On ∞-Lie

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Abstract

We discuss actions of Lie n-groups and the corresponding action Lie n-groupoids; discuss actions of Lie n-algebras (L_∞-algebras) and the corresponding action Lie n-algebroids; and discuss the relation between the two by integration and differentiation.

As an example of interest, we discuss the BRST complex that appears in quantum field theory. We describe it as the Chevalley-Eilenberg algebra of the Lie n-algebroid which linearizes the action n-groupoid (the homotopy quotient) of a gauge n-group acting on the space of fields. This identifies the ghosts-of-ghosts of degree k as the cotangents to the space of k-morphisms of this action n-groupoid.

Several separate aspects of what we say here are essentially "well known" to those who know it well. But a coherent description as attempted here is certainly missing in the literature and deserves to be better known.

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Unreasonable effectiveness of differential $\mathbb{N}$-graded algebra? In various applications of differential geometry – notably in quantum field theory – (super) differential graded-commutative algebras – "(s)DGCA"s – play a central role. The BRST-BV complex [31, 15, 18] is the prominent and also to some extent the universal example.

But why? What do DGCAs mean?

$$\begin{array}{c}\text{(s)DGCA}s\\ \text{why?}\end{array}$$

Part of the answer has a nice explanation in terms of supergeometry (compare also the discussion in section 2.3). As observed maybe first in [19] and emphasized for instance in [33], an action of $\mathrm{Aut}_{\mathrm{Supermanifolds}}(\mathbb{R}^{0|1})$ on any supermanifold X is the same as a differential $\mathbb{N}$-graded structure on $C^\infty(X)$.

DGCAs from smooth spaces: differential forms. These algebras are usually $\mathbb{N}$-graded (or $\mathbb{Z}$-graded but then with trivial cohomology in negative degrees), or $\mathbb{N} \times \mathbb{Z}_2$-graded if in the context of super-geometry. A source of $\mathbb{N}(\times\mathbb{Z}_2)$-graded commutative (super) algebras are algebras of differential (super) forms on smooth (super) spaces

$$\text{(s)SmoothSpaces} \xrightarrow{\Omega^\bullet(-)} \underset{\text{why?}}{\text{(s)DGCA}s} \;.$$

Notice that differential forms on a manifold X are the function algebra on the supermanifold called the odd tangent bundle $\Pi T X$ of X. This, in turn, is the mapping space from $\mathbb{R}^{0|1}$ into X

$$\Omega^\bullet(X) \simeq C^\infty(\Pi T X) \simeq C^\infty(\mathrm{hom}_{\mathrm{Supermanifolds}}(\mathbb{R}^{0|1}, X))$$

and hence naturally comes with an action of $\mathrm{Aut}(\mathbb{R}^{0|1})$, which indeed corresponds precisely to the $\mathbb{N}$-graded differential deRham structure on $\Omega^\bullet(x)$.

In a way this is already the universal example: the functor $\Omega^\bullet$ has an adjoint

$$\text{(s)SmoothSpaces} \underset{S}{\overset{\Omega^\bullet(-)}{\rightleftarrows}} \underset{\text{why?}}{\text{(s)DGCA}s}$$

which sends each (s)DGCA A to a smooth (super) space whose (s)DGCA of forms approximates A:

$$A \hookrightarrow \Omega^\bullet(S(A))\,.$$

DGCAs from L(ie)∞-algebras. We can understand $S(A)$ as the *classifying space* of "A-valued" differential forms, in the following sense [25]:

another source of (s)DGCAs are (super) $L(\mathrm{ie})_\infty$-algebras. These generalizes Lie algebras as ∞-group ("∞-categorical groups") generalize groups, which in turn follows the generalization of groupoids to ∞-groupoids:

categorical dimension	$n=1$	$n=\infty$
infinitesimal	Lie algebra	L_∞-algebra
finite	smooth group	smooth ∞-group

The most natural incarnation of a (super) L_∞-algebra $\mathfrak{g}$ is in terms of its Chevalley-Eilenberg (s)DGCA which we denote $\mathrm{CE}(\mathfrak{g})$:

an finite dimensional (super) L_∞-algebra is a $\mathbb{N}_+$-graded (super) vector space $\mathfrak{g}^*$ together with a degree $+1$ graded derivation (the "dual (higher) brackets")

$$d_{\mathrm{CE}(\mathfrak{g})} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

such that $d^2 = 0$ (the (dual of the) coherently weakened / strong homotopy Jacobi identity [20]);

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})})\,.$$

Hence

$$\begin{tikzcd} \text{(s)SmoothSpaces} \arrow[r, shift left, "\Omega^\bullet(-)"] & \text{(s)DGCA}s \arrow[l, shift left, "S"] & & \text{(s)}L_\infty \arrow[ll, "\mathrm{CE}(-)"'] \end{tikzcd}\,.$$

why?

L_∞-algebra valued differential forms. The following cocept turns out to be of importance: Given a smooth (super) space Y the notion of flat (super) Lie-algebra valued forms on Y generalizes to (super) L_∞-algebras $\mathfrak{g}$ [25] by setting:

$$\Omega^\bullet_{\mathrm{flat}}(Y,\mathfrak{g}) := \mathrm{Hom}_{\mathrm{(s)DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))\,.$$

A simple important class of examples are the L_∞-algebras $b^{n-1}\mathfrak{u}(1)$:

$$\Omega^\bullet(Y, b^{n-1}\mathfrak{u}(1)) = \Omega^n_{\mathrm{closed}}(Y)\,.$$

Another important class of examples are Chern-Simons (super) L_∞-algebras $\mathrm{cs}_P(\mathfrak{g})$ coming from transgressive (super) L_∞-algebra invariant polynomials P [25]:

$$\Omega^\bullet(Y, \mathrm{cs}_P(\mathfrak{g})) = \left\{ \begin{array}{l} A \in \Omega^\bullet(Y,\mathfrak{g}) \\ B \in \Omega^n(Y) \\ C = \mathrm{CS}_P(A) + dB \end{array} \right\}\,.$$

These come from the "String-like" extensions [25] which we will mention in a moment.

The fact that $S(-)$ is adjoint to $\Omega^\bullet(-)$ says that $S(\mathrm{CE}(\mathfrak{g}))$ is the classifying space for such forms:

$$\mathrm{Hom}_{\mathrm{(s)DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y)) \simeq \mathrm{Hom}_{\mathrm{(s)SmoothSpaces}}(Y, S(\mathrm{CE}(\mathfrak{g})))\,.$$

Path n-groupoids. Using this fact one can work out what the smooth (super) ∞-groupoids associated to an L_∞-algebra are: to any smooth space X we can associate its fundamental path n-groupoid $\Pi_n(X)$

$$\begin{tikzcd} \text{(s)Smooth}n\text{Grpd} & \text{(s)SmoothSpaces} \arrow[l, "\Pi_n(-)"'] \arrow[r, shift left, "\Omega^\bullet(-)"] & \text{(s)DGCA}s \arrow[l, shift left, "S"] & \text{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \end{tikzcd} .$$

why?

For instance we can model smooth (super) ∞-groupoids by Kan simplicial smooth (super) spaces, in which case $\Pi_\infty(-)$ is just the simplicial space of singular simplices:

$$(\Pi_\infty(X))_n = \mathrm{Hom}_{\mathrm{SmoothSpaces}}(\Delta^n, X)\,,$$

where Δ^n is the standard n-simplex.

The map from (s)DGCAs to smooth (super) ∞-groupoids thus obtained

$$\begin{tikzcd} \text{(s)Smooth}n\text{Grpd} & \text{(s)SmoothSpaces} \arrow[l, "\Pi_n(-)"'] \arrow[r, shift left, "\Omega^\bullet(-)"] & \text{(s)DGCA}s \arrow[l, shift left, "S"] \arrow[ll, bend left, "\text{Sullivan model}"] & \text{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \end{tikzcd}$$

is the construction of *Sullivan models* in rational homotopy theory [14].

