Twisted differential nonabelian cohomology

Twisted $(n-1)$-brane n-bundles and their Chern-Simons $(n+1)$-bundles with characteristic $(n+2)$-classes

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Abstract

We introduce nonabelian differential cohomology classifying ∞-bundles with smooth connection and their higher gerbes of sections, generalizing [138]. We construct classes of examples of these from lifts, twisted lifts and obstructions to lifts through shifted central extensions of groups by the shifted abelian n-group $\mathbf{B}^{n-1}U(1)$. Notable examples are String 2-bundles [9] and Fivebrane 6-bundles [133]. The obstructions to lifting ordinary principal bundles to these, hence in particular the obstructions to lifting Spin-structures to String-structures [13] and further to Fivebrane-structures [133, 52], are abelian Chern-Simons 3- and 7-bundles with characteristic class the first and second fractional Pontryagin class, whose abelian cocycles have been constructed explicitly by Brylinski and McLaughlin [35, 36]. We realize their construction as an abelian component of obstruction theory in nonabelian cohomology by ∞-Lie-integrating the L_∞-algebraic data in [132]. As a result, even if the lift fails, we obtain *twisted* String 2- and *twisted* Fivebrane 6-bundles classified in twisted nonabelian (differential) cohomology and generalizing the twisted bundles appearing in twisted K-theory. We explain the Green-Schwarz mechanism in heterotic string theory in terms of twisted String 2-bundles and its magnetic dual version – according to [133] – in terms of twisted Fivebrane 6-bundles. We close by transgressing differential cocycles to mapping spaces, thereby obtaining their volume holonomies, and show that for Chern-Simons cocycles this yields the action functionals for Chern-Simons theory and its higher dimensional generalizations, regarded as extended quantum field theories.

Handle with care. This is stuff we are still working on.

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Nonabelian cohomology generalizes the cocyclic description of fiber bundles (with or without connection) to bundles whose fibers are ∞-categories and whose sections form ∞-gerbes.

A familar example arises in the obstruction theory of $\mathrm{Spin}(n)$-bundles: An $\mathrm{SO}(n)$-principal bundle may be lifted to a $\mathrm{Spin}(n)$-bundle only if a well known obstruction class vanishes. But more can be said: there is a higher order bundle, a 2-bundle or gerbe, canonically associated with the original $\mathrm{SO}(n)$-bundle and the obstruction class is the characteristic class of this "lifting gerbe" [118]. In a context of nonabelian 2-bundles this picture can be refined one step further [138]: the original $\mathrm{SO}(n)$-bundle can always be lifted to a "twisted" $\mathrm{SO}(n)$-bundle – really a nonabelian 2-bundle – and the lifting gerbe is just the abelian component of that nonabelian structure.

We show that this phenomenon is just the lowest dimensional example of classes of examples of higher nonabelian bundles (with connection) which arise as

1. lifts;
2. twisted lifts;
3. obstructions to lifts

of structure ∞-groups ("gauge ∞-groups") G of G-principal bundles through shifted central *String-like* extensions $\mathbf{B}^{n-1}U(1) \to \hat{G} \to G$, where $\mathbf{B}^{n-1}U(1)$ is the n-group which is trivial everywhere except in degree $n-1$, where it has a copy of $U(1)$. Examples include

(a) ordinary central extensions of groups $U(1) \to \hat{G} \to G$, whose twisted lifts are the twisted vector bundles appearing in twisted K-theory (section 5.7.1) and the obstructions to the lift of which are line 2-bundles, or equivalently bundle gerbes;

(b) the *String extension* itself [9, 68] $\mathbf{B}U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n)$, with $\mathrm{String}(n)$ the String 2-group (section 5.2.3) the obstructions to the lift of which are abelian Chern-Simons 3-bundes (equivalently Chern-Simons bundle 2-gerbes [40]) and whose twisted nonabelian 2-bundle lifts (generalizing the twisted nonabelian 1-gerbes in [3]) we discuss in section 5.7.2;

(c) and the *Fivebrane-extension* $\mathbf{B}^5U(1) \to \mathrm{Fivebrane}(n) \to \mathrm{String}(n)$ which we obtain in section 5.2.4 by ∞-Lie-integrating the corresponding extension $b^5\mathfrak{u}(1) \to \mathfrak{fivebrane}(n) \to \mathfrak{string}(n)$ of L_∞-algebras [132].

We had discussed this lifting problem at the rationalized level in terms of L_∞-algebraic cocycles in [132]. Here we present a framework of ∞-Lie theory (section 4, motivated by [60, 68, 140, 44]), using which we integrate these L_∞-algebraic cocycles to cocycles in nonabelian cohomology. We find that at the level of abelian obstruction cocycles this general procedure reproduces central aspects of the abelian constructions presented in [35, 36], thus embedding that work in a more general nonabelian setting.

The main results of our concrete applications are

1. the diagram

$$\begin{tikzcd} \bar{H}(-,\mathrm{Spin}(n)) \arrow[r, "\mathrm{twistedLift}"] \arrow[rrr, bend right, "\mathrm{obstr}=\frac{1}{2}p_1"'] & \bar{H}(-,\mathrm{String}(n)//\mathbf{B}U(1)) \arrow[r, "\mathrm{twist}"] & \bar{H}(-,\mathbf{B}^2U(1)) \arrow[r, two heads] & H^4(-,\mathbb{Z}) \end{tikzcd}$$

which says that the twist of a twisted lift of a $\mathrm{Spin}(n)$-bundle with connection to a $\mathrm{String}(n)$-2-bundle with connection is a Chern-Simons $\mathbf{B}^2U(1)$-3-bundle with connection (an abelian Chern-Simons 2-gerbe) whose characteristic 4-class is the first fractional Pontryagin class $\frac{1}{2}p_1$ of the original $\mathrm{Spin}(n)$-bundle;

2. the diagram

$$
\begin{tikzcd}
\bar{H}(-,\mathrm{String}(n)) \arrow[r, "\mathrm{twistedLift}"] \arrow[rrr, bend right, "\mathrm{obstr}=\frac{1}{6}p_2"'] & \bar{H}(-,\mathrm{Fivebrane}(n)//\mathbf{B}^5U(1)) \arrow[r, "\mathrm{twist}"] & \bar{H}(-,\mathbf{B}^6U(1)) \arrow[r, two heads] & H^8(-,\mathbb{Z})
\end{tikzcd}
$$

which says that the twist of a twisted lift of a String(n)-2-bundle with connection to a Fivebrane(n)-6-bundle with connection is a Chern-Simons $\mathbf{B}^6U(1)$-7-bundle with connection (an abelian 6-gerbe) whose characteristic 8-class is the second fractional Pontryagin class $\frac{1}{6}p_2$ of the original String(n)-2-bundle.

We explicitly construct all the items appearing here.

We develop the differential nonabelian cohomology theory used to phrase these constructions in 3. The abstract nonsense prerequisites needed are treated in section 2.

The various ingredients of these lifting problems crucially appear in string theory, as discussed in [132] and [133], where they govern the higher gauge theoretic nature of the theory where the bulk and brane structures interact.

obstruction	**char. class**			**G-bundle**		**$\hat{G}$-bundle**
Line 2-bundles				$PU(H)$-bundles		$U(H)$-bundles
Line 3-bundles	with class	$\frac{1}{2}p_1$	obstruct the lift of	Spin(n)-bundles	to	String(n)-2-bundles
Line 7-bundles		$\frac{1}{6}p_2$		String(n) 2-bundles		Fivebrane(n)-6-bundles

Table 1: **Obstruction problems in nonabelian cohomology** appearing in string theory. A concise review of the relevant string-theoretic concepts is in section 6.7.

A brief account of the relevant string-theoretic concepts is given in section 6.7. The cohomological description of these phenomena has found a clean formulation in terms of generalized differential cohomology by Freed, Hopkins and Singer in [56, 74]. Nonabelian differential cohomology further refines this description in that not only the abelian obstructing fields (the Neveu-Schwarz field, the supergravity C-field as well as their magnetic dual) are represented, but also the nonabelian structures *twisted* by them are provided as well. Given the important role played by twisted $U(H)$ bundles in string theory, the twisted String 2-bundles and twisted Fivebrane 6-bundles which we introduce can be expected to be of comparable relevance. Further discussion of this application is in preparation [2].

1 Overview

We proceed as follows.

- To set ourselves up in a suitably general context of differential geometry we model Spaces as sheaves on CartesianSpaces, with DiffeologicalSpaces [10] and SmoothManifolds contained as subcategories of tame objects.

- To have manifest close contact to familiar constructions in homological algebra, differential geometry and physics which we want to reproduce and generalize, the model for ∞-categories which we choose is *strict* ∞-categories, known as ωCategories (figure 1). This turns out to be not only convenient, admitting all tools of *nonabelian algebraic topology* [34], but also sufficient.

- To handle the homotopy theoretic context of ω-categories internal to Spaces, equivalently: ω-category valued sheaves,
$$\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})\,,$$
we obtain from the known homotopy model category structure on ωCategories [28, 94] by stalkwise refinement the structure of a *category of fibrant objects* [26]. This yields a homotopy (bi-)category $\mathbf{Ho}(\omega\mathsf{Categories}(\mathsf{Spaces}))$ whose Hom-spaces realize cohomology in this context, analogous to [81].

- To establish contact with ordinary abelian Čech cohomology with coefficients in complexes of sheaves of abelian groups we consider descent for ω-category valued presheaves [151] and the corresponding notion of ω-stacks. Dually this leads to a notion of codescent for ω-category valued co-presheaves, which serves to translate from cohomology in terms of descent to cohomology in terms of the homotopy category.

- In this context we set up our central definition of twisted differential nonabelian cohomology:
  - nonabelian cohomology for structure ω-group G is cohomology with coefficients in $\hom(\Pi(-), \mathbf{B}G)$, for Π an ω-category valued copresheaf;
  - twisted cohomology is a refinement of the obstruction to lifting of cocycles through extensions $\mathbf{B}\hat{G} \twoheadrightarrow \mathbf{B}G$ (figure 6);
  - differential cohomology is a refinement of the obstruction to the extension of cocycles along the inclusion of copresheaves $\mathcal{P}_0(-) \hookrightarrow \Pi_\omega(-)$ of discrete ω-groupoids into fundamental ω-groupoids (figure 9);

$$\begin{tikzcd} & \mathbf{B}\hat{G} \arrow[d] \\ \mathcal{P}_0(X) \arrow[ur, dashed, "\text{lifting problem}"] \arrow[r, mapsto, "g"] \arrow[d, hook] & \mathbf{B}G \\ \Pi_\omega(X) \arrow[ur, dashed, "\text{extension problem}"'] & \end{tikzcd}$$

- As a tool for explicitly constructing (twisted, differential) nonabelian cocycles we describe ∞-Lie theory of smooth ω-groupoids and Lie ∞-algebroids, following [140, 60, 68, 141], as the theory of two consecutive adjunctions relating ωGroupoids(Spaces) to Spaces and Spaces to L_∞Algebroids. Both adjunctions are examples of Stone dualities induced by ambimorphic objects. The first adjunction is induced by the object of finite paths, $\Pi_\omega(-)$, while the second is induced by the object of infinitesimal paths, $\Omega^\bullet(-)$ (figure 7).

- Using these adjunctions we ∞-Lie integrate the L_∞-algebraic cocycles from [132] from L_∞Algebroids to ωGroupoids(Spaces) to obtain nonabelian differential cocycles when certain integrability conditions are met (figure 9).

- Applied to the String- and Fivebrane- L_∞-algebraic cocycles and their Chern-Simons obstructions of [132] this yields an explicit construction of twisted differential cocycles representing twisted String 2- and twisted Fivebrane 6-bundles with connection. The twist itself is the obstruction to obtaining untwisted such bundles and lives in abelian Deligne cohomology where it represents Chern-Simons connections and their higher analogs (figure 3).

- Finally we transgress the differential cocycles thus obtained to mapping spaces and show that the transgressed differential cocycles exhibit the holonomy [137, 138] and can be interpreted as the action functionals for extended Chern-Simons quantum field theories.

1.1 General subject of nonabelian cohomology

This article is primarily interested in the classification of geometric objects over *spaces*, which are locally related to constructions involving fixed (nonabelian) groups, called *coefficients*, as well as their higher categorical analogues. The gluing conditions of such objects (e.g. higher principal bundles) and the coefficients will also be of higher categorical nature: (higher) stacks rather than sheaves. One often conceives such objects by means of abstractly defined equivalence classes of data, called *cocycles*; sometimes one can realize these classes also as the homotopy classes of maps from the base space into some distingushed space, which is then said to *classify* the geometric objects in question.

A collection of related notions, comprising a subject called "nonabelian cohomology", allows, in various frameworks, all of the characters of the above story – spaces, coefficients, cocycles, (cohomology) classes, homotopy, gluing, higher categorical analogues, stacks – to be generalized in an appropriate sense; the word "nonabelian" pertains to the coefficients $\mathcal{C}$ in $H^*(\ ,\mathcal{C})$.

Geometric and topological objects are usually filtered by some notion of a dimension, and combinatorial devices such as simplicial or globular sets are suitable for inductive constructions. It is a fascinating fact, first observed in algebraic topology and nowdays prevalent in mathematics, that nicely structured functors into nice ("algebraic") categories usually respect dimension-like order; the appearance of chain-complexes and cohomology sequences are typical instances. When the target is in an abelian category (or something not far from it, e.g. a Quillen exact category), a systematic treatment of such functors follows familiar patterns. For example, one has cohomology classes for every degree n (not only in low dimension), the sheaf cohomology as a derived functor is, for paracompact spaces, the same as the Čech cohomology with coefficients in that sheaf; once we have the cocycles it is easy to pass to cohomology classes etc.

Digression: derived functors. This article is concerned with the generalizations to nonabelian cohomology of the Čech approach rather than the derived functor approach. Let us mention, however, that there are several approaches to *nonabelian derived functors*, i.e. nonabelian homological algebra (distinguish this phrase from the subject of nonabelian cohomology!). Quillen defined nonabelian derived functors in the setup of model categories; category theorists have studied the categories similar to abelian categories but with weakened axioms, with many elements of homological algebra (e.g. *semiabelian* and *homological* categories, studied by Bourn, Janelidze, Inassaridze [79]). In Abelian (and Quillen exact) categories one treat homology on the same footing with cohomology. It seems this is not possible in general: the two are not necessarily definable in the same category. With this observation in mind (suggested by the notion of suspended categories of B. Keller, which are a nonsymmetric generalization of triangulated categories), A. Rosenberg introduced [127, 129] right exact and left exact categories as categories with a distinguished class of morphisms (deflations, resp. inflations) suited for the theory of derived functors in the nonabelian setup.

Experience from low dimensions; other categories. Low-dimensional examples are usually driven by very concrete and tangible problems and give much insight. *Schreier's theory* of extensions of nonabelian groups ([139, 54, 25]) is a prime example. The classification of extensions $1 \to K \xrightarrow{i} G \xrightarrow{p} B \to 1$ is given in terms of equivalence classes of cocycles which are constructed using elementary constructions. One starts with a set-theoretic section $\sigma : B \to G$ of p and the corresponding cocycle is $B \times B \ni (a,b) \mapsto \sigma(a)\sigma(b)\sigma^{-1}(ab) \in i(K) \cong K$. The equivalence class will not depend on σ. This works because there is a forgetful functor from (the category of) groups to sets, while in sets one can always find a set-theoretical section. For topological groups one cannot do this: one cannot find continuous sections. Similarly, an extension of representable group functors may be not representable. Similarly, groups or Lie algebras in arbitrary monoidal category may have bad properties in this sense. Moral: *the nonabelian classification problem may exist even when the cocycles (in the more algebraic sense of maps given by cocycle equations) do not.* (Similarly a derived functor may exist even if no cocycle description exists) In this article, a "nonrepresentability" comes as the appearance of lifts which are given by integrated infinite-dimensional or even more general objects even when the original objects are finite-dimensional.

The categorical approach to nonabelian cohomology by Giraud, Breen, Street and others [63, 25, 122, 125, 149], provides a natural interpretation of the cocycles when the coefficient n-group is in the category of sets or in some topos. Similar cocycle conditions for group-like objects (e.g. Lie algebras, Hopf algebras, groups in other monoidal categories), are known and useful in some special cases, but a systematic general theory is missing (cf.6.6).

1.2 Descent

Suppose we are given some category of spaces in which each space X equipped with a *fiber*, i.e. a category C_X of objects of some type over it. For example, a space can be a smooth manifold and the fiber is the category of vector bundles over it; or a space is an object of the category dual to the category of rings and the fiber is its category of left modules. Given a map $f : Y \to X$, one often has an induced functor $f^* : C_X \to C_Y$ (pullback, inverse image functor, extension of scalars). The basic questions of classical descent theory are:

1. When an object E in C_Y is in the image of an object in C_X, what is the fiber $(f^*)^{-1}(E)$.

2. Classify all forms of object $G \in C_Y$, that is find all $E \in C_X$ for which $f^*(E) \cong G$.

Grothendieck introduced pseudofunctors and fiberd categories to formalize an ingenious method to deal with descent questions. He introduces additional data on an object E in C_Y to have a chance of determining an isomorphism class of an object in C_X. Such an enriched object over X is called a "descent datum". f is an *effective descent morphism* if the morphism f induces a canonical equivalence of the category of the descent data (for f over X) with C_X. It is a nontrivial result that in the case of rings and modules, the effective descent morphisms are precisely *pure morphisms* of rings. Grothendieck's flat descent theory tells a weaker result that faithfully flat morphisms are of effective descent. In algebraic situations one often introduces a (co)monad $T_f : C_X \to C_X$ (say with multiplication $\mu : T_f \circ T_f \to T_f$) induced by the morphism f ([37, 103, 128, 145]). The category of descent data is then nothing else than the "Eilenberg-Moore" category T_f-Mod of (co)modules (also called (co)algebras) over T_f. Then, by the definition, f is of an effective descent if and only if the *comparison map* (defined in (co)monad theory) between C_X and T_f-Mod is an equivalence. Several variants of *Barr-Beck theorem* give conditions ([102, 103, 129]) which are equivalent or sufficient to the comparison map for a monad induced by a pair of adjoint functors being an equivalence. Generically such theorems are called monadicity (or tripleability) theorems. One can describe most of (but not all) situations of 1-categorical descent theory via monadic approach; comparison of monadic descent with the approach of Grothendieck-Gabriel-Giraud via fiberd categories is made in a short note [20], where a so-called Beck-Chevalley condition is introduced. A version of Barr-Beck theorem for 2-categories has been studied in [97] (see also appendix to [69]), Barr-Beck theorem for $(\infty, 1)$-categories has been proved by Lurie in [101], and for triangulated categories by Kontsevich and Rosenberg (cf. [129]; the proof is via Verdier's abelianization functor).

A Grothendieck (pre)topology τ on a category with pullbacks is a collection of distinguished morphisms, called 'covers', which satisfy a list of 3 axioms. One of the conditions for a fiberd category to be a stack is that all covers in τ are of effective descent. Thus the basic theory of stacks (and ∞-stacks) may be partly viewed as a subset of descent theory. The equivariance data for a sheaf and generalizations (like 2-equivariant objects [146], Hopf modules [98] etc.) also correspond to a certain kind of descent data. Considering the pullback $f^*(G)$ as a "trivial" object over Y, one is concerned with identifying "trivial" structures with certain relations, the "gluing relations", on covers $Y \to X$ from space X to space Y which are such that they correspond to possibly nontrivial structures but without extra relations down on X: they *descend* from Y down to X. Since Y may be regarded as a local description of X, the structures on X thus obtained are "locally trivial". For categorically low-dimensional structures, i.e. for those which live in sets or at best in categories, this situation is described by the concept of *sheaves* and *stacks*, respectively, and is well understood; see for instance [87]. More generally, however, the structures in question will live in an ∞-category. The purpose of descent theory and the theory of ∞-stacks is to encode the right descent conditions on a structure on Y, which is an object in an ∞-category.

