in \tilde{C}$ is an object that admits an $\tilde{E}$-map $\tilde{Y}' \to Y$ to an object of C, then for any map $\tilde{Y}' \to \tilde{Y}$, the pullback $\tilde{X} \times_{\tilde{Y}} \tilde{Y}' \to \tilde{Y}'$ lies in $\tilde{E}$. To see this, note that by tameness there is a cover of $\tilde{Y}'$ by an $\tilde{E}$-map $\bigsqcup_i Y_i \to \tilde{Y}'$ of universal !-descent with $Y_i \in C$ and $Y_i \to X$ in E. As $\tilde{E}$ is local on the source and stable under disjoint unions, to check that $\tilde{X} \times_{\tilde{Y}} \tilde{Y}' \to \tilde{Y}'$ lies in $\tilde{E}$, it suffices to check after pullback to the Y_i. But here it is true by assumption.

Now let $C' \subset \tilde{C}$ be the full subcategory of those objects that admit an $\tilde{E}$-map towards an object of $C \subset \tilde{C}$. Let E' be the restriction of $\tilde{E}$ to C'. Then D restricts to a 6-functor formalism on (C', E'). Now we can apply [**Man22b**, Proposition A.5.16] to extend D uniquely from (C', E') back to $(\tilde{C}, \tilde{E}')$, as desired. (The uniqueness assertion of this extension, applied to the extension from (C', E') to $(\tilde{C}, \tilde{E})$, ensures that this really extends the formalism on $(\tilde{C}, \tilde{E})$.) It is clear that $\tilde{E}'$ stays tame.

Thus, given some $\tilde{E} \in A$ that is stable under disjoint unions and local on the source, we can find a minimal $\tilde{E}' \in A$ containing $\tilde{E}$ that is local on the target. In fact, this construction shows that it stays stable under disjoint unions and local on the source, giving the desired extension. □

5. Lecture V: Poincaré Duality

The goal of this lecture is to discuss, in the generality of abstract 6-functor formalisms, a notion of Poincaré Duality, and give a simple means for establishing it.

As discussed in the first lecture, Poincaré Duality should say that for a "smooth" morphism $f : X \to Y$, the right adjoint $f^!$ of $f_!$ (exists and) agrees with f^* up to a twist. Axiomatizing this desiratum leads to the following definition. Here, we fix (C, E) as before, and a 3-functor formalism

$$\mathcal{D} : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty.$$

DEFINITION 5.1. Let $f : X \to Y$ be a morphism in E. Then f is ($\mathcal{D}$-)cohomologically smooth if the following properties are satisfied.

(1) The right adjoint $f^!$ of $f_!$ exists, and the natural transformation

$$f^!(1_Y) \otimes f^*(-) \to f^!(-)$$

of functors $\mathcal{D}(Y) \to \mathcal{D}(X)$ is an isomorphism.

(2) The object $f^!(1_Y) \in \mathcal{D}(X)$ is $\otimes$-invertible.

(3) For any $g : Y' \to Y$ with base change $f' : X' = X \times_Y Y' \to Y'$ of f, properties (1) and (2) also hold for f', and moreover the natural map

$$g'^* f^!(1_Y) \to f'^!(1_{Y'})$$

is an isomorphism, where $g' : X' \to X$ is the base change of g.

Here, the map $f^!(1_Y) \otimes f^* \to f^!$ is adjoint to

$$f_!(f^!(1_Y) \otimes f^*(-)) \cong f_! f^!(1_Y) \otimes (-) \to (-)$$

using the projection formula and the counit map $f_! f^!(1_Y) \to 1_Y$; and $g'^* f^!(1_Y) \to f'^!(1_{Y'})$ is adjoint to

$$f'_! g'^* f^!(1_Y) \cong g^* f_! f^!(1_Y) \to g^*(1_Y) = 1_{Y'}$$

using base change and the same counit map.

REMARK 5.2. This notion was defined in [**Sch17**] when I failed to define a notion of smoothness for maps of perfectoid spaces. The issue there was that the topological definition ("locally looks like euclidean space") did not work as not all examples were locally isomorphic; and the algebraic-geometric definition ("lifting against nilpotent thickenings") did not work as perfectoid spaces have no nilpotents. Thus, in the end I simply characterized the desired cohomological properties, and then showed that all relevant examples have this property.

REMARK 5.3. In general, cohomological smoothness is weaker than smoothness; for example, any universal homeomorphism of schemes is cohomologically smooth with respect to étale cohomology, as étale sheaves are insensitive to universal homeomorphisms.

REMARK 5.4. It follows from the definition that the class of cohomologically smooth morphisms is stable under base change, composition, and contains all isomorphisms. Moreover, in case the 3-functor formalism is defined from classes of morphisms I and P as in the last lecture, all morphisms in I are cohomologically smooth. In fact, in that case $f_!$ is left adjoint to f^*, i.e. $f^* = f^!$, and the properties are clear.

Checking that $f : X \to Y$ is cohomologically smooth seems highly nontrivial. Indeed, for any base change of f, one needs to prove that some map is an isomorphism for all $B \in \mathcal{D}(Y)$; and the map involves $f^! B$ which is abstractly defined as an adjoint, so one has to compute the morphisms from any $A \in \mathcal{D}(X)$ towards $f^! B$.

The goal of this lecture is to show that in fact, it is enough to construct a surprisingly small amount of data (and check the commutativity of two diagrams), and this data involves only some very simple sheaves on X, Y and $X \times_Y X$.

To simplify the situation, we will in the following replace C by the slice $C_{/Y}$, so that we can assume that Y is the final object of C. Moreover, we will restrict to the subcategory of objects $X \in C$ such that $X \to Y$ lies in E, and assume that all morphisms $X \to X'$ over Y are still in E – this is notably satisfied in the setup of the last lecture. Thus, in fact all morphisms of C lie in E.

Theorem 5.5. *Assume that all morphisms of C lie in E, and let $f : X \to Y$ be a map to the final object $Y \in C$. Let $\Delta : X \to X \times_Y X$ be the diagonal. Then f is cohomologically smooth if and only if there is a $\otimes$-invertible object $L \in D(X)$ and maps*

$$\alpha : \Delta_! 1_X \to p_2^* L, \beta : f_! L \to 1_Y$$

such that the composite

$$1_X \cong p_{1!}\Delta_! 1_X \xrightarrow{p_{1!}\alpha} p_{1!}p_2^* L \cong f^* f_! L \xrightarrow{f^*\beta} 1_X$$

is the identity, as well as the composite

$$L \cong p_{2!}(p_1^* L \otimes \Delta_! 1_X) \xrightarrow{p_{2!}(p_1^* L \otimes \alpha)} p_{2!}(p_1^* L \otimes p_2^* L) \cong p_{2!} p_1^* L \otimes L \cong f^* f_! L \otimes L \xrightarrow{f^*\beta \otimes L} L.$$

Let us first prove the forward direction. For this, we take $L = f^!(1_Y)$, and $\beta : f_! L = f_! f^!(1_Y) \to 1_Y$ the counit of the adjunction. For α, we note that $p_2^* L = p_1^!(1_X)$ by property (3) in Definition 5.1, and then α is adjoint to $1_X = \Delta^! p_2^!(1_X)$.

We need to prove that these two composites are the identity. The first one is actually straightforward from the definition, but the second is more subtle. It could be done by a direct, but elaborate, diagram chase. Let us give a more abstract argument instead, one that will actually introduce the techniques that will be useful in the converse direction. Namely, note that if $X_1, X_2 \in C$ and $K \in D(X_1 \times_Y X_2)$ is an object, then it induces a "Fourier–Mukai functor with kernel K":

$$\mathcal{D}(X_1) \to \mathcal{D}(X_2) : A \mapsto p_{2!}(p_1^* A \otimes K).$$

In particular, taking $X_1 = X$ and $X_2 = Y$, the object $K = 1_X$ corresponds to the functor

$$F = f_! : \mathcal{D}(X) \to \mathcal{D}(Y),$$

while taking $X_1 = Y$ and $X_2 = X$, the object $K = L$ corresponds to the functor

$$G = f^* \otimes L : \mathcal{D}(Y) \to \mathcal{D}(X).$$

Cohomological smoothness ensures that these two functors are adjoint, so there are natural transformations

$$\alpha_0 : \mathrm{id}_{\mathcal{D}(X)} \to GF$$

and

$$\beta_0 : FG \to \mathrm{id}_{\mathcal{D}(Y)}.$$

Now note that a composite of two functors given by a kernel is itself induced by a kernel, through convolution of kernels. In particular FG is given by $f_! L \in \mathcal{D}(Y)$ and $\mathrm{id}_{\mathcal{D}(Y)}$ by $1_Y \in \mathcal{D}(Y)$, and β_0

is given by $\beta : f_!L \to 1_Y$. On the other hand, GF is given by the kernel $p_2^*L \in \mathcal{D}(X \times_Y X)$, while $\mathrm{id}_{\mathcal{D}(X)}$ is given by the kernel $\Delta_!(1_X) \in \mathcal{D}(X \times_Y X)$. We claim that

$$\alpha : \Delta_!(1_X) \to p_2^*L$$

induces

$$\alpha_0 : \mathrm{id}_{\mathcal{D}(X)} \to GF.$$

Let temporarily α_0' denote the transformation induced by α. Then

$$(\mathrm{Cat}, D(X), D(Y), f_!, f^* \otimes L, \beta_0, \alpha_0, \alpha_0')$$

satisfy the hypothesis of the following lemma, concerning a version of "left and right inverse agree" in the context of adjunctions in 2-categories.

Lemma 5.6. *Let $\mathcal{C}$ be a 2-category and $F : X \to Y$, $G : Y \to X$ be 1-morphisms in $\mathcal{C}$, together with 2-morphisms*

$$\beta : FG \to 1, \alpha : 1 \to GF, \alpha' : 1 \to GF$$

such that the composites

$$F \xrightarrow{F\alpha'} FGF \xrightarrow{\beta F} F, G \xrightarrow{\alpha G} GFG \xrightarrow{G\beta} G$$

are the identity. Then $\alpha = \alpha'$.

Proof. We have

$$\begin{aligned} 1 \xrightarrow{\alpha} GF &= 1 \xrightarrow{\alpha} GF \xrightarrow{GF\alpha'} GFGF \xrightarrow{G\beta F} GF \\ &= 1 \xrightarrow{\alpha'} GF \xrightarrow{\alpha GF} GFGF \xrightarrow{G\beta F} GF \\ &= 1 \xrightarrow{\alpha'} GF. \end{aligned}$$

□

Thus, the unit transformation $\alpha_0 : \mathrm{id}_{\mathcal{D}(X)} \to GF$ is indeed induced by $\alpha : \Delta_!(1_X) \to p_2^*L$. Now the composite

$$G \xrightarrow{\alpha_0 G} GFG \xrightarrow{G\beta_0} G$$

is the identity, and by unraveling the definitions one computes that this composite is induced by the map on kernels

$$L \cong p_{2!}(p_1^*L \otimes \Delta_! 1_X) \xrightarrow{p_{2!}(p_1^*L \otimes \alpha)} p_{2!}(p_1^*L \otimes p_2^*L) \cong p_{2!}p_1^*L \otimes L \cong f^*f_!L \otimes L \xrightarrow{f^*\beta \otimes L} L.$$

In general, passing from kernels to functors loses information, but in this case L encodes the functor $G = f^* \otimes L : \mathcal{D}(Y) \to \mathcal{D}(X)$ which sends $1_Y \in \mathcal{D}(Y)$ to $L \in \mathcal{D}(X)$. Thus, knowing that the induced transformation of functors is the identity implies that the transformation on kernels is the identity.

To prove the converse direction of Theorem 5.5, we formalize the preceding arguments through the introduction of a certain 2-category; some version of this idea is in work of Lu–Zheng [**LZ20**], and this was adapted slightly in [**FS21**], [**HS21**]. Informally, given a geometric setup (C, E) where E consists of all morphisms, and a 3-functor formalism $\mathcal{D}$, let

$$\mathrm{LZ}_{\mathcal{D}}$$

be the 2-category with:

- Objects given by objects of C;
- Morphism categories: $\mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X') = D(X \times X')$;

- Identity morphisms: $\mathrm{id}_X \in \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X) = D(X \times X)$ is given by $\Delta_!(1_X)$;
- Composition:

$$\mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X') \times \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X', X'') \to \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X'')$$

is given by the convolution

$$(A, B) \mapsto A \star B = p_{X,X''!}(p_{X,X'}^* A \otimes p_{X',X''}^* B).$$

To turn this into a definition, one has to supply isomorphisms between $(A\star B)\star C$ and $A\star(B\star C)$ satisfying the pentagon axiom for fourfould convolution (and similar isomorphisms for the identity morphisms). This becomes a bit cumbersome, so let us sketch a high-tech construction of $\mathrm{LZ}_{\mathcal{D}}$; see [**Zav23**] for an elaboration of this construction. One starts with the symmetric monoidal ∞-category $\mathrm{Corr}(C, E)$. All of its objects are dualizable (in fact, self-dual, using diagonal correspondences). Thus, this is a closed symmetric monoidal ∞-category, and hence enriched over itself, with internal mapping objects

$$\underline{\mathrm{Hom}}_{\mathrm{Corr}(C,E)}(X, X') = X \times X'.$$

Enrichments can be transferred along lax symmetric monoidal functors. Applying this to the functor $\mathcal{D}$, we turn $\mathrm{Corr}(C, E)$ into a Cat_∞-enriched ∞-category, with internal Hom-objects given by $\mathcal{D}(X \times X')$. But Cat_∞-enriched ∞-categories are a model for $(\infty, 2)$-categories, and in particular the homotopy 2-category gives the desired 2-category $\mathrm{LZ}_{\mathcal{D}}$.

Remark 5.7. It is worth pointing out how well the definition of a 3-functor formalism fits into this construction.

The idea behind the definition of $\mathrm{LZ}_{\mathcal{D}}$ is that there is a natural functor

$$\mathrm{LZ}_{\mathcal{D}} \to \mathrm{Cat}$$

taking any X to $D(X)$, and any $K \in D(X \times X')$ into the functor

$$D(X) \to D(X') : A \mapsto p_{2!}(K \otimes p_1^* A),$$

i.e. the "Fourier–Mukai functor with kernel K". In general, this passage from K to the induced functor $D(X) \to D(X')$ is very lossful, and working in $\mathrm{LZ}_{\mathcal{D}}$ amounts to working with kernels of functors directly, instead of with the induced functors.

To prove cohomological smoothness, we want to show that (up to twist) f^* is a right adjoint of $f_!$. What we will do instead is to prove the adjointness already in $\mathrm{LZ}_{\mathcal{D}}$. This makes sense, as adjunctions can be defined in any 2-category:

Definition 5.8. Let $\mathcal{C}$ be a 2-category and $F : X \to Y$ a 1-morphism in $\mathcal{C}$. A right adjoint of F is a triple (G, α, β) of a 1-morphism $G : Y \to X$ and 2-morphisms $\alpha : 1 \to GF$ and $\beta : FG \to 1$ such that the composites

$$F \xrightarrow{F\alpha} FGF \xrightarrow{\beta F} F, G \xrightarrow{\alpha G} GFG \xrightarrow{G\beta} G$$

are the identity.

Some basic properties of adjunctions are:

(1) Adjunctions are unique up to unique isomorphism. More precisely, if (G, α, β) and (G', α', β') are right adjoints of F, then we get a unique isomorphism $(G, \alpha, \beta) \cong (G', \alpha', \beta')$ making the obvious diagrams commute. For the construction, for example the map $G \to G'$ is defined as
$$G \xrightarrow{\alpha' G} G'FG \xrightarrow{G'\beta} G'.$$
(2) Any functor of 2-categories preserves adjunctions.
(3) Adjunctions in Cat are usual adjunctions of categories.

Properties (2) and (3) are immediate from the definition. With this preparation, we can give the proof of Theorem 5.5.

Proof of Theorem 5.5. We need to prove the backwards direction, so take (L, α, β). Consider X and Y as objects of $\mathrm{LZ}_{\mathcal{D}}$, and the morphisms
$$F = 1_X \in D(X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, Y), G = L \in D(X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, X).$$
(We recall that we assumed that Y is the final object of C.) Then F encodes the functor $f_! : D(X) \to D(Y)$ and G encodes the functor $f^* \otimes L : D(Y) \to D(X)$, and we wish to see that $f_!$ is the left adjoint of $f^* \otimes L$. We will in fact show that F is a left adjoint of G in $\mathrm{LZ}_{\mathcal{D}}$. Indeed, α and β translate into maps
$$\mathrm{id}_X = \Delta_!(1_X) \to GF = p_2^* L \in D(X \times_Y X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, X)$$
and
$$FG = f_! L \to \mathrm{id}_Y = 1_Y \in D(Y) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, Y),$$
and the required commutative diagrams translate into the ones in Theorem 5.5.

In particular, it follows that the right adjoint $f^!$ is given by a kernel, and in particular it is $D(Y)$-linear, so $f^!(1_Y) \otimes f^* \to f^!$ is an isomorphism. Moreover, we must have $L = f^!(1_Y)$, which we assumed to be $\otimes$-invertible.

Finally, for any base change of f along $g : Y' \to Y$, base change along g defines a functor of 2-category $\mathrm{LZ}_{\mathcal{D}} \to \mathrm{LZ}_{\mathcal{D}|_{C_{/Y'}}}$, which hence gives the same adjunction for the base change of f, with the base change of L as dualizing complex. □

Let us end the lecture by discussing the example of locally compact Hausdorff topological spaces.

Example 5.9. Consider the category C of finite-dimensional locally compact Hausdorff topological spaces, with the functor $\mathcal{D} : X \mapsto \mathcal{D}(\mathrm{Ab}(X))$. For the classes of open immersions I and proper maps P, this satisfies the conditions of the last lecture, giving us a 6-functor formalism (where E consists of all maps).

Proposition 5.10. *The map $f : \mathbb{R} \to *$ is cohomologically smooth.*

Note that by compatibility with composites and base change and open immersions, this implies that all manifold bundles are cohomologically smooth.

Proof. We take $L = \mathbb{Z}[1]$. We need to construct maps α and β. But $R\Gamma_c(\mathbb{R}, \mathbb{Z}[1]) \cong \mathbb{Z}$ giving us α; and using the triangle $0 \to \Delta_! \mathbb{Z} \to \mathbb{Z} \to j_! \mathbb{Z}$ for $j : \mathbb{R}^2 \setminus \Delta \to \mathbb{R}^2$ the complementary open, one computes
$$\mathrm{RHom}(\Delta_! \mathbb{Z}[-1], \mathbb{Z}) \cong \mathbb{Z},$$

giving us β. Indeed, the triangle shows that $\mathrm{RHom}(\Delta_!\mathbb{Z}[-1],\mathbb{Z})$ is the cone of

$$R\Gamma(\mathbb{R}^2,\mathbb{Z}) \to R\Gamma(\mathbb{R}^2\setminus\Delta,\mathbb{Z}),$$

and this map is the diagonal embedding $\mathbb{Z}\to\mathbb{Z}^2$.

To finish the proof, one would like to see that an endomorphism of the constant sheaf $\mathbb{Z}$ on $\mathbb{R}$ is the identity (if the signs of α and β are chosen compatibly). It is helpful that in this case, all intermediate maps are isomorphisms, so it has to be true up to sign. A priori, one might worry that these signs cannot be chosen compatibly, but here Lemma 5.6 comes to the rescue. Even better, one has the following general principle about adjunctions. □

Lemma 5.11. *Let $\mathcal{C}$ be a 2-category, $F: X\to Y$, $G: Y\to X$ be 1-morphisms in $\mathcal{C}$ as well as 2-morphisms $\alpha: 1\to GF$ and $\beta: FG\to 1$. Assume that the 2-morphisms*

$$F\xrightarrow{F\alpha} FGF\xrightarrow{\beta F} F, G\xrightarrow{\alpha G} GFG\xrightarrow{G\beta} G$$

are isomorphisms. Then one can find some $\alpha': 1\to GF$ such that (G,α',β) is a right adjoint of F.

Proof. Changing α by an isomorphism $G\cong G$, we can arrange that $G\xrightarrow{\alpha G} GFG\xrightarrow{G\beta} G$ is the identity. We can also find some $\alpha': 1\to GF$ such that $F\xrightarrow{F\alpha'} FGF\xrightarrow{\beta F} F$ is the identity. By Lemma 5.6, we necessarily have $\alpha=\alpha'$, which then does the trick. □

6. Lecture VI: Complements on the abstract formalism

This is the final lecture on the abstract theory of 6-functor formalisms; starting from the next lecture, we will discuss various examples. In this lecture, I want to discuss some complements that the 2-category from the previous lecture naturally leads to, and that give answers to the following natural questions:

(1) For which morphisms is f^* essentially $f^!$? For which morphisms is $f_!$ essentially f_*?
(2) What are reasonable finiteness conditions that one can put on objects $A \in \mathcal{D}(X)$? For example, which conditions ensure that A maps isomorphically to its double Verdier dual?