Notice for the following that if $X = S(\mathrm{CE}(\mathfrak{g}))$ happens to be the classifying space for flat $\mathfrak{g}$-valued forms, then the space of n-simplices of $\Pi_\infty(S(\mathrm{CE}(\mathfrak{g})))$ is that of flat $\mathfrak{g}$-valued k-forms on Δ^n.

$$(\Pi_\infty(S(\mathrm{CE}(\mathfrak{g}))))_n = \mathrm{Hom}_{\mathrm{SmoothSpaces}}(\Delta^n, S(\mathrm{CE}(\mathfrak{g}))) \simeq \Omega^\bullet_{\mathrm{flat}}(\Delta^n, \mathfrak{h})\,.$$

Integration of L_∞-algebras. While Sullivan models have been a standard tool for decades, it was only in [13] that it was noticed that – since the image of the Sullivan construction is not just a simplicial space, but actually a *Kan* simplicial space, hence an ∞-groupoid – this can be read as the process of integrating L_∞-algebras to ∞-groups

$$\begin{tikzcd} \text{(s)Smooth}n\text{Grpd} & \text{(s)SmoothSpaces} \arrow[l, "\Pi_n(-)"'] \arrow[r, shift left, "\Omega^\bullet(-)"] & \text{(s)DGCA}s \arrow[l, shift left, "S"] & \text{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \arrow[lll, bend left, "\text{integration}"] \end{tikzcd} .$$

We can use various models for higher groupoids, depending on taste and on convenience in certain applications. Remarkably, it is sufficient to probe a smooth space already by strict path n-groupoids [4, 29]: we write $\Pi_n(X)$ for the strict n-groupoid whose $(k < n)$-morphisms are $\mathrm{Hom}_{\mathrm{SmoothSpaces}}(D^k, X)$ modulo thin homotopy, and whose n-morphisms are $\mathrm{Hom}_{\mathrm{SmoothSpaces}}(D^n, X)$ modulo homotopy (details for $N = 2$ are in [29], the higher versions can be defined iteratively using paths-of n-paths).

Remark: Nonabelian differential cohomology: ∞-bundles with connection Working with strict smooth (super) n-groupoids throughout has the big advantage that it allows us to use Ross Street's theory of descent for ω-category valued presheaves [27], thus using the ∞-Lie theory we are discussing here in a theory of nonabelian differential cohomology [28, 30, 26].)

Examples for intgration of Lie n-algebras. In particular, if $\mathfrak{g}$ is an ordinary Lie algebra, then the integation procedure

$$\mathfrak{g} \mapsto \Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G$$

reproduces the "path method" for integration of Lie algebras [12] and produces the 1-object groupoid

$$\mathbf{B}G := \left\{ \bullet \xrightarrow{g} \bullet \, | g \in G \right\}$$

coming from the simply connected Lie group G integrating $\mathfrak{g}$.

If instead we take the universal example of a semisimple Lie 2-algebra, the String Lie 2-algebra $\mathfrak{g}_\mu$, we find [16] that

$$\mathfrak{g}_\mu \mapsto \Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu))) = \mathbf{B}\mathrm{String}(G)$$

is the strict String Lie 2-group [3] (whose realized nerve models the topological String group [34]) arising here in the form secretly appearing in [8, 9].

Gradings and categorical dimension. For our purposes, one important phenomenon to notice here is:

Observation. *The degree k generators in* $\mathrm{CE}(\mathfrak{g})$ *encode the degree k-morphisms in the integrating n-groupoid* $\Pi_n(S(\mathrm{CE}(\mathfrak{g})))$.

To some extent this is an implication of the Dold-Kan theorem in the form [5, 6, 7] which says that strict ∞-categories (aka ω-categories) internal to A-modules are equivalent to complexes of A-modules

$$\omega\mathrm{Cat}(A) \simeq \mathrm{Ch}^\bullet_+(A)\,;$$

namely L_∞-algebras can be regarded as ω-categories internal to vector spaces and equipped with a coherently Jacobi bracket ω-functor. Under this equivalence the A-module of k-morphisms maps to the degree k part of the cochain complex.

This is the context in which to understand that all L_∞-algebras with their underlying $\mathbb{N}_+$-graded vector spaces integrate to one-object ∞-groupoids, and hence to ∞-groups.

L_∞-algebroids. The way to generalize to the many-object versions is indicated by the concept of *Lie-Rinehart* pairs [22], see [17], which model *Lie algebroids* [12]: in a Lie-Rinehart pair a Lie algebra $\mathfrak{g}$ is accompanied by an associative commutative algebra A, with both being modules over each other in the obvious compatible way modeled on the archetypical example of the *tangent Lie algebroid*

$$(\mathfrak{g}, A) = (\Gamma(TX), C^\infty(X))$$

of a smooth space X.

From the nature of the Chevally-Eilenberg algebra of a Lie-Rinehart pair we deduce the general definition of CE-algebra for "homotopy Lie-Rinehart pairs" [17] or, as we shall say: L_∞-algebroids.

	single object	**many object**
finite: **Lie ∞-groupoids**	$\mathrm{Gr} = \mathbf{B}G$, G an ∞-group $C^\infty(\mathrm{Obj}(\mathbf{B}G)) = \mathbb{R}$	Gr arbitrary $C^\infty(\mathrm{Obj}(\mathrm{Gr})) = A$
infinitesimal: **Lie ∞-algebroids**	$\mathfrak{g} = (\mathfrak{g}, \mathbb{R})$ a Lie ∞-algebra, $\mathrm{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d)$	$(\mathfrak{g}, V)$ a Lie ∞-algebroid, $\mathrm{CE}_A(\mathfrak{g}) = (\wedge_A^\bullet \mathfrak{g}^*, d)$

Definition. *A Lie (super) ∞-algebroid $(\mathfrak{g}, A)$ consists of*

- *a commutative, associative, unital (super) algebra A;*
- *an $\mathbb{N}$-graded (or possibly $\mathbb{Z}$-graded) A-module $\mathfrak{g}^*$ such that $V_0 = A$;*
- *on $\wedge_A^\infty \mathfrak{g}^*$ regarded as a graded commutative algebra over the ground field (!) a graded degree +1 derivation*

$$d : \wedge_A^\infty \mathfrak{g}^* \to \wedge_A^\infty \mathfrak{g}^*$$

such that $d^2 = 0$. We address

$$\mathrm{CE}_A(\mathfrak{g}) := (\wedge_A^\bullet \mathfrak{g}^*, d)$$

as the Chevalley-Eileneberg (s)DGCA of $(\mathfrak{g}, A)$. If we demand all A-modules to be not just of finite rank but also projective, then this reproduces the notion of differential graded (super) manifold (DG manifold), sometime also called an "NQ-supermanifold" ($\mathbb{N}$-graded supermanifolds equipped with a degree +1 graded nilpotent derivation), e.g. [32].

Lie (1-)algebroids: Lie-Rinehart pairs. Lie-Rinehart pairs are the $n = 1$ examples ($\mathfrak{g}^*$ concentrated in degree 0 and 1) with

$$(d\omega)(x, y) = \rho(x)\omega(y) - \rho(y)\omega(x) + \omega([x, y])$$

for all $\omega \in \mathfrak{g}^*$, for all $x, y \in \mathfrak{g}$ and $\rho : \mathfrak{g} \otimes A \to A$ the action of $\mathfrak{g}$ on A.

Plugging in the tangent Lie algebroid Lie-Rinehart pair $(\Gamma(TX), C^\infty(X))$ with $\Gamma(TX)$ regarded as being in degree 1 and $C^\infty(X)$ in degree 0, we find that the Chevalley-Eilenberg algebra of the tangent Lie algebroid of a smooth space X is nothing but the deRham complex:

$$\mathrm{CE}(\Gamma(TX), C^\infty(X)) = \Omega^\bullet(X)\,.$$

For interpreting ∞-Lie theory it is helpful to notice that the tangent Lie algebroid of X integrates to the fundamental groupoid of X

$$(\Gamma(TX), C^\infty(X)) \mapsto \Pi_n(S(\mathrm{CE}_{C^\infty(X)}(\Gamma(TX)))) = \Pi_n(X)\,,$$

simply because the contravariant functor $\Omega^\bullet$ is full and faithful [**?**].

Conversely, we have a systematic way to differentiate smooth n-groupoids by making use of the functor S on the left of

$$\begin{tikzcd}
\mathrm{(s)Smooth}n\mathrm{Grpd} \arrow[r, shift left, "S"] & \mathrm{(s)SmoothSpaces} \arrow[l, shift left, "\Pi_n(-)"] \arrow[r, shift left, "\Omega^\bullet(-)"] & \mathrm{(s)DGCA}s \arrow[l, shift left, "S"] & \mathrm{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \arrow[lll, bend left, "\text{integration}"]
\end{tikzcd}$$

which probes any n-groupoid Gr with path groupoids:

$$S(\mathrm{Gr}) : U \mapsto \mathrm{Hom}_{\mathrm{Smooth}n\mathrm{Grpd}}(\Pi_n(U), \mathrm{Gr})\,.$$

Differentiation of smooth ∞-groupoids. The composite morphism from (super) smooth n-groupoids to (s)DGCAs we get this way is differentiation in ∞-Lie theory sending smooth n-groupoids to the CE-algebras of the L_∞-algebroids linearizing them.

$$\begin{tikzcd}
\mathrm{(s)Smooth}n\mathrm{Grpd} \arrow[r, shift left, "S"] \arrow[rr, bend left, "\text{differentiation}"] & \mathrm{(s)SmoothSpaces} \arrow[l, shift left, "\Pi_n(-)"] \arrow[r, shift left, "\Omega^\bullet(-)"] & \mathrm{(s)DGCA}s \arrow[l, shift left, "S"] & \mathrm{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \arrow[lll, bend left, "\text{integration}"]
\end{tikzcd}$$

This can be read as essentially being the procedure given in [33].