2 Underlying Machinery: Space and Quantity

Given a category S of *test objects* the most general notion of a *space* modeled on S is a presheaf on S, while the most general notion of a quantity modeled on S is a co-presheaf on S [95]: a space is something *probed* by mapping test spaces in S into it.

	space	**quantity**
general concept	presheaf	co-presheaf
nice version	smooth space (sheaf condition)	C^∞-algebra (monoid structure)
higher degree version	∞-groupoid internal to Spaces	differential graded-commutative C^∞-algebra

Table 2: **Space and quantity**. For a given category S of *test objects*, *spaces probeable by objects of* S are presheaves on S, whereas *quantities with values in* S are co-presheaves on S. Here we take $S :=$ CartesianSpaces which comes naturally with the structure of a site. Sheaves on this S are generalized smooth spaces. Monoids in co-presheaves on S are generalized smooth algebras. Higher categorical degree is obtained by passing to ∞-categories internal to Spaces and, respectively, passing to quasi-free differential graded-commutative algebras (qDGCAs) over C^∞-algebras.

For the differential geometric and Lie theoretic context that we are interested in we choose S to be the category of *cartesian spaces*. Other choices are possible without changing the essence of much of our discussion. In particular one could consider taking S to be Δ, the simplicial category, which is a popular choice in much of the literature in the context of cohomology theory. One advantage of using cartesian spaces instead is that these are also well suited as "co-probes" for function algebras and modules of sections, as discussed in section 2.3. This gives rise to dualities (Isbell duality, as in [95]) between spaces and quantities which are at the heart of the ∞-Lie theory in section 4.

Definition 2.1 (cartesian spaces) *Write* CartesianSpaces *for the full subcategory of* Manifolds *on the manifolds* $\mathbb{R}^k$ *equipped with their standard smooth structure, for all* $k \in \mathbb{N}$*. This category comes with the standard notion of cover that makes it a site.*

2.1 Spaces

Definition 2.2 (smooth spaces) *The category of smooth spaces is the category* $\mathsf{Spaces} := \mathrm{Sheaves}(\mathsf{CartesianSpaces})$ *of sheaves on* CartesianSpaces.

Terminology. For brevity and since for some of our applications CartesianSpaces could be replaced by some other site, we often write just *space* instead of *smooth space*. For $X \in \mathsf{Spaces}$ and $U \in \mathsf{CartesianSpaces}$ we say $X(U) \in \mathsf{Sets}$ is the set of plots from U into X or the set of probes of X by U. By the Yoneda lemma, $X(U) \simeq \mathrm{Hom}(U, X)$ is the set of maps of smooth spaces from U to X. See for instance [95] and [10] for some general background on the concept of sheaves on test domains as generalized spaces.

We have a canonical chain of inclusions

$$\mathsf{Manifolds} \hookrightarrow \mathsf{FrechetManifolds} \hookrightarrow \mathsf{ConcreteSpaces} \hookrightarrow \mathsf{Spaces}\ .$$

Concrete spaces / diffeological spaces. Since objects in Spaces are only required to be *probeable* by cartesian spaces, and not required to be locally isomorphic to cartesian spaces, they can be quite a bit more general than manifolds. Classes of spaces far from manifolds are in particular the classifying spaces of L_∞-algebra valued differential forms ([141]), smooth models of $K(G,1)$s for G a smooth ∞-group, discussed in section 4.1. A special class of spaces still more general than manifolds but more restrictive than general smooth spaces are *concrete spaces*, which are given by concrete sheaves that have an underlying set of points. As discussed in [10], these are the *diffeological spaces* [77] or *Chen-smooth spaces*. Reference [147] gives a comparative discussion of the various notions of generalized smooth spaces.

Definition 2.3 (indiscrete spaces) *For $S \in \mathsf{Sets}$ the indiscrete space over S is* $\mathrm{indiscrete}(S) := \mathrm{Hom}_{\mathsf{Sets}}(-, S)$. *This yields an injection* $\mathrm{indiscrete} : \mathsf{Sets} \hookrightarrow \mathsf{Spaces}$. *the image of which is the category* $\mathsf{IndiscreteSpaces} \subset \mathsf{Spaces}$.

Definition 2.4 (concrete spaces) *Concrete spaces X are the subobjects, in* Spaces *of indicrete spaces S,* $X \hookrightarrow \mathrm{indiscrete}(S)$.

Remark. In words this means that a concrete space X is a set S together with a rule which says which maps of sets from objects of $\mathsf{CartesianSpaces}$ into S are regarded as homomorphisms, i.e. as continuous or *smooth maps*. Notice the difference of the notion of concrete spaces to (possibly infinite-dimensional) manifolds: those are required to be locally *isomorphic* to some object in $\mathsf{CartesianSpaces}$. A concrete space is just required to be *probeable* by all objects of $\mathsf{CartesianSpaces}$.

Every space has an underlying concretization.

Definition 2.5 (concretization) *For $X, Y, Z \in \mathsf{Spaces}$ let*

$$\mathrm{post}(X,Y,Z) : \mathrm{Hom}_{\mathsf{Spaces}}(Y,Z) \to \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Hom}_{\mathsf{Spaces}}(X,Y), \mathrm{Hom}_{\mathsf{Spaces}}(X,Z))$$

be the image under the Hom-adjunction in Sets *of the composition operation*

$$\circ_{X,Y,Z} : \mathrm{Hom}_{\mathsf{Spaces}}(X,Y) \times \mathrm{Hom}_{\mathsf{Spaces}}(X,Y) \to \mathrm{Hom}_{\mathsf{Spaces}}(X,Z)$$

in Spaces. *Let*

$$\mathrm{ConcreteHom}_{\mathsf{Spaces}} := \mathrm{Image}(\mathrm{post}(\mathrm{pt}, -_2, -_1)) : \mathsf{Spaces}^{\mathrm{op}} \times \mathsf{Spaces} \to \mathsf{Sets}\,.$$

For $X \in \mathsf{Spaces}$ we say $\mathrm{ConcreteHom}(-, X) : \mathsf{CartesianSpaces}^{\mathrm{op}} \to \mathsf{Sets}$ *is the concretely representable presheaf of X. The concretization functor is the sheafification of the concrete representation*

$$\mathrm{concretize} := \mathrm{sheafify} \circ \mathrm{ConcreteHom}_{\mathsf{Spaces}}(-_2, -_1) : \mathsf{Spaces} \to \mathsf{Spaces}\,.$$

Remark. This means the set underlying the concretization $\mathrm{concretize}(X)$ of a space X is $\mathrm{Hom}_{\mathsf{Spaces}}(\mathrm{pt}, X)$.

Proposition 2.6 *The concretization operation produces concrete spaces*

$$\mathrm{concretize} : \mathsf{Spaces} \to \mathsf{ConcreteSpaces} \hookrightarrow \mathsf{Spaces}\,.$$

Remark. Notice that concretization is far from being injective. There are important objects in Spaces which have only a single underlying point but still have many nontrivial higher dimensional probes. These are notably the classifying spaces of L_∞-algebra valued forms discussed in section 4.1.

For an exhaustive description of operations on (concrete) spaces see [10]. We need the following operations.

Quotients.

Definition 2.7 (equivalence relation on an object in a category) *([1] and [15], beginning) Given an object X in some category $\mathcal{C}$ an equivalence relation on X is a triple (R, p_1, p_2) where R is an object in $\mathcal{C}$ and p_1, p_2 morphisms in $\mathcal{C}$ as in* $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$ *satisfying*

1. *reflexivity: There exist a morphism $j : X \to R$ which is a section of both p_1 and p_2: $p_2 \circ j = p_1 \circ j = \mathrm{id}_X$.*

2. *transitivity: A pullback*
$$\begin{tikzcd} \bar{R} \arrow[r, "q_1"] \arrow[d, "q_2"] & R \arrow[d, "p_2"] \\ R \arrow[r, "p_1"] & X \end{tikzcd}$$
exists, together with a morphism $t : R \to \bar{R}$ such that $p_1 \circ t = p_1 \circ q_1$ and $p_2 \circ t = p_2 \circ q_2$.

3. *symmetry: There exists $s : R \to R$ such that $p_1 = p_2 \circ s$ and $p_2 = p_1 \circ s$.*

If $\mathcal{C} = \mathsf{Set}$ we get the equivalence relation in usual sense. We use the definition however for $\mathcal{C} = \mathsf{Spaces}$.

Definition 2.8 (quotient by equivalence relation) *For $X \in \mathsf{Spaces}$ and $\sim = (R, p_1, p_2)$ an equivalence relation on X, we write $X/\sim$ for the pushout*

$$\begin{tikzcd} R \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\ X \arrow[r] & X/\sim \end{tikzcd} .$$

For details on quotient spaces of equivalence relations on concrete spaces see [10].

2.2 ∞-Categories

We choose here to model ∞-categories as *strict* ∞-categories – called ω-categories (as introduced in [149] – in terminology we follow section 2.2 of [47]). Their advantage is that in our examples and applications in section 5, they lead to comparably concrete structures familiar in differential geometry and physics, via the equivalence of ω-groupoids with crossed complexes of ordinary groups [34], recalled in section 2.2.1. ω-categories carry a natural homotopy model structure; using this model structure we can capture (sections 2.2.3 and 2.2.4) their weak and generalized ("Morita") morphisms what makes them behave like weak ∞-categories. This leads to a (weak) homotopy category of ω-categories, section 2.2.5, which is the home of the cohomology theory described in section 3. Furthermore, the correspondence sending a simplicial set to the ω-category freely generated from it, extends to a functor from the category of simplicial sets into the category of ω-categories; using this functor one can transport many constructions from simplicial sets to ω-categories. Notice that after a conjecture by Simpson [144], there has been growing evidence [90, 85, 121] that for the full generality it is indeed sufficient to extend strict ∞-categories by just allowing *weak units*.

The original definition of ω-categories is given in [149] (p. 305), recalled for instance in [47] and [94]. A conceptual introduction is provided in section 1.4 of [96]. The basic idea is simple: an ω-category has, for each $k \in \mathbb{N}$, a set of k-morphisms, each going from a source to a target $(k-1)$-morphisms which both, in turn, share the same source and target $(k-2)$-morphism. This means that k-morphisms are usefully thought of geometrically as k-dimensional disks – called *globes* in this context – as indicated in figure 1.

Ambient context. Fix once and for all a topos K equipped with a faithful functor $\mathsf{Sets} \to K$. Write $\emptyset$ for the initial and pt for the terminal object in K.

In our applications we usually set $K = \mathsf{Spaces}$. $\mathrm{const}(S) : \mathsf{Sets} \to K$ will denote the functor $U \mapsto S$ fro all U in Sets.

Definition 2.9 (globular object / ∞-graph) *The globular category G is the category whose objects are the integers and whose morphisms are generated from* $n \underset{\tau_n}{\overset{\sigma_n}{\underset{\longleftarrow \iota_n -}{\rightrightarrows}}} n+1$ *for all $n \in \mathbb{N}$ subject to the relations*

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$$

$$\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$$

$$\iota_n \circ \sigma_n = \mathrm{Id}$$

$$\iota_n \circ \tau_n = \mathrm{Id}\,.$$

A globular object S in K is a functor $S : G^{\mathrm{op}} \to K$. For $n \in \mathbb{N}$ we write

$$S_n \underset{t_n}{\overset{s_n}{\underset{- i_n \longrightarrow}{\leftleftarrows}}} S_{n+1} \quad := \quad S(\, n \underset{\tau_n}{\overset{\sigma_n}{\underset{\longleftarrow \iota_n -}{\rightrightarrows}}} n+1 \,)$$

and call s_n the $(n+1)$st source map and t_n the $(n+1)$st target map and i_n the nth identity assigning map. The relations

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$

$$t_n \circ s_{n+1} = t_n \circ t_{n+1}$$

$$s_n \circ i_n = \mathrm{Id}$$

$$t_n \circ i_n = \mathrm{Id}$$

which these satisfy by functoriality for all $n \in \mathbb{N}$ are called the globular identities.
A morphism of globular objects is a natural transformation of the corresponding functors. For the resulting category of globular objects in K we write $\mathsf{GlobularObjects}(K) := K^{G^{\mathrm{op}}}$ *or simply* $\mathsf{GlobularObjects}$.

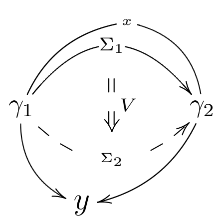


Figure 1: **A ("globular") 3-morphism** in an ω-category. The 3-morphism $V : \Sigma_1 \Rightarrow \Sigma_2$ goes between the 2-morphisms $\Sigma_1, \Sigma_2 : \gamma_1 \to \gamma_2$ which in turn both have as source the object x and as target the object y.

Definition 2.10 (standard n-globe) *The standard 0-globe is the point $\mathbb{R}^0$. The standard n-globe for $n \in \mathbb{N}$, $n \geq 1$ is the standard n-disk $D^n \subset \mathbb{R}^n$ regarded as a cell complex with two boundary components being the upper and the lower semi-$(n-1)$-sphere, both regarded as standard $(n-1)$-globes.*

See figure 1 for a picture of a 3-globe.

Remark. Globular objects are to standard globes as simplicial objects are to standard simplices. The standard n-globe, taken as a cell complex, is naturally regarded as a globular set concentrated in the first n degrees. This is definition 2.18 further below.

Notation for composite globular maps. The globular identities ensure that

- two sequences of boundary maps

$$f_n \circ \cdots \circ f_{n+m-1} \circ f_{n+m} : S_{n+m+1} \to S_n$$

  with $n, m \in \mathbb{N}$ and for $f_k, \in \{s_k, t_k\}$ are equal if and only if their last term f_n coincides;

- for all $n, m \in \mathbb{N}$ we have

$$s_n \cdots s_{n+1} \circ \cdots \circ s_{n+m} \circ i_{n+m} \circ \cdots \circ i_{n+1} \circ i_n = \mathrm{Id}$$

$$t_n \cdots t_{n+1} \circ \cdots \circ t_{n+m} \circ i_{n+m} \circ \cdots \circ i_{n+1} \circ i_n = \mathrm{Id}\,.$$

We therefore write

$$\begin{tikzcd} S_{n+m+1} \arrow[r, shift left=2, "s_n"] \arrow[r, shift right=2, "t_n"'] & S_n \arrow[l, "i_n" description] \end{tikzcd}$$

with i_n, s_n, t_m the sequence of m consecutive identity-assigning, source or target maps, respectively.

Definition 2.11 (ω-categories) *An $\underline{\omega\text{-category}}$ C internal to K is a globular K-object $\{C_k\}_{k\in\mathbb{N}}$ equipped for all $i > j$ with the structure of a category internal to K (definition 6.8) on* $\begin{tikzcd} C_i \arrow[r, shift left=2, "s_j"] \arrow[r, shift right=2, "t_j"'] & C_j \arrow[l, "i_j" description] \end{tikzcd}$ *, i.e. with a composition morphism $\circ_j : C_i \times_{t_j, s_j} C_i \to C_i$ satisfying the associativity and unity constraints, such that for all $i > j > k$ this makes*

$$\begin{tikzcd} C_i \arrow[r, shift left=2, "s_j"] \arrow[r, shift right=2, "t_j"'] & C_j \arrow[l, "i_j" description] \arrow[r, shift left=2, "s_k"] \arrow[r, shift right=2, "t_k"'] & C_k \arrow[l, "i_k" description] \end{tikzcd}$$

with horizontal composition $\circ_k$ and vertical composition $\circ_j$ a strict 2-category (definition 6.14) in that $\circ_k$ and $\circ_j$ satisfy the exchange law.

A morphism of ω-categories in K, called an $\underline{\omega\text{-functor}}$, is a morphism of the underlying globular K-objects preserving this extra structure and property. The $\underline{\text{category of } \omega\text{-categories}}$ in K obtained this way we call $\omega\mathsf{Categories}(K)$ and often just write $\omega\mathsf{Categories}$.

Terminology. We write $\mathrm{Obj}(C) := C_0$ for the K-object of $\underline{\text{objects}}$ and $\mathrm{Mor}_k(C) = C_k$ for the K-object of $\underline{k\text{-morphisms}}$ or $\underline{k\text{-cells}}$ of C. We say $\circ_k$ is $\underline{\text{composition along } k\text{-morphisms}}$.

Lemma 2.12 *Write $U : \omega\mathsf{Categories}(K) \to \mathsf{GlobularObjects}(K)$ for the obvious forgetful functor which sends every ω-category to its underlying globular object. This functor has a left adjoint $F : \mathsf{GlobularObjects}(K) \to \omega\mathsf{Categories}(K)$ which sends a globular object to the $\underline{\text{free } \omega\text{-category}}$ over it. And U is in fact* monadic.

We are grateful to Tom Leinster for discussion of this standard fact, which implies the following standard fact about monadic functors, useful for computations.

Corollary 2.13 (limits in ωCategories) *U preserves limits and all limits exist in $\omega\mathsf{Cat}$.*

So for D any small category and $f : D \to \omega\mathsf{Categories}$ any D-diagram in ω-categories, we have

$$U(\lim_D f) \simeq \lim_D (U \circ f)\,.$$

Lemma 2.14 (initial and final ω-category) *The* <u>*initial globular object*</u> *is the globular object constant on $\emptyset$, while the* <u>*terminal globular object*</u> *is the globular object constant on* pt*. The* <u>*initial ω-category*</u> *and the* <u>*terminal ω-category*</u> *are the unique ω-categories whose underlying globular objects are the initial and terminal globular object, respectively. We denote all these initial objects by $\emptyset$ and all these terminal objects by* pt.

Definition 2.15 (Hom-ω-category) *For $C \in \omega\mathsf{Categories}(K)$ and $a, b : \mathrm{pt} \longrightarrow C$ we write $C(a,b)$ for the* <u>*Hom ω-category*</u> *whose underlying globular object is given for all $k \in \mathbb{N}$ by the top front row of*

$$
\begin{tikzcd}
 & C_{k+2} \arrow[rr, shift left=2, "s_{k+1}"] \arrow[rr, shift right=2, "t_{k+1}"'] \arrow[dd, "s_0 \times s_0"] & & C_{k+1} \arrow[ll, "i_{k+1}"] \arrow[dd, "s_0 \times t_0"] \\
C(a,b)_{k+1} \arrow[ur] \arrow[rr, shift left=2, "s_k"] \arrow[rr, shift right=2, "t_k"'] \arrow[dd] & & C(a,b)_k \arrow[ll, "i_k"] \arrow[ur] \arrow[dd] & \\
 & C_0 \times C_0 \arrow[rr, "="] & & C_0 \times C_0 \\
\mathrm{pt} \arrow[ur, "a \times b"] \arrow[rr] & & \mathrm{pt} \arrow[ur, "a \times b"] &
\end{tikzcd},
$$

where the left and right squares are pullbacks and the top front morphisms are the universal morphisms induced by these pullbacks from the top rear globular morphisms. Similarly the composition operations $\circ_k$ on $C(a,b)$ are induced from $\circ_{k+1}$ of C.

Definition 2.16 (n-graphs and n-categories) *Globular objects S satisfying $S_k = S_{k-1}$ and $s_{k-1} = \mathrm{Id}$, $t_{k-1} = \mathrm{Id}$ for all $k \in \mathbb{N}$, $k > n$, for some $n \in \mathbb{N}$ are called* <u>*n-graphs*</u>*, forming the full subcategory $n\mathsf{Graphs}(K) \subset \mathsf{GlobularObjects}(K)$.*
ω-categories C whose underlying globular object is an n-graph are called <u>*n-categories*</u>*, forming the full subcategory $n\mathsf{Categories} \subset \omega\mathsf{Categories}$.*
There are obvious truncation functors $\mathsf{GlobularObjects}(K) \to n\mathsf{Graphs}(K)$ and $\omega\mathsf{Categories}(K) \to n\mathsf{Categories}(K)$.