As in the last lecture, we will fix some ∞-category C admitting all finite limits, and assume that E consists of all morphisms of C. Let $\mathrm{Corr}(C) := \mathrm{Corr}(C, \mathrm{all})$ be the resulting symmetric monoidal ∞-category of correspondences, and

$$\mathcal{D} : \mathrm{Corr}(C) \to \mathrm{Cat}_\infty$$

a lax symmetric monoidal functor. In this lecture, we will assume that it gives a 6-functor formalism, i.e. internal Homs, and the right adjoints f_* and $f^!$ (of f^* and $f_!$) exist for all f.

As in the last lecture, we get the 2-category $\mathrm{LZ}_{\mathcal{D}}$. Note that $\mathrm{LZ}_{\mathcal{D}}$ is equivalent to $\mathrm{LZ}_{\mathcal{D}}^{\mathrm{op}}$ (basically, as $\mathrm{Corr}(C) \cong \mathrm{Corr}(C)^{\mathrm{op}}$). Generalizing the idea of using adjunctions in $\mathrm{LZ}_{\mathcal{D}}$, we are led to consider the following notions.[9]

Definition 6.1. Let $Y \in C$ be the final object, $f : X \to Y$ some morphism, and $A \in D(X)$.

(1) The object $A \in D(X)$ is f-smooth if $A \in \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(X, Y)$ is a left adjoint in $\mathrm{LZ}_{\mathcal{D}}$.
(2) The object $A \in D(X)$ is f-proper if $A \in \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, X)$ is a left adjoint in $\mathrm{LZ}_{\mathcal{D}}$.

Remark 6.2. Both properties are stable under any base change, using that base change induces functors of 2-categories and hence preserves adjunctions. Here, if $f' : X' \to Y'$ is a morphism to some other object Y', then we first replace C by the slice $C_{/Y'}$ and then apply the previous definition to define f'-smooth and f'-proper objects $A' \in D(X')$.

Remark 6.3. If $\mathcal{D} : \mathrm{Corr}(C) \to \mathrm{Cat}_\infty$ is a 3-functor formalism, then also $\mathcal{D}^{\mathrm{op}} : \mathrm{Corr}(C) \to \mathrm{Cat}_\infty$, sending any X to the opposite ∞-category $\mathcal{D}(X)^{\mathrm{op}}$, is a 3-functor formalism (as passing to opposite ∞-categories is a (covariant!) symmetric monoidal self-equivalence of Cat_∞). This abstract procedure exchanges f-smooth and f-proper objects, and allows one to formally dualize some statements. Note, however, that this translation turns the right adjoints f_*, $f^!$, etc., into left adjoint functors; this explains some apparent asymmetries in the presentation below.

Remark 6.4. The second condition of "f-properness" was, to our knowledge, first suggested by Mann in [**Man22a**].

Concretely, A is f-smooth if and only if there is some $B \in D(X) = \mathrm{Hom}_{\mathrm{LZ}_{\mathcal{D}}}(Y, X)$ together with maps

$$\alpha : \Delta_!(1_X) \to p_1^* A \otimes p_2^* B, \beta : f_!(A \otimes B) \to 1_Y$$

such that the composite

$$B \cong p_{2!}(p_1^* B \otimes \Delta_!(1_X)) \xrightarrow{\alpha} p_{2!}(p_1^* B \otimes p_1^* A \otimes p_2^* B) \cong p_{2!} p_1^*(B \otimes A) \otimes B \cong f^* f_!(B \otimes A) \otimes B \xrightarrow{\beta} f^*(1_Y) \otimes B \cong B$$

[9] The names we give to these properties is nonstandard, but motivated by the idea that "smoothness" should imply that (twisted versions of) f^* and $f^!$ agree, while "properness" should imply that (twisted versions of) $f_!$ and f_* agree.

as well as the similar one with A and B exchanged, is the identity. (As in the last lecture, to check that A is f-smooth, it suffices to check that these composites are isomorphisms, not necessarily equal to the identity.)

We note that this data is in fact symmetric in A and B, which is a consequence of the self-duality of $\mathrm{LZ}_{\mathcal{D}}$ which ensures that the right adjoint of A is itself f-smooth, with right adjoint given again by A. We record some of these obversations below.

Proposition 6.5. *Let $A \in D(X)$ be f-smooth, with right adjoint $B \in D(X)$.*

(1) *The object $B \in D(X)$ is f-smooth, with right adjoint A.*
(2) *There is a natural isomorphism of functors*

$$B \otimes f^*(-) \cong \mathscr{H}\mathrm{om}(A, f^!(-)) : D(Y) \to D(X).$$

In particular, $B \cong \mathscr{H}\mathrm{om}(A, f^!(1_Y)) = \mathbb{D}_f(A)$ is the (relative) Verdier dual of A.
(3) *The Verdier biduality map*

$$A \to \mathbb{D}_f(\mathbb{D}_f(A))$$

is an isomorphism.
(4) *The formation of the Verdier dual $\mathbb{D}_f(A)$ commutes with any base change.*

In particular, (2) says that f-smooth objects A lead to isomorphisms between twisted versions of f^* and $f^!$. Also note that by (1), one also has

$$A \otimes f^*(-) \cong \mathscr{H}\mathrm{om}(B, f^!(-)) : D(Y) \to D(X).$$

Proof. We already saw (1). For (2), note that applying the functor $\mathrm{LZ}_{\mathcal{D}} \to \mathrm{Cat} : X \mapsto D(X)$, we see that $B \otimes f^*(-)$ is the right adjoint of $f_!(A \otimes -)$; but that right adjoint is $\mathscr{H}\mathrm{om}(A, f^!(-))$. For (3), apply (2) twice; note that the maps $A \to \mathbb{D}_f(B)$ and $B \to \mathbb{D}_f(A)$ are both induced by the pairing $\beta : f_!(A \otimes B) \to 1_Y$ (equivalently, $A \otimes B \to f^!(1_Y)$). For (4), use that the formation of right adjoints (when they exist) commutes with any functor of 2-categories, in particular with base change. □

We see that B is necessarily given by the Verdier dual $\mathbb{D}_f(A)$ of A. If one takes this as the definition of B, one automatically gets the map $\beta : f_!(A \otimes B) \to 1_Y$. One can then wonder what it takes to supply the map $\alpha : \Delta_!(1_X) \to p_1^* A \otimes p_2^* B$. This is analyzed in the next proposition.

Proposition 6.6. *Let $A \in D(X)$. Then A is f-smooth if and only if the natural map*

$$p_1^* A \otimes p_2^* \mathbb{D}_f(A) \to \mathscr{H}\mathrm{om}(p_2^* A, p_1^! A)$$

is an isomorphism. In fact, it suffices to see that it induces an isomorphism on global sections after applying $\Delta^!$.

Here, the natural map is the one adjoint to the natural map

$$p_1^* A \otimes p_2^* \mathbb{D}_f(A) \otimes p_2^* A \to p_1^* A \otimes p_2^* f^!(1_Y) \to p_1^* A \otimes p_1^!(1_X) \to p_1^! A.$$

Proof. If A is f-smooth, then for any base change $f' : X' \to Y'$ along $g : Y' \to Y$ (with base change $g' : X' \to X$) and any $B \in D(Y')$, the map

$$f'^* B \otimes g'^* \mathbb{D}_f(A) \to \mathscr{H}\mathrm{om}(g'^* A, f'^! B)$$

is an isomorphism. Applying this with $g : Y' \to Y$ given by $f : X \to Y$ and $B = A$ yields the desired isomorphism.

For the converse, we need to find the map $\alpha : \Delta_!(1_X) \to p_1^*A \otimes p_2^*\mathbb{D}_f(A)$. Conveniently, the hypothesis of the proposition shows that the target is naturally isomorphic to $\mathscr{H}\mathrm{om}(p_2^*A, p_1^!A)$, so it suffices to find a natural map

$$\Delta_!(1_X) \to \mathscr{H}\mathrm{om}(p_2^*A, p_1^!A).$$

By adjunction, this amounts to a map $\Delta_!(1_X) \otimes p_2^*A = \Delta_!(A) \to p_1^!A$, which by a further adjunction amounts to a morphism $p_{1!}\Delta_!A = A \to A$, where we take the identity map. It remains to see that two diagrams commute, which we leave as an exercise.

For the final sentence, note that the construction of the previous paragraph did not in fact require that the displayed map is an isomorphism, but only that it becomes one after applying global sections to $\Delta^!$ of this map. □

Remark 6.7. This discussion may seem like formal nonsense, but when writing [**FS21**], I was for a long time unsuccessfully trying to prove that certain $A \in D(X)$ satisfying the condition of Proposition 6.6 also satisfy Verdier biduality. The work of Lu–Zheng [**LZ20**] then trivialized this problem!

Let us now look at the "dual" case of f-proper $A \in D(X)$. In this case, the right adjoint $B \in D(X)$ comes with maps

$$\alpha : p_1^*A \otimes p_2^*B \to \Delta_!(1_X), \beta : 1_Y \to f_!(A \otimes B),$$

satisfying similar commutative diagrams. The analogue of Proposition 6.5 is the following.

Proposition 6.8. *Let $A \in D(X)$ be f-proper, with right adjoint $B \in D(X)$.*

(1) *The object $B \in D(X)$ is f-proper, with right adjoint A.*
(2) *There is a natural isomorphism of functors*

$$f_!(B \otimes -) \cong f_*\mathscr{H}\mathrm{om}(A, -) : D(X) \to D(Y).$$

(3) *The pairing*

$$\alpha : p_1^*A \otimes p_2^*B \to \Delta_!(1_X)$$

induces isomorphisms

$$B \cong p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)), A \cong p_{1*}\mathscr{H}\mathrm{om}(p_2^*B, \Delta_!(1_X)).$$

Of course, one could also formulate an analogue of Proposition 6.5 (4), that the dual in the sense of (3) commutes with any base change.

Proof. The proof is identical to the proof of Proposition 6.5. To get the (first) isomorphism in (3), apply the result of (2) to the base change of $f : X \to Y$ along the map $g : Y' \to Y$ given by $f : X \to Y$, and the sheaf $\Delta_!(1_X)$. □

We note that the dual in part (3) simplifies considerably in case $\Delta_!(1_X) = \Delta_*(1_X)$, as is often the case (and conditions for which we will discuss momentarily). Indeed, in that case

$$p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)) = p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_*(1_X)) = p_{2*}\Delta_*\mathscr{H}\mathrm{om}(\Delta^*p_1^*A, 1_X) = \mathscr{H}\mathrm{om}(A, 1_X)$$

is simply the naive dual of $A \in D(X)$. In particular, if $A = 1_X$ is the unit, then this naive dual is 1_X.

We also have an analogue of Proposition 6.6.

PROPOSITION 6.9. *Let $A \in D(X)$. Then A is f-proper if and only if the natural map*

$$f_!(A \otimes p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X))) \to f_*\mathscr{H}\mathrm{om}(A, A)$$

is an isomorphism on global sections.

In fact, one can construct a more general natural transformation

$$f_!(- \otimes p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X))) \to f_*\mathscr{H}\mathrm{om}(A, -)$$

of functors $D(X) \to D(Y)$. This is adjoint to

$$\begin{aligned} f^*f_!(- \otimes p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X))) &\cong p_{1!}(p_2^*(-) \otimes p_2^*p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X))) \\ &\to p_{1!}(p_2^*(-) \otimes \mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X))) \to \mathscr{H}\mathrm{om}(A, -) \end{aligned}$$

which in turn is adjoint to

$$p_2^*(-) \otimes \mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)) \to p_1^!\mathscr{H}\mathrm{om}(A, A) \cong \mathscr{H}\mathrm{om}(p_1^*A, p_1^!(-))$$

which in turn is adjoint to

$$p_2^*(-) \otimes \mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)) \otimes p_1^*A \to p_2^*(-) \otimes \Delta_!(1_X) \to p_1^!(-)$$

where the latter map is adjoint to the isomorphism

$$p_{1!}(p_2^*(-) \otimes \Delta_!(1_X)) \cong (-).$$

PROOF. The proof is similar to the proof of Proposition 6.6. □

Let us now come back to the question of when $f_!$ and f_* agree.

DEFINITION 6.10. A map $f : X \to Y$ in C is cohomologically proper if it is n-truncated for some n, the object $1_X \in D(X)$ is f-proper, and the diagonal $\Delta_f : X \to X \times_Y X$ is cohomologically proper (or an isomorphism).

This definition may seem self-referential, but as Δ_f is $n-1$-truncated, it works by induction on n (with Δ_f being an isomorphism as the base case).

PROPOSITION 6.11. *Let $f : X \to Y$ be a morphism in C such that Δ_f is cohomologically proper. Then there is a natural transformation $f_! \to f_*$ of functors $D(X) \to D(Y)$. It is an isomorphism if and only if f is cohomologically proper. To check that f is cohomologically proper, it suffices to check that $f_!(1_X) \to f_*(1_X)$ is an isomorphism on global sections.*

PROOF. We argue by induction on n such that f is n-truncated. Thus, we can assume that $\Delta_{f!} \cong \Delta_{f*}$. In particular, the map of Proposition 6.9 applied to $1_X \in D(X)$ reduces to a map $f_!(1_X) \to f_*(1_X)$, which is an isomorphism on global sections if and only if f is cohomologically proper; and the construction in fact gives a natural transformation $f_! \to f_*$ of functors $D(X) \to D(Y)$. If f is indeed cohomologically proper, then it follows that $f_! \cong f_*$ as functors $D(X) \to D(Y)$. □

In particular, we see that the natural transformation $f_! \to f_*$ that often exists in practice (for separated maps), and that often is encoded as extra data in a 6-functor formalism, notably in the approach of Gaitsgory–Rozenblyum, can in fact be constructed directly out of the data that we have fixed.

To restore the symmetry, let us also single out the corresponding class of cohomologically smooth morphisms, which we call cohomologically étale (as usually for étale morphisms, $f^* = f^!$ without twist).

Definition 6.12. A map $f : X \to Y$ in C is cohomologically étale if it is n-truncated for some n, the object $1_X \in D(X)$ is f-smooth, and the diagonal $\Delta_f : X \to X \times_Y X$ is cohomologically étale (or an isomorphism).

Proposition 6.13. *Let $f : X \to Y$ be a morphism in C such that Δ_f is cohomologically étale. Then there is a natural transformation $f^! \to f^*$ of functors $D(Y) \to D(X)$. It is an isomorphism if and only if f is cohomologically étale. To check that f is cohomologically étale, it suffices to check that $f^!(1_Y) \to 1_X$ is an isomorphism on global sections.*

Proof. This is the dual (in the sense of Remark 6.3) of Proposition 6.11. Note that Proposition 6.11 refers to the right adjoint f_* of f^*, which translates badly. But what used to be a transformation $f_! \to f_*$ becomes a transformation from a hypothetical left adjoint of f^* to $f_!$; but by passing to right adjoints, this manifests itself again as a transformation from $f^!$ to f^*.

Alternatively, follow the steps of the proof of Proposition 6.11 and translate them accordingly. □

As a final topic, let us discuss how the conditions of being f-smooth and f-proper relate to finiteness conditions one can put on $A \in D(X)$ as an object of the category $D(X)$. There are two notable conditions: As $D(X)$ is symmetric monoidal, there is the condition of being dualizable; and if $\mathcal{D}(X)$ has all colimits, one can ask that $A \in \mathcal{D}(X)$ is compact (i.e. $\mathrm{Hom}(A, -)$ commutes with all filtered colimits).

Proposition 6.14. *Let $f : X = Y \to Y$ be the identity. Then $A \in D(Y)$ is f-smooth if and only if A is f-proper if and only if A is dualizable.*

Proof. This is immediate from the definitions. □

Proposition 6.15. *Let $f : X \to Y$ be a morphism and let $A \in \mathcal{D}(X)$ be f-smooth. Assume that $\Delta_!(1_X) \in \mathcal{D}(X \times_Y X)$ is compact. Then $A \in \mathcal{D}(X)$ is compact.*

The condition that $\Delta_!(1_X)$ is compact is satisfied in many, but not all, situations, and there are important situations where f-smooth objects are quite far from compact objects.

Proof. For $B \in \mathcal{D}(X)$, we have an isomorphism

$$p_1^* B \otimes p_2^* \mathbb{D}_f(A) \cong \mathscr{H}\mathrm{om}(p_2^* A, p_1^! B).$$

We apply $\mathrm{Hom}(\Delta_!(1_X), -)$ to this, getting an isomorphism

$$\begin{aligned}\mathrm{Hom}(\Delta_!(1_X), p_1^* B \otimes p_2^* \mathbb{D}_f(A)) &\cong \mathrm{Hom}(\Delta_!(1_X), \mathscr{H}\mathrm{om}(p_2^* A, p_1^! B)) \\ &\cong \mathrm{Hom}(\Delta_!(1_X) \otimes p_2^* A, p_1^! B) \\ &\cong \mathrm{Hom}(\Delta_!(A), p_1^! B) \cong \mathrm{Hom}(p_{1!}\Delta_!(A), B) \cong \mathrm{Hom}(A, B).\end{aligned}$$

But by assumption, the functor $B \mapsto \mathrm{Hom}(\Delta_!(1_X), p_1^* B \otimes p_2^* \mathbb{D}_f(A))$ commutes with all filtered colimits. □

Again, there is an analogue for f-proper objects, this time under a much weaker assumption.

PROPOSITION 6.16. *Let $f : X \to Y$ be a morphism and let $A \in \mathcal{D}(X)$ be f-proper. Assume that $1_Y \in \mathcal{D}(Y)$ is compact. Then $A \in \mathcal{D}(X)$ is compact.*

PROOF. By f-properness, we have an isomorphism

$$f_!(p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)) \otimes B) \cong f_*\mathscr{H}\mathrm{om}(A, B).$$

Applying $\mathrm{Hom}(1_Y, -)$, we get

$$\mathrm{Hom}(1_Y, f_!(p_{2*}\mathscr{H}\mathrm{om}(p_1^*A, \Delta_!(1_X)) \otimes B)) \cong \mathrm{Hom}(A, B).$$

But, as a functor of B, the left-hand side commutes with all filtered colimits. □

Finally, we note the following stability of these conditions under retracts.

PROPOSITION 6.17. *Let $f : X \to Y$ and $f' : X' \to Y$ be morphisms, and assume that X' is a retract of X over Y, i.e. there are $i : X' \to X$ and $r : X \to X'$ over Y with $ri = \mathrm{id}_{X'}$. Let $A' \in \mathcal{D}(X')$ and $A \in \mathcal{D}(X)$ be such that there are maps $i^*A \to A'$ and $r^*A' \to A$ such that the composite $A' = i^*r^*A' \to i^*A \to A'$ is the identity; or else that there are maps $i_!A' \to A$ and $r_!A \to A'$ such that the composite $A' = r_!i_!A' \to r_!A \to A'$ is the identity. Then if $A \in \mathcal{D}(X)$ is f-smooth, also A' is f'-smooth.*

Again, there is a similar result in the f-proper case; as usual, the two cases are swapped under passing to opposite categories, so one should have maps $A' \to i^*A$ and $A \to r^*A'$ etc.

PROOF. We consider the case first scenario. Let $e = ir : X \to X$ be the idempotent endomorphism of X. We note that A comes equipped with a map $h : e^*A \to A$ given as the composite $r^*i^*A \to r^*A' \to A$; and this map is idempotent in the sense that the composite

$$e^*A = e^*e^*A \xrightarrow{e^*h} e^*A \xrightarrow{h} A$$

agrees with h. Conversely, given A with such an "idempotent" $h : e^*A \to A$, we get an actual idempotent endomorphism $i^*h : i^*A = i^*e^*A \to i^*A$, and letting A' be the corresponding retract of i^*A, we get a map $A' \to i^*A$ as well as a map $r^*A' \to r^*i^*A \to A$. This way, the datum of A and A' and the maps $i^*A \to A'$ and $r^*A' \to A$ is equivalently captured in the datum of A and $h : e^*A \to A$.

In other words, in $\mathrm{LZ}_{\mathcal{D}}$ (where as usual we assume that $\mathcal{C}$ has final object Y), we have the 1-morphism $A : X \to Y$ and $e : X \to X$ as well as the 2-morphism $h : Ae \Rightarrow A$. If B is the f-Verdier dual of A (i.e., the right adjoint morphism $B : Y \to X$), then h corresponds to a 2-morphism $e \Rightarrow BA$. If we denote the self-anti-equivalence of $\mathrm{LZ}_{\mathcal{D}}$ by a transpose $-^t$, then this is also the same as a morphism $e^t \Rightarrow A^tB^t$ (recall that $-^t$ is contravariant for 1-morphisms but covariant for 2-morphisms). Now A^t is right adjoint to B^t, so this can be further translated into $B^te^t \Rightarrow B^t$. Concretely, this is a morphism $h' : e_!B \to B$ in $\mathcal{D}(X)$. Again, this morphism is "idempotent", and can equivalently be described in terms of $B' \in \mathcal{D}(X')$ with maps $i_!B' \to B$ and $r_!B \to B'$ whose composite $B' = r_!i_!B \to r_!B \to B'$ is the identity. We note that this whole process could also be reversed, starting from the second type of data and producing the first on the Verdier dual side.