As an example of this, notice that an aspect of the central result of [28, 29] says that for G is a strict Lie 2-group and $\mathfrak{g}$ the corresponding strict Lie 2-algebra we have

$$\mathrm{Hom}_{\mathrm{Smooth2Grpd}}(\Pi_2(X), \mathbf{B}G) = \Omega^\bullet_{\mathrm{flat}}(X, \mathfrak{g}) = \mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(X)) = \mathrm{Hom}_{\mathrm{SmoothSpaces}}(X, S(\mathrm{CE}(\mathfrak{g})))\,.$$

But this means that

$$S(\mathbf{B}G) = S(\mathrm{CE}(\mathfrak{g}))$$

for G a strict Lie 2-group with strict Lie 2-algebra $\mathfrak{g}$. More general cases should work analogously.

Actions and their homotopy quotients. Finally, we want to understand *actions* of L_∞-algebras. Such actions have been defined in [20] and used in [17, 18] in our sense here, but in terms of codifferential coalgebras. We can reformulate this somewhat more conceptually by noticing the concept of *action n-groupoids* and their relation to universal bundles:

as a synthesis of [23] and [10] we observe that given an action of a group G in a set V, which is a functor

$$\rho : \mathbf{B}G \to \mathrm{Set}$$

we can pull back the "universal Set bundle" to obtain

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] & s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ V//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\ \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \end{tikzcd},$$

where the sequence of groupoids on the left characterizes the *action groupoid*

$$V//G := (\ G \times V \underset{\rho}{\overset{p_1}{\rightrightarrows}} V\ ),$$

which can be regarded as the weak quotient or homotopy quotient of V by the action of G. In particular, if ρ is the fundamental representation of G on itself, or of a subgroup of G, we get

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ G//H \arrow[r, hook] \arrow[d] & G//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\ \mathbf{B}H \arrow[r, hook] & \mathbf{B}G \arrow[r, "\rho_{\mathrm{fund}}"] & \mathrm{Set} \end{tikzcd}.$$

which, due to an old result of Segal (reviewed in [23]) is the groupoid incarnation of the universal G-bundle over BG, or its pullback to BH.

This diagrammatics straightforwardly generalizes to higher n. In [23] this situation was studied for 2-groups. (More details on the relation to universal

2-bundles are in preparation [24].) We can go further and think of action ∞-groupoids arising this way.

The important aspect for our purpose of this is that actions of G on V come from extensions

$$V \hookrightarrow V//G \twoheadrightarrow \mathbf{B}G \ .$$

Action L_∞-algebroids. This has an obvious analog for L_∞-algebras:

Definition. *For A an algebra as above, V^* a complex of finite rank A-modules such that $V_0 = A$ and $\mathfrak{g}$ an L_∞-algebra, an action of $\mathfrak{g}$ on V is a Lie ∞-algebroid with CE-algebra $\mathrm{CE}_A(\mathfrak{g}, V)$ such that there is an exact sequence of DGCAs*

$$\begin{tikzcd} \mathrm{CE}(V) & \mathrm{CE}(\mathfrak{g}, V) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} \ .$$

Example: the BRST complex. We claim that if V is a Koszul-Tate resolution (a "space of fields") and $\mathfrak{g}$ the corresponding ("gauge") L_∞-algebra, then the action L_∞-algebroid of $\mathfrak{g}$ acting on V $\mathrm{CE}(\mathfrak{g}, V)$ is the BRST complex [31, 15, 18].

Here the degree k generators of $\mathrm{CE}(\mathfrak{g})$ are what are called the k-fold "ghosts of ghosts". But we have a more geometrical and less ghostly interpretation of these generators now:

the k-fold ghosts-of-ghosts are the cotangents to the space of k-morphisms of the action n-groupoid abtained from the action of the gauge n-group on the space of fields. This action groupoid, in turn, is the weak quotient of the space of fields by the gauge group action.

1.1 Content

The heart of our discussion is definition 3 of the Chevalley-Eilenberg differential algebras $\mathrm{CE}(\mathfrak{g}, V)$ of "action Lie ∞-algebroids" which encode the action of a L(ie) ∞-algebra $\mathfrak{g}$ on a complex V of A-modules, for A an associative algebra. This generalizes the action of Lie algebras on vector spaces to higher categorical dimension and can be thought of as an ∞-zation of Lie-Rinehart pairs [22].

We motivate and apply this definition in the context of "∞-Lie theory": the relation between smooth ∞-groupoids and L_∞-algebroids by integration and differentiation. Various aspects of this have appeared in the literature. We give a unified description in terms of adjunctions between

- differential graded commutative algebras (DGCAs)
- smooth spaces
- smooth n-groupoids

in section 5.

Actions of L_∞-algebras on complexes of vector spaces have been defined before [20], as have their extension to what we we call here action L_∞-algebroids [17, 18]. Our definition is supposed to reproduce that, and add the aspect of actions on complexes of A-modules, for A any associative algebra; and add the dual aspect of the "∞-Chevalley-Eilenberg algebra", such that an important differential graded commutative algebra becomes an example of an action L_∞-algebroid: the BRST complex [15, 31] in quantum field theory.

Our definition 3 is of a different flavor than that in [20]: we do not define an L_∞-action in the "direct" way, but in terms of the corresponding *weak* or *homotopy quotient* that it induces on the thing being acted on – the corresponding L_∞ "action algebroid".

This construction is best illuminated by first considering it for the "finite" (as opposed to "infinitesimal") objects which integrate the "infinitesimal" Lie n-algebras: smooth n-groups. In section 3 we first look at action groupoids induced by actions of groups and discuss how they arise as groupoid-incarnations of (universal) principal bundles, an observation arising as a synthesis of [23] and [10].

The generalization to action n-groupoids induced by actions of n-groups is then conceptually straightforward. (**but at the moment only indicated very briefly).

The main point of this discussion is that it shows that for any action of an n-group G on an $(n-1)$-groupoid V we obtain a short exaxt sequence of n-groupoids

$$V \hookrightarrow V//G \twoheadrightarrow \mathbf{B}G$$

with the action n-groupoid $V//G$ arising as an "extension" of the one-object groupoid version $\mathbf{B}G = \bullet//G$ of the n-group G by the $(n-1)$-groupoid V that it acts on.

Such sequences are nicely amenable to the passage from smooth n-groupoids to Lie ∞-algebroids. The latter we conceive, as described in section 4, in terms of their dual Chevalley-Eilenberg algebras $\mathrm{CE}(-,-)$. The (CE-algebra of the) action L_∞-algebroid corresponding to the action of the L_∞-algebra $\mathfrak{g}$ on a cochain complex V is then essentially defined to be the middle piece in a sequence

$$\mathrm{CE}(V) \twoheadleftarrow \mathrm{CE}(\mathfrak{g}, V) \hookleftarrow \mathrm{CE}(\mathfrak{g})$$

of DGCAs.

In section 4.3 we list examples. In a sense, the most general example of $\mathrm{CE}(\mathfrak{g}, V)$ is what is known as the BRST complex in "higher gauge" quantum field theory (such as the quantum field theory of gauge fields that are locally given by differential n-forms, for some $n \in \mathbb{N}$): these QFTs have a "gauge n-group" and their space of fields is, in general, a complex of A-modules, where A is the algebra of functions on some space of field configurations. The BRST complex is nothing but the action L_∞-algebroid of the L_∞-algebra linearizing the gauge n-group acting on the "space" of fields. From roughly this point of view the BRST complex has also been considered in [18].

A special phenomenon here is that the BRST complex is in general generated not in non-negative but in arbitrary degree. This is noteworthy, since the degree k of a generator in the CE-algebra of an L_∞-algebroid corresponds to the degree k of morphisms in the n-groupoid integrating it – but there are no morphisms of negative degrees in n-groupoids.

But one important point about the BRST complex is that, while it has in general generators in negative degrees, it does not have nontrivial cohomology in negative degrees: the generators in negative degrees just serve to conventiently express a (weak) quotient in degree 0.

We show that after applying our integration procedure to the BRST complex, the result is a smooth n-groupoid whose smooth space of objects is that given by this weak quotient and which is otherwise the action n-groupoid of the gauge n-group acting on the fields.