There are two alternative equivalent perspectives on ω-categories internal to categories of sheaves, such as Spaces:

Proposition 2.17 *$\omega\mathsf{Categories}(\mathsf{Spaces})$ is equivalent to ω-category valued sheaves on $\mathsf{CartesianSpaces}$:*

$$
\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathrm{Sh}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories}(\mathsf{Sets}))\,.
$$

Proof. This is a special case of proposition 2.3 (iii) of [17]. $\square$

Remark. In the context of ∞-Lie theory the point of view of <u>smooth ω-categories</u> or <u>Lie ω-categories</u> as ω-categories internal to smooth spaces is useful. On the other hand, in the context of cohomology the point of view of sheaves with values in ∞-categories is useful.

Globes.

Definition 2.18 (n-globe) *The* <u>*(-1)-globe*</u> *G_{-1} is the initial ω-category $G_{-1} := \emptyset$. For $n \in \mathbb{N}$ the* <u>*n-globe*</u> *G_n is the unique ω-category on the globular set $G_n := \mathrm{Hom}_{\mathrm{Set}^{G^{\mathrm{op}}}}(-, n)$*

This is unqiue because there are no nontrivial compositions.

Remark. By Yoneda the ω-functors between the n-globe and the $(n+1)$-globe for $n \in \mathbb{N}$ are

$$\begin{tikzcd} G_n \arrow[r, shift left=2, "\sigma_n"] \arrow[r, shift right=2, "\tau_n"'] & G_{n+1} \arrow[l, "\iota_n" description] \end{tikzcd}$$

and satisfy the globular identities. It makes sense and is convenient in the following to write $\sigma_{-1} = \tau_{-1}$ for the unique morphism

$$\sigma_{-1}, \tau_{-1} : \; G_{-1} \longrightarrow G_0 \; .$$

The n-globe has a single nontrivial n-morphism. Discarding the single nontrivial n-morphism of the n-globe yields the boundary of the n-globe:

Definition 2.19 (boundary of the n-globe and generating cofibrations, [94]) *The boundary of the 0-globe is $\partial G_0 := G_{-1}$ and we write $\mathrm{i}_0 : \partial G_0 \to G_0$ for the unique morphism.*
By induction over $n \in \mathbb{N}$ the boundary of the $(n+1)$-globe, ∂G_{n+1}, is the pushout

$$\begin{tikzcd} \partial G_n \arrow[r, "\mathrm{i}_n"] \arrow[d, "\mathrm{i}_n"'] & G_n \arrow[d] \\ G_n \arrow[r] & \partial G_{n+1} \end{tikzcd} .$$

and $\mathrm{i}_{n+1} : \partial G_n \to G_n$ is the universal morphisms in

$$\begin{tikzcd} \partial G_n \arrow[r, "\mathrm{i}_n"] \arrow[d, "\mathrm{i}_n"'] & G_n \arrow[d] \arrow[ddr, bend left, "\sigma_n"] & \\ G_n \arrow[r] \arrow[drr, bend right, "\tau_n"'] & \partial G_{n+1} \arrow[dr, dashed, "i_{n+1}"] & \\ & & G_{n+1} \end{tikzcd} .$$

The $\{\mathrm{i_n}\}_{n \in \mathbb{N}}$ are the generating cofibrations of ω-categories.

Remark. G_n is a combinatorial model for the n-disk D^n and ∂G_n is a combinatorial model for the $(n-1)$-sphere S^{n-1}.

Discrete ω-categories.

Definition 2.20 (discrete ω-category) *For $X \in K$, write $\mathcal{P}_0(X)$ for the discrete ω-category with $\mathrm{Obj}(\mathcal{P}_0(X)) = X$ and every k-morphism for $k \geq 1$ being an identity.*

Remark. The notation $\mathcal{P}_0(X)$ is meant to allude to "0-dimensional paths in X". This is explained in section 4.2.1. When the context is clear we write just X for the ω-category $\mathcal{P}_0(X)$. Notice the different meaning of "discrete" in ω-categories and in spaces.

ω-Monoids.

Definition 2.21 (ω-monoids) *An ω-monoid is an ω-category of the form $C(a,a)$, definition 2.15, for C an ω-category and $a : \mathrm{pt} \to C$. For A an ω-monoid we write $\mathbf{B}A$ for the ω-category with $(\mathbf{B}A)_0\mathrm{pt}$ and such that $A = \mathbf{B}A(\bullet, \bullet)$.*
Morphisms of ω-monoids $K \to G$ are the morphisms of the corresponding ω-categories, $\mathbf{B}K \to \mathbf{B}G$. We write ωMonoids for the corresponding full subcategories of ωCategories.

Definition 2.22 (n-tuply monoidal ω-categories) *ω-Categories of the form $C = \mathbf{B}^k K$ for $k \in \mathbb{N}$ are called k-fold degenerate and K is then called a k-tuply monoidal ω-category.*

Remark. In particular, doubly monoidal ω-categories are abelian ω-monoids: the crossed complex, section 2.2.1, characterizing doubly monoidal ω-groupoids, i.e. monoidal ω-groups is a complex of abelian groups. For a discussion of the general phenomenon of k-tuply degenerate and k-tuply monoidal ω-categories and the *periodic table of n-categories* see [12] and [42].

Definition 2.23 (kernel and cokernel of morphisms of ω-monoids) *Write* 1 *for the* trivial ω-monoid, *such that* $\mathrm{pt} = \mathbf{B}1$*, with* pt *the terminal* ω*-category. For* K *an* ω*-monoid,* C *any* ω*-category and* $f : C \to \mathbf{B}K$ *an* ω*-functor, the* kernel *of* f *is the pullback*

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ \mathrm{pt} \arrow[r] & \mathbf{B}K \end{tikzcd}$$

and the cokernel *of* f *is the pushout*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & \mathrm{pt} \arrow[d] \\ \mathbf{B}K \arrow[r] & \mathrm{coker}(f) \end{tikzcd} .$$

Tensor product. The n-globe can be obtained from the n-cube by collapsing faces. Accordingly globular sets are special kinds of *cubical sets*, which are presheaves on the cubical category $\Box$ whose objects are the standard n-cubes $\Box_n$ and whose morphisms are the standard injection and collaps maps between these, see section 2 of [48].

The globular category G has the advantage of having the minimum of face and degeneracy maps, where $\Box$ has much more, but $\Box$ has the advantage of admitting an obvious monoidal structure with $\Box_k \otimes \Box_l = \Box_{k+l}$, modeled after the cartesian product of the standard k-cubes $[0,1]^k$ in topology. The *Day convolution product*, definition 6.20, canonically induces from this a biclosed monoidal structure on cubical sets which in turn induces one on globular sets and then on ω-categories. This is the Crans-Gray tensor product on ω-categories generalizing the Gray tensor product on 2-categories.

Theorem 2.24 ([48]) $\omega\mathsf{Categories}(\mathsf{Sets})$ *is monoidal biclosed, definition 6.11.*

A brief review is given in [151]; details are spelled out in [48].

Corollary 2.25 $\omega\mathsf{Categories}(\mathsf{Spaces})$ *is biclosed.*

Restricted to ω-groupoids and crossed complexes (section 2.2.1) the bicolosed structure becomes a symmetric monoidal closed structure and reproduces the tensor product treated in part II of [34] (see p. xv of [45]). Instead of reproducing the explicit description of $\otimes$, which involved tedious combinatorics, we list a handful of crucial properties from which all the facts we shall need follow. The crucial property of $\otimes$, which distinguishes it from the naive cartesian tensor product, is that it raises dimension in analogy to the way the product of an n-dimensional with an m-dimensional space is an $(n+m)$-dimensional space.

Definition 2.26 (cylinder, cone and globe) *The two boundary inclusions* $\sigma_0, \tau_0 : G_0 \to G_1$ *induce for any* $C \in \omega\mathsf{Categories}$ *two morphisms*

$$C \otimes \sigma_0 : \; C \xrightarrow{=} C \otimes G_0 \xrightarrow{\mathrm{Id}\otimes\sigma_0} C \otimes G_1$$

$$C \otimes \tau_0 : \; C \xrightarrow{=} C \otimes G_0 \xrightarrow{\mathrm{Id}\otimes\tau_0} C \otimes G_1 \; .$$

- *We call* $\mathrm{Cyl}(C) := C \otimes G_1$ *the* cylinder *over* C*;*

- *we call the pushout* $\mathrm{Cone}(C)$ *in*

$$\begin{tikzcd} C \otimes G_0 \arrow[r] \arrow[d, "C\otimes\sigma_0"] & G_0 \arrow[d] \\ C \otimes G_1 \arrow[r] & \mathrm{Cone}(C) \end{tikzcd}$$

the cone over C*;*

- *we call the pushout* $\mathrm{Globe}(C)$ *in*

$$\begin{tikzcd} C \otimes (G_0 \sqcup G_0) \arrow[r] \arrow[d, "C\otimes(\sigma_0\sqcup\sigma_1)"] & G_0 \sqcup G_0 \arrow[d] \\ C \otimes G_1 \arrow[r] & \mathrm{Globe}(C) \end{tikzcd}$$

the globe over C.

Remark. The cone construction for ω-groupoids is discussed in detail in section 9.9 of [34].

Lemma 2.27 (cylinders over globes) • *The 0-globe* $G_0 = \mathrm{pt}$ *is the strict unit under* $\otimes$*:* $C \otimes \mathrm{pt} = C$*. The (-1)-globe* $G_{-1} = \emptyset$ *is the strict zero under* $\otimes$*:* $C \otimes \emptyset = \emptyset$*.*

- *The cylinder over the 1-globe is the 2-category "free on a square" in that*

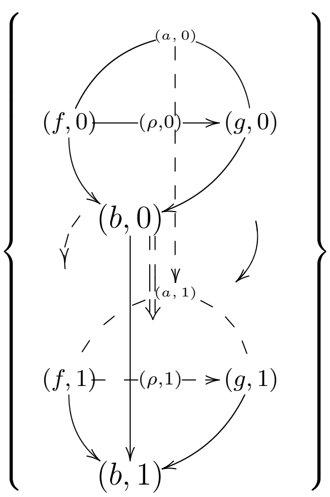
$$G_1 \otimes G_1 = \left\{ \begin{tikzcd} (a,0) \arrow[r] \arrow[d] & (a,1) \arrow[d] \arrow[dl, Rightarrow] \\ (b,0) \arrow[r] & (b,1) \end{tikzcd} \right\} .$$

- *The cylinder over the 2-globe*

$$G_2 = \left\{ \begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\rho"] & b \end{tikzcd} \right\}$$

is the 3-category "free on an ordinary cyclinder" in that

$G_2 \otimes G_1 =$



*[** need to adjust the relative orientation of arrows **]*

Proof. See [49] and [48]. □

[** need discussion here of left/right closed using lax or symmetric closed using pseudo **]

[** the following lemma needs attention – what we really need is corollary 2.29 below, and we really just need if for the ω-groupoid case, where it follows from stuff in [34] **]

Lemma 2.28 (globe over the n-globe) *The $(n+1)$-globe is the globe over the n-globe: for all $n \in \mathbb{N}$ this diagram is a pushout:*

$$\begin{tikzcd} G_n \sqcup G_n \arrow[r, "(G_n\otimes\sigma_0)\sqcup(G_n\otimes\tau_0)"] \arrow[d] & G_n \otimes G_1 \arrow[d] \\ \mathrm{pt} \sqcup \mathrm{pt} \arrow[r] & G_{n+1} \end{tikzcd} .$$

Proof. By induction on n: For $n = 0$ the left leg of the diagram is the identity, so that the claim is that $G_1 = G_0 \otimes G_1$. By lemma 2.27 G_0 is indeed the tensor unit.

Now assume the statement has been proven for $n \in \mathbb{N}$. Using that the boundary of G_{n+1} is two glued copies of G_n we get the pushout

$$\begin{tikzcd} \partial G_{n+1} \sqcup \partial G_{n+1} \arrow[r, "(\partial G_{n+1}\otimes\sigma_0)\sqcup(\partial G_{n+1}\otimes\tau_0)"] \arrow[d] & \partial G_{n+1} \otimes G_1 \arrow[d] \\ \mathrm{pt} \sqcup \mathrm{pt} \arrow[r] & \partial G_{n+2} \end{tikzcd} .$$

Then use that $G_{n+1} \otimes I$ as well as G_{n+2} have a single nontrivial $(n+2)$-morphism. [** complete details **] □

Corollary 2.29 *For C an ω-category, for all $a, b \in \mathrm{Obj}(C)$, the Hom-ω-category $C(a,b)$ is the pullback*

$$\begin{tikzcd} C(a,b) \arrow[r] \arrow[d] & C^I \arrow[d, "d_0 \times d_1"] \\ \mathrm{pt} \arrow[r, "a \times b"] & C \times C \end{tikzcd} .$$

Proof. Using lemma 2.13 and applying to proposition 2.28 the fact that the contravariant Hom-functor $\mathrm{Hom}(-, D)$ takes colimits to limits, we obtain for all $C \in \omega\mathsf{Categories}$ and $n \in \mathbb{N}$, that

$$\begin{tikzcd} \mathrm{Hom}(G_{n+1}, C) \arrow[r] \arrow[d] & \mathrm{Hom}(G_n \otimes I, C) \arrow[d, "d_0 \times d_1"] \\ \mathrm{Obj}(D) \times \mathrm{Obj}(C) \arrow[r, hook] & \mathrm{Hom}(G_n, C) \times \mathrm{Hom}(G_n, C) \end{tikzcd}$$

is a pullback. [...] □

Remark. In the context of ω-groupoids this is a consequence of the structure of the path object C^I in definition 15.1.1. There is a many-object version of proposition 7.1.19 and applied to C^I it yields the above statement.

We are grateful to Ronnie Brown for discussion of this point.

2.2.1 ω-Groupoids and crossed complexes

Of particular interest are ω-categories in which all cells have inverses: either strict inverses, in which case we speak of *ω-groupoids*, or weak inverses, in which case we speak of *weak ω-groupoids*.

Definition 2.30 (ω-equivalence, def. 4 in [94]) *For $C \in \omega\mathsf{Categories}(\mathsf{Sets})$ we call two k-morphisms f and g, $k \in \mathbb{N}$, parallel if either $k = 0$ or if they have the same source and source and target $(k-1)$-morphisms a and b*

$$\begin{tikzcd} a \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & b \end{tikzcd} .$$

There is an equivalence relation on parallel k-morphisms defined by coinduction as follows.

- *The k-morphisms f and g are ω-equivalent, $f \sim g$, if there exists a $k+1$-morphism*

  $$\begin{tikzcd} a \arrow[rr, bend left, "f", ""{name=U, below}] \arrow[rr, bend right, "g"', ""{name=D, above}] & & b \arrow[from=U, to=D, Rightarrow, "\rho"] \end{tikzcd}$$

  which is weakly invertible.

- *A k-morphism, $f \xrightarrow{\rho} g$, $k \geq 1$, is weakly invertible if there exists a k-morphism $g \xrightarrow{\lambda} f$ such that $\rho \circ \lambda \sim i(g)$ and $\lambda \circ \rho \sim i(f)$.*

Definition 2.31 (core of ω-category in Sets) *For $C \in \omega\mathsf{Categories}(\mathsf{Sets})$ let $\mathrm{Core}(C) \hookrightarrow C$, the core of C, denote the restriction of the ω-category to the globular subset of all weakly invertible morphisms of C.*

Proposition 2.32 $\mathrm{Core}(C)$ *is indeed an ω-category. Every ω-functor $C \to D$ in $\omega\mathsf{Categories}(\mathsf{Sets})$ restricts to an ω-functor $\mathrm{Core}(C) \to \mathrm{Core}(D)$. Hence the core construction extends to a functor*

$$\mathrm{Core} : \omega\mathsf{Categories}(\mathsf{Sets}) \to \omega\mathsf{Categories}(\mathsf{Sets}) .$$

Definition 2.33 (core of ω-category in Spaces) *For $C \in \omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathrm{Sh}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})$, let $\mathrm{Core}(C)$ be plot-wise the core from proposition 2.32:*

$$\mathrm{Core}(C) : \ \mathsf{CartesianSpaces}^{\mathrm{op}} \xrightarrow{C} \omega\mathsf{Categories}(\mathsf{Sets}) \xrightarrow{\mathrm{Core}} \omega\mathsf{Categories}(\mathsf{Sets}) \ .$$

Definition 2.34 (ω-groupoids) *A strict 2-groupoid is a strict 2-category such that the space of 2-morphisms is a groupoid under both composition operations.*

An ω-groupoid is an ω-category C such that for all $k > l$ the 2-category $C_{k+1} \rightrightarrows C_{l+1} \rightrightarrows C_l$ is a strict 2-groupoid. A weak ω-groupoid is an ω-category C equal to its core, definition 2.33, $C = \mathrm{Core}(C)$.

An ω-group G is an ω-monoid, definition 2.21, such that $\mathbf{B}G$ is an ω-groupoid.

We write $\omega\mathsf{Groupoids}(\mathsf{Spaces}) \subset \mathsf{Weak}\omega\mathsf{Groupoid}(\mathsf{Spaces}) \subset \omega\mathsf{Categories}(\mathsf{Spaces})$ for the full subcategoies on (weak) ω-groupoids.

The core construction from above maps ω-categories to the maximal weak ω-groupoids inside them $\mathrm{Core} : \omega\mathsf{Categories} \to \mathrm{Weak}\omega\mathrm{Groupoids}$.

Crossed complexes. Crossed complexes are a condensed but equivalent way to encode the information contained in ω-groupoids. They are a nonabelian generalization of complexes of abelian groups and hence give rise to a nonabelian generalization of homological algebra. Therefore crossed complexes often lend themselves to concrete computations more than the ω-groupoids to which they are equivalent. The theory of crossed complexes was developed by Ronnie Brown and his school. A comprehensive monograph is [34].