It is now some formal 2-categorical nonsense to obtain the adjunction between $A' : X' \to Y$ and $B' : Y \to X'$ as a "retract" of the adjunction between $A : X \to Y$ and $B : Y \to X$. □

Appendix to Lecture VI: Uniqueness of $f_!$

In Lecture IV, we stated a construction principle for 3-functor formalisms, or more precisely for constructing $f_!$ when given $\otimes$ and f^*. The results of this lecture should actually give a uniqueness result for it.

Let us assume that C is some ∞-category admitting all finite limits, and that all morphisms of C are n-truncated for some n (which may depend on the morphism). Let

$$\mathcal{D} : \mathrm{Corr}(C) \to \mathrm{Cat}_\infty$$

be a 3-functor formalism, where implicitly E is the class of all morphisms of C. We can then define the class of cohomologically proper morphisms, and the class of cohomologically étale morphisms. These classes are automatically stable under base change, composition, contain all isomorphisms, and satisfy the 2-out-of-3-property. In fact, given the others, the 2-out-of-3-property reduces to stability under diagonals, which is part of the definition of cohomologically étale (resp. proper). The only thing we did not verify in the lecture is stability under composition; we leave this as a simple exercise. If all morphisms of C factor as a cohomologically étale map followed by a cohomologically proper map, we are thus precisely in the setting of the construction from Lecture IV.

Slightly more abstractly, let us fix classes of morphisms I and P satisfying the conditions of Lecture IV (for the class E of all morphisms). We can then consider the following two ∞-categories.

(1) The ∞-category of functors

$$\mathcal{D}_0 : C^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Cat}_\infty)$$

(i.e., "2-functor formalisms") satisfying the properties from Lecture IV; and where morphisms $\mathcal{D}_0 \to \mathcal{D}_0'$ are those natural transformations that also commute with the functors $j_!$ for $j \in I$ and f_* for $f \in P$ (which is just a condition, as these are adjoints). Here, we recall that the morphisms of $\mathrm{CMon}(\mathrm{Cat}_\infty)$ are the symmetric monoidal natural transformations.

(2) The ∞-category of lax symmetric monoidal functors

$$\mathcal{D} : \mathrm{Corr}(C) \to \mathrm{Cat}_\infty$$

(i.e., 3-functor formalisms) with the property that all morphisms in I are cohomologically étale, and all morphisms in P are cohomologically proper; and morphisms $\mathcal{D} \to \mathcal{D}'$ are lax symmetric monoidal natural transformations.

Then there is a forgetful functor from (2) to (1), and the construction of Lecture IV gives a splitting of this functor, going from (1) to (2). We conjecture that these are inverse equivalences, and we believe that the methods we have introduced are sufficient to prove this. In other words, given the classes I and P, the functors $\otimes$ and f^* determine the functors $f_!$ uniquely.

Appendix to Lecture VI: Smooth or proper descent

Assume that $D(X)$ is a presentable ∞-category for all $X \in C$. Let $f : X \to Y$ be a map in E. If 1_X is f-smooth or f-proper, one can prove strong descent properties of $X \mapsto D(X)$. (Usually, when discussing things like Artin stacks, one uses a smooth topology; but the results here show that one could also allow proper covers, at least as far as the dicussion of 6 functors is concerned.)

Proposition 6.18. *Let $f : X \to Y$ be an E-morphism such that 1_X is f-smooth. Then*

$$f^* : D(Y) \to D(X)$$

is conservative if and only if the natural map

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! (1_Y) \to 1_Y$$

is an isomorphism (where $f_n : X^{n/Y} \to Y$ are the fibre products), and this condition passes to any base change. In that case, the pullback functors

$$(f_n^*)_n : D(Y) \to \lim_{[n]\in\Delta} D(X^{n/Y}), (f_n^!)_n : D(Y) \to \lim_{[n]\in\Delta} D(X^{n/Y})$$

are equivalences. In particular, if f is a canonical cover, then it is a D-cover, and of universal $!$-descent.

Recall here that if 1_X is f-smooth, then $f^! = f^* \otimes f^!(1_Y)$, and $f^* = \mathscr{H}\mathrm{om}(f^!(1_Y), f^!)$. In particular, f^* admits a left adjoint $f_!(f^!(1_Y) \otimes -)$, and the resulting counit transformation $f_!(f^!(1_Y) \otimes f^*) \to \mathrm{id}$ agrees with the counit transformation $f_! f^! \to \mathrm{id}$. These will be used implicitly in the proof.

Proof. Assume first that

$$(f_i^*)_i : D(Y) \to \prod_i D(X_i)$$

is conservative. If f has a section, then the map

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! (1_Y) \to 1_Y$$

is an isomorphism for formal reasons. But under the conservativity assumption, we can in general reduce to this situation, using that the formation of $f^!$ commutes with any pullback when 1_X is f-smooth.

In the converse direction, assume that this map is an isomorphism. In that case, the projection formula implies that

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! A \to A$$

is an isomorphism for all $A \in D(Y)$. In particular, if $f^*A = 0$, then $f^!A = f^*A \otimes f^!(1_Y) = 0$, and the isomorphism implies $A = 0$.

The pullback functor

$$(f_n^*)_n : D(Y) \to \lim_{[n]\in\Delta} D(X^{n/Y})$$

admits a left adjoint, such that the counit of the adjunction is precisely the map

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! A \to A$$

This is an equivalence, yielding fully faithfulness. On the other hand, both functors in this adjunction commute with any base change in Y, so to check that the unit transformation is also an equivalence, we can check after pullback along $f : X \to Y$. But there, the cover admits a splitting, and the claim is automatic. The similar argument applies to $!$-pullback. □

The f-proper analogue is the following result, related to the notion of descendability of Mathew [**Mat16**]. (Roughly speaking, the reason we have to impose stronger conditions here is that $D(X)$ is presentable, but $D(X)^{\mathrm{op}}$ is not, so we have to ensure that all limits and colimits that are implicit in the previous proof stay finite.)

PROPOSITION 6.19. *Assume that for all $X \in C$, the presentable ∞-category $D(X)$ is stable. Let $f : X \to Y$ be an E-morphism such that 1_X is f-proper, with $f_n : X^{n/Y} \to Y$ the fibre product. Assume that the map*

$$1_Y \to \text{“lim”}_{[n]\in\Delta}\, f_{n*} 1_{X^{n/Y}}$$

is an isomorphism in $\mathrm{Pro}(D(Y))$; equivalently (by base change and the projection formula), $f_* 1_X \in \mathrm{CAlg}(D(Y))$ is descendable. (This condition passes to any base change of f.)

Then the pullback functors

$$(f_n^*)_n : D(Y) \to \lim_{[n]\in\Delta} D(X^{n/Y}), (f_n^!)_n : D(Y) \to \lim_{[n]\in\Delta} D(X^{n/Y})$$

are equivalences. In particular, if f is a canonical cover, then it is a D-cover, and of universal $!$-descent.

Recall that if 1_X is f-proper, with dual object $A \in D(X)$, then $f_* = f_!(A \otimes -)$ and $f_! = f_* \mathscr{H}\mathrm{om}(A, -)$. In particular, f_* satisfies the projection formula and commutes with any $*$-base change. Similarly, $f_!$ commutes with any $!$-base change. Moreover, one has

$$f_! f^! = f_* \mathscr{H}\mathrm{om}(A, f^!) = \mathscr{H}\mathrm{om}(f_! A, -) = \mathscr{H}\mathrm{om}(f_* 1_X, -).$$

PROOF. The functor $(f_n^*)_n$ has a right adjoint taking a collection $(A_n)_n \in \lim_{[n]\in\Delta} D(X^{n/Y})$ to $\lim_n f_{n*} A_n$. The unit of the adjunction is the natural map

$$A \to \lim_n f_{n*} f_n^* A.$$

We claim that this is an isomorphism, by showing that in fact the map

$$A \to \text{“lim”}_n\, f_{n*} f_n^* A$$

in $\mathrm{Pro}(D(Y))$ is an isomorphism. But this latter map is obtained by tensoring the given isomorphism $1_Y \to \text{“lim”}_{[n]\in\Delta}\, f_{n*} 1_{X^{n/Y}}$ with A (by the projection formula).

In fact, more generally, the same argument applies to $A \in \mathrm{Pro}(D(Y))$, and shows that $f^* : \mathrm{Pro}(D(Y)) \to \mathrm{Pro}(D(X))$ is conservative. We now claim that the counit map is an isomorphism, for which we have to see that for all $(A_n)_n \in \lim_{[n]\in\Delta} D(X^{n/Y})$, the map

$$f^* \lim_n f_{n*} A_n \to A_0$$

is an isomorphism. Again, we in fact claim that the map

$$f^*\, \text{“lim”}_n\, f_{n*} A_n \to A_0$$

in $\mathrm{Pro}(D(X))$ is an isomorphism. But pullback under one projection map $\mathrm{Pro}(D(X)) \to \mathrm{Pro}(D(X \times_Y X))$ is conversative (we proved this above for f^*, but the hypotheses pass to any base change of f^*), so we can check this after base changing everything along $f : X \to Y$. But now the cover is split, so descent is automatic.

Now consider the case of $!$-descent; here, $(f_n^!)_n$ has a left adjoint taking $(A_n)_n \in \lim_{[n]\in\Delta} D(X^{n/Y})$ to $\mathrm{colim}_n f_{n!} A_n$. The counit of the adjunction is now given by

$$\mathrm{colim}_n f_{n!} f_n^! A \to A.$$

This is an equivalence by taking the internal Hom from the Pro-isomorphism $1_Y \to$ "$\lim$"$_n\, f_{n*}1_{X^{n/Y}}$ towards A. In fact, this shows the slightly more precise claim that the map

$$\text{“colim”}_n\, f_{n!}f_n^! A \to A$$

is an isomorphism in $\mathrm{Ind}(D(Y))$, and in fact this holds for all $A \in \mathrm{Ind}(D(Y))$. In particular, this implies that $f^! : \mathrm{Ind}(D(Y)) \to \mathrm{Ind}(D(X))$ is conservative.

To show that the unit of the adjunction is an isomorphism, we need to show that for all $(A_n)_n \in \lim_{[n]\in\Delta} D(X^{n/Y})$, the natural map

$$A_0 \to f^!(\mathrm{colim}_n f_{n!}A_n)$$

is an isomorphism. Again, we make the more precise claim that the map

$$A_0 \to f^!(\text{“colim”}_n\, f_{n!}A_n) = \text{“colim”}_n\, f^! f_{n!}A_n$$

is an isomorphism in $\mathrm{Ind}(D(X))$. We conclude as in the case of $*$-descent: It suffices to prove this after applying !-pullback to $X \times_Y X$, where it reduces to the similar descent for the split cover $X \times_Y X \to X$. □

7. Lecture VII: Topological spaces

Let us come back to the first example, of locally compact Hausdorff spaces. So C is the category of locally compact Hausdorff spaces; equivalently, these are those topological spaces that can be written as open subsets of compact Hausdorff spaces (for example, their one-point compactification). We take E to be the class of all morphisms.

In general, there are several slightly different possible definitions of the derived (∞-)category of abelian sheaves $D(X,\mathbb{Z})$ on X.

(1) One can literally take the derived ∞-category $D(\mathrm{Ab}(X))$ of the abelian category of abelian sheaves on X. This is obtained from the category of chain complexes by inverting weak equivalences (in the ∞-categorical sense).
(2) One can look at the ∞-category of sheaves $\mathrm{Shv}(X, D(\mathbb{Z}))$ (in the sense of Lurie [**Lur09**]) on X with values in $D(\mathbb{Z})$. This is the ∞-category of contravariant functors $\mathcal{F}$ from open subsets of X towards $D(\mathbb{Z})$, such that $\mathcal{F}(\emptyset) = *$ and the following two conditions are satisfied. First, if $U = U_1 \cup U_2$, then $\mathcal{F}(U) \to \mathcal{F}(U_1) \times_{\mathcal{F}(U_1\cap U_2)} \mathcal{F}(U_2)$ is an isomorphism. Second, if U is a filtered union of subsets $U_i \subset U$, then $\mathcal{F}(U) \to \lim_i \mathcal{F}(U_i)$ is an isomorphism (where all limits are taken in the ∞-category $D(\mathbb{Z})$).
(3) The ∞-category $\mathrm{HypShv}(X, D(\mathbb{Z}))$ of hypersheaves on X with values in $D(\mathbb{Z})$, which is the localization of $\mathrm{Shv}(X, D(\mathbb{Z}))$ at the morphisms that induce isomorphisms on all stalks.
(4) The versions bounded to the left $D^{\geq n}(\mathrm{Ab}(X))$ and $\mathrm{Shv}(X, D^{\geq n}(\mathbb{Z})) = \mathrm{HypShv}(X, D^{\geq n}(\mathbb{Z}))$ (as being bounded to the left implies being hypercomplete), and the left-completions

$$\hat{D}(\mathrm{Ab}(X)) = \lim_n D^{\geq n}(\mathrm{Ab}(X)), \widehat{\mathrm{Shv}}(X, D(\mathbb{Z})) = \lim_n \mathrm{Shv}(X, D^{\geq n}(\mathbb{Z})).$$

Let us compare the various possibilities.

Proposition 7.1. *The composite functor* $\mathrm{Ch}(\mathrm{Ab}(X)) \to \mathrm{Shv}(X, D(\mathbb{Z})) \to \mathrm{HypShv}(X, D(\mathbb{Z}))$ *factors over* $D(\mathrm{Ab}(X))$ *and induces a t-exact equivalence*

$$D(\mathrm{Ab}(X)) \cong \mathrm{HypShv}(X, D(\mathbb{Z})).$$

In particular, one gets an equivalence of left-completions

$$\hat{D}(\mathrm{Ab}(X)) \cong \widehat{\mathrm{Shv}}(X, D(\mathbb{Z})).$$

If X is paracompact and has finite covering dimension, then the functors

$$\mathrm{Shv}(X, D(\mathbb{Z})) \to \mathrm{HypShv}(X, D(\mathbb{Z})) \to \widehat{\mathrm{Shv}}(X, D(\mathbb{Z}))$$

are equivalences.

Remark 7.2. In general, the three variants

$$\mathrm{Shv}(X, D(\mathbb{Z})) \to \mathrm{HypShv}(X, D(\mathbb{Z})) \to \widehat{\mathrm{Shv}}(X, D(\mathbb{Z}))$$

are different. The latter two agree when X has "locally finite cohomological dimension". Hypersheaves always form a Bousfield localization of sheaves, but the left-completion in general does not. The functor $\mathrm{HypShv} \to \widehat{\mathrm{Shv}}$ (given by taking $\mathcal{F}$ to the system $(\tau_{\leq n}\mathcal{F})_n$) has a right adjoint, taking a truncation-compatible sequence $\mathcal{F}_n$ to $\mathcal{F} = \lim_n \mathcal{F}_n$. But in general the map $\tau_{\leq n}\mathcal{F} \to \mathcal{F}_n$ is not an isomorphism. For example, if $\mathcal{F}_n = \prod_{i=1}^n \mathbb{Z}[i]$, then $\mathcal{F} = \prod_{i=1}^\infty \mathbb{Z}[i]$ and $\pi_0\mathcal{F}$ is the sheafification of $U \mapsto \prod_{i=1}^\infty H^i(U, \mathbb{Z})$, which is in general nontrivial.

Proof. A map in $\mathrm{HypShv}(X, D(\mathbb{Z}))$ is an isomorphism if and only if it induces an isomorphism on all stalks. Indeed, passing to cones, it suffices to see that a hypercomplete sheaf $\mathcal{F}$ on X is trivial as soon as all of its stalks vanish. But if all stalks vanish, then in particular all sheaves $\pi_i\mathcal{F}$ vanish, and hence $\mathcal{F}$ is ∞-connective, and thus trivial if hypercomplete.

Thus, to show that $\mathrm{Ch}(\mathrm{Ab}(X)) \to \mathrm{HypShv}(X, D(\mathbb{Z}))$ factors over $D(\mathrm{Ab}(X))$, it suffices to see that quasi-isomorphisms induce isomorphisms on stalks (valued in $D(\mathbb{Z})$). But this is clear, as the stalks are unchanged under the sheafification implicit in the functor $\mathrm{Ch}(\mathrm{Ab}(X)) \to \mathrm{HypShv}(X, D(\mathbb{Z}))$: The functor $\mathrm{Ch}(\mathrm{Ab}(X)) \to \mathrm{HypShv}(X, D(\mathbb{Z}))$ is the composite of the functor to presheaves, taking any open $U \subset X$ to the value at U of the complex of sheaves, and hypersheafification.

When restricted to K-injective complexes, it turns out that the hypersheafification step is unnecessary. In other words, if $U \subset X$ has a hypercover $U_\bullet \to U$, and $C \in \mathrm{Ch}(\mathrm{Ab}(X))$ is K-injective, then the map

$$C(U) \to \lim C(U^\bullet)$$

in $D(\mathbb{Z})$ is an isomorphism. This follows by noting that the hypercover induces a resolution of $j_!\mathbb{Z}$ (where $j : U \subset X$ is the open immersion), and taking the associated $R\mathrm{Hom}$ into C (which vanishes by assumption that C is K-injective).

This implies that for any $j : U \subset X$ and any $C \in D(\mathrm{Ab}(X))$, the map

$$\mathrm{Hom}_{D(\mathrm{Ab}(X))}(j_!\mathbb{Z}, C) \to \mathrm{Hom}_{\mathrm{HypShv}(X,D(\mathbb{Z}))}(j_!\mathbb{Z}, \tilde{C})$$

is an isomorphism (where $\tilde{C} \in \mathrm{HypShv}(X, D(\mathbb{Z}))$ is the associated hypersheaf). Indeed, we can take a K-injective representative for C, and use that morphisms in $D(\mathrm{Ab}(X))$ can be computed via K-injective resolutions, as well as the previous paragraph.

As the objects $j_!\mathbb{Z}$ generate $D(\mathrm{Ab}(X))$ as well as $\mathrm{HypShv}(X, D(\mathbb{Z}))$ under colimits (and shifts), this proves the desired equivalence. Moreover, for all i, the i-th homology functor on $D(\mathrm{Ab}(X))$ corresponds under this equivalence to π_i on hypersheaves – on K-injective representatives, this is even true on the level of presheaves of abelian groups, and thus after sheafification. In particular, the equivalence is t-exact.

The equivalence on left-completions is a corollary. For the final statement, see [**Lur09**, Theorem 7.2.3.6, Proposition 7.2.1.10, Corollary 7.2.1.12]. □

In [**Sch17**], the author preferred left-completed versions. The reason are the very strong descent properties of $X \mapsto \widehat{\mathrm{Shv}}(X, D(\mathbb{Z}))$.

Proposition 7.3. *Endow C with the Grothendieck topology where a family of maps $f_i : X_i \to X$, $i \in I$, form a cover if for any compact subset $K \subset X$ there is some finite index set $J \subset I$ and compact subsets $K_j \subset X_j$ for $j \in J$ such that $K = \bigcup_j f_j(K_j)$.*

Then $X \mapsto \widehat{\mathrm{Shv}}(X, D(\mathbb{Z}))$ defines a hypersheaf of ∞-categories on C, which agrees with the sheafification of $X \mapsto \mathrm{Shv}(X, D(\mathbb{Z}))$.

Proof. As $\widehat{\mathrm{Shv}} = \lim_n \mathrm{Shv}(X, D^{\geq n}(\mathbb{Z}))$, it suffices to prove that $\mathrm{Shv}(X, D^{\geq n}(\mathbb{Z}))$ defines a hypersheaf of ∞-categories for all n; and by shifting we can assume $n = 0$. Let us write $D^{\geq 0}(X, \mathbb{Z}) = \mathrm{Shv}(X, D^{\geq 0}(\mathbb{Z}))$, recalling that in this case all possible definitions agree.

Let $f_\bullet : X_\bullet \to X$ be a hypercover of X. One reduces easily to the case that X and all X_n are compact. We get the pullback functor

$$f_\bullet^* : D^{\geq 0}(X, \mathbb{Z}) \to \lim D^{\geq 0}(X_\bullet, \mathbb{Z})$$

which has a right adjoint $f_{\bullet *}$ given by a totalization of the pushforwards along $f_n : X_n \to X$. The formation of these functors commutes with any base change in X, by proper base change (Theorem 7.6 below) and noting that the totalization is a finite limit in any bounded range of degrees (using critically that all sheaves lie in $D^{\geq 0}$). Thus, to check that the unit and counit transformations are equivalences, we can base change to points of X, where the hypercover splits and the desired descent is automatic.