1.2 Statement of the main differential algebraic point

We define an L_∞-algebroid to be

- an algebra $A = C^\infty(X)$;
- and a finite rank $\mathbb{Z}$-graded A-module $\mathfrak{g}$;
- together with a degree $+1$ graded derivation

  $$d_{\mathrm{CE}_A(\mathfrak{g})} : \wedge_A^\infty \mathfrak{g}^* \to \wedge_A^\infty \mathfrak{g}^*$$

  required to be linear over the ground field (not necessarily A-linear(!)) and such that

  $$(d_{\mathrm{CE}_A(\mathfrak{g})})^2 = 0\,.$$

Here

$$\wedge_A^\infty \mathfrak{g}^* := \mathrm{colim}_{n\in\mathbb{N}} \wedge_A^n \mathfrak{g}^* = \underbrace{A}_{\deg=0} \oplus \underbrace{\mathfrak{g}_1^*}_{\deg=1} \oplus \underbrace{\mathfrak{g}_1^* \wedge \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*}_{\deg=2} \oplus \cdots$$

The duals and all graded symmetric tensor products are over A. We address

$$\mathrm{CE}_A(\mathfrak{g}) := (\wedge_A^\infty \mathfrak{g}^*, d_{\mathrm{CE}_A(\mathfrak{g})})$$

as the corresponding Chevalley-Eilenberg DGCA.

This essentially reproduces the notion of CE-algebras for *homotopy Lie-Rinehart pairs* in [17, 18].

Notice that for $X = \mathrm{pt}$ and $\mathfrak{g}^*$ in non-negative degree this is an L_∞-algebra whose CE-algebra we write $\mathrm{CE}(\mathfrak{g})$ as in [25].

Given just a finite rank complex of A-modules V with $V_0 = A$ we write

$$\mathrm{CE}_A(V) := \wedge_A^\infty V\,.$$

We say that an *action* of an L_∞-algebra $\mathfrak{g}$ on V is an L_∞-algebroid $\mathrm{CE}(\mathfrak{g}, V)$ which sits in an extension

$$\begin{tikzcd}
\mathrm{CE}_A(V) & \mathrm{CE}_A(\mathfrak{g}, V) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

and we address $\mathrm{CE}(\mathfrak{g}, V)$ as the CE-algebra of the *action L_∞-algebroid* of $\mathfrak{g}$ acting on V.

In the case that V is a Koszul-Tate resolution and $\mathfrak{g}$ a compatible L_∞-algebra we claim that $\mathrm{CE}(\mathfrak{g}, V)$ is the corresponding BRST complex, also essentially reproducing [18].

The details are in section 4. The reader not interested in the smooth ∞-groupoids corresponding to L_∞-algebroids should just look at section 4.

1.3 Statement of the main ∞-categorical point

What should be called "∞-Lie theory" – the relation between smooth ∞-groupoids and L_∞-algebroids – is, at its core, the transport back and forth across two adjunctions that relate smooth *spaces* X with

- *finite* paths in these spaces – $\Pi_n(X)$

on the one hand and

- *infinitesimal paths* in these spaces – $\Omega^\bullet(X)$

on the other:

$$\begin{tikzcd}
 & (\mathrm{s})\mathrm{SmoothSpaces} \arrow[dl, shift left, "\Pi_n(-)"] \arrow[dr, shift left, "\Omega^\bullet(-)"] & \\
(\mathrm{s})\mathrm{Smooth}n\mathrm{Grpds} \arrow[ur, shift left, "S"] \arrow[rr, shift left, "\text{differentiation}"] & & (\mathrm{s})\mathrm{DGCAs} \arrow[ul, shift left, "S"] \\
 & & (\mathrm{s})L_\infty \arrow[u, "\mathrm{CE}(-)"'] \arrow[llu, shift left, "\text{integration}"]
\end{tikzcd} .$$

This gives us a "geometric" or "space-wise" interpretation of differential algebraic structures. In particular, it identifies the $\mathbb{N}$-grading prevalent in many contexts with the categorical dimension of higher morphisms.

Of course, as we discuss, saying so is essentially saying "Sullivan model" and saying "Dold-Kan theorem", but it deserves to be said this explicitly.

2 Space and quantity

Since we want to talk about smooth ("Lie") n-groups and their Lie n-algebras, we need a suitably general notion of "smooth spaces".

Our attitude is: find a large enough nice category SmoothSpaces of sufficiently general "smooth spaces" such that all operations in our application that ought to exist do exist. Then *after* the formalism is up and running the way it should, it is time to ask whether any given generalized smooth space appearing in an application is "particularly nice", for instance a quasi-representable sheaf on S, also known as a *diffeological space*, or even a Fréchet manifold, or even a finite dimensional manifold

$$\mathrm{Manifolds} \hookrightarrow \mathrm{FréchetManifolds} \hookrightarrow \mathrm{DiffeologicalSpaces} \hookrightarrow \mathrm{SmoothSpaces} \qquad .$$

$$\xleftarrow{\text{labour is involved in going in this direction}} \quad \begin{array}{c}\text{here abstract nonsense}\\ \text{gives prescriptions}\\ \text{for all desired operations}\end{array}$$

For instance, as discussed in section 5.3, the abstract nonsense gives a prescription for integrating a Lie algebra to a group internal to SmoothSpaces. That prescription happens to reproduce the known "method of integration by paths". For this one proves, see [12], that the resulting group internal to smooth spaces is actually even a group internal to manifolds.

For Lie 1-algebroids abstract nonsense still gives a 1-groupoid internal to smooth spaces by a prescription which coincides with the known integration of Lie algebroids by "A-paths". But now, as also discussed in [12], one finds that not all of the smooth 1-groupoids obtained this way are actually groupoids internal to manifolds.

Similarly, the bulk of the work in [13] is to restrict the abstract integration procedure for L_∞-algebras to those for which the result is an ∞-groupoid internal to manifolds. Alternatively, [16] tries to realize the result internal to Banach spaces.

2.1 Smooth spaces

So we take "smooth spaces" to be sheaves on the "abstract" site S (abstract in that it is not a site of open subsets of a fixed space) whose objects are $\mathbb{R}^n$s for all $n \in \mathbb{N}$, and whose morphisms are all smooth maps between these. Covers in S are the obvious covers of Euclidean spaces by smooth images of Euclidean spaces.

$$\mathrm{Obj}(\mathrm{S}) = \mathbb{N}$$

$$\mathrm{S}(n,m) = \mathrm{SmoothManifolds}(\mathbb{R}^n, \mathbb{R}^m)\,.$$

One could take various slightly different sites (for instance open subsets of Euclidean spaces) and still get an equivalent category of sheaves. But using full Euclidean spaces has a certain conceptual economy to it and nicely generalizes to the case of *super*-smooth spaces:

We take sS to tbe the site whose objects are super-Euclidean spaces and whose morphisms are all smooth maps of supermanifolds between these

$$\mathrm{Obj}(\mathrm{sS}) = \mathbb{N} \times \mathbb{N}$$

$$\mathrm{sS}(n|n', m|m') = \mathrm{SmoothSuperManifolds}(\mathbb{R}^{n|n'}, \mathbb{R}^{m|m'})\,.$$

The category sSmoothSpaces of smooth superspaces is taken to be that of sheaves on sS. Here a cover in sS is a collection of morphisms of $\mathbb{R}^{p|q}$s such that the underlying morphisms of $\mathbb{R}^p$s is a cover.

Of central importance is the smooth (super) space $\Omega^\bullet$ of (super) differential forms, given by the sheaf which assigns

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U)$$

for U any test domain, i.e. any object in S or sS. Here $\Omega^\bullet(U)$ on the right is the *set* of (super)differential forms on U. But of course this set carries the structure of a differential super $\mathbb{N}$-graded commutative algebra (hence $\mathbb{N}$-graded in the ordinary and $\mathbb{N}\times\mathbb{Z}_2$-graded in the super case, with the signs coming from the total degree $\deg_{\mathbb{N}} + \deg_{\mathbb{Z}_2} : \mathbb{N} \times \mathbb{Z}_2 \to \mathbb{Z}_2$). This makes $\Omega^\bullet$ an *ambimorphic* object: it is both a smooth (super) space as well as a differential (super) graded-commutative algebra ((s)DGCA). As such it gives rise to a (contravariant) adjunction between smooth spaces and DGCAs:

$$\text{(s)SmoothSpaces} \underset{S}{\overset{\Omega^\bullet(-)}{\rightleftarrows}} \text{(s)DGCAs}\ . \qquad (1)$$

Here the top morphism sends any (super) smooth space X to the set $\Omega^\bullet(X) := \text{(s)SmoothSpaces}(X, \Omega^\bullet)$ which is naturally equipped with the structure of a (super) DGCA.