Definition 2.35 (crossed complex of an ω-category) *To every ω-category $C \in \omega\mathsf{Categories}(K)$ we assign its crossed complex*

$$[C]_\bullet = \left(\begin{tikzcd} \cdots \arrow[r, "\delta"] & {[C]_3} \arrow[r, "\delta"] \arrow[d] & {[C]_2} \arrow[r, "\delta"] \arrow[d] & {[C]_1} \arrow[r, shift left, "\delta_t"] \arrow[r, shift right, "\delta_s"'] \arrow[d, "\delta_s"] & {[C]_0} \arrow[d, "="] \\ \cdots \arrow[r, "="'] & {[C]_0} \arrow[r, "="'] & {[C]_0} \arrow[r, "="'] & {[C]_0} \arrow[r, "="'] & {[C]_0} \end{tikzcd} \right)$$

given by the sequence of dashed horizontal morphisms in

$$\begin{tikzcd} \cdots \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & C_5 \arrow[rr, shift left, "t"] \arrow[rr, shift right, "s"'] & & C_4 \arrow[rr, shift left, "t"] \arrow[rr, shift right, "s"'] & & C_3 \arrow[rr, shift left, "t"] \arrow[rr, shift right, "s"'] & & C_2 \arrow[rr, shift left, "t"] \arrow[rr, shift right, "s"'] & & C_1 \arrow[rr, shift left, "t"] \arrow[rr, shift right, "s"'] & & C_0 \arrow[rr, "\mathrm{Id}"] & & C_0 \\ {} \arrow[rr, dashed, "\delta_6"] & & C_5 \times_{i(C_4)} C_0 \arrow[ul] \arrow[dr] \arrow[rr, dashed, "\delta_5"] & & C_4 \times_{i(C_3)} C_0 \arrow[ul] \arrow[dr] \arrow[rr, dashed, "\delta_4"] & & C_3 \times_{i(C_2)} C_0 \arrow[ul] \arrow[dr] \arrow[rr, dashed, "\delta_3"] & & C_2 \times_{i(C_1)} C_0 \arrow[ul] \arrow[dr] \arrow[rr, dashed, "\delta_2"] & & C_1 \arrow[ul] \arrow[dr] \arrow[rr, dashed, shift left, "\delta_t"] \arrow[rr, dashed, shift right, "\delta_s"'] & & C_0 \arrow[ul] \arrow[dr] & \\ \cdots \arrow[r] & C_0 \arrow[uu, "i"] \arrow[rr, "="'] & & C_0 \arrow[uu, "i"] \arrow[rr, "="'] & & C_0 \arrow[uu, "i"] \arrow[rr, "="'] & & C_0 \arrow[uu, "i"] \arrow[rr, "="'] & & C_0 \arrow[uu, "i"] \arrow[rr, "="'] & & C_0 \arrow[uu, "\mathrm{Id}"] \arrow[rr, "="'] & & C_0 \arrow[uu, "\mathrm{Id}"] \end{tikzcd} ,$$

where all objects in the center row are pullbacks using the source maps s and all dashed morphisms are universal morphisms induced thereon from the target maps t. When C is an ω-groupoid $[C]$ is called a crossed complex of groupoids. In this case

- $\begin{tikzcd} {[C]_1} \arrow[r, shift left, "\delta_t"] \arrow[r, shift right, "\delta_s"'] & {[C]_0} \end{tikzcd}$ *is a 1-groupoid and the* $\begin{tikzcd} {[C]_k} \arrow[r] & {[C]_0} \end{tikzcd}$ *, for all $k \geq 2$, are skeletal 1-groupoids (bundles of groups), abelian for $k \geq 2$;*
- *the groupoid $[C_1]$ acts by conjugation on the $[C_k]$, $k \geq 2$,*

$$\begin{tikzcd} & & {[C]_1 \times_{[C]_0} [C]_k} \arrow[dl] \arrow[dr] \arrow[dddl, bend left, "\mathrm{Ad}"] & \\ & {[C]_1} \arrow[dl, "\delta_t"'] \arrow[dr, "\delta_s"] & & {[C]_k} \arrow[dl] \\ {[C]_0} & & {[C]_0} & \\ & {[C]_k} \arrow[ul] & & \end{tikzcd} ;$$

- *the maps δ_k, $k \geq 2$ are morphisms of groupoids over $[C_0]$ compatible with the action by $[C_1]$;*
- $\mathrm{im}(\delta_2) \subset [C_1]$ *acts by conjugation on $[C_2]$ and trivially on $[C_k]$, $k \geq 3$;*
- $\delta_{k-1} \circ \delta_k = 0$*; $k \geq 3$.*

A morphism of crossed complexes of groupoids is a collection of degreewise morphisms respecting all this structure This yields a category $\mathsf{CrossedComplexes}(K)$ *and the above construction extends to a functor*

$$[-] : \omega\mathsf{Groupoids}(K) \to \mathsf{CrossedComplexes}(K)\,.$$

Remark. For $C \in \omega\mathsf{Groupoids}(\mathsf{Sets})$ its crossed complex $[C] \in \mathsf{CrossedComplexes}(\mathsf{Sets})$ is a crossed complex of groupoids in the sense of [29], see p. 8 of [30]. An exhaustive treatment is in section 7 of [34]. The fact that for $[C]$ a crossed complex and $k \geq 3$ the skeletal groupoid $[C]_k$ is necessarily a bundle of *abelian* groups can be traced back to an Eckmann-Hilton argument, by which endomorphisms of a *strict* identity endomorphism form a commutative monoid. It is the same kind of argument which shows that higher homotopy groups of spaces are necessarily abelian.

Notice that the conjecture in [144], evidence for which is given in [90, 85, 121], says that a general ∞-groupoid is an ω-groupoid in which identity k-morphisms satisfy their defining laws only up to ω-equivalence. It is conceivable that these generalized ω-groupoids correspond to generalized crossed complexes which may be non-abelian in all degrees.

Theorem 2.36 (Brown-Higgins [30, 32, 33]) *The functor* $[-] : \omega\mathsf{Groupoids}(\mathsf{Sets}) \to \mathsf{CrossedComplexes}(\mathsf{Sets})$ *is an equivalence of categories.*

Remark. A comprehensive discussion of this equivalence is in section 13 of [34].

Corollary 2.37 *Also* $[-] : \omega\mathsf{Groupoids}(\mathsf{Spaces}) \to \mathsf{CrossedComplexes}(\mathsf{Spaces})$ *is an equivalence of categories.*

Proof. [** check **] □

Theorem 2.38 *[** adjunction between crossed complexes and chain complexes **]*

Remark. This is section 5 of [34]. Combined with corollary 2.37 proposition 2.17 this yields the inclusion

$$\mathsf{Sheaves}(\mathsf{CartesianSpaces}, \mathsf{ChainComplexes}_+(\mathsf{AbelianGroups}(\mathsf{Sets}))) \subset \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \mathsf{CrossedComplexes}(\mathsf{Sets}))$$
$$\cdots \simeq \omega\mathsf{Groupoids}(\mathsf{Spaces})\,.$$

Definition 2.39 (homotopy groups and homology groups of a crossed complex) *For* $[C]$ *a crossed complex we say*

- $\pi_0([C])$ *is the space of* <u>*connected components*</u>*, i.e. the pushout*

$$\begin{tikzcd} C_1 \arrow[r, "s"] \arrow[d, "t"'] & C_0 \arrow[d] \\ C_0 \arrow[r] & \pi_0([C]) \end{tikzcd} \;;$$

- $\pi_1([C]) := \mathsf{cokernel}(\delta_2) = \frac{C_1}{\mathrm{Im}(\delta_2)}$ *is the* <u>*fundamental groupoid*</u> *of* $[C]$*;*

- *for* $k \geq 2$ *and* $x \in [C_0]$*,* $\pi_k([C], x) = \frac{\ker\delta_k(x)}{\mathrm{im}\delta_{k+1}(x)}$ *is the* k*th* <u>*homotopy group*</u> *of* $[C]$.

Remark. This is the notation used for instance in [28] and it is well adapted for all cases where one wants to think of C as modelling a space, in particular if $C = \Pi_\omega(X)$ for X a space and $\Pi_\omega(X)$ the fundamental ω-groupoid of X from section 4.2.1. In cases where the alternative point of view of $[C]$ as a nonabelian generalization of a chain complex is preferable, it is more suggestive to write $H_k([C], x)$ for $\pi_k([C], x)$ for $k \geq 2$, and speak of the kth <u>homology group</u> of $[C]$. This is the notation used in [34]. We shall use both notations interchangeably, as convenient.

Weak equivalences, fibrations and cofibrations. In sections 2.2.2, 2.2.3 and 2.2.4 we discuss weak equivalences, fibrations and cofibrations in the general context of $\omega\mathsf{Categories}(\mathsf{Spaces})$. Their restriction to $\omega\mathsf{Groupoid}$, which is sufficient in many of our applications, yields the following notions from [28].

Definition 2.40 (weak equivalence) *A morphism $[f] : [C] \to [D]$ of crossed modules is a* <u>*weak equivalence*</u> *if it induces isomorohisms on the π_0, π_1, H_k.*

Remark. See section 7.1.4 of [34] for details. Notice that from the point of view of crossed complexes as nonabelian generalizations of chain complexes, this says that weak equivalences of crossed complexes are indeed *quasi-isomorphisms* in the sense of homological algebra, namely morphisms which induce an isomorphism on homology. This point of view is useful when comparing the relation between the homotopy model structure on $\omega\mathsf{Groupoids}(\mathsf{Spaces})$ with that on $L_\infty\mathsf{Algebroids}$ (section 2.4) under the ∞-Lie integration and differentiation maps, section 4.

2.2.2 Weak and surjective equivalences

Being "∞-structures", ω-categories should live not just in the 1-category ωCategories from definition 2.11 but in some kind of ∞-category, for instance an $(\infty, 1)$-category [21], giving rise to the *homotopy theory* of ω-categories. A "presentation" for such $(\infty, 1)$-categories is well known to be given by a Quillen model structure [64, 72] on the 1-category $\omega\mathsf{Categories}$. A Quillen model structure can be regarded as a convenient way to handle morphisms of arbitrary degree just in terms of 1-morphisms with extra properties, the possible extra properties going by the name *weak equivalences*, *fibrations* and *cofibrations*.

A model category structure on $\omega\mathsf{Groupoids}(\mathsf{Sets})$ is described in [28], a generalization to $\omega\mathsf{Categories}(\mathsf{Sets})$ in [94]. We want to generalize this, at least in parts, to

$$\omega\mathsf{Categories}(\mathsf{Spaces}) = \omega\mathsf{Categories}(\mathsf{Sheaves}(\mathsf{CartesianSpaces})) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})\,.$$

The problem of lifting a model structure on a category of certain structures to the category of sheaves with values in these structures is a familiar one for which various strategies and recipes exist. The main issue is whether or not one takes the weak equivalences of sheaves to be *globally* or just *locally* (stalkwise) to be the weak equivalences of the given structures.

Definition 2.41 (stalkwise property) *Let P be a statement about diagrams in $\omega\mathsf{Categories}$. A diagram D in $\mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories})$ is said to satisfy P* <u>*locally*</u> *or* <u>*stalkwise*</u> *precisely if for all $U \in \mathsf{CartesianSpaces}$ its component diagram $D(U)$ in $\omega\mathsf{Categories}$ has the property that for every point $x \in U$ there is an open neighbourhood $V \subset U$ of x such that the restriction $D(V)$ of $D(U)$ to V satisfies P.*

Much general theory has been developed for the case of global weak equivalences [46, 17, 18] but for the purposes of cohomology theory the local choice is the natural one. This has been studied in detail for the case of presheaves with values in simplicial sets [80] and for presheaves with values in spectra [26], where the latter uses a slight variant of a Quillen model category structure: that of a *category of fibrant objects*.

We now exhibit on $\omega\mathsf{Categories}(\mathsf{Spaces})$ the structure of a category of fibrant objects in this sense, for which the weak equivalences are locally those of $\omega\mathsf{Categories}(\mathsf{Sets})$ while the fibrations are globally the fibrations of $\omega\mathsf{Categories}(\mathsf{Sets})$. The local acyclic fibrations play an auxiliary role as the *hypercovers* which crucially enter the discussion of *ω-anafunctors* in section 2.2.5, following [105], and then of cocycles in section 3, following [26] and [81].

The following definition is that of weak equivalences and of acyclic fibrations in $\omega\mathsf{Categories}(\mathsf{Sets})$ from [94], but formulated diagrammatically and hence internally in a way that is applicable to $\omega\mathsf{Categories}(K)$ for all contexts K.

Definition 2.42 ((essential) k-surjectivity) *An ω-functor $F : C \longrightarrow D$ is 0-surjective if*

$$F_0 : C_0 \to D_0$$

is an epimorphism. It is k-surjective for $k \in \mathbb{N}$, $k \geq 1$ if the universal morphism $C_k \dashrightarrow (F_{k-1} \times F_{k-1})^* D_k =: P_k$ in

$$\begin{tikzcd}
C_k \arrow[dr, dashed] \arrow[drrr, bend left, "F_k"] \arrow[ddr, bend right, "s_k \times t_k"'] & & & \\
& P_k \arrow[rr] \arrow[d] & & D_k \arrow[d, "s_k \times t_k"] \\
& C_{k-1} \times C_{k-1} \arrow[rr, "F_{k-1} \times F_{k-1}"] & & D_{k-1} \times D_{k-1}
\end{tikzcd}$$

is an epimorphism. The ω-functor is essentially k-surjective for $k \in \mathbb{N}$ if the composite

$$C_k \dashrightarrow P_k \longrightarrow P_k/_\sim$$

is an epimorphism, for $P_k/_\sim$ the quotient space, definition 2.8, of ω-equivalence classes defined as follows: Define Q_k for $k \in \mathbb{N}$ by $Q_0 := \mathrm{Core}(D)_1$, where $\mathrm{Core}(D)$ is the core of D as in definition 2.33, and for $n \geq 1$ as the pullback Q_k in

$$\begin{tikzcd}
Q_k \arrow[r] \arrow[d] & \mathrm{Core}(D)_{k+1} \arrow[d, "s_k \circ s_{k+1} \times t_k \circ t_{k+1}"] \\
C_{k-1} \times C_{k-1} \arrow[r, "F_{k-1} \times F_{k-1}"] & D_{k-1} \times D_{k-1}
\end{tikzcd},$$

where the two morphisms $Q_k \overset{\sigma}{\underset{\tau}{\rightrightarrows}} P_k$ are for $k = 0$ given by $\sigma = s$ and $\tau = t$ and are for $k \geq 1$ by the two universal morphisms in

$$\begin{tikzcd}
& Q_k \arrow[dl, dashed, "\tau"'] \arrow[rr] \arrow[ddl] & & \mathrm{Core}(D)_{k+1} \arrow[dl, "t"'] \arrow[ddl, "s \circ s \times t \circ t"] \\
P_k \arrow[rr] \arrow[d] & & D_k \arrow[d, "s \times t"] & \\
C_{k-1} \times C_{k-1} \arrow[rr, "F_{k-1} \times F_{k-1}"'] & & D_{k-1} \times D_{k-1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Q_k \arrow[dl, dashed, "\sigma"'] \arrow[rr] \arrow[ddl] & & D_{k+1} \arrow[dl, "s"'] \arrow[ddl, "s \circ s \times t \circ t"] \\
P_k \arrow[rr] \arrow[d] & & D_k \arrow[d, "s \times t"] & \\
C_{k-1} \times C_{k-1} \arrow[rr, "F_{k-1} \times F_{k-1}"'] & & D_{k-1} \times D_{k-1} &
\end{tikzcd}.$$

The quotient space $P_k/_\sim$ in question is the coequalizer of these, i.e. the pushout

$$\begin{tikzcd}
Q_k \arrow[r, "\tau"] \arrow[d, "\sigma"] & P_k \arrow[d] \\
P_k \arrow[r] & P_k/_\sim
\end{tikzcd}.$$

Remark. A detailed description of essential k-surjectivity and its meaning can be found in [12], around definition 4, discussed there in the context $K = \mathsf{Sets}$. Recall from lemma **??** that in the context $K = \mathsf{Spaces}$ the epimorphisms are the *local* sections admitting maps, hence the local epimorphisms of Sets.

Definition 2.43 (weak equivalences in $\omega\mathsf{Categories}(K)$) *An ω-functor $f : C \to D$ in $\omega\mathsf{Categories}(K)$ which is essentially k-surjective for all $k \in \mathbb{N}$ is a weak equivalence.*

We write

$$f : C \xrightarrow{\simeq} D \iff f \text{ is a weak equivalence}.$$

Remark. Weak equivalences of 1-categories are the familiar fully faithful and essentially surjective functors; compare theorem 6.17. This follows from the elementary but remarkable fact, amplified in [12], that faithfulness in the highest nontrivial degree is fullness in one degree higher.

Lemma 2.44 *The weak equivalences in* $\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories}(\mathsf{Sets}))$ *are those morphism of sheaves which are locally weak equivalences of* $\omega\mathsf{Categories}(\mathsf{Sets})$ *in the sense of [94].*

Lemma 2.45 (weak equivalence of ω-groupoids and crossed complexes) *A morphism $f : C \to D$ of ω-groupoids is a (local) weak equivalence if and only if the induced morphism of crossed complexes is a (local) weak equivalence of crossed complexes, definition 2.40.*

Proof. The quotient $\ker\delta_k/\mathrm{im}\delta_{k+1}$ realizes precisely the space of ω-equivalence classes of k-automorphisms of identity $(k-1)$-morphisms. Hence surjectivity of $H_k([f]) : H_k([C]) \to H_k([D])$ is essential k-surjectivity. The map is injective if and only we have essential $(k+1)$-surjectivity. [** polish and give more details **] $\square$

Definition 2.46 (surjective equivalences) *An morphism in* $\omega\mathsf{Categories}(K)$ *which is k-surjective for all $k \in \mathbb{N}$ is a* <u>*surjective equivalence*</u>

Definition 2.47 (cofibrations and fibrations) *Fix a faithful functor* $\mathsf{Sets} \to K$*.*

- *The inclusions of globular sets $\mathcal{I} := \{\mathrm{i}_n : \partial G^n \longrightarrow G^n\}$ from definition 2.47 become morphisms of globular K-objects. These i_n are the* <u>*generating cofibrations*</u> *in* $\omega\mathsf{Categories}(K)$*.*
- *The morphisms $f : C \to D$ with the right lifting property with respect to the generating cofibrations are the* <u>*I-fibrations*</u>*.*

$$\begin{tikzcd} \partial G^n \arrow[r] \arrow[d, "i_n"] & C \arrow[d, "f"] \\ G^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

- *The morphisms with the left lifting property with respect to the $\mathcal{I}$-fibrations are the* <u>*cofibrations*</u>*.*
- *The morphism with the right lifting property with respect to acyclic cofibrations (cofibrations which are also weak equivalences) are the* <u>*fibrations*</u>*.*

For $K = \mathsf{Sheaves}$ we speak of <u>*local $\mathcal{I}$-fibrations*</u>*,* <u>*local cofibrations*</u>*,* <u>*local fibrations*</u> *if the respective lifting properties hold* locally*.*

For $K = \mathsf{Sheaves}$ we take $\mathsf{Sets} \to \mathsf{Sheaves}$ to be the functor which sends sets to the sheaves constant on them.

Lemma 2.48 (surjective equivalences and $\mathcal{I}$-fibrations) *We have*

- *surjective equivalences, definition 2.46, in* $\omega\mathsf{Categories}(\mathsf{Sets})$ *are precisely the $\mathcal{I}$-fibrations for $K = \mathsf{Sets}$.*
- *surjective equivalences in* $\omega\mathsf{Categories}(\mathsf{Spaces})$ *are precisely local $\mathcal{I}$-fibrations for $K = \mathsf{Spaces}$.*

Theorem 2.49 ([94]) *There is a cofibrantly generated model structure on* $\omega\mathsf{Categories}(\mathsf{Sets})$ *with the generating cofibrations and the weak equivalences as in definition 2.47 for $K = \mathsf{Sets}$.*

Remark. Two types of model structure on categories and higher categories are known: a "topological" type going back to Thomason [154] [67], later generalized to 2-categories [161] and cubical n-categories [55], which relates under the nerve construction to the standard model structure on simplicial sets, and the "categorical" or "folklore" one used here, generalized to 2-categories [91] (notice the erratum in [92]) and ω-categories [94], in which the weak equivalences are the actual categorical equivalences.

Lemma 2.50 ([114]) *With respect to the model structure on* $\omega\mathsf{Categories}(\mathsf{Sets})$ *from theorem 2.49, surjective equivalences are precisely the acyclic fibrations.*

Proof. For S any set of morphisms, write $\mathrm{rlp}(S)$ for the set of morphisms having the right lifting property with respect to S and $\mathrm{llp}(S)$ for the set of morphisms having the left lifting property with respect to S. With $\mathcal{I}$ the set of generating cofibrations from definition 2.47 we have by definition $\mathrm{SurjEqu} = \mathrm{rlp}(\mathcal{I})$ and $\mathrm{Cof} = \mathrm{llp}(\mathrm{SurjEqu}) = \mathrm{llp}(\mathrm{rlp}(\mathcal{I}))$ and want to show that $\mathrm{Fib} \cap \mathrm{WEqu} = \mathrm{SurjEqu} = \mathrm{rlp}(\mathcal{I})$. Notice that

$$\mathrm{rlp}(\mathcal{I}) = \mathrm{rlp}(\mathrm{Cof})$$

since $\mathcal{I} \subset \mathrm{Cof}$ which implies $\mathrm{rlp}(\mathrm{Cof}) \subset \mathrm{rlp}(\mathcal{I})$, and since generally $S \subset \mathrm{rlp}(\mathrm{llp}(S))$ for all S which implies $\mathrm{rlp}(\mathcal{I}) \subset \mathrm{rlp}(\mathrm{Cof})$ for $S = \mathrm{rlp}(\mathcal{I})$.