For the last part, it suffices to show that the functor $\mathrm{Shv}(X, D(\mathbb{Z})) \to \widehat{\mathrm{Shv}}(X, D(\mathbb{Z}))$ is an equivalence locally in the Grothendieck topology. But any X admits a cover by a disjoint union of profinite sets (which have covering dimension 0), where it is an equivalence by Proposition 7.1. (Note that to get an equivalence of sheafifications, one should check more precisely that this also gives an equivalence on all terms of the corresponding Cech cover; but the terms there are all closed inside such disjoint unions of profinite sets, and thus themselves disjoint unions of profinite sets.) □

Now we state the theorem on the existence of a 6-functor formalism. Note that taking a topos to its ∞-category of $D(\mathbb{Z})$-valued sheaves defines a functor to symmetric monoidal ∞-categories, so we have

$$C^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty) : X \mapsto \mathrm{Shv}(X, D(\mathbb{Z})).$$

For the classes I and P, we take the open immersions and the proper maps, respectively.

THEOREM 7.4. *In this situation, the conditions of Theorem 4.6 are satisfied, and one gets a resulting* 6*-functor formalism*

$$X \mapsto D(X) := \mathrm{Shv}(X, D(\mathbb{Z})).$$

REMARK 7.5. We work with $D(\mathbb{Z})$-coefficients here, but everything works in a similar way with coefficients in the ∞-category of spectra (or other coefficients). See Volpe's thesis [**Vol21**] for a much more detailed and general discussion. Moreover, Kashiwara–Shapira have done a lot of work on topological 6-functor formalisms, see for example [**KS90**].

PROOF. Condition (1) on the morphisms in I and P is easy to check. Note that if $f : X \to Y$ is any map in C, then by choosing a compactification $\overline{X}_0$ and and letting $\overline{X} \subset \overline{X}_0 \times Y$ be the closure of the graph of f, one gets factorization of f into an open immersion $j : X \to \overline{X}$ and a proper map $\overline{f} : \overline{X} \to Y$.

Condition (2) is essentially automatic. Roughly speaking, whenever $D(X)$ is some category of sheaves on X with respect to a topology for which the morphisms $j \in I$ are open immersions, the pullback functors j^* automatically acquire left adjoints $j_!$ having good properties.

Condition (3) is the hardest work and stated separately as Theorem 7.6 below. Condition (4) is then actually automatic by Remark 4.2. □

THEOREM 7.6 (Proper Base Change). *Let $f : X \to Y$ be a proper map of locally compact Hausdorff spaces.*

(1) *The functor $f_* : D(X) \to D(Y)$ commutes with all colimits.*
(2) *For $A \in D(X)$ and $B \in D(Y)$, the projection formula map*

$$f_*A \otimes B \to f_*(A \otimes f^*B)$$

is an isomorphism.

(3) *For any other map $g : Y' \to Y$ of locally compact Hausdorff spaces and pullback $f' : X' = X \times_Y Y' \to Y'$, $g' : X' \to X$, the base change map*
$$g^* f_* \to f'_* g'^* : D(X) \to D(Y')$$
is an isomorphism.

Proof. Choosing compactifications, we can assume that Y (and hence X) is compact Hausdorff. For part (1), it is convenient to give a different description of $D(X) = \mathrm{Shv}(X, D(\mathbb{Z}))$. Given $\mathcal{F} \in D(X)$, to any closed subset $K \subset X$ one can associate $\mathcal{F}(K)$ which are the global sections of the pullback to K; equivalently,
$$\mathcal{F}(K) = \mathrm{colim}_{U \supset K} \mathcal{F}(U)$$
is the colimit over all open neighborhoods U of K. Then $D(X)$ is equivalent to the ∞-category of contravariant functors $\mathcal{F}$ taking closed subsets K of X to $D(\mathbb{Z})$, with the properties that if $K = K_1 \cup K_2$, then
$$\mathcal{F}(K) \to \mathcal{F}(K_1) \times_{\mathcal{F}(K_1 \cap K_2)} \mathcal{F}(K_2)$$
is an isomorphism, and if $K = \bigcap_i K_i$ is a cofiltered intersection, then
$$\mathcal{F}(K) = \mathrm{colim}_i \mathcal{F}(K_i).$$
Indeed, one can recover $\mathcal{F}(U) = \lim_{K \subset U} \mathcal{F}(K)$, noting that relatively compact open subsets are cofinal with relatively compact closed subsets. See also [**Lur09**, Corollary 7.3.4.10].

In this alternative description, the pushforward is given by $(f_*\mathcal{F})(K) = \mathcal{F}(f^{-1}(K))$, noting that this does in fact satisfy all required conditions. Moreover, arbitrary colimits in $\mathcal{F}$ also respect the conditions, yielding (1).

Now for (2), as all functors commute with colimits, it suffices to check the result on generators, so we can assume $B = j_! \mathbb{Z}$ for an open immersion $j : Y' \to Y$. The result is clearly true over Y' (as base change to open subsets always holds true), so we need to show that $f_*(A \otimes f^* j_! \mathbb{Z})$ vanishes on the closed complement $Z \subset Y$ of Y'. But $j_! \mathbb{Z}$ can be written as a colimit of $j_{i!} \mathbb{Z}$ for open immersions $Y'_i \to Y$ such that Z admits an open neighborhood Z_i that does not meet Y'_i. This implies that $f_*(A \otimes f^* j_{i!} \mathbb{Z})$ vanishes upon restriction to Z_i, and in particular to Z; and hence the colimit also vanishes on Z.

For (3), we can also assume that Y' is compact. If g is injective (i.e., a closed immersion), then the result is part of the description of pushforward given in (1). In general, to prove the result, it suffices to check that the sections over all compact subsets of Y' agree. Replacing Y' by such a compact subset, we see in particular that it suffices to prove the result after applying g_*. Now
$$g_* g^* f_* \cong f_* \otimes g_* \mathbb{Z}$$
by the projection formula (i.e., part (2)) for g; while
$$g_* f'_* g'^* \cong f_* g'_* g'^* \cong f_*(- \otimes g'_* \mathbb{Z})$$
by the projection formula for g'. Finally, the projection formula for f reduces us to showing
$$g'_* \mathbb{Z} \cong f^* g_* \mathbb{Z},$$
i.e., up to switching the factors, the isomorphism of (3) restricted to the constant sheaf. But now $\mathbb{Z} \in D^{\geq 0}$, so to check this isomorphism, we can check on stalks. This means we can reduce to the case that Y' is a point, which we have already handled. $\square$

Finally, we discuss a few things regarding cohomological smoothness, and f-proper and f-smooth objects. The following proposition was observed before:

Proposition 7.7. *Let $f : X \to Y$ be a topological manifold bundle. Then f is cohomologically smooth.*

Proof. Cohomological smoothness can be checked locally on the source, and is stable under pullback and passage to open subsets. This reduces the question to $\mathbb{R} \to *$, which we handled directly in Proposition 5.10. $\square$

Conversely, one can ask the following question. Say X is some compact Hausdorff space. When is $f : X \to *$ cohomologically smooth? Recall that cohomological smoothness is equivalent to $\mathbb{Z}$ being f-smooth, with invertible dualizing sheaf. The first condition is actually satisfied in large generality.

Proposition 7.8. *Let X be a locally compact Hausdorff space that is a euclidean neighborhood retract, i.e. a retract of an open subset of $\mathbb{R}^n$ for some n. Then $\mathbb{Z}$ is f-smooth for $f : X \to *$.*

Proof. This is a consequence of the stability under retracts, see Proposition 6.17. $\square$

In particular, if X is a euclidean neighborhood retract, then $f : X \to *$ is cohomologically smooth if and only if the dualizing complex $f^!\mathbb{Z}$ is invertible. This can actually be tested pointwise. Note first that if X is a euclidean neighborhood retract, then the homology of X is given by $f_! f^! \mathbb{Z}$ (where $f : X \to *$ denotes the projection). Indeed, this is true for topological manifolds, and then passes to retracts. (Also note that the dual of this is given by cohomology $f_*\mathscr{H}\mathrm{om}(f^!\mathbb{Z}, f^!\mathbb{Z}) = f_*\mathbb{Z}$ (using Proposition 6.5 (3)).) It follows that the stalk $(f^!\mathbb{Z})_x$ at a point $x \in X$ is given by the relative homology of the pair $(X, X \setminus \{x\})$.

Proposition 7.9. *Let X be a euclidean neighborhood retract. Assume that there is some d such that for all $x \in X$, the relative homology group $H_i(X, X \setminus \{x\}; \mathbb{Z})$ vanishes for $i \neq d$, and is given by $\mathbb{Z}$ for $i = d$. Then $f^!\mathbb{Z}$ is locally isomorphic to $\mathbb{Z}[d]$, and hence invertible.*

Proof. The assumption ensures that $f^!\mathbb{Z} = \mathcal{F}[d]$ for some sheaf of abelian groups $\mathcal{F}$ all of whose stalks are isomorphic to $\mathbb{Z}$. Moreover, Proposition 6.5 (3) ensures that (the derived) $\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{F}) = \mathbb{Z}$. We will show that these properties force $\mathcal{F}$ to be locally isomorphic to $\mathbb{Z}$. Take any $x \in X$ and a local section $s \in \mathcal{F}(U)$ that is a generator at the stalk at some $x \in U \subset X$. We can assume that U is connected (as X is locally connected, as a euclidean neighborhood retract); our goal is to show that s induces an isomorphism $\mathbb{Z} \to \mathcal{F}|_U$. We can replace X by U. Moreover, it suffices to see that it induces an isomorphism modulo p for any prime p. The image of the map $\mathbb{F}_p \to \mathcal{F}/p$ is given by some quotient of the constant sheaf $\mathbb{F}_p$, and hence is of the form $i_*\mathbb{F}_p$ for some closed immersion $i : Z \to X$. We get an exact sequence

$$0 \to i_*\mathbb{F}_p \to \mathcal{F}/p \to j_! j^* \mathcal{F}/p \to 0;$$

note that indeed the quotient is of the form $j_!\mathcal{F}'$ for some sheaf $\mathcal{F}'$, which must be the restriction of $\mathcal{F}/p$ to U. But the projection map splits naturally, so $i_*\mathbb{F}_p$ is a summand of $\mathcal{F}/p$. But the endomorphisms of $\mathcal{F}/p$ are just $\mathbb{F}_p$, so in particular it is irreducible, and thus $i_*\mathbb{F}_p \to \mathcal{F}/p$ is an isomorphism, so that $Z = X$ and $\mathcal{F}/p = \mathbb{F}_p$. $\square$

Such X are classically known as (ENR) homology manifolds, and they have been intensely studied. Let me highlight some of their fascinating theory.

(1) If X is any (connected) homology manifold, one can define an invariant $I(X) \in 1 + 8\mathbb{Z}$ known as the Quinn index, [**Qui87**]. (The strange value group $1 + 8\mathbb{Z}$ is justified by the multiplicativity property $I(X \times Y) = I(X)I(Y)$.)
(2) If X is a topological manifold, then $I(X) = 1$.
(3) Conversely, if $I(X) = 1$ and satisfies the "disjoint disks property" (any two maps $D^2 \to X$ admit small perturbations that are disjoint), then X is a topological manifold.
(4) A theorem of Bryant–Ferry–Mio–Weinberger [**BFMW96**] states that (in dimension ≥ 6) there are homology manifolds with any given Quinn index.

Unfortunately, the construction of homology manifolds with $I(X) \neq 1$ is extremely indirect, relying on a lot of (high-dimensional) surgery theory. It is an open problem to give explicit constructions of homology manifolds, and whether there are local "model spaces" like euclidean space.

A consequence of Proposition 7.8 is that "constructible" sheaves are f-smooth.

Proposition 7.10. *Let X be a euclidean neighborhood retract, and let $A \in D(X)$ be such that there is a locally finite stratification of X into locally closed subsets $X_i \subset X$ whose closures $\overline{X}_i \subset X$ are neighborhood retracts, and such that $A|_{X_i}$ is a constant sheaf on a perfect complex of abelian groups. Then A is f-smooth for $f : X \to *$.*

We assume here that a closure of a stratum is a union of strata.

Proof. The condition of being f-smooth is local on X, and stable under triangles. Now A is locally in the stable subcategory generated by the pushforwards of perfect complexes under the closed immersion $p_i : \overline{X}_i \to X$. (This uses that the closure of a stratum is a union of strata.) Now $\overline{X}_i$ is a euclidean neighborhood retract, hence the constant sheaf and therefore any perfect complex is f_i-smooth for $f_i : \overline{X}_i \to *$; and proper pushforwards preserve this property. □

It seems hard to give a complete characterization of the f-smooth objects. In the f-proper case, a complete characterization can be given.

Proposition 7.11. *Let X be a locally compact Hausdorff space and $A \in D(X)$, and denote $f : X \to *$. Then A is f-proper if and only if A is compact if and only if A is locally isomorphic to the constant sheaf on a perfect complex and supported on a compact subset of X.*

Thus, this class of sheaves is very small, which also explains why that notion has not been much studied before (in contrast to f-smooth objects). In fact, even the relative setting of a map $f : X \to Y$ can be analyzed completely, and f-proper objects are exactly those that are locally isomorphic to the constant sheaf on a perfect complex, and supported on a subset of X that is compact over Y. Indeed, by base change compatibility, one can assume that Y is compact. Then Proposition 6.16 ensures that any f-proper A must be compact, and Proposition 7.11 applies.

Proof. By Proposition 6.16, if A is f-proper, then A is compact. The hard part is to show that A being compact implies that A is locally isomorphic to the constant sheaf on a perfect complex and supported on a compact subset of X. Indeed, such objects are easily seen to be f-proper.

Thus, let A be compact. It is enough to show that A is dualizable: Indeed, the dualizable objects are known to be locally isomorphism to the constant sheaf on a perfect complex, and the support must also be compact as A is compact. (One way to show that any dualizable A must be locally constant is to show this first on profinite sets, where sheaves are equivalent to modules over $C(X, \mathbb{Z})$, and then descend.)

As the support of A must be compact, we can assume that X is compact. Then $D(X)$ is rigid dualizable as a presentable symmetric monoidal stable ∞-category, and in such categories dualizable and compact objects agree. Let us explain one direct way of arguing: One way to characterize "rigid dualizable" is that the unit is compact, it is ω_1-compactly generated, and that any ω_1-compact B can be written as a sequential colimit of objects B_n such that for all other objects C, the natural map

$$\mathscr{H}\mathrm{om}(C, \mathbb{Z}) \otimes B \to \mathrm{colim}_n \mathscr{H}\mathrm{om}(C, B_n)$$

is an isomorphism. This is in fact not hard to see in our case; the ω_1-compact generators can be taken as $B = j_! \mathbb{Z}$ for open immersions $j : U \to X$ that are sequential unions of open subsets $j_n : U_n \to X$ along embeddings $\overline{U}_n \subset U_{n+1}$; then $B_n = j_{n!}\mathbb{Z}$ works. Applied to a compact B, the sequential colimit of B_n's must necessarily be pro-constant, and then taking $C = B$ shows that B is dualizable. $\square$

Appendix to Lecture VII: Étale Sheaves

Étale sheaves on schemes behave in many ways quite similar to the case of abelian sheaves on locally compact Hausdorff spaces, at least with torsion coefficients (prime to the characteristic). Let us recall these results.

For the category C, we take qcqs schemes. Most naturally, one would take for E the separated maps of finite type, but actually one can be a tiny bit more general, following Hamacher [**Ham19**]: we take for E the class of separated morphisms $f : X \to Y$ of "finite expansion", i.e. such that on open affines $\mathrm{Spec}(A) \subset X$, mapping to $\mathrm{Spec}(B) \subset Y$, there is a finite set of elements $X_1, \ldots, X_n \in A$ such that the map $B[X_1, \ldots, X_n] \to A$ is integral. (In particular, this includes separated maps of finite type, but it is slightly more general, and in particular it allows perfections of maps of finite type. It also includes things like pro-(finite étale) maps.) As I, we take the open immersions, while for P we take the proper maps in E, i.e. the ones that satisfy the valuative criterion of properness (equivalently, are universally closed).

The Nagata compactification theorem extends to this setting; this ensures that the condition on morphisms in Theorem 4.6 is satisfied.

Proposition 7.12 ([**Ham19**, Proposition 1.8, Theorem 1.17]). *Let $f : X \to Y$ be a morphism in E.*

1. *The morphism f can be written as the composite of an integral map $X \to X'$ and a separated map of finite presentation $X' \to Y$.*
2. *If $f \in P$, then f can be written as the composite of an integral map $X \to X'$ and a proper map of finite presentation $X' \to Y$.*
3. *The morphism f can be written as the composite of an open immersion $X \hookrightarrow \overline{X}$ and a morphism $\overline{X} \to Y$ in P.*

Following the discussion for topological spaces, one can again consider several slightly different versions of the derived category of étale sheaves; effectively,

1. sheaves $\mathrm{Shv}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z}))$ in the sense of Lurie;
2. hypersheaves $\mathrm{HypShv}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z}))$; this agrees with the derived category of the abelian category of étale sheaves of Λ-modules;
3. the left-completion $\widehat{\mathrm{Shv}}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z}))$, which agrees with the left-completion of hypersheaves.

Under mild assumptions, all three notions agree by the following theorem of Clausen–Mathew.

Theorem 7.13 ([**CM21**, Corollary 1.10, Corollary 4.40]). *Let X be a qcqs scheme of finite Krull dimension that has a uniform bound on the virtual cohomological dimension of its residue fields; e.g., X is of finite type over $\mathbb{Z}$ or an algebraically closed field. Then*

$$\mathrm{Shv}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z})) = \mathrm{HypShv}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z})) = \widehat{\mathrm{Shv}}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z})).$$

Most sources in the literature either work with D^+, i.e. sheaves bounded to the left (as in the original SGA setting), or with left completions (for example this is what we did in our own work [**Sch17**]). This has the advantage of guaranteeing extremely strong descent theorems, most generally v-descent or even descent for universal submersions, see [**HS21**, Theorem 5.7]. On the other hand, considering all sheaves has the advantage that one can reduce to schemes of finite type:

Proposition 7.14. *The contravariant functor $X \mapsto \mathrm{Shv}(X_{\mathrm{\acute{e}t}}, D(\mathbb{Z}))$ from qcqs schemes to presentable stable ∞-categories takes cofiltered limits $X = \lim_i X_i$ along affine transition maps to* *filtered colimits. Equivalently, cf. Lemma 7.17 (1), the functor of ∞-categories*

$$\mathrm{Shv}(X_{\text{ét}}, D(\mathbb{Z})) \to \lim_i \mathrm{Shv}(X_{i,\text{ét}}, D(\mathbb{Z})),$$

induced by pushforward along the maps $X \to X_i$, is an equivalence.

Proof. This follows easily from the presentation of $\mathrm{Shv}(X_{\text{ét}}, D(\mathbb{Z}))$: It is generated by the free abelian sheaves on quasicompact separated étale maps to X, and for any étale cover between such, one gets a corresponding relation given by Cech descent. The generators and relations both satisfy noetherian approximation. □

In the case of schemes, some finiteness conditions on the morphisms have to be enforced in any case (as there is no scheme-theoretic compactification of infinite-dimensional affine space), and it turns out that proper base change holds for any choice. But as usual in the étale setting, we have to restrict to torsion sheaves, or at least profinitely complete sheaves. Thus, we take coefficients in the full subcategory

$$\widehat{D}_{\mathrm{pf}}(\mathbb{Z}) \subset D(\mathbb{Z})$$

of profinitely complete complexes, i.e. all A such that $A \to \lim_n A/^L n$ is an isomorphism. This is also a Verdier quotient of $D(\mathbb{Z})$ by $D(\mathbb{Q})$, and acquires a symmetric monoidal structure. It is also equivalent to the full subcategory $D_{\mathrm{tor}}(\mathbb{Z}) \subset D(\mathbb{Z})$ of torsion complexes.

Theorem 7.15. *Any of the functors*

$$X \mapsto \mathrm{Shv}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z})), X \mapsto \mathrm{HypShv}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z})), X \mapsto \widehat{\mathrm{Shv}}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z}))$$

satisfies the hypotheses of Theorem 4.6 with respect to the classes I and P, and hence defines a 6-functor formalism. Moreover, the natural functors

$$\mathrm{Shv}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z})) \to \mathrm{HypShv}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z})) \to \widehat{\mathrm{Shv}}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z}))$$

commute with the respective $\otimes$, f^* and $f_!$ functors.

Proof. We already discussed the conditions (1) on the classes of morphisms I and P. For condition (2) regarding open immersions j, one has indeed the left adjoint $j_!$ of extension by zero, which satisfies base change and projection formula formally. Condition (3) is Theorem 7.16 below. Finally, for (4), we note that if $j : U \to X$ is an open immersion with closed complement $i : Z \to X$, then one gets an excision triangle

$$j_! j^* A \to A \to i_* i^* A$$

for any $A \in D(X)$ (for any choice of $D(X)$). Indeed, the projection formula and commutation with colimits reduce to the image of $\mathbb{Z}$ in profinitely complete sheaves (which is also the colimit of $1/n\mathbb{Z}/\mathbb{Z}[-1]$), where all objects are bounded to the left, and the exactness can be checked after pullback to geometric points, where it is evident. In particular Remark 4.2 applies to establish (4).