The lower morphism sends any DGCA A to the (super) smooth space

$$A \mapsto (S(A) : U \mapsto \mathrm{Hom}_{\mathrm{DGCAs}}(A, \Omega^\bullet(U)))\,.$$

The fact that we have a (contravariant) adjunction means that

$$\mathrm{Hom}_{\text{(s)SmoothSpaces}}(X, S(A)) \simeq \mathrm{Hom}_{\text{(s)DGCAs}}(A, \Omega^\bullet(X))\,. \qquad (2)$$

In section 5.1 we use this in the case that $A = \mathrm{CE}(\mathfrak{g})$ is the Chevalley-Eilenberg algebra of an L_∞-algebra to interpret $S(\mathrm{CE}(\mathfrak{g}))$ as the smooth *classifying space* for smooth $\mathfrak{g}$-valued forms.

This relation between L_∞-algebras and the classifying spaces of the differential forms with values in them, together with the analogous adjunction between smooth spaces and smooth n-groupoids discussed in section 2.2, is seen in section 5 to be at the heart of n-Lie theory: the relation between L_∞-algebras and Lie ∞-groups.

$$\begin{tikzcd}
\text{(s)Smooth}n\text{Grpd} \arrow[r, shift left, "S"] \arrow[rrr, bend left, "\text{differentiation}"] & \text{(s)SmoothSpace} \arrow[l, shift left, "\Pi_n(-)"] \arrow[r, shift left, "\Omega^\bullet(-)"] & \text{(s)DGCAs} \arrow[l, shift left, "S"] & \text{(s)}L_\infty \arrow[l, "\mathrm{CE}(-)"'] \arrow[lll, bend left, "\text{integration}"]
\end{tikzcd}$$

Figure 1: **The heart of n-Lie theory.** The two *ambimorphic* objects $\Omega^\bullet$ (the infinitesimal paths) and Π_n (the finite paths) induce adjunctions between (super) smooth spaces on the one hand and (super) DGCAs respectively smooth (super) n-groupoids on the other. Integration of (super) Lie n-algebras is passing along these adjunctions from right to left. Differentiation is passing from left to right. This is discussed in section 5.

2.2 Smooth n-categories

By n-category we mean, throughout, *strict* n-categories. Strict ∞-categories are usually called ω*-categories*. A strict n-category is an ω-category with only identity k-morphisms for $k > n$.

Definition 1 *A* smooth (super) ω-category *is an ω-category internal to smooth (super) spaces.*

We write (s)SmoothωCat for the ωCat-category of smooth (super) n-categories [11].

Remark. The concept of a super n-groupoid as an n-groupoid internal to superspaces is immediate, but apparently has not been considered much in the literature (except for $n = 1$ and a single object, in which case we have a super group). 1-Groupoids internal to super*manifolds* have been considered in [21].

The central examples are fundamental n-groupoids

$$\Pi_n : \text{SmoothSpaces} \to \text{Smooth}n\text{Categories}$$

of smooth spaces: the $(k < n)$-morphisms of $\Pi_n(X)$ are thin homotopy classes of smooth images of the standard k-disk in X, the n-morphisms are homotopy classes of the n-disk in X.

The fundamental path n-groupoid Π_n is also an ambimorphic object: the assignment

$$\Pi_n : U \mapsto \Pi_n(U)$$

is a smooth n-groupoid valued co-presheaf. It hence [**URS: need to check this, careful here] induces an adjunction between smooth spaces and smooth n-groupoids

$$\text{(s)Smooth}n\text{Grpd} \underset{\Pi_n(-)}{\overset{S}{\rightleftarrows}} \text{(s)SmoothSpace} \; .$$

Here the map from smooth n-groupoids to smooth spaces is obtained by probing a given smooth n-groupoid by path n-groupoids.

$$\begin{array}{llcl} S: & \text{(s)Smooth}n\text{Grp} & \xrightarrow{\quad S \quad} & \text{(s)SmoothSpace} \\ & C & \longmapsto & (U \mapsto \text{(s)Smooth}n\text{Grpd}(\Pi_n(U), C)) \end{array}$$

2.3 Categorification and superification

There is a curious similarity between supermanifolds and L_∞-algebroids. In fact, in a big part of the literature what we address as L_∞-algebroids here are addressed as "NQ-supermanifolds".

A very popular definition of supermanifolds is as topological spaces together with a sheaf of $\mathbb{Z}_2$-graded algebras. One then derives that every supermanifold X can be identified with a smooth vector bundle $E \to |X|$ over the underlying manifold $|X|$ and every morphism of supermanifolds $X \xrightarrow{f} X'$ with a morphism

$$\wedge^\bullet_{C^\infty(|X|)} \underbrace{\Gamma(E)}_{\text{deg=odd}} \xleftarrow{f^*} \wedge^\bullet_{C^\infty(|X|)} \underbrace{\Gamma(E)}_{\text{deg=odd}}$$

between the $\mathbb{Z}_2$-graded commutative algebras of exterior powers of sections of this vector bundle. The main point here being that there are more such GCA morphism than morphisms of the underlying vector bundles.

We can reformulate this by saying that the algebras of functions on a supermanifold are the $\mathbb{Z}_2$-graded-commutative algebras of the form

$$C^\infty(X) = \wedge^\infty_A(\underbrace{A}_{\text{deg=even}} \oplus \underbrace{V}_{\text{deg=odd}}) = \wedge^\bullet_A \underbrace{V}_{\text{deg=odd}} = A \oplus V \oplus V \wedge_A V \oplus \cdots \; ,$$

where $A = C^\infty(|X|)$ for some manifold $|X|$ and where V is a projective module of finite rank over A.

If the projective A-module V is $\mathbb{N}_+$-graded, we can form the $\mathbb{N}$-graded commutative algebra

$$\wedge^\infty_A(A \oplus V) = \wedge^\bullet_A V \; .$$

This is often addressed as (the algebra of functions on) an "N-supermanifold".

If, furthermore, there is a degree $+1$ derivation, linear over the ground field on this $\mathbb{N}$-graded commutative algebra $\wedge^\bullet_A V$

$$d : \wedge^\bullet_A V \to \wedge^\bullet_A V$$

categorification		superification	
combinatorially	**algebraically**	**algebraically**	**geometrically**
higher morphisms	$\mathbb{N}$-grading	$\mathbb{Z}_2$-grading	
composition	differential	$\mathrm{Aut}(\mathbb{R}^{0\|1})$-action	odd flow
many objects	underlying algebra A		underlying manifold

Table 1: **Categorification and superification** have, while different, striking similarities.

this structure is often addressed as a differental graded supermanifold or "NQ-supermanifold".

But this is then also the same structure that deserves to be addressed as (the Chevalley-Eilenberg algebra of) an L_∞-algebroid, our definition 2: the infinitely categorified (higher morphisms) and oid-ified (many objects) version of a Lie algebra.

The main point connecting the two parts of table 1 is the observation, reviewed in [33] that differential $\mathbb{N}$-graded manifolds are $\mathrm{Diff}(\mathbb{R}^{0|1})$-modules.

3 Lie n-group actions and action Lie n-groupoids

We make some observations on the nature of the familiar elementary concept of group actions that will be helpful for understanding the generalization to actions of n-groups and for the passage from smooth n-group actions to L_∞-algebra actions.

When a group G has an action ρ on a set V, the quotient V/G is the *set* of orbits: the collection of equivalence classes under the equivalence relation $s \sim_{V/G} s' \Leftrightarrow \exists g \in G. s' = \rho(g)(s)$.

This set of orbits is the decategorification of the *weak* quotient,

$$V//G := \left(V \times G \underset{\rho}{\overset{p_1}{\rightrightarrows}} V \right) ,$$

called the *action groupoid* of G acting on V: the objects of $V//G$ are the elements of V, but instead of two different elements on the same orbit being equal in $V//G$, they are just *isomorphic*:

$$s \sim_{V/G} s' \Leftrightarrow \exists \; s \xrightarrow{g} s' \; \in \mathrm{Mor}(V//G) .$$

A nice introduction to action groupoids is in [1].

The following characterization of action groupoids will be useful when passing to Lie n-algebras in section 4. It combines an insight from [23] with an observation in [10].

For G any group, regarded as a set equipped with a product, we write $\mathbf{B}G$ for the corresponding groupoid with a single object and one morphism per element of G.

$$\mathbf{B}G := \left\{ \bullet \xrightarrow{g} \bullet | g \in G \right\} .$$

Note that $\mathbf{B}G$ is the action groupoid of the trivial action of G on the singleton set:

$$\mathbf{B}G = \bullet//G .$$

The $\mathbf{B}$-notation here is such that it commutes with the operation

$$|\cdot| : \mathrm{Cat} \to \mathrm{Top}$$

of geometric realizations of nerves of categories: we have

$$|\mathbf{B}G| = B|G| = BG .$$

With this notation, an action ρ of a group G on a set V is a functor

$$\rho : \mathbf{B}G \to \mathrm{Set}$$

$$(\bullet \xrightarrow{g} \bullet) \mapsto (V \xrightarrow{\rho(g)} V) .$$

The action groupoid $V//G$ turns out to be a pullback of something along this functor.