Now the fact that we do have a model structure by theorem 2.49 says that $\mathrm{Fib} \cap \mathrm{WEqu} \subset \mathrm{rlp}(\mathrm{Cof})$ which with the previous statement says that $\mathrm{Fib} \cap \mathrm{WEqu} \subset \mathrm{SurjEqu}$. Since the converse $\mathrm{Surj} \subset \mathrm{Fib} \cap \mathrm{WEqu}$ holds trivially this yields the desired result. □

Reflecting this we introduce the notation

Definition 2.51 *For f a morphism in* $\omega\mathsf{Categories}(K)$ *we write*

$$f : C \xrightarrow{\simeq} D \quad \Longleftrightarrow \quad f \textit{ is an } \mathcal{I}\textit{-fibration}.$$

Remark. To emphasize: for $K = \mathsf{Sets}$ the notation $C \xrightarrow{\simeq} D$ denotes precisely an acyclic fibration, while for $K = \mathsf{Spaces}$ it denotes not necessarily an acyclic fibration but a *local* acyclic fibration (which may be but need not be an acyclic fibration globally).

2.2.3 Cofibrations and pseudo-∞-functors

A detailed discussion of cofibrations in $\omega\mathsf{Categories}(\mathsf{Sets})$ and cofibrant replacements (free resolutions) has been given in [112], [93] and [113] following the concept of "polygraphs" in [39] (which is equivalent to the much earlier introduced concept of *computads* by Ross Street).

[** free resolutions should be closely related to the left adjoint of the ω-nerve functor discussed at the end of [149] and reviewed in section 3.1.1 – still needs to be discussed **]

Cofibrant ω-categories. An ω-category C is *cofibrant* if the unique morphism $\emptyset \to C$ is a cofibrations, i.e. if all ω-functors out of C into codomains of surjective equivalences can be lifted

$$\begin{tikzcd} & A \arrow[d, "\simeq"] \\ \mathsf{C} \arrow[r, "F"] \arrow[ur, dashed, "\hat{F}"] & B \end{tikzcd} .$$

We recall from [112, 113] how cofibrant ω-categories are precisely those that are *degreewise freely generated.*

Recall n-graphs and n-categories from definition 2.16. Let $n\mathsf{Categories}^+$ be the pullback in

$$\begin{tikzcd} n\mathsf{Categories}^+ \arrow[r] \arrow[d] & (n+1)\mathsf{Graphs} \arrow[d, "V_n"] \\ n\mathsf{Categories} \arrow[r, "U_n"] & n\mathsf{Graphs} \end{tikzcd} .$$

An object in $n\mathsf{Categories}^+$, which we denote $(S_{n+1} \overset{s}{\underset{t}{\rightrightarrows}} C)$, is an $(n+1)$-graph $S_\bullet$ equipped with the structure of an n-category C on its truncation to an n-graph. There is an obvious forgetful functor $W_{n+1} : (n+1)\mathsf{Categores} \to n\mathsf{Categories}^+$ arising as the universal morphism in

$$\begin{tikzcd} (n+1)\mathsf{Categories} \arrow[drr, bend left, "U_{n+1}"] \arrow[dr, dashed, "W_{n+1}"] \arrow[ddr, bend right] & & \\ & n\mathsf{Categories}^+ \arrow[r] \arrow[d] & (n+1)\mathsf{Graphs} \arrow[d, "V_n"] \\ & n\mathsf{Categories} \arrow[r, "U_n"] & n\mathsf{Graphs} \end{tikzcd} .$$

Lemma 2.52 *This functor W_{n+1} has a left adjoint $L_{n+1} : n\mathsf{Categories}^+ \to (n+1)\mathsf{Categories}$.*

Remark. Acting on $(S_{n+1} \overset{s}{\underset{t}{\rightrightarrows}} C)$ the functor L_{n+1} is the identity on C and sends the space S_{n+1} to the space of all possible *pasting diagrams* of elements of S_{n+1} (whiskered in all possibly ways by morphisms in C).

Definition 2.53 (polygraph, [112]) *Every 0-category is a 0-polygraph. For $n \in \mathbb{N}$ and $n \geq 1$ an n-category $C^{(n)}$ is an n-polygraph if it is of the form $C^{(n)} = L_n(S_n \overset{s}{\underset{t}{\rightrightarrows}} W_{n-1}(C^{(n-1)}))$ for $C^{(n-1)}$ an $(n-1)$-polygraph. An ω-polygraph is an ω-category C such that for all $n \in \mathbb{N}$ its truncation $\mathrm{truncate}_n(C)$ to an n-category is an n-polygraph.*

Remark. An ω-polygraph is an ω-category obtained from "generators and relations", where each relation in degree n is itself a generator subject to relations in degree $n+1$.

Theorem 2.54 ([113]) *Cofibrant objects in $\omega\mathsf{Categories}(\mathsf{Sets})$ are precisely the ω-polygraphs.*

Proof. Let C be a polygraph, $A \xrightarrow{\simeq} B$ a surjective equivalence and $f : C \to B$ any morphism. We need to exhibit a lift $\hat{f}$

$$\begin{tikzcd} & A \arrow[d, "\simeq"] \\ C \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"] & B \end{tikzcd} .$$

Construct this recursively: choose any lift $\hat{f}_0$ of f_0. This exist since $A \to B$ is surjective on objcts. Then, assume a lift $\hat{f}_k$ has been found. Use that $C_{k+1} = L_{k+1}(S_{k+1} \overset{s}{\underset{t}{\rightrightarrows}} C_k)$, choose a lift of maps of $(n+1)$-graphs from S_{k+1}, which exists because $A \to B$ is $(k+1)$–surjective and then use the freeness property of L_{k+1} to extend this uniquely to a lfit $\hat{f}_{k+1}$. □

Cofibrant replacements.

Definition 2.55 *A* <u>*good cofibrant replacement*</u> *of an ω-category C is a surjective equivalence* $\hat{C} \xrightarrow{\simeq} C$ *with $\hat{C}$ a cofibrant ω-category.*

Proposition 2.56 (functorial cofibrant replacement) *There is a functor*

$$(-)_{\mathrm{cof}} : \omega\mathsf{Categories}(\mathsf{Sets}) \to \omega\mathsf{Categories}(\mathsf{Sets})$$

and a natural transformation

$$\begin{tikzcd} \omega\mathsf{Categories}(\mathsf{Sets}) \arrow[r, bend left, "(-)_{\mathrm{cof}}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow\rho"] & \omega\mathsf{Categories}(\mathsf{Sets}) \end{tikzcd}$$

such that the components of ρ are good cofibrant replacements $\emptyset \hookrightarrow C_{\mathrm{cof}} \xrightarrow{\rho_C} C$.

Proof. We follow section 4.2 of [112]. Given $C \in \omega\mathsf{Categories}(\mathsf{Sets})$ define $\rho : C_{\mathrm{cof}} \to C$ with C_{cof} a polygraph, definition 2.53, inductively as follows. First let $C^0_{\mathrm{cof}} := C_0$ be the 0-category over the space of objects of C and let $\rho^0_C :\ C^0_{\mathrm{cof}} \xrightarrow{=} C_0$ be the identity. Then assume the k-category C^k_{cof} and the k-functor $\rho^k_C : C^k_{\mathrm{cof}} \to \mathrm{truncate}_k(C)$ have been defined, as well as a section $w^k : C_k \to (C_{\mathrm{cof}})_k$ of the component $(\rho^k_C)_k$, and set

$$C^{k+1}_{\mathrm{cof}} := L_{k+1}(\ D_{k+1} \underset{\tau}{\overset{\sigma}{\rightrightarrows}} C^k_{\mathrm{cof}}\ )\,,$$

with L_{k+1} the functor from lemma 2.52, and where D_{k+1} is given by the pullback square

$$\begin{tikzcd} D_{k+1} \arrow[r, "f"] \arrow[d] & C_{k+1} \arrow[d, "s\times t"] \\ (C^k_{\mathrm{cof}})_k \times (C^k_{\mathrm{cof}})_k \arrow[r, "(\rho^k_C)_k \times (\rho^k_C)_k"] & C_k \times C_k \end{tikzcd} .$$

Then let $\rho^{k+1}_C : C^{k+1}_{\mathrm{cof}} \to \mathrm{truncate}_{k+1}(C)$ be the image of the canonical morphism (f, ρ^k_C) in $k\mathsf{Categories}^+$ (with f from the above pullback diagram) under the isomorphism

$$\mathrm{Hom}((\ D_{k+1} \underset{\tau}{\overset{\sigma}{\rightrightarrows}} C^k_{\mathrm{cof}}\ ), W_{k+1}(\mathrm{truncate}_k(C))) \simeq \mathrm{Hom}(L_{k+1}(\ D_{k+1} \underset{\tau}{\overset{\sigma}{\rightrightarrows}} C^k_{\mathrm{cof}}\ ), \mathrm{truncate}_k(C))\,.$$

Finally take the directed limit over n: C_{cof} is the unique ω-category whose truncation at degree k is C^k_{cof} and $\rho : C_{\mathrm{cof}} \to C$ the unique ω-functor whose restriction to degree k is ρ^k_C, for all $k \in \mathbb{N}$. We have by construction, using theorem 2.54, that C_{cof} is cofibrant. It remains to be shown that ρ_C is a surjective equivalence.

To check this, we need to check, by definition 2.42, whether the universal morphism in

$$\begin{tikzcd} (C_{\mathrm{cof}})_{k+1} \arrow[dr, dashed] \arrow[drr, bend left, "(\rho_C)_{k+1}"] \arrow[ddr, bend right, "s\times t"'] & & \\ & P_{k+1} \arrow[r, "f"] \arrow[d] & C_{k+1} \arrow[d, "s\times t"] \\ & (C^k_{\mathrm{cof}})_k \times (C^k_{\mathrm{cof}})_k \arrow[r, "(\rho^k_C)_k \times (\rho^k_C)_k"] & C_k \times C_k \end{tikzcd}$$

admits local sections. But the pullback P_{k+1} appearing here is precisely the D_{k+1} in the above, while $(C_{\mathrm{cof}})_{k+1}$ is the underlying set of $(k+1)$-morphisms of the free $(k+1)$-category on D_{k+1}. So the section in question is the unit of the adjunction

$$D_{k+1} \hookrightarrow W(L(D_{k+1})) \; .$$

[** complete the end of the argument **]

$\square$

Remark. This construction of C_{cof} has the advantage of being systematic, but in applications C_{cof} may be an unconviently big realization of a cofibrant replacement of C. Notice, for instance, that $(C_{\mathrm{cof}})_{\mathrm{cof}}$ is also a cofibrant replacement of C which is even "bigger" than C_{cof}. To break down C_{cof} in special cases to something more tractable, notice that the model category structure on $\omega\mathsf{Categories}$ induces a model category structure on $n\mathsf{Categories}$ by the inclusion $n\mathsf{Categories} \hookrightarrow \omega\mathsf{Categories}$. A cofibrant replacement $\hat{C}_{(n)}$ of the n-category C in $n\mathsf{Categories}$ is not the same as a cofibrant replacement $\hat{C}$ of C in $\omega\mathsf{Categories}$. But ω-functors out of $\hat{C}_{(n)}$ are special cases of ω-functors out of $\hat{C}$.

We now describe the restriction of the above discussion to cofibrant replacements of 2-categories. This reproduces the construction appearing in the proof of prop. 4.2 of [91].

Definition 2.57 *For C a strict 2-category define a strict 2-category C_{cof_2} as follows:*

- *The objects of C_{cof_2} are those of C.*
- *The morphisms of C_{cof_2} are finite sequences of composable morphisms consisting of*
  - *the morphisms of C;*
  - *one new endomorphism i_a for each object a of C.*
- *The 2-morphisms of C_{cof} are generated from finite sequences of horizontally composable 2-morphisms of C together with new generators*

$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\simeq\Downarrow c_{f,g}"] & \\ a \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & {} & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, bend left=60] \arrow[rr, "i_a"'] \arrow[rr, phantom, bend left=30, "\simeq\Downarrow u_a"] & & a \end{tikzcd}$$

for all composable 1-morphisms f, g and all objects a of C

subject to the relations [...] [tetrahedron, pillow and roll and compatibility identities as usual].

Lemma 2.58 *In* $2\mathsf{Categories}$ *the above 2-category C_{cof_2} is a cofibrant replacement of C.*

Proof. To see that C_{cof_2} is weakly equivalent to C notice that the 2-functor on the right is 0-, 1- and 2-surjective. Injectivity on 2-morphisms follows from the relations divided out in the construction of C_{cof_2}. To see that C_{cof_2} is a cofibrant replacement in $2\mathsf{Categories}$ of C consider

$$\begin{tikzcd} & A \arrow[d] \\ \hat{C} \arrow[ur, dashed, "\hat{F}"] \arrow[r, "F"] & B \end{tikzcd}$$

with $A \xrightarrow{\simeq} B$ a surjective equivalence and F any 2-functor, and find a lift $\hat{F}$ as follows:

1. choose lifts $\hat{F}(a) \in \mathrm{Obj}(A)$ of objects for all $a \in \mathrm{Obj}(\hat{C}) = \mathrm{Obj}(C)$;
2. choose lifts $(\hat{F}(a) \xrightarrow{\hat{F}(f)} \hat{F}(b)) \in 1\mathrm{Mor}(A)$ of 1-morphisms, for all $f \in 1\mathrm{Mor}(C)$;
3. choose lifts of 2-cell generators

$$\begin{tikzcd}
\hat{F}(a) \arrow[r, bend left=50, "\hat{F}(f)"] \arrow[r, bend right=50, "\hat{F}(g)"'] \arrow[r, phantom, "\Downarrow \hat{F}(\rho)"] & \hat{F}(b)
\end{tikzcd}$$

for all $\rho \in 2\mathrm{Mor}(C)$ and

$$\begin{tikzcd}
& \hat{F}(b) \arrow[dr, "\hat{F}(g)"] \arrow[d, phantom, "\Downarrow \hat{F}(c_{f,g})"] & \\
\hat{F}(a) \arrow[ur, "\hat{F}(f)"] \arrow[rr, "\hat{F}(g\circ f)"'] & {} & \hat{F}(c)
\end{tikzcd}$$

for all additional 2-cell-generators in $\hat{C}$.

All these lifts exists by the fact that $A \twoheadrightarrow B$ is assumed to be a surjective equivalence. That $\hat{F}$ defined this way is indeed 2-functorial follows from the fact that all images of identities between 2-cells in $\hat{C}$ have to have lifts, by 3-surjectivity of $A \twoheadrightarrow B$ to 3-morphisms of A. But since A is a 2-category this makes them identities in A, too. (In other words: 3-surjectivity is 2-injectivity in $2\mathsf{Categories}$). For instance the identity 3-morphism

$$\begin{tikzcd}[column sep=large]
F(a) \arrow[r, bend left, "F(f_1)"] \arrow[r, bend right, "F(g_1)"'] \arrow[r, phantom, "\Downarrow F(\rho_1)"] \arrow[rr, bend left=70, "F(f_2\circ f_1)"] \arrow[rr, bend right=70, "F(g_2\circ g_1)"'] \arrow[rr, phantom, shift left=8, "\Downarrow F(c^{-1}_{f_1,f_2})"] \arrow[rr, phantom, shift right=8, "\Downarrow F(c^{-1}_{g_1,g_2})"] & F(b) \arrow[r, bend left, "F(f_2)"] \arrow[r, bend right, "F(g_2)"'] \arrow[r, phantom, "\Downarrow F(\rho_2)"] & F(c) \arrow[r, "="] & F(a) \arrow[r, bend left=50, "F(f_2\circ f_1)"] \arrow[r, bend right=50, "F(g_2\circ g_1)"'] \arrow[r, phantom, "\Downarrow F(\rho_1\cdot\rho_2)"] & F(c)
\end{tikzcd}$$

in B is guaranteed to have a lift

$$\begin{tikzcd}[column sep=large]
\hat{F}(a) \arrow[r, bend left, "\hat{F}(f_1)"] \arrow[r, bend right, "\hat{F}(g_1)"'] \arrow[r, phantom, "\Downarrow \hat{F}(\rho_1)"] \arrow[rr, bend left=70, "\hat{F}(f_2\circ f_1)"] \arrow[rr, bend right=70, "\hat{F}(g_2\circ g_1)"'] \arrow[rr, phantom, shift left=8, "\Downarrow \hat{F}(c^{-1}_{f_1,f_2})"] \arrow[rr, phantom, shift right=8, "\Downarrow \hat{F}(c^{-1}_{g_1,g_2})"] & \hat{F}(b) \arrow[r, bend left, "\hat{F}(f_2)"] \arrow[r, bend right, "\hat{F}(g_2)"'] \arrow[r, phantom, "\Downarrow \hat{F}(\rho_2)"] & \hat{F}(c) \arrow[r, "\hat{F}(=)"] & \hat{F}(a) \arrow[r, bend left=50, "\hat{F}(f_2\circ f_1)"] \arrow[r, bend right=50, "\hat{F}(g_2\circ g_1)"'] \arrow[r, phantom, "\Downarrow \hat{F}(\rho_1\cdot\rho_2)"] & \hat{F}(c)
\end{tikzcd} ,$$

which can only be an identity 3-morphism in A. $\square$

ω-Functors out of cofibrant replacements allow to obtain *weak* morphisms between ω-categories, pseudo ω-functors, those that respect all compositions and units only up to higher coherent equivalence:

Definition 2.59 (pseudo ω-functor) *For C, D ω-categories a pseudo ω-functor is a span of the form*

$$\begin{tikzcd} & C_{\mathrm{cof}} \arrow[dl, "\simeq"'] \arrow[dr] & \\ C & & D \end{tikzcd}$$

with C_{cof} the cofibrant ω-category from definition 2.56.

To justify our use of the term "weak ω-functors" for ana-ω-functors out of cofibrant replacements, notice that for $n = 2$, where a well-known version of weak n-functors between strict n-categories is available, this concept is indeed reproduced:

Proposition 2.60 (pseudo 2-functors, [91]) *For C and D strict 2-categories, pseudo 2-functors $F : C \to D$ are in bijective correspondence with strict 2-functors*

$$\begin{tikzcd} & C_{\mathrm{cof}_2} \arrow[dl, "\simeq"'] \arrow[dr, "\hat{F}"] & \\ C \arrow[rr, "|" marking] & & D \end{tikzcd} .$$

Proof. The components of the compositor and unitor of F are the images under $\hat{F}$ of the new 2-cell generators $c_{f,g}$ and i_a in $\hat{C}$, respectively. The coherence condition for compositor and unitor are the relations built into $\hat{C}$. $\square$

Remark. Pseudo 2-functors as strict 2-functors out of cofibrant replacements are treated in more detail in section 4.1 of [91]. restricted to the case of one-object 2-groupoids is discussed in [120].

2.2.4 Fibrations and ∞-bundles

Fibrations in the category CrossedComplexes and hence, by the equivalence theorem 2.36, in the category ωGroupoids, have been defined before the more general fibrations in ωCategories:

Definition 2.61 (fibration of crossed complexes of groupoids, [76]) *A fibration of 1-groupoids is a functor $f : C \to D$ which is 1-surjective on source fibers in that the dashed universal morphism in*

$$\begin{tikzcd} D_1 \arrow[dr, dashed] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "s"'] & & \\ & C_0 \times_{D_0} D_1 \arrow[r] \arrow[d] & D_1 \arrow[d, "s"] \\ & C_0 \arrow[r, "f_0"] & D_0 \end{tikzcd}$$

has local sections. A fibration of crossed complexes of groupoids is a morphism $f : [C] \to [D]$ which restricts to a fibration of 1-groupoids on all the groupoids

$$\begin{tikzcd} {[C]_k} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & {[C]_0} \end{tikzcd}$$

for $k \in \mathbb{N}$.

See also [28].