The functors between the different theories are by definition compatible with $\otimes$ and f^*, and it is easy to see that they commute with $j_!$ for open immersions, while the proof of Theorem 7.16 shows that proper pushforwards also commute with these functors. □

Again, the key input is the proper base change theorem.

Theorem 7.16 (Proper Base Change). *Let $f : X \to Y$ be a universally closed separated map of finite expansion. Let $D(X)$ denote any of three options in Theorem 7.15.*

(1) *The functor $f_* : D(X) \to D(Y)$ commutes with all colimits.*

(2) *For $A \in D(X)$ and $B \in D(Y)$, the projection formula map*

$$f_*A \otimes B \to f_*(A \otimes f^*B)$$

is an isomorphism.

(3) *For any other map $g : Y' \to Y$ of schemes and pullback $f' : X' = X \times_Y Y' \to Y'$, $g' : X' \to X$, the base change map*

$$g^*f_* \to f'_*g'^* : D(X) \to D(Y')$$

is an isomorphism.

Proof. We first handle the case that $D(X) = \mathrm{Shv}(X_{\text{ét}}, \widehat{D}_{\mathrm{pf}}(\mathbb{Z}))$. We can write f as the composite of an integral map and a finitely presented proper map, and it suffices to handle both cases separately. By Lemma 7.17 (1), one reduces to the case that f is finitely presented proper. Then f arises via base change from a similar map where Y is of finite type over $\mathbb{Z}$, and Lemma 7.17 (2) reduces us to the case that Y is of finite type. Then all possible notions of $D(X)$ agree by Theorem 7.13. Moreover, f_* has finite cohomological dimension (bounded by twice the dimension of the geometric fibres of f), and one can reduce to $D^+(X)$ by Postnikov limits, or then to sheaves of abelian groups. Part (1) is then a general consequence of the étale sites of X and Y being coherent. Part (2) can be checked on geometric fibres, where it reduces to part (3) and part (1) (noting that $D(Y)$ is generated by the unit when Y is a geometric point). Part (3) is then the usual proper base change theorem in étale cohomology.

In the case of hypercomplete sheaves, it suffices to prove all isomorphisms after pullbacks to geometric points. After base change to geometric points, sheaves and hypersheaves agree, and the assertions reduce to the case of sheaves. For example, for part (1), take any collection $A_i \in \mathrm{Shv}(X, D(\mathbb{Z}))$, $i \in I$, of sheaves, and assume that all A_i are already hypersheaves. Let $\bigoplus_i A_i$ denote the direct sum as sheaves, and $\hat{\bigoplus}_i A_i$ its hypercompletion. We want to know whether the map

$$\hat{\bigoplus_i} f_*A_i \to f_*(\hat{\bigoplus_i} A_i)$$

is an isomorphism, where we note that f_* preserves hypersheaves. To check this, take the base change to any geometric point $g : \overline{y} \to Y$. Note that base change taken in hypersheaves commutes with direct sums taken in hypersheaves. Also, pullback commutes with pushforwards as taken in sheaves (by the case of sheaves already established), but sheaves and hypersheaves agree after pullback. In the end, this reduces the displayed isomorphism to the case where Y is a geometric point, where it was already established.

In the case of left-completed sheaves, one uses that f_* has bounded cohomological dimension to show that it commutes with Postnikov limits, and then any question can be reduced to the case of bounded sheaves by passing to suitable truncations. □

Lemma 7.17. *Let I be an ∞-category.*

(1) *Let $i \mapsto D_i$ be a covariant functor to presentable ∞-categories, so that all transition functors $f_{ij} : D_i \to D_j$ have right adjoints $g_{ij} : D_j \to D_i$. Then the colimit $D = \mathrm{colim}_i D_i$ in presentable ∞-categories is given by the limit $\lim_i D_i$ in ∞-categories, where the limit is taken along those right adjoint functors.*

If I is filtered and all g_{ij} commute with filtered colimits, then the induced endofunctors $D_i \to D \to D_i$ are given by the colimit over $j \geq i$ of the composite functors $D_i \xrightarrow{f_{ij}} D_j \xrightarrow{g_{ij}} D_i$.

(2) *Let* $i \mapsto (f_i : C_i \to D_i)$ *be a covariant functor to the* ∞*-category of maps of presentable* ∞*-categories. Assume that* f_i *admits a colimit-preserving right adjoint* $g_i : D_i \to C_i$ *that commutes with the transition functor* $C_i \to C_j$, $D_i \to D_j$. *Then the functor* $f : C = \mathrm{colim}_i C_i \to \mathrm{colim}_i D_i$ *given by the colimit of the* f_i *admits the right adjoint* $g : C = \mathrm{colim}_i C_i \to \mathrm{colim}_i D_i$ *given as the colimit of the* g_i.

Proof. The first assertion of (1) is the general description of colimits of presentable ∞-categories, see [**Lur09**, Theorem 5.5.3.18, Corollary 5.5.3.4]. The other assertion is also standard, but we do not know the correct reference, so we sketch the argument. Namely, one shows that the left adjoint to $D = \lim_j D_j \to D_i$ is the functor $D_i \to D = \lim_j D_j$ taking X_i to $\mathrm{colim}_{j' \geq i,j} g_{j'j}(f_{ij}(X_i))$, noting that this indeed lands in $\lim_j D_j$, and that one can easily write down the unit and counit of the adjunction.

Part (2) again follows by noting that one can write down the unit and counit of the adjunction between f and g, as the colimit of the ones for f_i and g_i. □

Let us also prove Poincaré duality in this setting (modulo identifying the dualizing complex).

Proposition 7.18. *In any of the* 6*-functor formalisms from Theorem 7.15, all étale morphisms of schemes are cohomologically étale.*

Proof. By construction, open immersions are cohomologically étale. If $f : X \to Y$ is étale, then its diagonal is an open immersion, so Δ_f is cohomologically étale. As all categories satisfy étale descent, checking whether f is cohomologically étale is an étale local question. We can thus base change f along itself, yielding $X \times_Y X \to X$. This decomposes into two components, one of which is an isomorphism (the diagonal $X \hookrightarrow X \times_Y X \to X$). By induction on the maximal cardinality of a geometric fibre of f, we can thus prove the desired result. □

We note that this implies that $f_!$ is canonically a left adjoint of f^* when f is étale, even if we did not make this part of the definition.

Theorem 7.19. *Let* $f : X \to Y$ *be a smooth morphism of schemes. Assume that a prime* ℓ *is invertible on* Y. *Then* f *is cohomologically smooth in the* ℓ*-completions of the* 6*-functor formalisms from Theorem 7.15.*

In Zavyalov's paper [**Zav23**], it is explained how one can moreover canonically identify the dualizing complex as a Tate twist (using a deformation to the normal cone).

Proof. The question is étale local on the source, so using Proposition 7.18 one reduces to the case of affine space, and then to the affine line $f : \mathbb{A}^1_{\mathbb{Z}[\frac{1}{\ell}]} \to \mathrm{Spec}\mathbb{Z}[\frac{1}{\ell}]$. We take the sheaf $L = \mathbb{Z}_\ell(1)[2]$ on $\mathbb{A}^1$. One constructs a map

$$\alpha : \Delta_* \mathbb{Z}_\ell \to p_2^* L$$

using the first Chern class of the line bundle $\mathcal{O}(\Delta)$ on $\mathbb{A}^2$. (Indeed, $\mathcal{O}(\Delta)$ defines a $\mathbb{G}_m$-torsor trivialized outside Δ, i.e. a map $\Delta_*\mathbb{Z} \to \mathbb{G}_m[1]$. Passing to derived ℓ-completions gives the desired map α.) It is a standard computation that $f_! L \cong \mathbb{Z}_\ell$, where the isomorphism comes from the first Chern class of the Cartier divisor of a section. Combined, this suffices to construct the desired datum for Theorem 5.5. See Zavyalov's paper [**Zav23**] for a more detailed account. □

In [**HS21**] it is shown how the classical arguments of Deligne on Verdier duality and nearby cycles can be rephrased in terms of f-smooth objects. In particular, f-smooth objects are exactly the "universally locally acyclic" (ULA) objects in the sense of Deligne. Over a geometric point, this coincides with the constructible sheaves, i.e. the compact objects. In particular, Verdier duality is a perfect duality on constructible sheaves on finite type schemes X over an algebraically closed field k.

8. Lecture VIII: Coherent sheaves

Let us now consider a rather different kind of sheaves, namely coherent sheaves. As our category of geometric objects, we would like to take schemes X, and as $D(X)$ the quasicoherent derived category $D_{\mathrm{qc}}(X)$. There are two possible definitions: Either this is the full subcategory of the derived category of sheaves of $\mathcal{O}_X$-modules such that all cohomology sheaves are quasicoherent; or, ∞-categorically, it is defined via descent from affine schemes, i.e.

$$D_{\mathrm{qc}}(X) = \lim_{R,\mathrm{Spec}(R)\to X} D(R).$$

There is of course a version of Poincaré duality in coherent cohomology, which is Grothendieck–Serre duality.

THEOREM 8.1. *Let $f : X \to Y$ be a proper smooth map of schemes of relative dimension d. Then*

$$f_* : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$$

has a right adjoint given by $\Omega^d_{X/Y}[d] \otimes f^*$.

This indicates that proper smooth maps should be cohomologically smooth in the formalism we seek. Moreover, the dualizing object has a description that is local on X; this would most naturally be explained if in fact all smooth morphisms are cohomologically smooth. We will see, however, that in the coherent setting things behave somewhat differently. In the next lecture, we will explain a way to modify D_{qc} so as to recover a 6-functor formalism that feels much closer to the topological ones.

Before getting started, we must note that base change theorems for coherent sheaves tend to have some flatness or Tor-independence assumptions in them. The underlying reason is that when forming fibre products of schemes, on affine pieces one takes the corresponding tensor product of rings, but really this should be a derived tensor product. For this reason, any discussion of a coherent 6-functor formalism has to work with derived schemes.

8.1. Reminders on derived schemes. As our model, we take here the ones modeled on animated commutative rings, but one could also work with (connective) E_∞-rings. This is the model also studied by Toën–Vezzosi [**TV08**]. Recall that the ∞-category of animated commutative rings is freely generated under sifted colimits by polynomial algebras $\mathbb{Z}[X_1, \ldots, X_n]$; equivalently, it is obtained from the category of simplicial commutative rings by inverting weak equivalences. (This procedure of animation is a classical operation, first introduced by Quillen as a non-abelian derived category (but see also the work of Illusie), and ∞-categorically the theory has been written up by Lurie [**Lur09**, Section 5.5.8]. The name animation is due to Clausen, and a general discussion of this procedure in this language is in [**ČS19**, Section 5.1.4].)

DEFINITION 8.2. A derived scheme is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of animated commutative rings $\mathcal{O}_X$ such that $(X, \pi_0\mathcal{O}_X)$ is a scheme, and each $\pi_i\mathcal{O}_X$ is a quasicoherent $\pi_0\mathcal{O}_X$-module. A morphism of derived schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of topological spaces $f : X \to Y$ along with a map $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of animated commutative rings that induces a map of classical schemes upon passing to π_0 (i.e., induces local maps on local rings).

A derived scheme $(X, \mathcal{O}_X)$ is affine if the classical scheme $(X, \pi_0\mathcal{O}_X)$ is affine.

(To define this rigorously as an ∞-category, one first defines an ∞-category of topological spaces equipped with a sheaf of animated commutative rings, and then restricts the objects and the 1-morphisms as above.) Then the ∞-category of affine derived schemes is equivalent to the ∞-category of animated commutative rings, and general derived schemes are glued from affine derived schemes along open covers. Also, there is an alternative definition of derived schemes in terms of their functor of points.

One can define flat maps of derived schemes (e.g., as those where any base change to a classical scheme becomes classical), and then étale resp. smooth maps as those maps that are flat and induce étale resp. smooth maps on the classical truncations.

Thus, we take for C the ∞-category of derived schemes; for simplicity, we restrict to qcqs objects, i.e. $|X|$ is a quasicompact and quasiseparated topological space. For any $X \in C$, one can define the quasicoherent ∞-category $D_{\mathrm{qc}}(X)$, for example via descent (along open covers) from the affine case. This defines a contravariant functor from C to symmetric monoidal presentable stable ∞-categories. We will have occasion to consider some subcategories of $D_{\mathrm{qc}}(X)$.

Theorem 8.3 (Thomason–Trobaugh, [**TT90**]). *The ∞-category $D_{\mathrm{qc}}(X)$ is compactly generated, and the compact objects agree with the dualizable objects, which are also the perfect complexes* $\mathrm{Perf}(X)$*, i.e. on open affine subsets in the subcategory generated by A under direct sums, cones, and retracts.*

Proof. Let us recall the argument briefly. First, if $X = \mathrm{Spec}(A)$ is affine, then A is a compact generator of $D_{\mathrm{qc}}(X) = D(A)$, and thus all compact objects are perfect complexes, and thus dualizable. Conversely, as the unit is compact, all dualizable objects are compact. Now in general one can classify the dualizable objects as the perfect complexes, as dualizable objects satisfy Zariski descent. Moreover, the unit on $D_{\mathrm{qc}}(X)$ is compact, and thus all dualizable objects are compact. It remains to show that the dualizable (i.e., perfect) objects generate $D_{\mathrm{qc}}(X)$. This follows from the following general categorical lemma (applied first to separated X with basis of open affines, and then to general X with basis of separated open subsets), together with the observation that for $X = \mathrm{Spec}(A)$ affine and a constructible closed subset Z of X, given as the vanishing locus of some $f_1, \ldots, f_n \in A$, the subcategory $D_{\mathrm{qc}}(X \text{ on } Z) \subset D_{\mathrm{qc}}(X)$ of those complexes that vanish outside Z is compactly generated by the Koszul complex

$$A/^L(f_1, \ldots, f_n) = A \otimes^L_{\mathbb{Z}[X_1,\ldots,X_n]} \mathbb{Z} \in \mathrm{Perf}(X \text{ on } Z). \qquad \square$$

Lemma 8.4. *Let X be a spectral space equipped with basis B of quasicompact open subsets stable under finite intersections. Consider a sheaf of presentable stable ∞-categories $U \mapsto \mathcal{C}_U$ on X. Assume that for all $U \in B$, the ∞-category $\mathcal{C}_U$ is compactly generated, that for all $U' \subset U$ with $U' \in B$, the functor $\mathcal{C}_U \to \mathcal{C}_{U'}$ is a Bousfield localization that preserves compact objects, with compactly generated kernel.*

Then for all quasicompact open $U \subset X$, the ∞-category $\mathcal{C}_U$ is compactly generated, and for all quasicompact open $U' \subset U$, the functor $\mathcal{C}_U \to \mathcal{C}_{U'}$ is a Bousfield localization that preserves compact objects, and its kernel is compactly generated.

Proof. Note first that $U \mapsto \mathcal{C}_U^\omega \subset \mathcal{C}_U$ is also a sheaf of ∞-categories on B (as an object that is locally compact is in fact compact, by computing Hom's via descent, which is finitary on a spectral space). It thus extends to a sheaf $U \mapsto \mathcal{C}^\omega(U)$ for all quasicompact open $U \subset X$, which comes with a functor

$$\mathrm{Ind}(\mathcal{C}^\omega(U)) \to \mathcal{C}_U$$

which is in fact fully faithful.

One proves that this is an equivalence by induction on B, the starting point being the case that U is a quasicompact open subset of X that can be written as a union $U = U_1 \cup U_2$ with $U_1, U_2 \in B$. To show that the functor is an equivalence, it suffices to show that for any $A \in \mathcal{C}^\omega_{U_1}$ the object $A \oplus A[1]$ lifts to an object of $(\mathcal{C}^\omega)(U)$. This is equivalent to the similar lifting from $U_1 \cap U_2$ to U_2. But by assumption $\mathcal{C}_{U_2} \to \mathcal{C}_{U_1 \cap U_2}$ is a left Bousfield localization of compactly generated presentable stable ∞-categories whose kernel is also compactly generated; thus, the compact objects of $\mathcal{C}_{U_1 \cap U_2}$ are the idempotent completion of the Verdier quotient

$$\mathcal{C}^\omega_{U_2}/\ker(\mathcal{C}^\omega_{U_2} \to \mathcal{C}^\omega_{U_1 \cap U_2}).$$

(Indeed, taking the Verdier quotient on the level of compact objects first and passing to Ind-categories produces the Verdier quotient on the level of the big categories. But the compact objects of the Ind-category give the idempotent completion.) By Proposition 8.5 below we see that the desired lifting is possible for any object with trivial K_0-class, such as $A \oplus A[1]$. The rest of the proof is some routine verification. □

Proposition 8.5 (Thomason–Trobaugh). *Let $\mathcal{C}$ be a small stable ∞-category and let $\mathcal{C}'$ be its idempotent completion. Then $K_0(\mathcal{C})$ injects into $K_0(\mathcal{C}')$, and an object $X \in \mathcal{C}'$ lies in $\mathcal{C} \subset \mathcal{C}'$ if and only if its K_0-class $[X] \in K_0(\mathcal{C}')$ lies in $K_0(\mathcal{C}) \subset K_0(\mathcal{C}')$.*

Here, $K_0(\mathcal{C})$ is generated by elements $[X]$ for all $X \in \mathcal{C}$, subject to $[X] = [X'] + [X'']$ for any distinguished triangle $X' \to X \to X''$ in $\mathcal{C}$.

Proof. Let us first show what we really need, namely that for any $A' \in \mathcal{C}'$, the object $A' \oplus A'[1]$ lies in $\mathcal{C}$. By definition, A' is determined by an object $A \in \mathcal{C}$ with an idempotent endomorphism $e : A \to A$. But then the cone of $A \xrightarrow{1-e} A$ is $A' \oplus A'[1]$, as desired.

This implies that the relations defining $K_0(\mathcal{C}')$ are generated by the relations $[X] + [X'] = [X \oplus X']$, and the relations coming from $K_0(\mathcal{C})$. Indeed, any distinguished triangle $X' \to X \to X''$ can be modified into one of the form $X' \oplus X'[1] \to X \oplus X'[1] \oplus X''[1] \to X'' \oplus X''[1]$ making the first and last term in $\mathcal{C}$, and thus also the middle term. This easily implies that $K_0(\mathcal{C}) \to K_0(\mathcal{C}')$ is injective. Moreover, it shows that for $X \in \mathcal{C}'$, one has $[X] \in K_0(\mathcal{C})$ if and only if there is some $X' \in \mathcal{C}$ such $X \oplus X' \in \mathcal{C}$. But then $X \in \mathcal{C}$ as the cone of $X' \to X \oplus X'$ is. □

Definition 8.6. Let A be an animated commutative ring. An object $K \in D(A)$ is pseudocoherent if for any n there is a perfect complex K_n and a map $K_n \to K$ whose cone sits in homological degrees $\geq n$. An object $K \in D(A)$ is coherent if it is pseudocoherent and bounded.

These notions satisfy Zariski descent and thus globalize to conditions on $K \in D_{\mathrm{qc}}(X)$. On general derived schemes, one always has many perfect and pseudocoherent complexes, but not always coherent ones. One would like to obtain them by truncating pseudocoherent complexes, but this may not preserve pseudocoherent complexes. It is true, however, under mild conditions on the derived scheme $(X, \mathcal{O}_X)$.

Proposition 8.7. *Assume that $(X, \pi_0\mathcal{O}_X)$ is a coherent scheme, i.e. locally the spectrum of a coherent ring, and $\pi_i\mathcal{O}_X$ is a coherent $\pi_0\mathcal{O}_X$-module for all i. Then $A \in D_{\mathrm{qc}}(X)$ is pseudocoherent if and only if all $\pi_i A$ are coherent $\pi_0\mathcal{O}_X$-modules. In particular, any truncation of A is again pseudocoherent.*

Finally, we have the following finiteness condition on maps $A \to B$.

Definition 8.8. A map $f : A \to B$ of animated commutative rings is almost of finite presentation if there is some factorization $A \to A[X_1, \ldots, X_n] \to B$ such that B is pseudocoherent as $A[X_1, \ldots, X_n]$-module.

One can check that this condition is independent of the chosen map $A[X_1, \ldots, X_n] \to B$ as long as it induces a surjection on π_0. Moreover, one can show that it globalizes to a notion for maps $f : X \to Y$ of derived schemes. We will use the following terminology.

Definition 8.9. A map $f : X \to Y$ of derived schemes is proper if it is almost of finite presentation and the map of underlying classical schemes is proper.