This something is the category of *pointed* sets (sets with basepoint), which we denote $T_{\mathrm{pt}}\mathrm{Set}$: its objects are sets with a chosen element and its morphisms are maps of sets repsecting the basepoint. The obvious forgetful functor

$$\begin{tikzcd} T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\ \mathrm{Set} \end{tikzcd} \qquad (3)$$

simply forgets the choice of basepoint. One sees that the action groupoid $V//G$ is the strict pullback

$$\begin{tikzcd} V//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\ \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \end{tikzcd} .$$

The vertical morphism on the left forgets the elements on an orbit and just remembers the group action on them.

There is a sense in which (3) is the "universal Set-bundle" (compare the discussion in [23]). In that sense, the fiber of this bundle is the set of (small) pointed sets (no morphisms between them). This we write $s^{-1}(\mathrm{pt})$. Then we have a sequence of categories

$$\begin{tikzcd} s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ T_{\mathrm{pt}}\mathrm{Set} \arrow[d, two heads] \\ \mathrm{Set} \end{tikzcd} .$$

Forming pullbacks along an action $\rho : \mathbf{B}G \to \mathrm{Set}$ we obtain the sequence of groupoids

$$\begin{tikzcd} V \arrow[r, hook] & V//G \arrow[r, two heads] & \mathbf{B}G \end{tikzcd} , \qquad (4)$$

where the set V is regarded as a groupoid with only identity morphisms, as the left column of

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] & s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ V//G \arrow[r] \arrow[d, two heads] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d, two heads] \\ \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \end{tikzcd} .$$

It is useful to consider this for the important special case that $\rho = \rho_{\mathrm{fund}} : \mathbf{B}G \to \mathrm{Set}$ is the *fundamental* representation of G on itself by right multiplication: in this case the left column

$$\begin{tikzcd} G \arrow[r] \arrow[d, hook] & s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ G//G \arrow[r] \arrow[d, two heads] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d, two heads] \\ \mathbf{B}G \arrow[r, "\rho"] & \mathrm{Set} \end{tikzcd}$$

is the groupoid incarnation of the universal G-bundle, in that it becomes the universal G-bundle under $|\cdot|$ (this is an old result of Segal, reviewed in [23]):

$$\begin{tikzcd} G \arrow[r, mapsto, "|\cdot|"] \arrow[d, hook] & G \arrow[d, hook] \\ G//G \arrow[r, mapsto, "|\cdot|"] \arrow[d, two heads] & EG \arrow[d, two heads] \\ \mathbf{B}G \arrow[r, mapsto, "|\cdot|"] & BG \end{tikzcd} .$$

Given a subgroup inclusion $H \hookrightarrow G$ we can regard it as the representation of H on G given by the composite functor

$$\mathbf{B}H \hookrightarrow \mathbf{B}G \xrightarrow{\rho_{\mathrm{fund}}} \mathrm{Set} \ .$$

The corresponding quotient is G/H, the collection of cosets. The weak quotient is the action groupoid $G//H$, which is the groupoid incarnation of the canonical principal G-bundle over BH

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & s^{-1}(\mathrm{pt}) \arrow[d, hook] \\ G//H \arrow[r, hook] \arrow[d] & G//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}\mathrm{Set} \arrow[d] \\ \mathbf{B}H \arrow[r, hook] & \mathbf{B}G \arrow[r, "\rho_{\mathrm{fund}}"] & \mathrm{Set} \end{tikzcd} \ .$$

All these considerations generalize to higher groups and higher groupoids: given some notion of n-groupoids, an n-group is just a one-object n-groupoid, which we always denote $\mathbf{B}G$ to indicate that it is the one-object n-groupoid version of its Hom-object

$$G := \mathrm{End}_{\mathbf{B}G}(\bullet)\,,$$

(usually an $(n-1)$-groupoid itself), that has a monoidal structure.

An action of such an n-group then is an n-functor

$$\rho : \mathbf{B}G \to (n-1)\mathrm{Grpd}$$

and we obtain the corresponding action n-groupoid from this by pulling back the "universal $(n-1)$Grpd-n-bundle

$$\begin{tikzcd} V//G \arrow[r] \arrow[d] & T_{\mathrm{pt}}(n-1)\mathrm{Grpd} \arrow[d] \\ \mathbf{B}G \arrow[r, "\rho"] & (n-1)\mathrm{Grpd} \end{tikzcd} \ .$$

And so on.

4 Lie n-algebra actions and action Lie n-algebroids

Lie n-algebras are supposed to be to Lie n-groups as Lie algebras are to Lie groups. Aspects of a "n-Lie theory" making this precise have begun to appear in the literature. Aspects of this we shall discuss below.

Under a Lie n-algebra (more properly: "semistrict" Lie n-algebra) we understand an L_∞-algebra whose underlying graded vector space is concentrated in degree 1 through n. For a review of the basics of L_∞-algebras and pointers to the literature we refer the reader to section 6.1 of [25].

We will see in section 5 that the generators in degree k of a Lie n-algebra correspond to the k-morphisms of the one-object Lie n-groupoid integrating it. The single object of the n-groupoid here is a direct reflection of the absence of degree 0 elements in the Lie n-algebra.

Therefore an L_∞-algebra with concentration in degree 0 through n is to be addressed as a Lie *n-algebroid*. Their nature is one of our main concerns here.

4.1 Lie ∞-algebras

Following [25] we think of L_∞-algebras almost exclusively in terms of their dual Chevalley-Eilenberg differential graded commutative algebras:

a finite dimensional L_∞-algebra $\mathfrak{g}$ is a finite dimensional $\mathbb{N}_+$-graded vector space $\mathfrak{g}^*$ together with a graded degree $+1$ derivation

$$d : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

on the free graded symmetric tensor algebra $\wedge^\bullet \mathfrak{g}^*$, such that $d^2 = 0$.

The differential graded-commutative algebra

$$\mathrm{CE}(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^*, d)$$

thus obtained we address as the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$.

An ordinary Lie algebra is the special case with $\mathfrak{g}$ taken to be concentrated in degree 1. The derivation is the dual of the Lie bracket, extended as a graded derivation by the graded Leibnitz rule. In this case $\mathrm{CE}(\mathfrak{g})$ is the ordinary Chevalley-Eilenberg algebra of a Lie algebra.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The fact that an ordinary Lie algebra $\mathfrak{g}$ is taken to be in degree 1 in the above is directly related to the fact that G is the collection of (1-)morphisms in the groupoid $\mathbf{B}G$.

4.2 Lie ∞-algebroids

Similarly, as further discussed below, the degree k-part of an L_∞-algebra $\mathfrak{g}$ controls the space of k-morphisms in the one-object Lie n-group $\mathbf{B}G$ integrating it.

This makes it clear that generalizing the above definition of L_∞-algebras from $\mathbb{N}_+$-graded to $\mathbb{N}$-graded vector spaces corresponds to passing from Lie n-groups to Lie n-groupoids.

To amplify this point, we notice that with $\mathbb{R}$ denoting the ground field (the tensor unit of vector spaces) we have

$$\wedge^\bullet \mathfrak{g}^* = \wedge^\infty(\mathbb{R} \oplus \mathfrak{g}^*) := \mathrm{colim}_{n \in \mathbb{N}} \wedge^n (\mathbb{R} \oplus \mathfrak{g}^*) \, .$$

So instead of thinking as an L_∞-algebra as a structure on a $\mathbb{N}_+$-graded vector space, we can think of it as a structure on a $\mathbb{N}$-graded vector space which is restricted to have the ground field in degree 0.

This should remind us strongly of the situation for n-groups: an n-group is best thought of as an n-groupoid that is restricted to have a single element in degree 0.

Indeed, in section 5 we will find that the n-groupoid integrating an L_∞-algebra $\mathfrak{g}$ has as space of objects a space whose algebra of functions is $\mathrm{CE}(\mathfrak{g})_0 = \mathbb{R}$. Hence the n-groupoid integrating an L_∞-algebra has a single object, as it should be.

Therefore, we expect the many object version, an L_∞-algebroid, to have a CE algebra which in degree 0 has the algebra $A = C^\infty(X)$ of functions over the space X of objects. That is what the following definition formalizes.

Definition 2 (Lie ∞-algebroids) *A Lie ∞-algebroid $(\mathfrak{g}, A)$ consists of*

- *a commutative, associative, unital algebra A;*
- *a $\mathbb{N}$-graded A-module $\mathfrak{g}^*$ such that $V_0 = A$;*
- *on $\wedge_A^\infty \mathfrak{g}^*$ regarded as a graded commutative algebra over the ground field (!) a graded degree +1 derivation*

$$d : \wedge_A^\infty \mathfrak{g}^* \to \wedge_A^\infty \mathfrak{g}^*$$

such that $d^2 = 0$.