Conjecture. It should be true that the fibrations of crossed complexes in [76] are precisely those of [94] restricted along the inclusion $\mathsf{CrossedComplexes} \simeq \omega\mathsf{Groupoids} \hookrightarrow \omega\mathsf{Categories}$. This still needs to be checked. For the time being we will assume that it is true.

Corollary 2.62 *All ω-groupoids are fibrant.*

[** It ought to be true that even all ω-categories are fibrant. This is true for the restriction of the model structure on ω-cat to 1Cat and 2Cat. **]

We now describe important examples of fibrations arising from pullbacks of path objects.

Definition 2.63 (interval object) *Write $I := \{ a \longrightarrow b \}$ for the interval category consisting of two objects and precisely one nontrivial 1-morphism between them.*

Remark. Another common symbol for I is $\mathbf{2}$. I is also known as the *second oriental* $I = O(\Delta^1)$ (see section 3.1.1) as well as the *1-globe* $I = O(G^1)$, definition 2.18. In the context of $\omega\mathsf{Groupoids}$ the interval object is $I_\simeq := \{ a \xrightarrow{\simeq} b \}$. We will be mainly interested in ω-categories $\mathrm{hom}(I, C)$ for C an ω-groupoid. In this context one can equivalently take I to be $I_\simeq$.

Lemma 2.64 (path objects for ω-categories) *For every $C \in \omega\mathsf{Groupoids}$ the ω-category $C^I = \mathrm{hom}(I, C)$ is the path object of C in that we have that*

$$
\begin{tikzcd}
 & & C \\
C \arrow[r, hook, "\simeq"] & C^I \arrow[ur, two heads, "d_0", "\simeq"'] \arrow[r, two heads, "d_0 \times d_1"] \arrow[dr, two heads, "d_1"', "\simeq"] & C \times C \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
 & & C
\end{tikzcd} .
$$

Proof. This is essentially lemma 2.5 in [28]. □
See also p. 421 of [26].

Definition 2.65 (tangent ω-category [126]) *For $C \in \omega\mathsf{Categories}(\mathsf{Spaces})$ and $x : \mathrm{pt} \to C$ an object in C we call the pullback $T_x C$ in*

$$
\begin{tikzcd}
T_x C \arrow[r] \arrow[d, two heads] & C^I \arrow[d, two heads, "d_0"] \\
\mathrm{pt} \arrow[r, "x"] & C
\end{tikzcd}
$$

the tangent ω-category of C at x. It comes equipped with the canonical map $p : T_x C \longrightarrow C^I \xrightarrow{d_1} C$.

Lemma 2.66 *For all ω-groupoids C, this morphism $p : T_x C \twoheadrightarrow C$ is a fibration.*

Proof. This is a special case of one part of the proof of the "factorization lemma" in [26]: By corollary 2.62 all ω-groupoids are fibrant. Using that pullbacks of fibrations are again fibrations, we obtain for all fibrant objects C and D that projections out of their product are fibrations

$$
\begin{tikzcd}
C \times D \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"'] & D \arrow[d, two heads] \\
C \arrow[r, two heads] & \mathrm{pt}
\end{tikzcd}
$$

and for all morphisms $f : C \to D$ that the top left vertical morphisms in the double pullback square

$$\begin{tikzcd}
C \times_D D^I \arrow[r] \arrow[d, two heads, "\mathrm{id}\times d_1"'] & D^I \arrow[d, two heads, "d_0 \times d_1"'] \arrow[dd, two heads, bend left, "d_0"] \\
C \times D \arrow[r, "f\times \mathrm{Id}"] \arrow[d, two heads, "\mathrm{pr}_1"] & D \times D \arrow[d, two heads, "\mathrm{pr}_1"] \\
C \arrow[r, "f"] & D
\end{tikzcd} .$$

is a fibration. Since composites of two fibrations are fibrations, it follows that p in

$$\begin{tikzcd}
C \times_D D^I \arrow[r] \arrow[d, two heads, "\mathrm{id}\times d_1"'] \arrow[drr, two heads, "p"] & D^I \arrow[d, two heads, "d_0 \times d_1"] & \\
C \times D \arrow[r, "f\times \mathrm{Id}"] & D \times D \arrow[r, two heads, "\mathrm{pr}_2"'] & D
\end{tikzcd}$$

is a fibration. Taking f to be $\mathrm{pt} \xrightarrow{x} C$ this yields the desired statement. □

Remark. As described in the following, tangent ω-categories play the role of universal ω-bundles [126, 71]. The term "tangent" alludes to their construction in terms of morphisms emanating at one object, which corresponds to the fact, discussed in **??**, that they arise from ∞-Lie integration of shifted tangent bundles.

There is a close relation between the notion of fiberd categories and the above fibrations:

Proposition 2.67 ((split op-)fiberd categories) *For B a 1-category and* Cat *the 1-category of 1-categories (split op-)fiberd categories $p : E \to B$ are precisely those functors arising from pullbacks of the* <u>*universal category bundle*</u> $p_{\mathrm{Cat}} : T_{\mathrm{pt}}\mathrm{Cat} \to \mathrm{Cat}$.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, two heads, "p"] & T_{\mathrm{pt}}\mathrm{Cat} \arrow[d, two heads, "p_{\mathrm{Car}}"] \\
B \arrow[r] & \mathrm{Cat}
\end{tikzcd} .$$

Proof. After noticing [124] that $T_{\mathrm{pt}}\mathrm{Cat} \simeq \mathrm{Cat}_*$ is the "category of pointed categories" as defined in [71], this is the corresponding theorem in [71]. □

Principal ω-bundles. For G an ω-monoid, definition 2.21, the global structures classified by $H(-, \mathbf{B}G)$ are principal G ω-bundles. We observe the characterization of G-principal bundles for G a 1- or a 2-group as pullbacks of the universal G-principal bundle [126] and take that as the definition of G-principal bundles for general ω-groups G. We show that every pullback of the universal G-principal ω-bundle is a locally trivializable G-torsor.

Definition 2.68 (universal G-principal bundle, [126]) *For G an ω-monoid, we write $\mathbf{E}G := T_\bullet \mathbf{B}G$ for the tangent ω-category, definition 2.65, of the one-object ω-groupouid $\mathbf{B}G$. We address the morphism $p : \mathbf{E}G \to \mathbf{B}G$ as the* <u>*universal G-bundle*</u>.

Remark. In [126] $\mathbf{E}G$ was denoted $\mathrm{INN}(G)$ to indicate its relation to inner automorphisms of G.

Proposition 2.69 (exact sequence of universal ω-bundle) *For G an ω-monoid, the ω-category $\mathbf{E}G$ fits into a short exact sequence of ω-monoids*

$$G \xrightarrow{i} \mathbf{E}G \overset{p}{\twoheadrightarrow} \mathbf{B}G$$

in that i is the kernel, definition 2.23, of p.

Proof. Consider the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & T_\bullet^{\mathrm{op}}\mathbf{B}G \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathbf{E}G = T_\bullet\mathbf{B}G \arrow[r] \arrow[d] & \mathbf{B}G^I \arrow[r, "d_1"] \arrow[d] & \mathbf{B}G \\
\mathrm{pt} \arrow[r] & \mathbf{B}G &
\end{tikzcd} .$$

The right and bottom square are pullback squares by definition. The top left square is a pullback by proposition 2.29. Therefore the pasting composite of the two top squares is a pullback square. This says that i is the kernel of p. □

Remark. For 2-groups this is in [126]. See there for some illustrative diagrams.

Definition 2.70 (G-action on $\mathbf{E}G$) *The inclusion $G \hookrightarrow \mathbf{E}G$ naturally induces a G-action $\mathbf{E}G \times G \to \mathbf{E}G$ [** write out details **].*

Definition 2.71 (G-principal bundles) *For G an ω-group and $X \in \mathsf{Spaces}$, a G-principal bundle is a morphism $p : P \to X$ of ω-categories together with an action $r : P \times G \to P$ such that there is a G-cocycle*

$$\begin{tikzcd}
\mathbf{Y} \arrow[r] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \\
X &
\end{tikzcd}$$

and a G-equivariant weak equivalence $g^*\mathbf{E}G \overset{\simeq}{\twoheadrightarrow} P$, where $p : g^*(\mathbf{E}G) \to X$ is the G-principal bundle obtained as the pullback in

$$\begin{tikzcd}
g^*\mathbf{E}G \arrow[r] \arrow[d] \arrow[dd, bend right, "p"'] & \mathbf{E}G \arrow[d] \\
\mathbf{Y} \arrow[r] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \\
X &
\end{tikzcd} .$$

Proposition 2.72 *Let*
$$\begin{tikzcd}
\mathbf{Y}' \arrow[rr, two heads, "w", "\simeq"'] \arrow[dr, two heads, "\simeq"] & & \mathbf{Y} \arrow[dl, two heads, "\simeq"] \\
& X &
\end{tikzcd}$$
be a refinement of covers of X and let $\mathbf{Y} \longrightarrow \mathbf{B}G$ be a cocycle on $\mathbf{Y}$. Then the G-principal bundles defined by g and by $g\circ w$ are weakly equivalent, $w^*g^*\mathbf{E}G \xrightarrow{\simeq} g^*\mathbf{E}G$.

Proof. Consider the double pullback diagram

$$\begin{tikzcd} w^* g^* \mathbf{E}G \arrow[r, "\simeq"] \arrow[d, two heads] & g^* \mathbf{E}G \arrow[r] \arrow[d, two heads] & \mathbf{E}G \arrow[d, two heads] \\ \mathbf{Y}' \arrow[r, "w", "\simeq"'] & \mathbf{Y} \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} .$$

The right vertical morphism is a fibration by lemma 2.66. Since fibrations are closed under pullback, so are the other two vertical morphisms. This means that the top left horizontal morphism in question is the pullback of a weak equivalence along a fibration. According to corollary 2.76 further below, this implies that it is also a weak equivalence. □

Morphisms of G-principal bundles are <u>concordances</u> induced from morphisms of cocycles. Suppose that two cocycles g and g' are defined on the same cover and are homotopic in that there is a transformation $\begin{tikzcd} \mathbf{Y} \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{B}G \end{tikzcd}$. This is equivalent to a *left homotopy*, namely a morphism $[g, g'] : \mathbf{Y} \otimes_{\mathsf{Gray}} I \to \mathbf{B}G$ out of the cylinder object $\mathbf{Y} \otimes I$, where the Gray-tensor product is that which raises categorical dimension.

This gives rise to a <u>concordance</u> of the corresponding 2-bundles over the interval

$$\begin{tikzcd} {[g,g']^*(\mathbf{E}G)} \arrow[r] \arrow[d] \arrow[dd, bend right=50, "\pi"'] & \mathbf{E}G \arrow[d] \\ \mathbf{Y} \otimes I \arrow[r, "{[g,g']}"] \arrow[d, two heads] & \mathbf{B}G \\ X \otimes I \end{tikzcd}$$

.

[** so G-principal ω-bundles with G-equivariant ω-anafunctors between them are classified by G-cohomology etc. pp. **] We are grateful for discussion with Konrad Waldorf about this point.

[** close discussion, **]

Proposition 2.73 *For G an n-group with $n = 1$ or $n = 2$, the above notion of G-principal bundles is equivalent to that of ordinary G-principal bundles and G-principal 2-bundles [15, 14, 160], respectively.*

Proof. This is discussed in section 5.4.2. □

Remark. G-principal bundles with connection are obtained in section 3.3.4 by refining the cocycles $g : \mathbf{Y} \to \mathbf{B}G$. The general picture is illustrated in figure 2 where the pulled back G-bundles just discussed appear in the top part of the diagram, whereas the remainder of the diagram encodes the differential refinement of the cocycle and its characteristic forms.

2.2.5 The homotopy category

Given a notion of weak equivalences in a category C, the homotopy category Ho is the universal category containing C in which all weak equivalences in C becomes isomorphisms. Using theorems by K.-S. Brown and Jardine, we can characterize the morphisms in the homotopy category of $\omega\mathsf{Categories}(\mathsf{Spaces})$ conveniently as colimits over homotopy classes of morphisms out of a cover of the domain ω-category. These hom-spaces of

Ho are the home of the cohomology theory described in section 3. If we instead do not divide out homotopy of morphisms this procedure generalizes the notion of *anafunctors* [105] to ω-anafunctors and yields a bicategory **Ho** which we address as the *weak homotopy category* of $\omega\mathsf{Categories}(\mathsf{Spaces})$. Composition operations in this weak homotopy category allow useful operations on nonabelian cocycles, section 3.2.

Proposition 2.74 (homotopy structure on $\omega\mathsf{Categories}(\mathsf{Spaces})$) *The category* $\omega\mathsf{Groupoids}(\mathsf{Spaces})$ *[** probably also* $\omega\mathsf{Categories}(\mathsf{Spaces})$*, check**] becomes a* category of fibrant objects *in the sense of [26] by setting*

- weak equivalences *are the local weak equivalences from definition 2.43 for* $K = \mathsf{Spaces}$*;*
- fibrations *are the global fibrations, i.e. the morphisms with the right lifting property with respect to the generating cofibrations of definition 2.47.*

Proof.

- Weak equivalences satisfy 2-out-of-3 since they do so locally, using [28, 94].
- Fibrations are closed under composition because they are given by a right lifting property by definition.
- Fibrations and acyclic fibrations are closed under pullback because they are both (locally) given by a right lifting property by lemma 2.50.
- The path object C^I of C is $\mathrm{hom}(I, C)$ for I the interval category and hom the internal hom. By the discussion in section 2.2.4. [** check details **]
- Objects are fibrant by corollary 2.62.

$\square$

Definition 2.75 (right proper model structure) *A category with weak equivalences and fibrations is* <u>*right proper*</u> *if weak equivalences are closed under pullback along fibrations.*

Corollary 2.76 *The category* $\omega\mathsf{Groupoids}(\mathsf{Spaces})$ *is right proper.*

Proof. Follows by lemma 2 in [26] from proposition 2.74. $\square$

For such model categories there is a relatively explicit description of their homotopy categories, due to [26] and [81]. We discuss that in terms of the *weak* homotopy category enriched in ω-categories which yields the ordinary strict homotopy category after passing to equivalence classes.

Definition 2.77 (ω-covers) *Given* $C \in \omega\mathsf{Categories}(\mathsf{Spaces})$*, write* $\omega\mathsf{Covers}(C)$ *for the category of* <u>*ω-covers*</u> *(*<u>*hypercovers*</u> *of ω-categories) of C whose objects are local acyclif fibrations* $\pi : \mathbf{Y} \xrightarrow{\simeq} C$ *and whose morphisms are commuting triangles*

$$\begin{tikzcd} \mathbf{Y} \arrow[rr] \arrow[dr, two heads, "\pi", "\simeq"'] & & \mathbf{Y}' \arrow[dl, two heads, "\pi'"', "\simeq"] \\ & C & \end{tikzcd} .$$

Let similarly $\mathsf{Replacements}(C)$ *be the category whose objects are just required to be local weak equivalences.* $\pi : \mathbf{Y} \xrightarrow{\simeq} C$

Definition 2.78 (ω-anafunctors) *A span in ωCategories*

$$\begin{tikzcd} \hat{D} \arrow[r, "\hat{f}"] \arrow[d, two heads, "\simeq"'] & D \\ C \end{tikzcd}$$

with left leg a local acyclic fibration is an *<u>ω-anafunctor</u>* $f : C \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; D$ *.*

We regard ordinary ω-functors as ω-anafunctors whose left leg is an identity.

Remark. The anafunctor terminology follows [105] where the the concept was introduced (without the model-theoretic interpretation) in 1Categories(Sets) and 2Categories(Sets) and [15] which followed [105] and considered internal anafunctors in 1Categories(ConcreteSpaces).

Definition 2.79 (composition of ω-anafunctors) *Given two ω-anafunctors $f : C \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; D$ and $g : D \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; E$ their composite*

$$g \circ f : C \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; E$$

is given by the span

$$\begin{tikzcd} (\hat{f})^* \hat{D} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{D} \arrow[r, "\hat{g}"] \arrow[d, two heads, "\simeq"] & E \; . \\ \hat{C} \arrow[r, "\hat{f}"] \arrow[d, two heads, "\simeq"] & D \\ C \end{tikzcd}$$

Proposition 2.80 *This composition of ω-anafunctors is well defined.*

Proof. We need to check that the composite morphism $f^*\hat{D} \longrightarrow \hat{C} \xrightarrow{\simeq} C$ is again a local acyclic fibration. This is since local acyclic fibrations are closed under pullback and under compoition (using locally that this is true for acyclic fibrations). □

Remark. This composition is not strictly associative, as usual when composition involves pullbacks. There is instead a bicategory (a weak 2-category) **Ho** of ω-anafunctors, a "weak homotopy category". See below.

Theorem 2.81 (ana-invertibility of weak equivalences) *Let C and D be weak ω-groupoids, definition* **??**. *Then every local weak equivalence $f : C \xrightarrow{\simeq} D$ has a weak ana-inverse $f^{-1} : D \longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\; C$ in that $C \xrightarrow{f^{-1}} D \overset{f}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} C$ is right homotopic to the identity.*

Proof. On the level ωCategories(Sets) this is a direct consequence of proposition 5 in [94] after observing that for C any ω-category the ω-category $\Gamma(C)$ from their theorem 2 is $C^I := \hom(I, C)$.

On pages 5 and 6 of [94] in total the following diagram is considered.

$$
\begin{tikzcd}
C \arrow[r, "f", "\simeq"'] \arrow[d, dashed, "\tilde{f}", "\simeq"'] \arrow[dd, bend right=60, "\mathrm{Id}"'] & D \arrow[d] \arrow[dd, bend left=60, "\mathrm{Id}"] & \\
C \times_D D^I \arrow[r, "f'"] \arrow[d, two heads, "\widehat{f^{-1}}", "\simeq"'] \arrow[rr, bend right, two heads, "p", "\simeq"'] & D^I \arrow[r, two heads, "\pi_2", "\simeq"'] \arrow[d, two heads, "\pi_1", "\simeq"'] & D \\
C \arrow[r, "\simeq", "f"'] & D &
\end{tikzcd}
.
$$

The notation is adapted to our needs here.

Corollary 2 in [94] says that π_1 and π_2 are acyclic fibrations, remark 9 says that $\tilde{f}$ and $\widehat{f^{-1}}$ are weak equivalences and proposition 5 says that p is an acyclic fibration. Hence we can take the inverse ω-anafunctor to be given by the span

$$
\begin{tikzcd}
C \times_D D^I \arrow[r, "\widehat{f^{-1}}"] \arrow[d, two heads, "p", "\simeq"'] & C \\
D &
\end{tikzcd}
.
$$

Then the above diagram, using the interpretation of D^I as the path object of D, says that f' exhibits a right homotopy:

$$
\begin{tikzcd}
& C \times_D D^I \arrow[dl, "\widehat{f^{-1}}"'] \arrow[dr, "p"] \arrow[d, phantom, "\Swarrow"] & \\
C \arrow[rr, "f"'] & \, & D
\end{tikzcd}
\quad :\Leftrightarrow \quad
\begin{tikzcd}
C \times_D D^I \arrow[r, "p \times (f \circ \widehat{f^{-1}})"] \arrow[dr, "f'"'] & D \times D \\
& D^I \arrow[u, "\pi_1 \times \pi_2"']
\end{tikzcd}
.
$$

[** discuss generalization to ωCategories(Spaces) **] ☐

Theorem 2.82 ([26], theorem 1) *There is an isomorphism, natural in $C, D \in \omega$Categories(Spaces), between the set of morphisms from C to D in the homotopy category* Ho *of ωCategories(Spaces) and the equivalence classes of the Hom-ω-categories in the weak homotopy category*

$$\mathrm{Ho}(C, D) \simeq \mathrm{colim}_{\mathbf{Y} \in \mathsf{Replacements}(C)} \hom(\mathbf{Y}, D)/_\sim .$$

A similar characterization is available for right proper model categories from theorem 2 in [81].