This definition is in particular relevant for the finiteness results in coherent cohomology:

Theorem 8.10. *Let $f : X \to Y$ be a proper map of derived schemes. Then f_* takes pseudocoherent objects of $D_{\mathrm{qc}}(X)$ to pseudocoherent objects of $D_{\mathrm{qc}}(Y)$, and coherent objects to coherent objects. If f is of finite Tor-dimension, then f_* takes $\mathrm{Perf}(X)$ to $\mathrm{Perf}(Y)$.*

We will give a direct proof in the next lecture, using the formalism of solid modules. It can also be deduced from the usual finiteness results in coherent cohomology.

8.2. Right and left adjoints to pullback. We would like to extend $X \mapsto D_{\mathrm{qc}}(X)$ to a 6-functor formalism. To do so, we need to define classes of morphisms I and P, and these classes are restrained by the existence of suitable left and right adjoints to pullback. First, right adjoints actually always satisfy base change:

Proposition 8.11. *Let $f : X \to Y$ be any map in C, i.e. a map of qcqs derived schemes. Then f^* admits a colimit-preserving right adjoint f_* satisfying projection formula and base change.*

Proof. Once one has proved the result for affine Y, it follows in general (as the then locally defined right adjoints commute with base change and thus globalize to the desired adjoint). Thus, assume that Y is affine. One can then moreover reduce to the case that X is affine, by first deducing the case of separated X, and then general X, using finite covers by open affine subsets. If $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine, then f^* corresponds to the tensor product functor $-\otimes_B A : D(B) \to D(A)$, whose right adjoint $D(A) \to D(B)$ is the forgetful functor, and commutes with all colimits, and satisfies the projection formula. Base change is also immediate, noting that everything is suitably derived. $\square$

Thus, in principle we could take for P the class of all maps. Before turning to the discussion of such possibilities, let us analyze a case when f^* admits a left adjoint.

Proposition 8.12. *Let $f : X \to Y$ be a proper map of finite Tor-dimension. Then f^* admits a left adjoint $f_\sharp$ satisfying projection formula and base change. Concretely, it is the functor taking a filtered colimit $\mathrm{colim}_i P_i$ of perfect complexes $P_i \in D_{\mathrm{qc}}(X)$ to $\mathrm{colim}_i (f_* P_i^\vee)^\vee$ where $P_i^\vee = \mathscr{H}\mathrm{om}(P_i, \mathcal{O}_X)$ denotes the naive dual (and the second $-^\vee$ denotes the naive dual on Y).*

Proof. We have the adjunction between f^* and f_* on $D_{\mathrm{qc}} = \mathrm{Ind}(\mathrm{Perf})$. But both functors preserve Perf, so we get an adjunction there. On the other hand, Perf is selfdual (with the naive duality) and so one gets an adjunction the other way between the naive duals of f^* and f_* on Perf. But f^* commutes with the duality, so one gets a left adjoint of $f^* : \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$. Passing to Ind-categories again, we get the desired left adjoint, with the given formula. This shows that it commutes with base change and satisfies the projection formula. $\square$

Something weird happens here: In the other formalisms, it were the open immersions where f^* admits a left adjoint; here is it is the proper maps (of finite Tor-dimension). And indeed, for open immersions like $\mathrm{Spec}(A[\frac{1}{g}]) \hookrightarrow \mathrm{Spec}(A)$, the pullback functor

$$- \otimes_A A[\tfrac{1}{g}] : D(A) \to D(A[\tfrac{1}{g}])$$

does not admit a left adjoint, as it does not commute with products. We will see in the next lecture a way to change the setting so that it does, by working with some kind of topological modules, and remembering the product topology.

Today, however, we will stick with abstract modules, and see what we can do.

8.3. Option 1: All maps are proper. The first option is to simply allow all maps to lie in P (and hence in E) and for I just take the isomorphisms. By Proposition 8.11 and Theorem 4.6, this defines a 6-functor formalism on C with values in presentable stable ∞-categories.

THEOREM 8.13. *In this 6-functor formalism, for any proper map $f : X \to Y$ of finite Tor-dimension the sheaf $\mathcal{O}_X$ is f-smooth.*

In particular, we get a dualizing object $\omega_{X/Y} \in D_{\mathrm{qc}}(X)$ as the Verdier dual of $\mathcal{O}_X$, i.e. $f^!\mathcal{O}_Y$ with respect to the !-functor of this formalism. (We are reluctant to denote this by $f^!$ in general, as $f^!$ has a standard meaning in coherent cohomology, but for proper maps it is the correct functor.)

PROOF. Using Proposition 6.6, it suffices to see that the formation of $f^!\mathcal{O}_Y$ commutes with any base change. But $f^!$ commutes with all colimits by Theorem 8.10 and is thus $D_{\mathrm{qc}}(Y)$-linear, and the resulting $D_{\mathrm{qc}}(Y)$-linear adjunction between f_* and $f^!$ base changes to a similar adjunction after any base change $Y' \to Y$, noting that $D_{\mathrm{qc}}(X \times_Y Y') = D_{\mathrm{qc}}(X) \otimes_{D_{\mathrm{qc}}(Y)} D_{\mathrm{qc}}(Y')$. □

If f is smooth, we can say more:

PROPOSITION 8.14. *Let $f : X \to Y$ be a proper smooth map of derived schemes. Then f and its diagonal Δ are cohomologically smooth. The dualizing sheaf $\omega_{X/Y} = f^!\mathcal{O}_Y$ is inverse to the sheaf $\Delta^!\mathcal{O}_{X\times_Y X}$ and thus local on X.*

This argument goes back to Verdier [**Ver69**].

PROOF. Let $p_1, p_2 : X \times_Y X \to X$ be the two projections. We compute, using the base change compatibility of $f^!\mathcal{O}_X$ and Theorem 8.13 applied to Δ (which is still of finite Tor-dimension):

$$\mathcal{O}_X = \Delta^! p_1^! \mathcal{O}_X \cong \Delta^! p_2^* f^! \mathcal{O}_Y \cong \Delta^* p_2^* f^! \mathcal{O}_Y \otimes \Delta^! \mathcal{O}_{X\times_Y X} \cong f^!\mathcal{O}_Y \otimes \Delta^!\mathcal{O}_{X\times_Y X}. \qquad \square$$

If X and Y are classical, it is not hard to define a canonical isomorphism $\Delta^!\mathcal{O}_{X\times_Y X} \cong (\Omega^d_{X/Y}[d])^\vee$ by working with local coordinates, and checking that the isomorphism does not depend on the choice of coordinates. If Y is derived, this does not work, but instead one can use a deformation to the normal cone as in [**CS22**, Lecture 13] to define this isomorphism.

8.4. Option 2: I consists of the proper maps of finite Tor-dimension, P consists of the open immersions. If we want to allow some morphisms in I, then it has to be the proper maps of finite Tor-dimension. We restrict P to consist of the open immersions. The class E should then consist of the separated maps that are almost of finite presentation and of finite Tor-dimension. (We note that the discussion of Lecture 4 also applies if instead of asking that all maps in E factor as $P \circ I$, they factor as $I \circ P$. In fact, one can formally deduce it by replacing all $D(X)$ by $D(X)^{\mathrm{op}}$.)

For this discussion, one needs a derived version of Nagata compactifications, see also [**GR17**, Chapter 5, Proposition 2.1.6].

THEOREM 8.15. *Let $f : X \to Y$ be a separated map of qcqs derived schemes that is almost of finite presentation. Then f can be factored as the composite of an open immersion $j : X \hookrightarrow \overline{X}$ and a proper map $\overline{f} : \overline{X} \to Y$.*

More precisely, any such Nagata compactification on the level of classical schemes can be lifted to the derived level, up to universal homeomorphism.

PROOF. Let $f_0 : X_0 \to Y_0$ be the map of underlying classical schemes, and fix a compactification $X_0 \hookrightarrow \overline{X}_0$. We will lift the given compactification to the derived level, up to universal homeomorphism. This problem is local on $\overline{X}_0$. (More precisely, we make X_0 inductively larger, and the resulting extension is local on $\overline{X}_0$.) We can thus assume that Y and $\overline{X}_0$ are affine. Replacing Y by $\mathbb{A}^n_Y$, we can moreover arrange that the map $\overline{X}_0 \to Y_0$ is finite. Now we apply the following lemma. □

LEMMA 8.16. *Let A be an animated commutative ring, and let U be a derived scheme over* $\mathrm{Spec}(A)$ *of almost finite presentation together with an open embedding of the corresponding classical scheme U_0 into* $\mathrm{Spec}(B_0)$ *for some classical $\pi_0 A$-algebra B_0 that is finitely presented as $\pi_0 A$-module. Then there is a pseudocoherent A-algebra B with a map $\pi_0 B \to B_0$ that is a universal homeomorphism, and an open embedding of U into* $\mathrm{Spec}(B)$ *lifting the given open embedding of classical schemes.*

PROOF. One can argue by inductively modifying A by first adjoining further elements, so as to generate $\pi_0 \mathcal{O}_U$ and make $\pi_0 A \to B_0$ surjective up to universal homeomorphism, and then quotienting by elements in degree 0 (so that U_0 becomes an open subscheme of $\mathrm{Spec}(\pi_0 A)$), and then by elements in higher homotopy groups, giving the desired B in the limit. □

However, in the current setting, we actually need to preserve the property of finite Tor-dimension. We do not know whether this property can be preserved under Nagata compactifications, so for the moment we do not know how to construct a 6-functor formalism of the desired type. This would actually have the further problem that P does not satisfy the 2-out-of-3 property, and neither does E. This is not actually really required for the machinery of Liu–Zheng [**LZ12b**], but it is required for some of our machinery like the notions of cohomologically étale (or proper) maps.

Still, let us check that the maps I and P would satisfy axiom (4) from Lecture IV:

PROPOSITION 8.17. *Consider a cartesian diagram of derived schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y \end{array}$$

where f is an open immersion, and g is a proper map of finite Tor-dimension. Then the natural map

$$g_\natural f'_* \to f_* g'_\natural : D_{\mathrm{qc}}(X') \to D_{\mathrm{qc}}(Y)$$

is an isomorphism.

Proof. The claim is local on Y, so we can assume that $Y = \mathrm{Spec}(B)$ is affine. Moreover, we can use descent along open immersions in Y' to reduce to the case that $Y' = \mathrm{Spec}(B[\frac{1}{h}])$ is a standard open affine. All functors commute with colimits, so it suffices to check the isomorphism on perfect complexes on X'; moreover, we can assume that they arise via pullback from X. In this case, it follows from the base change compatibility of $g_\sharp$ upon inverting h. □

Thus, if we would have the required Nagata compactifications of finite Tor-dimension (and suitably generalize the construction principle of Lecture IV to allow situations where P does not satisfy the 2-out-of-3 property), we would get a 6-functor formalism

$$X \mapsto D_{\mathrm{qc}}(X)$$

on all derived schemes, with values in presentable stable ∞-categories. The class E would conists of separated maps of finite Tor-dimension.

This time, Grothendieck–Serre duality is encoded not in terms of f-smoothness, but in terms of f-properness.

Theorem 8.18. *In this* 6*-functor formalism, any map $f : X \to Y$ in E that is a local complete intersection has the property that $\mathcal{O}_X$ is f-proper, with f-proper dual being invertible, and the inverse of $\omega_{X/Y}$.*

One could wonder whether in fact $\mathcal{O}_X$ is f-proper for all $f \in E$, but this turns out to be wrong. In Option 5, we will later correct this.

There is something very mind-bending about this situation. There is a way to make things more intuitive by passing to opposite categories, so we will prove the theorem after a reinterpretation.

8.5. Option 3: Replace Ind(Perf) **by** Pro(Perf)**.** Recall that replacing $D(X)$ by $D(X)^{\mathrm{op}}$ replaces cohomologically étale maps by cohomologically proper maps. As in Option 2, there is an apparent mismatch between geometric notions and the abstract notions defined in terms of D, it may be psychologically useful to switch from

$$D_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$$

to its opposite, which is $\mathrm{Pro}(\mathrm{Perf}(X))$ (noting that $\mathrm{Perf}(X)$ is self-dual, via the naive duality functor). A formalism of this type (or maybe rather the next option below) was first investigated by Deligne in the appendix of [**Har66**]. This has the disadvantage that $\mathrm{Pro}(\mathrm{Perf}(X))$ is no longer presentable, so we will generally have to be content with a 3-functor formalism (with occasionally defined right adjoints).

The formalism from Option 2 then dualizes to a 3-functor formalism on $\mathrm{Pro}(\mathrm{Perf}(X))$. Let us use the standard notation $f_!$ for the functors for $f \in E$. Then $f_! = \overline{f}_* j_!$ as usual, where $j_!$ is a left adjoint of j^* and $\overline{f}_*$ is a right adjoint of $\overline{f}^*$ (and j is an open immersion and $\overline{f} \in E$ is proper).

Example 8.19. Assume that $Y = \mathrm{Spec}(k)$ is the spectrum of a field. Then the category of Pro-objects of finite-dimensional k-vector spaces is the category of linearly compact k-vector spaces. For any separated scheme of finite type X over k and any $E \in \mathrm{Perf}(X)$, the object

$$R\Gamma_c(X, E) := f_! E \in \mathrm{Pro}(\mathrm{Perf}(k))$$

defines a bounded complex of linearly compact k-vector spaces. Concretely, to compute it take any compactification $\overline{X} \supset X$ whose boundary is a Cartier divisor $D \subset \overline{X}$. Assume that E extends to

$\overline{E}$ on $\overline{X}$ (which always happens up to retracts). Then

$$R\Gamma_c(X,E) = \lim_n R\Gamma(\overline{X}, \overline{E}(-nD)) \in \mathrm{Pro}(\mathrm{Perf}(k)).$$

In this language, Theorem 8.18 becomes the following result.

THEOREM 8.20. *Let $f : X \to Y$ be any map in E that is a local complete intersection. Then f is cohomologically smooth.*

We can then define $\omega_{X/Y} = f^! \mathcal{O}_Y \in \mathrm{Perf}(X)$ (which is then defined). We note that this is automatically local on X.

PROOF. In this formalism, open immersions are cohomologically étale, so we can localize (on X and Y), and hence assume that X and Y are affine. By the assumption that f is a local complete intersection, and using base change compatibility, one can in fact reduce further to the case of $\mathbb{A}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ or its zero section. The case of $\mathbb{A}^1$ can be reduced to $\mathbb{P}^1$. In those cases, the right adjoint $f^!$ to $f_! = f_*$ actually exists, preserves perfect complexes, and commutes with any base change (using Proposition 8.14, upon restricting from Ind(Perf) to Perf, and extending back to Pro(Perf)). Thus, $\mathcal{O}_X$ is f-smooth, but by Proposition 8.14 the object $f^! \mathcal{O}_X$ is invertible. □

As we observed above, it is very annoying that the maps in E have to be restricted to be of finite Tor-dimension; this precludes in particular the 2-out-of-3 property. The underlying reason is that in Theorem 8.10 this restriction appears in relation to perfect complexes.

8.6. Option 4: Replace Pro(Perf) **by** Pro(Coh)**.** However, we can artificially replace Perf by Coh, the coherent complexes, at least for noetherian schemes X (i.e. the underlying classical scheme is noetherian, and $\pi_i \mathcal{O}_X$ is a coherent $\pi_0 \mathcal{O}_X$-module for all i).

REMARK 8.21. I believe the theory of this section also works with general derived schemes, if one uses the following interpretation of $\mathrm{Pro}(\mathrm{Coh}(X))$ that makes sense in general.

PROPOSITION 8.22. *Assume that X is a noetherian derived scheme. Let $\mathrm{Coh}(X) \subset D^+_{\mathrm{qc}}(X)$ be the subcategory of coherent complexes. Then the image of the fully faithful functor*

$$\mathrm{Pro}(\mathrm{Coh}(X))^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(D^+_{\mathrm{qc}}(X))^{\mathrm{op}} \cong \mathrm{Fun}^{\mathrm{ex}}(D^+_{\mathrm{qc}}(X), \mathrm{Sp})$$

is given by those exact functors $D^+_{\mathrm{qc}}(X) \to \mathrm{Sp}$ of stable ∞-categories that commute with all filtered colimits in $D^{\geq 0}_{\mathrm{qc}}(X)$.

PROOF. By formal nonsense, the image consists of those functors $D^+_{\mathrm{qc}}(X) \to \mathrm{Sp}$ that are left Kan extended from $\mathrm{Coh}(X)$. All such commute with filtered colimits in $D^{\geq 0}$ as for $K \in \mathrm{Coh}(X)$ the functor $\mathrm{Hom}(K, -)$ commutes with filtered colimits in $D^{\geq 0}$. In the converse direction, it suffices to observe that $D^+_{\mathrm{qc}}(X)$ is generated under filtered colimits by $\mathrm{Coh}(X)$, in fact by such filtered colimits that stay bounded to the left. □

Restricting to noetherian derived schemes, we send any X to the stable ∞-category $\mathrm{Pro}(\mathrm{Coh}(X))$. We note that by restricting the symmetric monoidal ∞-category $D_{\mathrm{qc}}(X)$ to $\mathrm{Coh}(X)$, one still has an ∞-operad, which after passage to $\mathrm{Pro}(\mathrm{Coh}(X))$ is again a symmetric monoidal ∞-category. Concretely, the tensor product of two objects of $\mathrm{Coh}(X)$ need not itself lie in $\mathrm{Coh}(X)$ (in general, it just lies in $\mathrm{PCoh}(X)$), but it still defines a pro-object of $\mathrm{Coh}(X)$ (the Pro-system of its truncations $\tau_{\leq n}$). Similar remarks apply to pullback functoriality. We note that the resulting pullback and tensor product functors on Pro(Coh) automatically commute with all limits.

We let I be the class of open immersions, and P the class of proper maps; then E is the class of separated maps of almost finite type (of noetherian derived schemes).

THEOREM 8.23. *The functor $X \mapsto \mathrm{Pro}(\mathrm{Coh}(X))$ satisfies the hypotheses of Theorem 4.6 with respect to I and P, and hence defines a 3-functor formalism on noetherian derived schemes.*

The proof is a variation on the same observations as before. Compared to the previous option, there are many more f-smooth objects. In particular, we get a stronger result on biduality.

PROPOSITION 8.24. *Let $f : X \to Y$ be a separated map of noetherian derived schemes of almost finite type and assume that $K \in \mathrm{Coh}(X) \subset \mathrm{Pro}(\mathrm{Coh}(X))$ is of finite Tor-dimension over Y. Then K is f-smooth.*

In particular, if Y is a regular classical scheme, then all objects $K \in \mathrm{Coh}(X)$ are f-smooth, and $\mathrm{Coh}(X)$ is self-dual with respect to the Verdier duality $\mathbb{D}_f(K) = \mathscr{H}\mathrm{om}(K, f^!\mathcal{O}_Y)$.

We note in particular that if f is itself of finite Tor-dimension, we get the dualizing complex $\omega_{X/Y} = f^!\mathcal{O}_Y \in \mathrm{Coh}(X)$, in a way that localizes on X, fulfilling a promise made in Option 1 to show that the dualizing complex is of local nature in general.

PROOF. The assertion is local on X and Y, so we can assume that they are affine. Replacing Y by an affine space over Y, which is cohomologically smooth, one can then assume that f is a closed immersion. In that case K becomes perfect as an object of $\mathrm{Coh}(Y)$, and its naive Y-dual (with induced $\mathcal{O}_X$-module structure) defines the desired f-smooth dual. □

8.7. Option 5: Replace $\mathrm{Pro}(\mathrm{Coh})$ **by** $\mathrm{Ind}(\mathrm{Coh})$**.** Finally, we can dualize again, and replace $\mathrm{Pro}(\mathrm{Coh}(X))$ by its opposite ∞-category $\mathrm{Ind}(\mathrm{Coh}(X)^{\mathrm{op}})$. If one restricts to schemes almost of finite type over a classical regular base scheme Y (e.g., a field k), one can use the self-duality from Proposition 8.24 to identify $\mathrm{Ind}(\mathrm{Coh}(X)^{\mathrm{op}})$ with $\mathrm{Ind}(\mathrm{Coh}(X))$. Now the ∞-categories are again presentable (and all 3 functors commute with colimits), so one gets a full 6-functor formalism. This gives exactly the 6-functor formalism constructed by Gaitsgory–Rozenblyum [**GR17**].

Working over a classical regular base scheme Y, we note that under this identification

$$\mathrm{Ind}(\mathrm{Coh}(X)^{\mathrm{op}}) \cong \mathrm{Ind}(\mathrm{Coh}(X)),$$

the pullback, tensor product, and exceptional functors on $\mathrm{Ind}(\mathrm{Coh}(X))$ are Verdier dual to the usual pullback, tensor product, and lower shriek functors; they are thus denoted by $f^!$, $\otimes^!$, and f_* in the work of Gaitsgory–Rozenblyum, even while they play the role of f^*, $\otimes$ and $f_!$ in this 6-functor formalism.