Remark. No harm is done to this definition by allowing $\mathbb{Z}$-grading instead of $\mathbb{N}$-grading. But $\mathbb{N}$-grading is "natural", compare the remark *Grading and categorical dimension* in section 1. The point is that it may be convenient to have $\mathbb{N}$-graded $\mathfrak{g}^*$ being *resolved* by a $\mathbb{Z}_2$-graded $\mathfrak{g}^*$, i.e. with the latter having generators in negative degree, but no cohomology there. This is what happens in the example of the BRST complex.

Definition 3 (action Lie ∞-algebroid) *For A an algebra as above, V^* a complex of finite rank A-modules such that $V_0 = A$ and $\mathfrak{g}$ an L_∞-algebra, an action of $\mathfrak{g}$ on V is a Lie ∞-algebroid with CE-algebra $\mathrm{CE}_A(\mathfrak{g}, V)$ such that there is an exact sequence of DGCAs*

$$\begin{tikzcd} \mathrm{CE}(V) & \mathrm{CE}(\mathfrak{g}, V) \arrow[l, two heads] & \mathrm{CE}(\mathfrak{g}) \arrow[l, hook'] \arrow[ll, bend left, "0"] \end{tikzcd} .$$

Remark. Notice how we *generate* the GCAs freely over A, but then regard the result as GCAs over the ground field and in particular demand the derivations to be linear over the ground field, not over A. The following examples will clarify the point of this.

$$V \twoheadrightarrow V//G \hookrightarrow \mathbf{B}G \qquad \text{action Lie } n\text{-groupoid}$$

$$\mathrm{CE}(V) \twoheadleftarrow \mathrm{CE}(\mathfrak{g}, V) \hookleftarrow \mathrm{CE}(\mathfrak{g}) \qquad \text{action Lie } n\text{-algebroid}$$

Table 2: **Actions** in terms of their weak/homotopy quotients in integral (i.e. finite) and differential (i.e. infinitesimal) incarnation.

Remark. This definition is supposed to provide the linearized version of the sequence of action Lie n-groupoids (4), compare table 2. The precise relation is the topic of section 5.

4.3 Examples

Lie 1-algebroids: Lie-Rinehart pairs. For $n = 1$ Lie algebroids are the same as Lie-Rinehart pairs [22]. The definition can be found reviewed for instance as Def. 1.1 in [17].

Definition 4 (Lie-Rinehart pair) *A Lie-Rinehart pair $(A, \mathfrak{g})$ is a pair consisting of an associative algebra A and a Lie algebra $\mathfrak{g}$, such that both are modules over each other in the obvious compatible way modeled on the archetypical example $(A, \mathfrak{g}) = (C^\infty(X), \Gamma(TX))$, for X any manifold.*

If we denote the Lie action of $\mathfrak{g}$ on A by

$$\rho : \mathfrak{g} \otimes A \to A$$

and regard A itself as a complex of A-modules concentrated in degree 0, we get the DGCA

$$\mathrm{CE}(\mathfrak{g}, V) = \left(\wedge_A^\infty (\underbrace{A}_{\deg=0} \oplus \underbrace{\mathfrak{g}^*}_{\deg=1}), d_{\mathrm{CE}(\mathfrak{g},A)} \right)$$

with

$$(d_{\mathrm{CE}(\mathfrak{g},A)}\omega)(x, y) = \rho(x)(\omega(y)) - \rho(y)(\omega(x)) + \omega([x, y])$$

for all $\omega \in \mathfrak{g}^*$ and $x, y \in \mathfrak{g}$; and

$$d_{\mathrm{CE}(\mathfrak{g},A)} f = \rho(\cdot)(f) \in \mathfrak{g}^*$$

for all $f \in A$.

Beware that the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is the dual over A.

The tangent Lie algebroid. The archetypical examples of Lie algebroids are tangent Lie algebroids. We will see in section 5 that they correpond to the fundamental Lie groupoids of spaces. See exercise 30 in [12].

Consider the archetypical Lie-Rinehart pair $(A, \mathfrak{g}) := (C^\infty(X), \Gamma(TX))$. We have

$$\wedge^\bullet_A(\Gamma(TX)^*) = \Omega^\bullet(X)\,,$$

as graded commutative algebras and

$$\mathrm{CE}(\wedge^\bullet_A(\Gamma(TX)^*), d_{\mathrm{CE}(\Gamma(TX),C^\infty(X))}) = \Omega^\bullet(X)$$

as DGCAs. So the deRham DGCA is nothing but the Chevalley-Eilenberg algebra of the action Lie algebroid of the Lie algebra of vector fields on X acting on functions on X. This point of view is originally due to Rinehart.

Adjoint representation of L_∞-algebras on themselves Given an L_∞-algebra $\mathfrak{g}$ consider the DGCA

$$\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g}) := \left(\wedge^\bullet(\mathfrak{g}^*[-1] \oplus \mathfrak{g}^*), d_{\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g},\mathfrak{g})}\right)$$

with

$$d_{\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g},\mathfrak{g})}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})}$$

and

$$d_{\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g},\mathfrak{g})}|_{\mathfrak{g}^*[-1]} = \sigma \circ d_{\mathrm{CE}(\mathfrak{g})} \circ \sigma^{-1}\,,$$

where

$$\sigma : \mathfrak{g}^*[-1] \to \mathfrak{g}^*$$

is the canonical isomorphism of vector spaces that just shifts the grading, extended as a degree +1 graded derivation.

For $\mathfrak{g}$ an ordinary Lie algebra this is the ordinary CE-algebra of the adjoint representation of $\mathfrak{g}$ on itself.

Notice that $\mathrm{CE}_{\mathrm{ad}}(\mathfrak{g}, \mathfrak{g})$ is related to the Weil algebra $\mathrm{W}(\mathfrak{g})$ essentially (but not quite) by replacing the degree -1 shift with a degree +1 shift. This will be relevant for the interpretation of the BRST complex in terms of physical phase space (**to be discussed somewhere, sometime)

5 n-Lie theory: integration and differentiation

The integration method from Lie n-algebroids to Lie n-groupoids which we present is based on the same general idea as [13, 16], but uses strict smooth (super) n-groupoids instead of the weak n-groupoids conceived as Kan complexes used in [13, 16].

Similarly, the differentiation method from Lie n-groupoids to Lie n-algebras which we present is analogous to the basic idea of [33], but is formulated in terms strict smooth (super) n-groupoids instead of in terms of Kan complexes.

We observe that the basic underlying mechanism made us of in [13, 16, 32, 33] is the general situation described in section 2:

$$\begin{tikzcd}
(\mathrm{s})\mathrm{Smooth}n\mathrm{Grpd} \arrow[r, shift left, "S"] \arrow[rrr, bend left, "\text{differentiation}"] & (\mathrm{s})\mathrm{SmoothSpace} \arrow[l, shift left, "\Pi_n(-)"] \arrow[r, shift left, "\Omega^\bullet(-)"] & (\mathrm{s})\mathrm{DGCAs} \arrow[l, shift left, "S"] & (\mathrm{s})L_\infty \arrow[l, "(-)^*"'] \arrow[lll, bend left, "\text{integration}"]
\end{tikzcd} .$$

This relation between (super) DGCAs and smooth (super) n-groupoids is essentially the basic idea of Sullivan models in rational homotopy theory [14]. While in the theory of Sullivan models one has only be interested in turning DGCAs into simplicial spaces, the observation that these simplicial spaces actually happen to be Kan complexes and hence qualify as weak ∞-groupoids is the starting point of [13]. The only difference here is that we replace weak ∞-groupoids by strict n-groupoids.

The preference of strict n-groupoids over weak ∞-groupoids here is purely for practical, not for conceptual reasons: strict n-groupoids are convenient, useful and sufficient for our purposes. In particular, their usage allows to plug our constructions here into the theory of nonabelian differential cohomology [26] based on Ross Street's descent theory.

For $n = 1$ our integration procedure reproduces on the nose the integration of Lie (1-)algebroids (and hence in particular of Lie (1-)algebras) to smooth (1-)groupoids (Lie groups, in particular) in terms of the "A-path method" which is very nicely reviewed in [12].

5.1 L_∞-algebra valued forms and their classifying spaces

For $\mathfrak{g}$ any (super) L_∞-algebra and Y any smooth (super) space, we say that

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}_{\mathrm{DGCAs}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))$$

is the collection of flat $\mathfrak{g}$-valued forms on Y. For $\mathfrak{g}$ an ordinary (super) Lie algebra this coincides with the ordinary notion of flat $\mathfrak{g}$-valued 1-forms. More details and examples can be found in [25]. There also non-flat $\mathfrak{g}$-valued forms are discussed, which however shall not concern us here.