[** discuss the following: for ωGroupoids equivalence classes in $\hom(C, D)$ are in bijection to homotopy classes of morphisms $C \to D$. Moreover, adapting [81] to ω-categories we expect that we can pass from left legs being local weak equivalences to left legs being local acyclic fibrations. Then writing **Ho** for the bicategory of ω-anafunctors the above should say that morphisms in the homotopy category are equivalence classes of morphisms in the ω-anafunctor bicategory

$$\mathrm{Ho}(C, D) \simeq \mathbf{Ho}(C, D)/_\sim$$

**]

Following [26] and [81] we address the Hom in the weak homotopy category as *cohomology*.

Definition 2.83 (cohomology on ωCategories with coefficients in ωCategories) *For $C, A \in \omega\mathsf{Categories}(\mathsf{Spaces})$ we call*

$$H(C, A) := \mathbf{Ho}(\omega\mathsf{Categories}(\mathsf{Spaces}))[C, A]$$

the cohomology ω-category of C with coefficients in A.

Alternatively, we call this the equivariant cohomology of the space C_0 with respect to $\mathrm{Mor}(C)$*.*

Terminology.
- cocycles are the objects of $H(C, A)$;
- coboundaries are the 1-morphisms of $H(C, A)$;
- higher coboundaries are the higher morphisms of $H(C, A)$;
- cohomology classes are the ω-equivalence classes in $H(C, A)$.

Remark. In section 3.1.2 we give a definition of cohomology on spaces with coefficients in ω-category valued presheaves in terms of descent. The relation of that notion to the definition above is provided by the notion of codescent in section 3.1.3.

2.3 Quantities

Following [95] we address co-presheaves on our site CartesianSpaces, definition 2.1, as the *quantities* corresponding to the smooth spaces given by sheaves, definition 2.2. These quantities include and generalize algebras of C^∞-functions and modules over these of C^∞-sections of vector bundles on smooth spaces. In analogy to how Spaces are refined to higher structures by passing to their graded refinement $\omega\mathsf{Categories}(\mathsf{Spaces})$, quantities are refined to higher structures by considering differential graded commutative algebras over C^∞-functions. These are naturally identified with (the duals of) ∞-Lie algebroids and provide the linearized version of smooth ω-categories.

Definition 2.84 (Quantities [95]) *Write* $\mathsf{Quantities} := \mathsf{Sets}^{\mathsf{CartesianSpaces}}$ *for the co-presheaf category of quantities with values in* CartesianSpaces*.*

As a co-presheaf category there is a canonical monoidal structure on Quantities, where for $A, B \in \mathsf{Quantities}$ we have $A \times B : \mathbb{R}^k \mapsto A(\mathbb{R}^k) \times B(\mathbb{R}^k)$.

Definition 2.85 (C^∞-monoids) *A monoid internal to the monoidal category* Quantities *is a C^∞-monoid, equivalently a copresheaf on* CartesianSpaces *with values in* Monoids*. The chain of canonical inclusions of categories* $\mathsf{Algebras} \hookrightarrow \mathsf{Modules} \hookrightarrow \mathsf{VectorSpaces}$ *induces a chain of types of smooth quantities*

$$\begin{array}{ccccccccc}
C^\infty\mathsf{VectorSpaces} & \hookleftarrow & C^\infty\mathsf{Modules} & \hookleftarrow & C^\infty\mathsf{Algebras} & \hookrightarrow & C^\infty\mathsf{Monoids} & \hookrightarrow & \mathsf{Quantities} \\
\Big\| {:=} & & \Big\| {:=} & & \Big\| {:=} & & \Big\| {=} & & \Big\| {:=} \\
\mathsf{VectorSpaces}^{\mathsf{CartesianSpaces}} & \hookleftarrow & \mathsf{Modules}^{\mathsf{CartesianSpaces}} & \hookleftarrow & \mathsf{Algebras}^{\mathsf{CartesianSpaces}} & \hookrightarrow & \mathsf{Monoids}^{\mathsf{CartesianSpaces}} & \hookrightarrow & \mathsf{Sets}^{\mathsf{CartesianSpaces}}
\end{array} .$$

Notice that for $A \in C^\infty\mathsf{Monoids}$ the monoid structure $p_A : A \times A \to A$ comes with associative and unital component maps $p_A(\mathbb{R}^k) : A(\mathbb{R}^k) \times A(\mathbb{R}^k) \to A(\mathbb{R}^k)$ for all $k \in \mathbb{N}$ which equips each set $A(\mathbb{R}^k)$ with the structure of a monoid.

Remark. The notion of C^∞-monoids is essentially the same concept as that considered in [116], from where we borrow the terminology. [** give more details on precise relation**]

Important examples of C^∞-algebras and C^∞-modules over them comes from function algebras and sections of vector bundles over smooth spaces.

Definition 2.86 (C^∞ function algebras) *For $X \in \mathsf{Spaces}$ the functor*

$$C^\infty(X) := \mathrm{Hom}(X, -) : \mathsf{CartesianSpaces} \to \mathsf{Sets}$$

naturally inherits the structure of an object in $\mathsf{Algebras}$*. This is the* *algebra of smooth functions* *on the space X.*

The component morphism of the product is induced from the entry-wise multiplication $\cdot^k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$:

$$p_{C^\infty(X)}(\mathbb{R}^k) : \ \mathrm{Hom}(X, \mathbb{R}^k) \times \mathrm{Hom}(X, \mathbb{R}^k) \xrightarrow{\simeq} \mathrm{Hom}(X, \mathbb{R}^k \times \mathbb{R}^k) \xrightarrow{\mathrm{Hom}(-, \cdot^k)} \mathrm{Hom}(X, \mathbb{R}^k)\ .$$

Definition 2.87 (C^∞ sections) *For $p : E \to X$ a vector bundle in* Spaces *the* *set of sections* $\Gamma_{\mathsf{Sets}}(E) \in \mathsf{Sets}$ *of E is the pullback in*

$$\begin{tikzcd}
\Gamma_{\mathsf{Sets}}(E) \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "\mathrm{const(Id)}"] \\
\mathrm{Hom}(X, E) \arrow[r, "p^*"] & \mathrm{Hom}(X, X)
\end{tikzcd} .$$

This set becomes a C^∞-vector space and in fact a $C^\infty(X)$-module $\Gamma(E)$, the *C^∞-module of sections* *by setting*

$$\Gamma(E) : \mathbb{R}^k \mapsto \Gamma_{\mathsf{Sets}}(E \otimes \mathbb{R}^k)\,.$$

2.4 ∞-Lie algebroids

Definition 2.88 (C^∞-qDGCAs) *For $X \in \mathsf{Spaces}$, a* *quasi-free differental graded-commutative algebra* *over $A := C^\infty(X)$, or* *qDGCA* *for short, is a non-positively graded cochain complex $\mathfrak{g}$ of A-modules together with a degree +1 derivation $d : \wedge^\bullet_A \mathfrak{g}^* \to \wedge^\bullet_A \mathfrak{g}^*$ (linear over the ground field) squaring to 0, $d^2 = 0$. Write* qDGCAs *for the category of such algebras with morphisms the morphisms of C^∞*$\mathsf{Algebras}$ *which respect the differential.*

Here $\mathfrak{g}^* := \mathrm{Hom}_{A\mathsf{Modules}}(\mathfrak{g}, A)$ is the dual over A and $\wedge^\bullet_A \mathfrak{g}^* := \mathrm{Sym}_A(\mathfrak{g}^*[1])$ is the graded Grassmann algebra of $\mathfrak{g}^*$ over A.

Remark. More explicitly, the graded C^∞-vector space underlying $\mathrm{CE}(\mathfrak{g})$ is $\wedge^\bullet_A \mathfrak{g}^* = \mathrm{Sym}_A(\mathfrak{g}^*[1]) = \underbrace{A}_{0} \oplus \underbrace{\mathfrak{g}_0^*}_{1} \oplus \underbrace{\mathfrak{g}_0^* \wedge \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*}_{2} \oplus \cdots$. This differs from the DGCAs familiar from rational homtopy theory [70] (only) in that tensor products are not over the ground field but over the commutative C^∞-algebra A.

Definition 2.89 (differential forms on Spaces) *Write $\Omega^\bullet \in \mathsf{Spaces}$ for the deRham sheaf $\Omega^\bullet : U \mapsto (\Omega^\bullet(U), d_{\mathrm{dR}})$ which assigns to each $U \in \mathsf{CartesianSpaces}$ the differential graded commutative algebra of differential forms on U.*

Remark. In rational homotopy theory this corresponds to the map given for instance in definition 1.20 of [70].

Proposition 2.90 *For $X \in \mathsf{Spaces}$ the space $\Omega^\bullet(X) := \mathrm{hom}(X, \Omega^\bullet)$ naturally carries the structure of a qDGCA over $\Omega^0(X) = C^\infty(X)$. This is the qDGCA of* *differential forms* *on X. The construction extends to a contravariant functor $\Omega^\bullet : \mathsf{Spaces} \to \mathsf{qDGCAs}$.*

Definition 2.91 (∞-Lie algebroid) *Given $X \in \mathsf{Spaces}$ and $\mathfrak{g}$ a non-positively graded cochain complex of $(A := C^\infty(X))$-modules a qDGCA-structure $(\wedge^\bullet_A \mathfrak{g}^*, d)$ equips $\mathfrak{g}$ with a family of n-ary brackets. Equipped with these brackets we call $(\mathfrak{g}, A)$ a* *∞-Lie algebroid* *or* *L_∞-algebroid* *over X and*

$$\mathrm{CE}_A(\mathfrak{g}) := (\wedge^\bullet_A \mathfrak{g}^*, d)$$

the *Chevalley-Eilenberg algebra* *of the $(\mathfrak{g}, A)$.*

By definition L_∞Algebroids are equivalent to C^∞qDGCAs with the equivalence being induced by forming the Chevalley-Eilenberg algebra

$$C^\infty\mathsf{qDGCAs} \xleftarrow[\simeq]{\mathrm{CE}(-)} L_\infty\mathsf{Algebroids}\ .$$

Types of L_∞-algebroids. The following special cases are distinguished:

- A Lie n-algebroid is an L_∞-algebroid with $\mathfrak{g}$ concentrated in the first n degrees.
- An L_∞-algebra is an L_∞-algebroid with $X = \mathrm{pt}$.
- A Lie n-algebra is a Lie n-algebroid with $X = \mathrm{pt}$.
- A strict L_∞-algebroid is an L_∞-algebroid with $d : \mathfrak{g}^* \to \mathfrak{g}^* \oplus \mathfrak{g}^* \wedge_A \mathfrak{g}^*$.
- A dg-Lie algebra is a strict L_∞-algebra.

Definition 2.92 (Weil qDGCA) *For $\mathrm{CE}_A(\mathfrak{g}) = (\wedge_A^\bullet \mathfrak{g}^*, d_\mathfrak{g})$ the Chevalley-Eilenberg qDGCA of an L_∞-algebroid $(A = C^\infty(X), \mathfrak{g})$ over the space X, the Weil algebra is the C^∞ qDGCA*

$$\mathrm{W}_A(\mathfrak{g}) := \left(\wedge_A^\bullet (\Gamma(TX)^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*[1]),\ d_{\mathrm{W}(\mathfrak{g})} := \begin{pmatrix} d_\mathfrak{g} & 0 \\ \sigma & -\sigma \circ d_\mathfrak{g} \circ \sigma^{-1} \end{pmatrix} \right),$$

where the matrix on the right is the schematic action of the differential $d_{\mathrm{W}(\mathfrak{g})}$ on generators defined as follows: define $\sigma : \mathrm{W}_{C^\infty(X)}(\mathfrak{g}) \to \mathrm{W}_{C^\infty(X)}(\mathfrak{g})$ by letting $\sigma|_{\mathfrak{g}^*} : \mathfrak{g}^* \xrightarrow{\simeq} \mathfrak{g}^*[1]$ be the canonical isomorphism and letting $\sigma|_{\wedge^\bullet_{C^\infty(X)}\Gamma(TX)^*} = d_{\mathrm{dR}}$ be the deRham differential on X and extending σ uniquely as a graded degree +1 derivation. Then $d_{\mathrm{W}(\mathfrak{g})}$ is defined by $d_{\mathrm{W}(\mathfrak{g})}|_{\wedge^\bullet_{C^\infty(X)}\wedge^\bullet\mathfrak{g}^*} := d_\mathfrak{g} + \sigma$ and $d_{\mathrm{W}(\mathfrak{g})}(\sigma a) := -\sigma(d_\mathfrak{g} a)$ for all $a \in C^\infty(X) \oplus \mathfrak{g}^*[1]$.

Since $\mathrm{W}_A(\mathfrak{g})$ is itself a qDGCA it is itself the Chevalley-Eilenberg algebra of some L_∞-algebroid. Following [132], we call this the L_∞-algebroid $\mathrm{inn}(\mathfrak{g})$ of inner derivations of $\mathfrak{g}$ and write

$$W_A(\mathfrak{g}) = \mathrm{CE}_A(\mathrm{inn}(\mathfrak{g}))\,.$$

Remark. In parts of the literature L_∞-algebroids are conceived in the context of supermanifolds whose $\mathbb{Z}_2$-grading is refined to an $\mathbb{N}$-grading and which are equipped with a *homological vector field* (an odd and nilpotent vector field): the GCA underlying $\mathrm{CE}_A(\mathfrak{g})$ is interpreted as the algebra of smooth functions on the supermanifold and the differential $d_\mathfrak{g}$ is identified with the homological vector field. In this context the Weil algebra $\mathrm{W}_A(\mathfrak{g})$ is the algebra of functions on the shifted tangent bundle of this supermanifold. A comprehensive discussion of this point of view is in [109].

Definition 2.93 (algebra of invariant polynomials) *For $\mathfrak{g}$ an L_∞-algebroid, the DGCA $\mathrm{inv}(\mathfrak{g})$ of invariant polynomials or* basic forms *on $\mathfrak{g}$ is as a GCA $(\wedge^\bullet \mathfrak{g}^*[1])/d^{-1}_{\mathrm{W}(\mathfrak{g})}(\mathfrak{g}^* \wedge \mathrm{W}(\mathfrak{g}))$ equipped with the differential obtained by restricting $d_{\mathrm{W}(\mathfrak{g})}$ to this quotient.*

Proposition 2.94

$$\begin{tikzcd} \mathrm{CE}(\mathfrak{g}) & \mathrm{W}(\mathfrak{g}) \arrow[l] & \mathrm{inv}(\mathfrak{g}) \arrow[l] \arrow[ll, bend left, "0"] \end{tikzcd}\ .$$

[** state the universal property of this sequence, relate to obstruction problem from cocycles to flat differential cocycles **]

[** review of the necessary concepts from [132] **]

$$
\begin{tikzcd}
 & & & g^*\mathbf{E}G \arrow[rr] \arrow[d] & & \mathbf{E}G \arrow[d] \\
Y \arrow[dd, "\pi"] & & & \mathrm{Codesc}(Y,\mathcal{P}_0) \arrow[rr, "g"] \arrow[dl, hook] \arrow[dd, "\simeq"] & & \mathbf{B}G \arrow[dl, hook] \arrow[dd] \\
 & & \mathrm{Codesc}(Y,\Pi_\omega) \arrow[urrr, dashed, "\bar{g}_{\mathrm{flat}}"] \arrow[rr, "\bar{g}"] \arrow[dd, "\simeq"] & & \mathbf{BE}G \arrow[dd] & \\
X & & & X \arrow[rr] \arrow[dl, hook] & & \mathrm{pt} \arrow[dl, hook] \\
 & & \Pi_\omega(X) \arrow[rr, "P"] & & \mathbf{B}[\mathbf{B}G] &
\end{tikzcd}
$$

Figure 2: **Nonabelian differential cocycles for principal ω-bundles with connection**. The cocycle g mapping the codescent ω-groupoid $\mathrm{Codesc}(Y, \mathcal{P}_0)$ (the Čech groupoid) of a cover $\pi : Y \to X$ of base space X to the one-object ω-groupoid $\mathbf{B}G$ defines a G-principal ω-bundle $g^*\mathbf{E}G$, the pullback of the universal G-principal ω-bundle $\mathbf{E}G \to \mathbf{B}G$ (section 2.2.4). A flat connection on this is an extension $\bar{g}_{\mathrm{flat}}$ of g to the differential codescent ω-groupoid $\mathrm{Codesc}(Y, \Pi_\omega)$ which is surjectively equivalent to the fundamental ω-groupoid $\Pi_\omega(X)$ of X. In general such a flat connection does not exist, but a non-flat connection given by a morphism $\bar{g}$ to the ω-groupoid $\mathbf{BE}G$ does. Its non-vanishing curvature is measured by the characteristic classes P, closed differential forms represented by an ω-functor from the fundamental ω-groupoid to $\mathbf{B}[\mathbf{B}G]$, where $[\mathbf{B}G]$ is an ω-groupoid providing a rational approximation to $\mathbf{B}G$. The ω-groupoids $\mathbf{BE}G$ and $\mathbf{B}[\mathbf{B}G]$ are obtained from ∞-Lie integration of L_∞-algebras (section 3.3.4).

3 Homotopy and Cohomology

As recalled by Ross Street in [152], it is originally an insight due to John Roberts [125] (arrived at, remarkably, through a study of algebraic quantum field theory) that *cohomology* is about coloring simplices by ∞-categories.

Notice that categories generalize groups, ∞-categories generalize complexes of abelian groups and sheaves of ∞-categories generalize sheaves of complexes of abelian groups. Nonabelian Čech cohomology (of spaces) generalizes (abelian) Čech cohomology by allowing sheaves of complexes of abelian groups to be replaced by sheaves (rectified stacks) of ∞-categories.[1]

We define

- cohomology with coefficients in ω-category valued presheaves $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}$ in terms of descent: $H(X, \mathbf{A})$ is the ω-category of objects of $\mathbf{A}$ on a hypercover of X which *glue*;

- homotopy with coefficients in ω-category valued co-presheaves $\mathbf{A} : \mathsf{Spaces} \to \omega\mathsf{Categories}$ in terms of codescent: $\pi(X, \mathbf{A})$ is the ω-category of objects of $\mathbf{A}$ on a hypercover of X which *co-glue.*

The definition of cohomology follows that introduced by Ross Street in [149] and further discussed in [151].

We define *ω-stacks* to be presheaves $\mathbf{A}$ which are equivalent to cohomology with coefficients in them. We define *ω-costacks* to be co-presheaves $\mathbf{A}$ which is equivalent to homotopy with coefficients in them.

[1]It has been argued elsewhere (e.g. [100]) that more appropriate coefficients of the cohomology are (stacks of) *homotopy types* (e.g. cohomology does not detect the automorphisms of the coefficient group); and other *model categories* for homotopy theory may be substituted. The approach in [100] gives a satisfactory approach for obtaining cohomology *sets* with such coefficients, but it does not supply more structure than a pointed set on the cohomology, while our examples tell us that the nonabelian cocycles make a higher category themselves.

3.1 ∞-Stacks and ∞-costacks

3.1.1 ω-Categories and simplicial objects

Cohomology and homotopy arise from gluing values of ω-category valued presheaves and co-presheaves, respectively. The glue is provided by ω-categories modelling the n-simplex, for all $n \in \mathbb{N}$, and arranging themselves into cosimplicial ω-categories, definition 3.3. Depending on how much invertibility one demands, there are the following three choices:

symbol	name	description
$O(\Delta^n)$	the nth *oriental*	the free ω-category on the n-simplex
$U(\Delta^n)$	the nth *unoriental*	the free weak ω-groupoid on the n-simplex
$\Pi_\omega(\Delta^n)$	the *fundamental ω-groupoid* on the filtered n-simplex	the free ω-groupoid on the n-simplex

All three constructions are natural in n and give rise to cosimplicial ω-categories $O, U, \Pi_\omega : \Delta \to \omega\mathsf{Categories}(\mathsf{Sets})$.