As we have seen, there are many possible 6-functor formalisms in the coherent setting. Our notational choice is thus to explicitly include the 6-functor formalism in the notation, i.e. we denote these functors as f^*_{IndCoh}, $\otimes_{\mathrm{IndCoh}}$, $f_{!,\mathrm{IndCoh}}$, so

$$f^*_{\mathrm{IndCoh}} = f^!_{\mathrm{GR}}, \otimes_{\mathrm{IndCoh}} = \otimes^!_{\mathrm{GR}}, f_{!,\mathrm{IndCoh}} = f_{*,\mathrm{GR}},$$

using a subscript GR to refer to Gaitsgory–Rozenblyum's choice of notation.

Let us end with a remark on the relation between D_{qc} and $\mathrm{Ind}(\mathrm{Coh})$. As in [**GR17**], take as our category C the affine derived schemes almost of finite type over a field k (of any characteristic, for now). We have two functors to symmetric monoidal presentable stable ∞-categories

$$X \mapsto D_{\mathrm{qc}}(X), X \mapsto \mathrm{Ind}(\mathrm{Coh}(X))$$

from Option 2 and Option 5, respectively. We note that both theories satisfy Zariski descent, so in the limit above it suffices to restrict to affine X. In [**GR17**], sometimes a restriction to truncated X is made. This does not change $\mathrm{Ind}(\mathrm{Coh}(X))$, in the following sense: If $X = \mathrm{Spec}(A)$ is affine, then the functor

$$\mathrm{Ind}(\mathrm{Coh}(A)) \to \lim_n \mathrm{Ind}(\mathrm{Coh}(\tau_{\leq n} A))$$

is an equivalence. (To see this, note that the pullback functors here are concretely realized as !-functors, and have the forgetful functors $\mathrm{Coh}(\tau_{\leq n} A) \to \mathrm{Coh}(A)$ as left adjoints. Thus, this limit can also be computed as the colimit of presentable stable ∞-categories along these left adjoint functors that preserve compact objects. Then it follows from

$$\mathrm{colim}_n \mathrm{Coh}(\tau_{\leq n} A) \to \mathrm{Coh}(A)$$

being an equivalence of small stable ∞-categories, i.e. on a bounded complex the A-action always factors (uniquely) over $\tau_{\leq n} A$ for large enough n.) This means that in the limit defining $\mathrm{IndCoh}(\tilde{X})$, one can even restrict to affine $X = \mathrm{Spec}(A)$ for which A is n-truncated for some n.

We note that there is a symmetric monoidal natural transformation

$$D_{\mathrm{qc}}(X) \to \mathrm{Ind}(\mathrm{Coh}(X))$$

given by

$$D_{\mathrm{qc}}(X) = \mathrm{Ind}(\mathrm{Perf}(X)) \cong \mathrm{Ind}(\mathrm{Perf}(X)^{\mathrm{op}}) \to \mathrm{Ind}(\mathrm{Coh}(X)^{\mathrm{op}}) \cong \mathrm{Ind}(\mathrm{Coh}(X)) = \mathrm{IndCoh}(X),$$

where the first equivalence is naive duality, and the last equivalence is Serre duality (over k). (The composite functor is actually just tensoring with the dualizing complex of X, which is the tensor unit of $\mathrm{IndCoh}(X)$.) If X is truncated (i.e. $\pi_i X = 0$ for $i \gg 0$) then this comparison functor is fully faithful.

Remark 8.25. We note that the arguments of this lecture give in particular a construction of the 6-functor formalism IndCoh of Gaitsgory–Rozenblyum that does not rely on difficult properties of the Gray tensor product of $(\infty, 2)$-categories; in fact, it makes use of no $(\infty, 2)$-categorical machinery at all.

Appendix to Lecture VIII: D-modules

As in the work of Gaitsgory–Rozenblyum [**GR17**], one can recover the 6-functor formalism of D-modules (on algebraic varieties over a field k of characteristic 0), originally developed by Sato, Kashiwara, Bernstein, ... [10]

Before starting to use the formalism from the lecture, let us announce directly a way to introduce the formalism. Fix a field k of characteristic 0,[11] and consider the category C of separated schemes of finite type over k, with all morphisms allowed in E. To any $X \in C$, we can associate the category of crystals on the infinitesimal site; concretely,

$$\mathrm{Crys}(X) = \lim_{R,\mathrm{Spec}(R_{\mathrm{red}})\to X} D(R)$$

as R runs over k-algebras of finite type.[12] This gives a contravariant functor from C to symmetric monoidal presentable stable ∞-categories.

THEOREM 8.26. *The functor $X \mapsto \mathrm{Crys}(X)$ satisfies the hypotheses of Theorem 4.6 with respect to the classes I of proper maps and P of open immersions, and hence extends to a 6-functor formalism.*

Our approach in this appendix is actually to construct the 6-functor formalism directly and verify that proper maps are cohomologically étale while open immersions are cohomologically proper; this proves Theorem 8.26. To do so, we use the extension of the coherent 6-functor formalism to stacks. By Theorem 4.20, the formalism $X \mapsto \mathrm{Ind}(\mathrm{Coh}(X))$ extends to stacks on C (i.e. sheaves of anima), sending any stack $\tilde{X}$ to

$$\mathrm{IndCoh}(\tilde{X}) = \lim_{X\in C, X\to\tilde{X}} \mathrm{Ind}(\mathrm{Coh}(\tilde{X})).$$

In this generality, this is not anymore compactly generated, so for stacks $\mathrm{IndCoh} \neq \mathrm{Ind}(\mathrm{Coh})$.

We can similarly define $D_{\mathrm{qc}}(\tilde{X})$, and note that there is a natural symmetric monoidal transformation

$$D_{\mathrm{qc}}(X) \to \mathrm{IndCoh}(X),$$

which is usually fully faithful.

One way to motivate the upcoming definition of D-modules is to observe that one respect in which the coherent 6-functor formalisms differ from the previous ones is that they are not nil-invariant: If $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a nilpotent immersion, i.e. $\pi_0 B \to \pi_0 A$ is surjection with nilpotent kernel, then $D(B) \to D(A)$ is not usually an equivalence.

There is a universal way to make a theory nil-invariant. Namely, for any derived scheme X, let X_{dR} denote the prestack taking an animated commutative ring A to

$$\mathrm{colim}_{I\subset\pi_0 A} X(\pi_0 A/I)$$

as I runs over nilpotent ideals of $\pi_0 A$ (i.e., there is some n such that $I^n = 0$). If X is almost of finite type over k, then X_{dR} is also left Kan extended from k-algebras A that are almost of finite

[10]Caveat: I am only just learning about D-modules, and (as the rest of the notes) this section is still being revised, and further material will be added.

[11]We actually do not use the assumption of characteristic 0 for a long time. It is only used in the identification with modules over the ring of differential operators, and in the discussion of Poincaré duality – in characteristic p the resulting theory is a 6-functor formalism but does not satisfy Poincaré duality.

[12]One can also allow animated k-algebras almost of finite type; it is not hard to see that the two results are equivalent, via a cofinality argument.

type, so there is no harm restricting to our category C of affine derived schemes almost of finite type over k as above. Note also that if A is almost of finite type, then the nilradical of $\pi_0 A$ is nilpotent, and hence

$$X_{\mathrm{dR}}(A) = X(A_{\mathrm{red}})$$

where A_{red} is the reduced quotient of $\pi_0 A$. For a nil-invariant theory, the pullback functor $D(X_{\mathrm{dR}}) \to D(X)$ is an equivalence; and in general $X \mapsto D(X_{\mathrm{dR}})$ is nil-invariant.

Definition 8.27. Let k be a field of characteristic 0 and let X be a derived scheme almost of finite type over k. The symmetric monoidal presentable stable ∞-category of left D-modules on X is

$$\mathrm{Dmod}^L(X) := D_{\mathrm{qc}}(X_{\mathrm{dR}}),$$

while that of right D-modules is

$$\mathrm{Dmod}^R(X) := \mathrm{IndCoh}(X_{\mathrm{dR}}).$$

Thus, $\mathrm{Crys}(X) = \mathrm{Dmod}^L(X)$ for the above definition of $\mathrm{Crys}(X)$. Regarding the names "left/right D-modules", we follow here the terminology of [**GR17**]. From the general comparison map $D_{\mathrm{qc}} \to \mathrm{IndCoh}$, we get a colimit-preserving symmetric monoidal functor

$$\mathrm{Dmod}^L(X) \to \mathrm{Dmod}^R(X)$$

and we will see below that this is actually an equivalence. We can already see now that it is fully faithful – this follows as in the limit defining X_{dR}, classical schemes are cofinal.

Remark 8.28. From the definition it follows that if X is a derived scheme almost of finite type with underlying reduced classical scheme X_{red}, the map $X_{\mathrm{red,dR}} \to X_{\mathrm{dR}}$ is an isomorphism, hence $\mathrm{Dmod}^L(X) = \mathrm{Dmod}^L(X_{\mathrm{red}})$ (and similarly for Dmod^R). Thus, in the following we can often restrict attention to (reduced) classical schemes.

Our first goal is to prove the following theorem. Together with the equivalence $\mathrm{Dmod}^L(X) = \mathrm{Dmod}^R(X)$, it will prove Theorem 8.26.

Theorem 8.29. *For any map $f : X \to Y$ of schemes of finite type over k, the $!$-functors on* IndCoh *are defined for $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$. Moreover, if f is étale, then f_{dR} is cohomologically proper, while if f is proper, then f_{dR} is cohomologically étale.*

To get started, we have the following proposition regarding "formal completions".

Proposition 8.30. *Let X be a derived scheme almost of finite type over k and let $Z \subset X$ be a closed subscheme. Let $X^\wedge_Z \subset X$ be the subfunctor of those maps* $\mathrm{Spec}(A) \to X$ *that set-theoretically factor over Z. Then the map $j : X^\wedge_Z \hookrightarrow X$ admits $!$-functors in the* IndCoh*-formalism, and is cohomologically étale. In fact, one can identify $X^\wedge_Z \subset X$ as the image of the cohomologically étale map $Z \to X$.*

Proof. The question is local on X, so we can assume that X is affine, and Z is the vanishing locus of some functions $f_1, \ldots, f_n$; and by induction on n and stability under pullback, we can assume that $n = 1$. In that case, the situation comes via pullback from $X = \mathrm{Spec}(k[T])$ and $Z = \mathrm{Spec}(k)$ embedded as $T = 0$. The map $Z \to X$ is cohomologically étale (like any proper map), and hence its image (in the sheaf-theoretic sense) admits a $!$-functor and is a cohomologically étale monomorphism. Indeed, this image admits a cohomologically étale surjection from Z (as any base change agrees with a base change of $Z \to X$) and $Z \to X$ is cohomologically étale.

We claim that the image of $Z \to X$ is given by $X_Z^\wedge \subset X$. It is clearly contained in $X_Z^\wedge \subset X$. Conversely, if $Y = \mathrm{Spec}(B)$ is some affine derived scheme mapping towards $X_Z^\wedge$, then it factors over $\mathrm{Spec}(k[T]/T^n)$ for some n, so it suffices to see that $\mathrm{Spec}(k[T]/T^n) \to X = \mathrm{Spec}(k[T])$ is contained in the image of $Z \to X$, i.e. $\mathrm{Spec}(k) \to \mathrm{Spec}(k[T]/T^n)$ is a cover in the D-topology. But this follows from Proposition 6.18 noting that $\mathrm{Ind}(\mathrm{Coh}(k[T]/T^n)) \to \mathrm{Ind}(\mathrm{Coh}(k)) = D(k)$ is conservative. But $k[T]/T^n$ has an n-step filtration whose graded pieces are k. □

It may be useful to describe what happens in terms of the derived categories.

Proposition 8.31. *In the situation of Proposition 8.30, the functor*

$$j_! : \mathrm{IndCoh}(X_Z^\wedge) \to \mathrm{IndCoh}(X)$$

is fully faithful and its essential image is given by $\mathrm{IndCoh}(X \text{ on } Z)$*, i.e. those objects that vanish outside* Z*.*

Proof. For any cohomologically étale monomorphism j, the functor $j_!$ is fully faithful as $j^* j_! = \mathrm{id}$ by base change. As $Z \to X_Z^\wedge$ is surjective, the kernel of j^* agrees with the kernel of $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Z)$; and $\mathrm{IndCoh}(X \text{ on } Z) \to \mathrm{IndCoh}(Z)$ is known to be conservative. (The point is that if the base change of a complex to the vanishing locus of some function f is zero, then f acts invertibly on that complex, and hence lives on the open subset where f is invertible.) □

Proposition 8.32. *For a finite type k-scheme X, the map $X \to X_{\mathrm{dR}}$ admits !-functors in the* IndCoh*-formalism and is cohomologically étale and surjective.*

Proof. The question is local on X, so we can restrict first to affine X, but then by a compactification we can also restrict to proper X. We have to see that for any derived scheme Y almost of finite type over k with a map $Y \to X_{\mathrm{dR}}$, the fibre product $X \times_{X_{\mathrm{dR}}} Y \to Y$ admits !-functors, and is cohomologically smooth and surjective. We note that as we work everywhere with sheaves of anima for the D-topology, the map $Y \to X_{\mathrm{dR}}$ is a map to the sheafified version of X_{dR}; but because the condition of admitting !-functors is local on source and target, we can assume that $Y = \mathrm{Spec}(B)$ is affine and the map $Y \to X_{\mathrm{dR}}$ arises from a map $Y_{\mathrm{red}} \to X$.

Now we first claim that the monomorphism $X \times_{X_{\mathrm{dR}}} Y \to X \times_k Y$ admits !-functors and is cohomologically étale. In fact, the graph of the map $Y_{\mathrm{red}} \to X$ gives a closed subset $Z \subset X \times_k Y$ and then

$$X \times_{X_{\mathrm{dR}}} Y = (X \times_k Y)_Z^\wedge,$$

so the result follows from Proposition 8.30. But the projection $X \times_k Y \to Y$ is also cohomologically étale (as X is proper). It is easy to see that the map $X \to X_{\mathrm{dR}}$ is surjective. □

At this point, we can prove Theorem 8.29.

Proof of Theorem 8.29. First, $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ admits !-functors. Indeed, it suffices to check this over the cohomologically étale cover $X \to X_{\mathrm{dR}}$, and then it is the composite of $X \to Y$ and $Y \to Y_{\mathrm{dR}}$, both of which are allowed. Moreover, if f is proper, then f_{dR} is cohomologically étale. Indeed, $X \to X_{\mathrm{dR}}$ is a cohomologically étale cover, and over this cover it is the composite of the cohomologically étale maps $X \to Y$ and $Y \to Y_{\mathrm{dR}}$. Finally, if f is étale, then f_{dR} is cohomologically proper. Indeed, this can be checked after pullback along the cohomologically étale surjection $Y \to Y_{\mathrm{dR}}$, and then we get the cohomologically proper map $f : X \to Y$, as $X = X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y$ if $f : X \to Y$ is étale (using the infinitesimal lifting property of étale maps). $\square$

This finishes the construction of the D-module 6-functor formalism, but it stays a bit abstract, and the relation to actual D-modules remains unclear. Let us address these issues.

First, the D-module 6-functor formalism actually has a property stronger than nil-invariance: Namely, it satisfies excision. This result is classically known as "Kashiwara's lemma".

Proposition 8.33. *Let X be a finite type k-scheme with a closed subscheme $j : Z \subset X$ and open complement $i : U \subset X$.*[13] *For any $A \in \mathrm{Dmod}^R(X)$, the triangle*

$$j_! j^* A \to A \to i_* i^* A$$

is exact, yielding a semi-orthogonal decomposition of $\mathrm{Dmod}^R(X)$ into $\mathrm{Dmod}^R(Z)$ and $\mathrm{Dmod}^R(U)$.

Proof. It suffices to check this after pullback along the cohomologically étale surjection $X \to X_{\mathrm{dR}}$, and then the strata become $X \times_{X_{\mathrm{dR}}} U_{\mathrm{dR}} = U$ and $X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}} = X^{\wedge}_Z$. The result follows from Proposition 8.31. $\square$

Using this, we can finally show the equivalence $\mathrm{Dmod}^L \cong \mathrm{Dmod}^R$. This also finishes the proof of Theorem 8.26.

Theorem 8.34. *For all finite type k-schemes X, the natural symmetric monoidal functor*

$$\mathrm{Dmod}^L(X) \to \mathrm{Dmod}^R(X)$$

is an equivalence.

Proof. We claim that if X is smooth, then the map of prestacks $X \to X_{\mathrm{dR}}$ is of universal descent for D_{qc}. We can assume $X = \mathrm{Spec}(A)$ is affine. Consider any affine $Y = \mathrm{Spec}(B)$ with a map $Y \to X_{\mathrm{dR}}$; we want to show that $Y \times_{X_{\mathrm{dR}}} X \to Y$ is of universal D_{qc}-descent. But $Y \to X_{\mathrm{dR}}$ factors over some classical Y, so we can assume that Y is classical. The map $Y \times_{X_{\mathrm{dR}}} X \to Y$ splits over Y_{red}. But $Y_{\mathrm{red}} \to Y$ is of universal D_{qc}-descent as it is descendable in the sense of Mathew [**Mat16**, Proposition 3.35]; and if a refinement over a morphism satisfies universal descent, so does the original morphism.

Now start with any X, without loss of generality affine, and embed into a smooth $\tilde{X}$. By descent along $\tilde{X} \to \tilde{X}_{\mathrm{dR}}$, it suffices to show that the image of

$$\mathrm{Dmod}^R(X) = \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}} \times_{\tilde{X}_{\mathrm{dR}}} \tilde{X})$$

lies in $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{\tilde{X}_{\mathrm{dR}}} \tilde{X})$. But $X_{\mathrm{dR}} \times_{\tilde{X}_{\mathrm{dR}}} \tilde{X} = \tilde{X}^{\wedge}_X$, and

$$\mathrm{IndCoh}(\tilde{X}^{\wedge}_X) = \mathrm{IndCoh}(\tilde{X} \text{ on } X) = D_{\mathrm{qc}}(\tilde{X} \text{ on } X) = D_{\mathrm{qc}}(\tilde{X}^{\wedge}_X),$$

where the middle equation uses smoothness of $\tilde{X}$. $\square$

Finally, we can explain the name of left and right D-modules. Their names really only make sense when one restricts to smooth X. In the case of left D-modules, the proof of the theorem shows that one can describe objects of $\mathrm{Dmod}^L(X) = D_{\mathrm{qc}}(X_{\mathrm{dR}})$ in terms of their pullback to an object $M \in D_{\mathrm{qc}}(X)$ together with a descent datum, which exactly amounts to a left $\mathcal{O}_X$-linear action on M of the algebra D_X of differential operators on X.

[13] We apologize for the nonstandard notation; it is however dictated by the 6-functor formalism, where the closed immersion j is cohomologically étale and the open immersion i is cohomologically proper.

On the other hand, if one describes objects of $\mathrm{Dmod}^R(X) = \mathrm{IndCoh}(X_{\mathrm{dR}})$ in the similar way, one gets by pullback an object $M \in \mathrm{Ind}(\mathrm{Coh}(X))$, with a similar-looking action. If as in the proof one identifies $\mathrm{Ind}(\mathrm{Coh}(X))$ with $D_{\mathrm{qc}}(X)$ via twisting with the dualizing sheaf, this action is precisely a left action of D_X again, yielding the equivalence. But if one uses the more naive equivalence $\mathrm{Ind}(\mathrm{Coh}) = D_{\mathrm{qc}}(X)$ coming from $\mathrm{Coh}(X) = \mathrm{Perf}(X)$, then this action unravels to a right $\mathcal{O}_X$-linear D_X-action.

Moreover, we have the following important result on duality.

Theorem 8.35. *For all finite type k-schemes X, the presentable stable ∞-category* $\mathrm{Dmod}(X)$ *is compactly generated, and the compact objects agree with the f_{Dmod}-proper objects for $f : X \to$* $\mathrm{Spec}(k)$*. In particular, duality for f_{Dmod}-proper objects gives a selfduality on the compact objects of* $\mathrm{Dmod}(X)$*.*

Moreover:

(1) *If X is smooth of dimension d, then the f_{Dmod}-proper dual of $1_{\mathrm{Dmod(X)}}$ is isomorphic to* $1_{\mathrm{Dmod}(X)}[-2d]$*.*
(2) *If $f : X \to Y$ is a proper map, then $f_{!,\mathrm{Dmod}}$ preserves compact objects, and commutes with proper duality.*

Under the identification of $\mathrm{Dmod}(X)$ with the derived category of left D_X-modules (for X smooth), the compact objects are of course the D_X-coherent objects. Assertion (1) is encoding Poincaré duality.