Using the adjunction (1) and hence the bijection (2) we find that the smooth (super) space $S(\mathrm{CE}(\mathfrak{g}))$ is the classifying space for flat $\mathfrak{g}$-valued forms

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) = \mathrm{Hom}_{(\mathrm{s})\mathrm{SmoothSpaces}}(Y, S(\mathrm{CE}(\mathfrak{g}))) .$$

5.2 Integration

Given a Lie n-algebra $\mathfrak{g}$, we say that the Lie n-group G universally integrating it is that given by the fundamental n-path groupoid of the classifying space of flat $\mathfrak{g}$-valued forms:

$$\mathbf{B}G := \Pi_n(S(\mathrm{CE}(\mathfrak{g})))\,.$$

Similarly, given any Lie n-algebroid $(\mathfrak{g}, V)$, we say that the Lie n-groupoid $V//G$ universally integrating it is

$$V//G := \Pi_n(S(\mathrm{CE}(\mathfrak{g}, V)))\,.$$

5.3 Examples

Integration of ordinary Lie (1-)algebras and Lie (1-)algebroids. Lie's third theorem, that every Lie algebra comes from a Lie group, is usually proven by relating everything to matrix Lie algebras using Ado's theorem.

That there is a more elegant and more conceptual method which identitfies the simply connected Lie group integrating a given Lie algebra with a certain quotient of based *paths* in the Lie algebra, and identifies the product in the Lie group with *compisition of paths* has apparently been well known to a chosen few for a long time (I am being told that Bott taught it his students this way) but was certainly not widely appreciated. It received renewed attention only when people started thinking about the more general problem of the integration of Lie algebroids to Lie groupoids. In that case the more conceptual path method is the only sensible one.

A beautiful and exhaustive review of the theory of integration of Lie algebroids is [12]. In section 3.2 the reader can find a discussion of the path-method for integrating Lie algebras, which then in section 3.3 is generalized to the integration of Lie algebroids.

The discussion in [12] is not exactly formulated in the language used here, but is easily translated into it:

- It is well known that Lie-Rinehart pairs $(\mathfrak{g}, A)$ for $A = C^\infty(X)$ are equivalent to Lie algebroids over X. Hence so are our DGCAs $\mathrm{CE}(\mathfrak{g}, A)$ from definition 4.
- Accordingly, DGCA morphisms

  $$\mathrm{CE}(\Gamma(TX), C^\infty(X)) = \Omega^\bullet(U) \longleftarrow \mathrm{CE}(\mathfrak{g}, A)$$

  are in bijection with Lie algebroid morphisms

  $$TU \longrightarrow (\mathfrak{g}, A)\ .$$

- By the Yoneda lemma, the morphisms of

  $$\Pi_1(S(\mathrm{CE}(\mathfrak{g}, A)))$$

are hence precisely Lie algebroid morphisms

$$TI \to (\mathfrak{g}, A)$$

– this are the “A-paths” of [12] (see definition 2.13 and exercise 27 there)
– modulo Lie algebra homotopies

$$T(I \times I) \to (\mathfrak{g}, A)$$

– this are the “A-homotopies” of [12] (see definition 3.18 there).

Therefore the smooth Lie groupoid $\Pi_1(S(\mathrm{CE}(\mathfrak{g}, A)))$ which we define to be the smooth Lie groupoid integrating the Lie algebroid $(\mathfrak{g}, A)$ coincides with the corresponding groupoid in [12]. In particular, under the integrability conditions discussed there, it is actually a groupoid internal to manifolds.

All this is essentially indicated, without details, at the beginning of [32].

Integration of the String Lie 2-algebra. The fundamental (and in some sense the universal) example of a Lie 2-algebra is the *String Lie 2-algebra* $\mathfrak{g}_\mu$ first considered as such in [2] and considered in the general context of String-like extensions of L_∞-algebras in [25].

Here $\mathfrak{g}$ is an ordinary semisimple Lie algebra and $\mu \in \wedge^3\mathfrak{g}^* \subset \mathrm{CE}(\mathfrak{g})$, $d_{\mathrm{CE}(\mathfrak{g})}\mu = 0$ is a multiple of the canonical Lie 3-cocycle

$$\mu = \langle \cdot, [\cdot, \cdot] \rangle \,.$$

Then $\mathfrak{g}_\mu$ is defined to have as Chevalley-Eilenberg algebra that of $\mathfrak{g}$, but with the 3-cocycle μ “killed”:

$$\mathrm{CE}(\mathfrak{g}_\mu) := \left(\wedge^\bullet (\underbrace{\mathfrak{g}^*}_{\mathrm{deg}=1} \oplus \underbrace{\langle c \rangle}_{\mathrm{deg}=2}) \right)$$

with

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})}$$

and

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)} c = \mu \,.$$

The (strict) Lie 2-group $\mathrm{String}(G)$ integrating $\mathfrak{g}_\mu$ was found in [3] by guessing and then differentiating to reobtain $\mathfrak{g}_\mu$. In its weak (Kan complex) form it was obtained in [16] by using essentially the integration pocess we are discussing here. In both cases one shows that, if μ is nowmalized such that it extends left-invariantly over the compact, simple, simply connected Lie group G integrating $\mathfrak{g}$ to the generator of $H^3(G, \mathbb{Z})$, the realization of the nerve of G_μ is a topological group which is a model for the topological String (1-)group [34].

If we don’t worry about the (Fréchet- or Banach-)manifold structure on $\mathrm{String}(G)$ for the time being, the derivation of the strict Lie 2-group $\mathrm{String}(G)$ using our integration procedure proceeds as follows.

As a warmup, notice that we could have, in the previous example, formed the 2-groupoid $\Pi_2(S(\mathrm{CE}(\mathfrak{g})))$ instead of the 1-groupoid $\Pi_1(S(\mathrm{CE}(\mathfrak{g})))$. This yields a "puffed up" 2-group version of the 1-group G: now 1-morphisms are just thin-homotopy classes of paths in G, starting at the identity, and 2-morphisms are homotopy classes of disks in D interpolating between two paths with the same endpoint.

Here we are using the fact that a flat $\mathfrak{g}$-valued 1-form A on a contractible space Y is the same as a choice of point in Y and a functor $g : Y \to G$, using $A = gdg^{-1}$.

Since $\pi_2(G) = 1$ we get

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}(\Omega G \to PG)\,.$$

Compare with [3].

In a similar manner the String Lie 2-algebra $\mathfrak{g}_\mu$ for $\mathfrak{g}$ a semisiple Lie algebra and $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ the canonical 3-cocycle is integrated (compare [16]): choose the normalization of μ such that it yields the integral 3-form representing $H^3(G, \mathbb{Z})$ for the compact, simple, simply connected group G.

Then 1-morphisms in $\Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu)))$ are thin homotopy classes of path in G, starting at the identity. Thin homotopy classes of 2-paths in $S(\mathrm{CE}(\mathfrak{g}_\mu))$ are disks in G as before, but now equipped with a 2-form B on the disk, of which only the integral $\int_{D^2} B$ survises dividing out thin homotopy.

A non-thin homotopy between a pair $(g : D^2 \to G, \int B)$ and a pair $(g' : D^2 \to G, \int B')$ is an extension

$$\tilde{g} : D^3 \to G$$

such that

$$\int B - \int B' = \int_{D^3} \tilde{g}^* \mu\,.$$

We recognize the construction of the "tautological bundle gerbe on G" which is the central extension of the loop group. Hence

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}_\mu))) = \mathbf{B}(\hat{\Omega} G \to PG) =: \mathbf{B}\mathrm{String}(G)\,.$$

This is essentially the integration found in [3], only that the horizontal composition is now by concatenation of paths in G. This reproduces actually the construction in [8, 9]

Integration of quotients. A Lie n-groupoid has morphisms only in non-negative degree, clearly. But we allowed Lie ∞-algebroids to have generators in negative degree.

Let $A = C^\infty(X)$, K an A-module and

$$V := (\ 0 \longrightarrow K_{\deg=-1} \xrightarrow{f} A_{\deg=0} \longrightarrow 0\ )$$

a complex with an A-module K in degree -1.

Then notice that for any smooth space U

$$\mathrm{Hom}_{\mathrm{DGCA}}(\mathrm{CE}(V), \Omega^\bullet(U))$$

contains all those maps $U \to X$ which hit the zero set of the functions in the image of f.

5.4 Differentiation

Given any Lie n-group $\mathbf{B}G$, we say that the Lie n-algebra $\mathfrak{g}$ universally differentiating it is that given by

$$\mathrm{CE}(\mathfrak{g}) := \Omega^\bullet(S(\mathbf{B}G)) .$$

Similarly, given any Lie n-groupoid $V//G$, we say that the Lie n-algebroid universally differentiating it is that given by

$$\mathrm{CE}(\mathfrak{g}, V) := \Omega^\bullet(S(V//G)) .$$

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