Proof. The map $X \to X_{\mathrm{dR}}$ induces a cohomologically étale map, and hence the (colimit-preserving) pullback $\mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X)$ has a left adjoint, which thus preserves compact objects. As $\mathrm{IndCoh}(X)$ is compactly generated (and pullback is conservative), so is $\mathrm{Dmod}(X)$. Moreover, any $K \in \mathrm{Coh}(X)$ is f-proper in the IndCoh-formalism for $f : X \to *$. As cohomologically étale pushforwards (like $g : X \to X_{\mathrm{dR}}$) preserve proper objects, this implies that $g_{!,\mathrm{IndCoh}} K \in \mathrm{IndCoh}(X_{\mathrm{dR}})$ is f_{dR}-proper, i.e. as an object of $\mathrm{Dmod}(X)$ it is f_{Dmod}-proper. These generate all compact objects, so all compact objects are f_{Dmod}-proper. The converse follows from Proposition 6.16.

In part (1), one first shows that it is locally isomorphic to $1_{\mathrm{Dmod}(X)}[-2d]$; this reduces to a computation on $\mathbb{A}^1$ (and is the only place we use characteristic 0). Then there are various ways to conclude a global isomorphism, for example using a deformation to the normal cone. Part (2) follows from the preservation of proper objects under cohomologically étale pushforward (and the preservation of proper duality). $\square$

Remark 8.36. There is yet another interpretatation of $\mathrm{Dmod}(X)$, or rather of its opposite $\mathrm{Dmod}(X)^{\mathrm{op}}$. Namely,

$$\begin{aligned}\mathrm{Dmod}(X)^{\mathrm{op}} &= \mathrm{ProPerf}(X_{\mathrm{dR}}) = \lim_{R,\mathrm{Spec}(R_{\mathrm{red}})\to X} \mathrm{Pro}(\mathrm{Perf}(R))\\ &\cong \mathrm{ProCoh}(X_{\mathrm{dR}}) = \lim_{R,\mathrm{Spec}(R_{\mathrm{red}})\to X} \mathrm{Pro}(\mathrm{Coh}(R)),\end{aligned}$$

using the formalisms from Option 3 and 4 in the lecture. In this interpretation, the étale maps are cohomologically étale and the proper maps are cohomologically proper. Moreover, the excision triangle takes the usual form: If X has an open subscheme $j : U \subset X$ with closed complement $i : Z \subset X$, then for all $A \in \mathrm{ProPerf}(X_{\mathrm{dR}})$, one has an exact triangle

$$j_! j^* A \to A \to i_* i^* A.$$

Another advantage of this realization is that the comparison between ProPerf and ProCoh is naive. Moreover, in this realization, Theorem 8.35 says that all co-compact objects of $\mathrm{Dmod}^{\mathrm{op}}$ are f-smooth over the point, the duality becomes actual Verdier duality, and part (1) says that the map $X \to *$ is cohomologically smooth if X is smooth (with dualizing complex $1_{\mathrm{Dmod}^{\mathrm{op}}(X)}[2d]$).

Example 8.37. Let us give some examples of D-modules, and discuss their different realizations. Let us take the affine line $X = \mathbb{A}^1_k = \mathrm{Spec}(k[T])$ with its closed point $Z = \mathrm{Spec}(k) = \{0\} \subset \mathbb{A}^1_k = X$; let $U = \mathrm{Spec}(k[T^{\pm 1}])$ be its open complement. Following our previous terminology, we write $j : Z \to X$ and $i : U \to X$ for the morphisms, and we want to understand the different realizations of the exact triangle

$$j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)}) \to 1_{\mathrm{Dmod}(X)} \to i_{*,\mathrm{Dmod}}(1_{\mathrm{Dmod(U)}}).$$

Here $i_{*,\mathrm{Dmod}}$ is, as in any 6-functor formalism, the right adjoint to i^*_{Dmod}; but because i is open and thus cohomologically proper, it also agrees with $i_{!,\mathrm{Dmod}}$. The functor $j_{!,\mathrm{Dmod}}$ is the left adjoint of j^*_{Dmod}, as j is proper and thus cohomologically étale.

Let us first realize under the functor $\mathrm{Dmod}(X) \to D_{\mathrm{qc}}(X)$; this is the usual realization as left D-modules. This functor is symmetric monoidal for the usual tensor product. Also, $i_{*,\mathrm{Dmod}}$ pulls back to the the functor $i_{*,D_{\mathrm{qc}}}$ as $U = X \times_{X_{\mathrm{dR}}} U_{\mathrm{dR}}$. Thus, we can identify the last two terms of the sequence above as $k[T] \to k[T^{\pm 1}]$, and thus the whole exact triangle must be

$$(k[T^{\pm 1}]/k[T])[-1] \to k[T] \to k[T^{\pm 1}].$$

In particular,

$$j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)}) = (k[T^{\pm 1}]/k[T])[-1].$$

We note that this module can also be written as the local cohomology of the structure sheaf.

Now let us analyze the realization via IndCoh. More precisely, we consider the realization

$$\mathrm{Dmod}(X) \cong \mathrm{Dmod}^R(X) = \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X) = D_{\mathrm{qc}}(X)$$

where the functor is the general functoriality of IndCoh, and the last equality comes from the equality $\mathrm{Coh}(X) = \mathrm{Perf}(X)$. This functor sends $1_{\mathrm{Dmod}(X)}$ to the dualizing complex $\omega_X = (k[T]dT)[1]$. The functor i^*_{Dmod} for an open immersion i is actually the naive pullback in any possible realization (in Gaitsgory–Rozenblyum's notation, $i^*_{\mathrm{IndCoh}} = i^!_{\mathrm{GR}} = i^*_{\mathrm{GR}}$), and so its right adjoint $i_{*,\mathrm{Dmod}}$ is the naive pushforward. Thus, the triangle above becomes

$$(k[T^{\pm 1}]/k[T])dT \to (k[T]dT)[1] \to k[T^{\pm 1}]dT[1].$$

In fact, in general this realization is just the first realization tensored by ω_X (which is $(k[T]dT)[1]$ in this case).

Finally, let us analyze the realization via ProPerf = ProCoh. In this realization, maps within the category go the other way, so the triangle rather becomes a triangle

$$j_{!,\mathrm{Dmod}^{\mathrm{op}}}(1_{\mathrm{Dmod}^{\mathrm{op}}(U)}) \to 1_{\mathrm{Dmod}^{\mathrm{op}}(X)} \to i_{*,\mathrm{Dmod}^{\mathrm{op}}}(1_{\mathrm{Dmod}^{\mathrm{op}}(Z)}),$$

where also now switched to $j : U \subset X$ and $i : Z \subset X$. Under $\mathrm{Dmod}^{\mathrm{op}}(X) = \mathrm{ProPerf}(X_{\mathrm{dR}}) \to \mathrm{ProPerf}(X)$, this realizes to

$$(k[[T]]/k[T])[-1] \to k[T] \to k[[T]],$$

where $k[[T]]$ denotes the pro-object $(k[T]/T^n)_n$.

EXAMPLE 8.38. Continuing our previous example, we note that all objects involved are D-coherent, so we also get a triangle of their duals. Let us write $\mathbb{D}_X$ for the duality on the compact objects of $\mathrm{Dmod}(X)$. We have $\mathbb{D}(1_{\mathrm{Dmod}(X)}) = 1_{\mathrm{Dmod}(X)}[-2]$ and

$$\mathbb{D}(j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)})) = j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)}).$$

(This equation may look confusing, but recall that $\mathbb{D}$ is proper duality, and this commutes with $j_!$ for cohomologically étale maps. Alternatively, think in terms of the ProPerf-picture, where $j_!$ would be denoted $i_! = i_*$, and Verdier duality commutes with (cohomologically) proper pushforward.) Thus, the triangle dualizes to a triangle

$$\mathbb{D}(i_{*,\mathrm{Dmod}}(1_{\mathrm{Dmod}(U)})) \to 1_{\mathrm{Dmod}(X)}[-2] \to j_{!,\mathrm{Dmod}}(1_{\mathrm{Dmod}(Z)}).$$

In the first realization, this becomes a triangle

$$M \to k[T][-2] \to (k[T^{\pm 1}]/k[T])[-1]$$

for some $M \in D(k[T])$. In fact, any such triangle is split, so $M = (k[T] \oplus k[T^{\pm 1}]/k[T])[-2]$, but the D-module structure is more subtle. More precisely, on any basis element $(T^n, 0)$ or $(0, T^{-n})$ with $n > 0$ the derivative ∂_T does the same as the direct sum of the two D-modules, but instead of killing $(T^0, 1)$, it sends it to $(0, T^{-1})$.

The second realization via IndCoh is again just the twist by $\omega_X = (k[T]dT)[1]$. Finally, the ProPerf-realization becomes a triangle

$$k[[T]] \to k[T][2] \to N.$$

REMARK 8.39. We should also translate between the functors f^*_{Dmod}, $\otimes_{\mathrm{Dmod}}$ and $f_{!,\mathrm{Dmod}}$ from the 6-functor formalism Dmod constructed above, and the usual functors from the literature. We take Bernstein's notes [**Ber**] as our reference. Bernstein takes the perspective of left D-modules as the primary one, and we would like to follow him. However, Bernstein also identifies left and right D-modules on any smooth variety X, but he does so by twisting with Ω^d_X, not by $\omega_X = \Omega^d_X[d]$. This means that his identification between left and right D-modules differs from our identification by a shift by $d = \dim(X)$. Unfortunately, naively identifying left D-modules in our theory and in Bernstein's theory is not compatible with the functors he defines – they will involve degree shifts. Thus, for the purposes of the comparison between the theories, we use the perspective of right D-modules.

This way, the functor $f^*_{\mathrm{Dmod}} = f^!_{\mathrm{GR}}$ matches the functor $f^!_{\mathrm{Be}}$ (Be for Bernstein). For him, this is a functor of left D-modules that is a shift by $\dim(X) - \dim(Y)$ of the naive pullback functor, but his identification of left and right D-modules undoes this twist. Moreover, the functor $f_{!,\mathrm{Dmod}} = f_{*,\mathrm{GR}}$ agrees with the functor $f_{*,\mathrm{Be}}$. We note that Bernstein also occasionally writes $f_{+,\mathrm{Be}}$, but this is just $f_{*,\mathrm{Be}}$ on the level of abelian categories instead of derived categories. The tensor product on D-modules introduced in [**Ber**, p. 28] agrees with $\otimes$.

If we would use the more natural identification between $\mathrm{Dmod}(X)$ and left D-modules, then f^*_{Dmod} would correspond to Bernstein's naive pullback functor $f^{\triangle}_{\mathrm{Be}}$, while $f_{!,\mathrm{Dmod}}$ corresponds to $f_{*,\mathrm{Be}}[\dim(Y) - \dim(X)]$.

REMARK 8.40. Continuing the previous remark, the usual discussion of D-module 6 functors proceeds in the following way. First, one has the big category of all D-modules, on which functors $f^!$ and f_* are introduced. Then one restricts to the coherent D-modules, which have a selfduality $\mathbb{D}$, and then tries to dualize $f^!$ and f_* to get functors f^* and $f_!$. These are in general only well-defined as functors to the Pro-category, yielding $\mathrm{Pro}(\mathrm{Dmod}(X)^\omega)$ with $\otimes$, f^* and $f_!$-functors. We note that this exactly agrees with the $\otimes$, f^* and $f_!$ we defined on $\mathrm{Dmod}(X)^{\mathrm{op}} \cong \mathrm{Pro}(\mathrm{Dmod}(X)^\omega)$.

Then one defines the class of holonomic D-modules and shows that this is stable under $\otimes$, f^*, $f_!$ and their right adjoints (which, for abstract reasons, are also Verdier duals of $f_!$ and f^*). The resulting 6-functor formalism of holonomic D-modules will then satisfy excision, étale maps are cohomologically étale, proper maps are cohomologically proper, and smooth maps are cohomologically smooth, so is a "usual" 6-functor formalism. In particular, in our above example of $X = \mathbb{A}^1_k$ with $i : Z = \{0\} \subset X$ and open complement $j : U \to X$ (using standard notation now!), the triangle

$$j_!(1) \to 1 \to i_*(1)$$

realizes in the IndCoh-realization to

$$(k[T] \oplus k[T^{\pm 1}]/k[T])dT[-1] \to k[T]dT[-1] \to (k[T^{\pm 1}]/k[T])dT,$$

with the D-module structure on $k[T] \oplus k[T^{\pm 1}]/k[T]$ discussed in Example 8.38 (with $\partial_T(T^0, 0) = (0, T^{-1})$). We note that all modules are concentrated in one degree, so the shifted sequence

$$i_*(1) \to j_!(1)[1] \to 1[1]$$

is exact for the standard t-structure on IndCoh. That same sequence would also be exact in the perverse t-structure on constructible sheaves; this is of course related to the original emergence of the perverse t-structure in the context of D-modules.

As another example, the Verdier dual sequence

$$i_* i^!(1) \to 1 \to j_*(1)$$

realizes to

$$(k[T^{\pm 1}]/k[T])dT[-2] \to k[T]dT[-1] \to k[T^{\pm 1}]dT[-1].$$

Note that $i^!(1) \cong 1[-2]$; we see that the shifted sequence

$$1[1] \to j_*(1)[1] \to i_* i^!(1)[2]$$

is exact for the standard t-structure on IndCoh.

9. Lecture IX: Solid Modules

10. Lecture X: Ring Stacks

11. Lecture XI: Motivic sheaves

12. Lecture XII: A formalism related to arithmetic D-modules

13. Miscellaneous

This section consists of some remarks and examples of 6-functor formalisms that did not really fit anywhere.

13.1. $C = *$**.** As a first example, let us take for $C = *$ the trivial category with just one object (and one morphism). A 3-functor formalism is a symmetric monoidal ∞-category D. There seems to be nothing to say in this case, but actually there is something to say once one passes to stacks. Namely, the category of sheaves of anima on C is just the ∞-category of anima An, and if D is presentable, then D determines uniquely a 6-functor formalism on An, taking any $X \in \mathrm{An}$ to

$$D(X) = \mathrm{Fun}(X, D),$$

i.e. "X-parametrized objects of D". If D is the symmetric monoidal ∞-category of spectra, this is the notion of parametrized spectrum of May-Sigurdsson [**MS06**].

Exercise 13.1. Show that the class of morphisms $\tilde{E}$ allowed by Theorem 4.20 is the class of morphisms $f : X \to Y$ of anima that on every connected component of X are n-truncated for some n.

Exercise 13.2. Show that in the 6-functor formalism from Theorem 4.20, all maps in $\tilde{E}$ are cohomologically étale.

One can in fact directly construct a 6-functor formalism on An by sending X to $D(X) = \mathrm{Fun}(X, D)$ and taking for I all maps (so also E consists of all maps), while P consists only of isomorphisms. This makes $f_!$ in general the left adjoint of f^*, and it extends the formalism coming from Theorem 4.20 to not necessarily truncated morphisms.

The class of cohomologically proper morphisms depends on a lot on D. In general, if D is stable (or just preadditive), then all 0-truncated maps with finite fibres are D-cohomologically proper, and this is a way of saying that finite coproducts agree with finite products in D. This means that 1-truncated maps whose fibres have finite automorphism groups are "cohomologically separated", yielding a natural transformation $f_! \to f_*$ for such f. In particular, for a finite group G and $f : BG \to *$, one has an identification between $D(BG)$ and G-equivariant objects in D (in practice, this is the derived category of G-representations), and $f_!$ is G-homology, f_* is G-cohomology, and $f_! \to f_*$ yields the norm map from G-homology to G-cohomology. If the order of G is invertible in $\mathrm{End}_D(1_D)$, this is an isomorphism, but there are also other situations, notably if D is the ∞-category of $K(1)$-local spectra, where this is related to the telescopic Tate vanishing, see for example [**CM17**] for a quick proof. If this map is an isomorphism, one gets resulting comparison maps for 2-truncated maps, etc.

In particular, a phenomenon known as "ambidexterity" in algebraic topology gives the following theorem:

Theorem 13.3 (Hopkins–Lurie, [**HL13**]). *Let D be the symmetric monoidal ∞-category of $K(n)$-local spectra for $n \geq 1$ (and some implicit prime p). Then any n-truncated map $f : X \to Y$ of anima all of whose fibres have finite π_i, for $i = 0, \ldots, n$, is D-cohomologically proper.*

On another note, one can wonder for which $X \in \mathrm{An}$ the object $1_X \in D(X)$ is f-proper for the projection $f : X \to *$. Note that after Proposition 6.9, we constructed a general transformation

$$f_!(- \otimes \mathbb{D}_X) \to f_*$$

for a certain object $\mathbb{D}_X \in D(X)$ which is in fact exactly the Spivak–Klein dualizing object when $D = \mathrm{Sp}$, and the map is exactly the twisted norm map; see [**NS18**, Section I.4] for an account. Here, the left-hand side is in fact the colimit-preserving approximation to f_*, and so the map is an isomorphism if and only if f_* preserves all direct sums. This is the case, in particular, if X is a compact object of An. More generally, for any "finitely dominated" map of anima $f : X \to Y$, the sheaf 1_X is f-proper.

A lot of work in (parametrized) homotopy theory can then be recast in this language, and I will not attempt to do the vast literature justice due to ignorance on my side. Let me just cite, as an one example with a similar point of view to these notes, the paper by Cnossen [**Cno23**]; see also its introduction and extended bibliography.

Remark 13.4. Generalizing this example, one can take for C the category of transitive G-sets, for some (abstract, say) group G, where one again takes for I all morphisms (so also E consists of all morphisms), while P consists only of isomorphisms. In that case, Elmendorf's theorem [**Elm83**] says that the ∞-category of presheaves of anima on C is exactly the ∞-category of "G-spaces". Moreover, 6-functor formalisms on C with values in Pr^L are G-symmetric monoidal presentable ∞-categories, i.e. functors from C^{op} to symmetric monoidal presentable ∞-categories. One can then again extend such 6-functor formalisms to all "G-spaces". I believe much of the previous discussion can then be generalized, and is related to a lot of work on (genuine) G-equivariant homotopy theory.

13.2. $C = \mathrm{ProFin}$**.** Next, let us consider $C = \mathrm{ProFin}$. As morphisms, we take for P all maps (so also E consists of all maps), while for I we could in principle allow open immersions, but we could also just restrict to isomorphisms. There are many possible 6-functor formalisms, so let us restrict to the case of the functor taking any $S \in \mathrm{ProFin}$ to the stable ∞-category $D(S, k)$ of sheaves on S with values in modules over some E_∞-ring k. Note that sheaves on S are just functors taking any open closed subset $U \subset S$ to the value on S, subject to taking finite disjoint unions to finite products. This is very much a finitary condition, and using this it is trivial to prove proper base change, so this functor indeed extends to a 6-functor formalism.

Moreover, the functor $S \mapsto D(S, k)$ is a hypersheaf for the Grothendieck topology used in condensed mathematics, i.e. covers are generated by finite families of jointly surjective maps. We can thus extend $X \mapsto D(X, k)$ to all condensed anima; and if one possibly slightly restricts the Grothendieck topology (we did not know whether all covers in condensed mathematics satisfy universal !-descent) we also get an extended 6-functor formalism. This applies in particular to locally profinite sets, and for open immersions of such, the functor $f_!$ is defined and agrees with the left adjoint of f^* (i.e. they are cohomologically étale). To see this, cover locally profinite sets via open and closed subsets of profinite sets, and use that $f_!$ commutes with all direct sums to see that it must be the expected functor.[14]

We also note that one can show that the morphism from the Cantor set to the interval is of universal !-descent, in fact it satisfies the hypotheses of Proposition 6.19. Thus, for finite-dimensional compact Hausdorff spaces, the !-functors are defined, and behave as expected. On the other hand, for a Hilbert cube, the !-functors are not defined.

13.3. $C = \mathrm{CHaus}$**.** Now consider $C = \mathrm{CHaus}$, compact Hausdorff spaces. All morphisms are in P, but only isomorphisms in I, and we again use the functor $X \mapsto D(X, k)$, the ∞-category of

[14]Thanks to Clausen for explaining this!

sheaves on X with values in $D(k)$, for some E_∞-ring k. Again, we could also start with locally compact Hausdorff spaces, but on the level of sheaves of anima, the distinction disappears.

In this case, the resulting D-topology in fact depends on k, and certainly not all covers in the sense of condensed mathematics are allowed; for example, the cover of the Hilbert cube by a Cantor set is not allowed. However, if X is finite-dimensional, then one can cover it by a Cantor set in the D-topology. Thus, the formalism from this section and the previous section agree when specialized to finite-dimensional spaces, but in general they differ.

We note that in both formalisms, with $C = \mathrm{ProFin}$ and $C = \mathrm{CHaus}$, the ∞-category of sheaves of anima mixes purely homotopy-theoretic spaces with actual topological spaces. So for a manifold X, one also has its associated anima $|X|$, with a map $f : X \to |X|$, and then $D(X)$ consists of all sheaves on X, while $D(|X|)$ is equivalent, via pullback along f, to the locally constant sheaves in $D(X)$. Moreover, the truncated map $f_n : X \to \tau_{\leq n}|X|$ is also such that $f_{n!}$ is defined. Unfortunately, we do not currently see how to define $f_!$.

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