The orientals were introduced by Ross Street [149]. They provide the fundamental relation between the simplicial and globular structures and give rise to an adjunction.

Proposition 3.1 ([149]) *There is an adjunction*

$$\omega\mathsf{Categories}(\mathsf{Sets}) \underset{F}{\overset{N}{\rightleftarrows}} \mathsf{SimplicialSets}$$

with F left adjoint to N, where the <u>ω-nerve</u> $N(C)$ of an ω-category C is obtained by mapping orientals into C

$$N(C):\ \Delta(-) \xrightarrow{O} \omega\mathsf{Categories} \xrightarrow{\mathrm{Hom}(-,C)} \mathsf{Sets}$$

and where the <u>free ω-category</u> $F(S)$ on the simplicial set S is given by the coend $F(S) = \int^{[n]\in\Delta} O(\Delta^n)\cdot S^n$.

The explicit description of higher orientals quickly becomes unwieldy as n-grows. Compare table 5. If C is a weak ω-groupoid or even an ω-groupoid, definition 2.34, then one can map equivalently the free weak or, respectively, the free ω-groupoids over the orientals into C. The free weak ω-groupoids over orientals we call *unorientals* and describe below. The free ω-groupoid on the n-oriental is the fundamental ω-groupoid on the filtered n-simplex which is described in section 9.9 of [34].

Fundamental ω-groupoid of filtered n-simplex.

Remark. The fundamental ω-groupoid $\Pi_\omega(\Delta^n)$ of the standard n-simplex regarded as a filtered space has as objects the vertices of Δ^n, as invertible 1-morphisms the edges, as invertible 2-morphisms the triangular faces and, generally, as invertible k-morphisms the k-face of Δ^n.

Recall from section 2.2.1 the notation $[\Pi_\omega(\Delta^n)]$ of the crossed complex underlying $\Pi_\omega(\Delta^n)$. Notice that in in [34] what we write $\Pi_\omega(\Delta^n)$ is denoted $\rho(\Delta^n)$ and what we write $[\Pi_\omega(\Delta^n)]$ is $\Pi\Delta^n$ there.

Proposition 3.2 (homotopy addition lemma, [34]) *The crossed complex $[\Pi_\omega(\Delta^n)]$ underlying the fundamental ω–groupoid of the standard filtered n-simplex is*

$$\cdots \xrightarrow{\delta_4} F(\Delta^n_3) \xrightarrow{\delta_3} F(\Delta^n_2) \xrightarrow{\delta_2} F(\Delta^n_1) \underset{\delta_t}{\overset{\delta_s}{\rightrightarrows}} F(\Delta^n_0)$$

*where $F(\Delta^n_0) = [n] = \{0, 1, \cdots, n\}$ is the free 0-groupoid over the set of vertices of Δ^n, i.e. just that set of vertices, $F(\Delta^n_1)$ is the free 1-groupoid over the graph of edges of Δ^n, and $F(\Delta^n_k)$ for $k \geq 2$ is the bundle of groups over $[n]$ which over each point is the free group on the set of k-faces of Δ^n. [** is that said well and right? improve **] The maps δ_k, $k \geq 2$ are give as follows:*

$$\delta_2 : \sigma^2 \mapsto\ 2 \xrightarrow{(\partial_1\sigma^2)^{-1}} 0 \xrightarrow{\partial_2\sigma^2} 1 \xrightarrow{\partial_0\sigma^2} 2$$

$$\partial_3\sigma^3 = \mathrm{Ad}_{u_3}(\partial_3\sigma^3) - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3$$

$$\delta_k\sigma^k = \mathrm{Ad}_{u_n}(\partial_n\sigma^n) + \sum_{i=0}^{n-1}(-1)^{n-i}\partial_i\sigma^n \qquad \text{for } k \geq 3\,.$$

Here σ^k is the unique k-cell of the standard k-simplex and ∂_i are the ordinary face maps. u_n is the edge ...

Proof. This is section 9.9 of [34]. □

Remark. Once recognizes the familiar formulas for boundaries of abelian chains, but there is a nonabelian twist given by the adjoint action of the 1-morphisms on one of the elements.

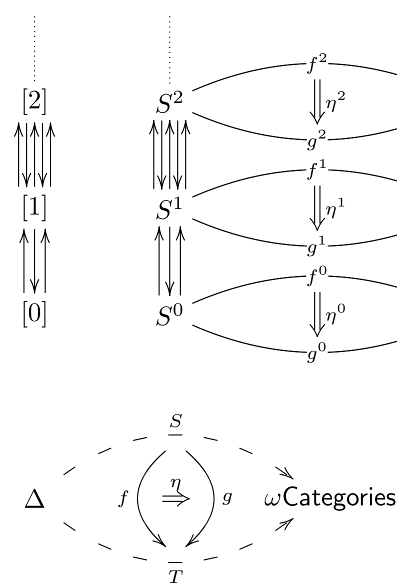


Figure 3: **Higher morphisms between cosimplicial ω-categories**. The diagram illustrates the enrichment in ωCategories of cosimplicial ω-categories from lemma 3.4. Here $S, T : \Delta \to \omega$Categories are two cosimplicial ω-categories internal to Spaces, $f, g : S \to T$ are two 1-morphisms between them and $\eta : f \Rightarrow G$ a 2-morphism betwen these. The upper diagram shows the component ω-functors and their naturality condition. All parallel diagrams on the right strictly commute.

Cosimplicial ω-categories.

Definition 3.3 (cosimplicial ω-categories) *A cosimplicial ω-category is a functor $\Delta \to \omega$Categories.*

Lemma 3.4 (enrichment of $\omega\mathsf{Categories}^\Delta$ over $\omega\mathsf{Categories}$) *Using the closed monoidal structure on $\omega\mathsf{Categories}$, the category $\omega\mathsf{Categories}^\Delta$ of cosimplicial ω-categories is naturally enriched over $\omega\mathsf{Categories}$ by taking the internal hom to be the* end *(see section 6.2)*

$$\mathrm{Hom}(S([-]), T([-])) := \int_{[n]\in\Delta} \mathrm{Hom}(S([n]), T([n])\,.$$

Proof. This follows from general facts in enriched category theory [89] (section 2.2) since both Δ and $\omega\mathsf{Categories}$ can be regarded as enriched over $\omega\mathsf{Categories}$. □

We are grateful to Dominic Verity for discussion of this point. The component-wise enrichment mechanism is illustrated in figure 3.

Unorientals.

Definition 3.5 (codiscrete groupoid) *The codiscrete groupoid $\mathrm{Codisc}(S)$ over a set S is the 1-groupoid with S as its set of objects and $S \times S$ as its set of morphisms, with composition being $(b,c) \circ (a,b) = (a,c)$ for all $a,b,c \in S$.*

The codiscrete groupoid over a set can be thought of as a model for a discrete contractible space, since obviously there is a weak equivalence $\mathrm{Codisc}(S) \xrightarrow{\simeq_w} \mathrm{pt}$. We need "bigger" resolutions of the point, built using the cofibrant replacements constructed in definition 2.56.

Definition 3.6 (fundamental ω-category of a discrete contractible space) *For S a set and $n \in \mathbb{N}$, we write*

$$P_n(S) := (\mathrm{Codisc}(S))_{\mathrm{cof}_n}$$

for the cofibrant replacement, definition 2.56, in $n\mathsf{Categories}$ of the codiscrete groupoid over S and write

$$P_\omega(S) := (\mathrm{Codisc}(S))_{\mathrm{cof}}$$

for the cofibrant replacement in $\omega\mathsf{Categories}$ of the codiscrete groupoid over S.

Remark. The notation here is alluding to the concept of fundamental ω-groupoids of smooth spaces in section 4.2.1. This will make manifest the phenomenon which enters crucially in section 4.4, that nonabelian cocycles are akin to ω-functors out of fundamental ω-groupoids of (fiberwise) contractible spaces.

Notice the following way to look at the simplicial category Δ, which is particularly suggestive in the present context:

Definition 3.7 (simplicial category) *The category Δ is the full subcategory of $\mathsf{Categories}$ on categories $[n]$ freely generated from linear graphs of length n.*

$$[0] = \{0\}$$

$$[1] = \{\ 0 \longrightarrow 1\ \}$$

$$\begin{tikzcd} {[2] = \{\ 0} \arrow[r] \arrow[rr, bend right] & 1 \arrow[r] & {2\ \}} \end{tikzcd}$$

$$\vdots$$

Let $P_\omega^{\geq}([n])$ be the full sub-ω-category of $P_\omega([n])$ on those 1-morphisms along which the sequences of objects are non-decreasing.

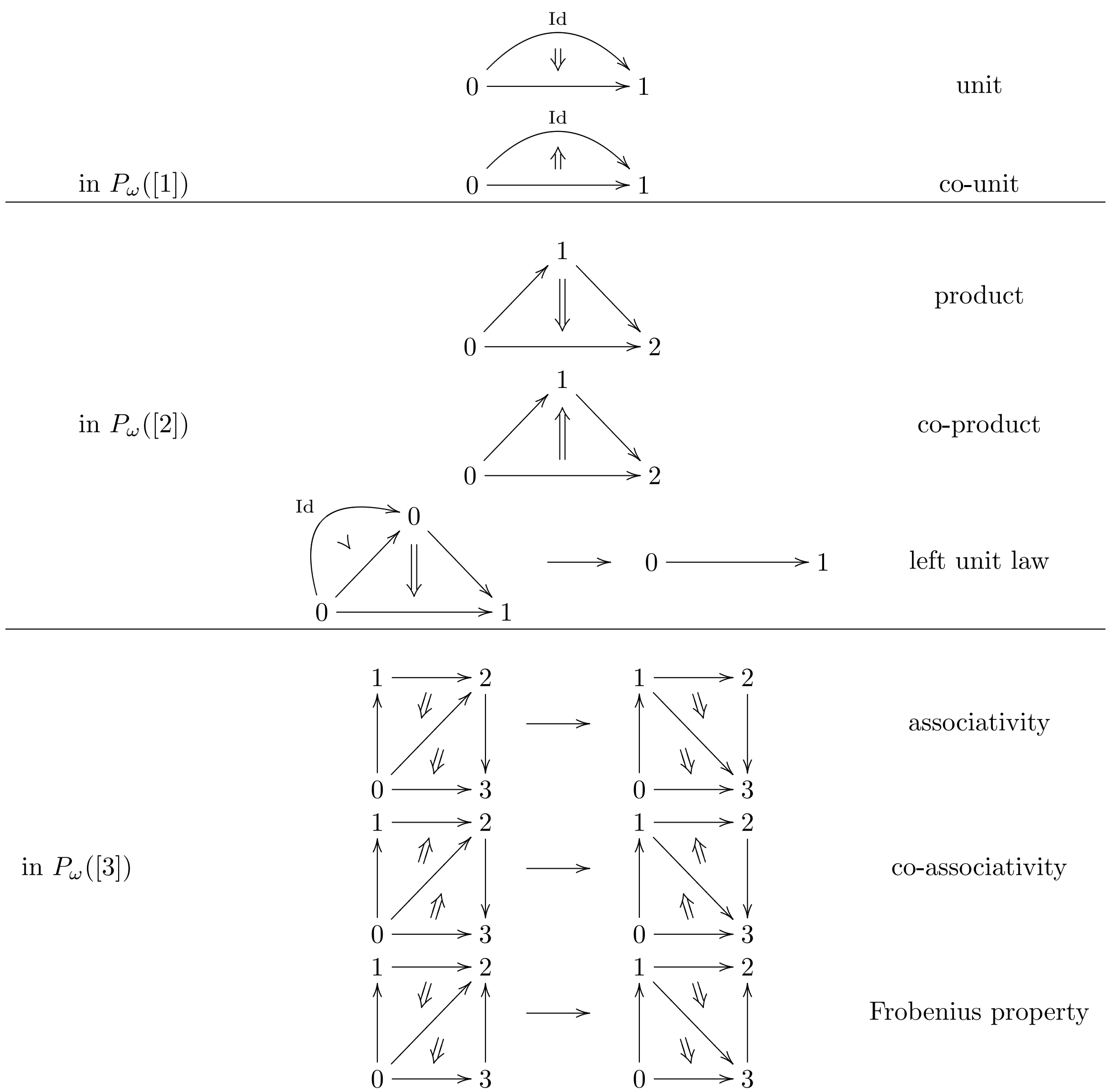


Figure 4: **Unorientals**. The nth unoriental is the universal cofibrant resolution of the codiscrete 1-groupoid on $n+1$ objects. ω-Functors out of unorientals map paths of paths in $P_\omega([n])$ to Frobenius algebroids (monoidoids, in general) with invertible product and with coproduct the inverse of the product. Compare with proposition 3.10.

Lemma 3.8 *For all $n \in \mathbb{N}$ there is an ω-anafunctor $[n] \longrightarrow\!\!\!\!\!\!\!\!|\;\;\; P_\omega([n])$ given by*

$$\begin{tikzcd} & P_\omega^{\geq}([n]) \arrow[dl, "\simeq"'] \arrow[dr, hook] & \\ {[n]} \arrow[rr, "|" marking] & & P_\omega([n]) \end{tikzcd}.$$

Lemma 3.9 (unorientals) *This ana-embedding of $[n]$ into $P_\omega([n])$ uniquely induces the structure of a cosimplicial ω-category P_ω, called here the unorientals:*

$$P_\omega : \Delta \to \omega\mathsf{Categories}\,.$$

Remark. As we discuss now, *un*orientals are like Street's orientals but such that every morphisms has an inverse.

Orientals. The bridge between simplicial methods and globular ω-categories is usually established by Ross Street's *orientals* [150]. The n-th oriental $O(\Delta^n) \in \omega\mathsf{Categories}(\mathrm{FinSet})$ is to be thought of as the free n-category on a single n-simplex.

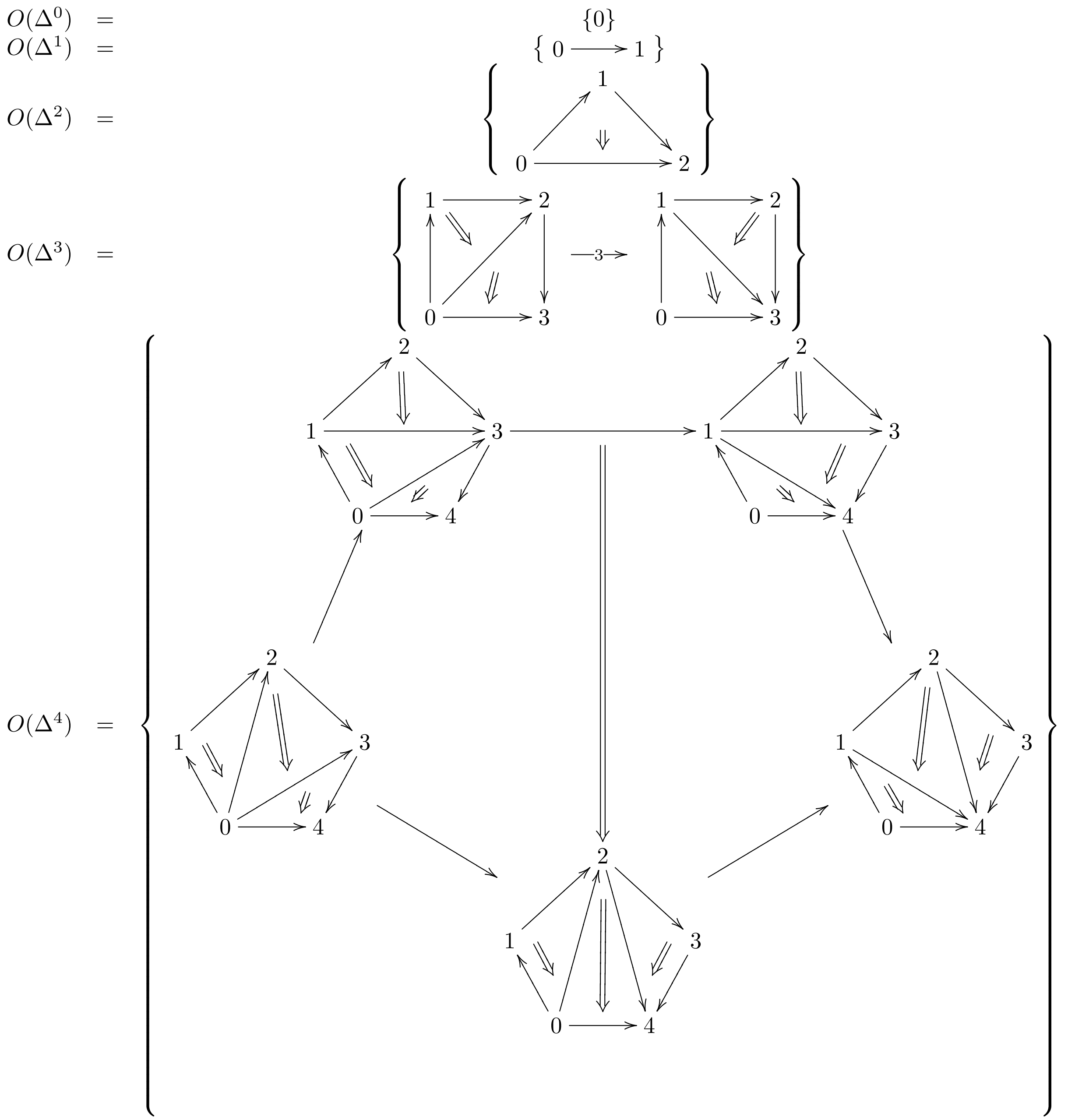


Figure 5: **Orientals.** The n-th oriental $O(\Delta^n) \in n\mathsf{Categories} \subset \omega\mathsf{Categories}$ is the free n-category on a single n-simplex. The first five orientals are shown explicitly. Here $O(\Delta^3)$ is to be thought of as a tetrahedron, filled by the 3-morphism $\xrightarrow{3}$, which we have depicted after slicing it open. Similarly for $O(\Delta^4)$. Diagrams for the orientals $O(\Delta^5)$ and $O(\Delta^6)$ can be found in [149].

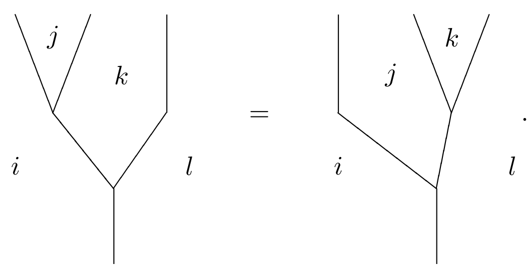
The precise definition for all n needs a bit of combinatorics [149, 150], but the basic idea is clear from looking at the first few orientals for low n, shown in figure 5. By design, orientals arrange themselves into a cosimplicial ω-category

$$\begin{aligned} O(\Delta^{(-)}) : \Delta &\to \omega\mathsf{Categories}(\mathsf{FinSet}) \subset \omega\mathsf{Categories}(\mathsf{Spaces}) \\ [n] &\mapsto O(\Delta^n)\,. \end{aligned}$$

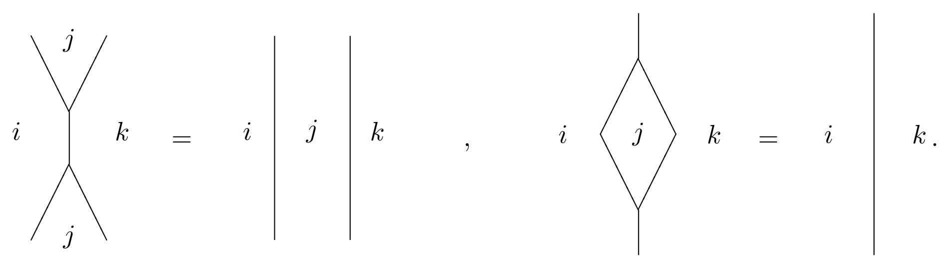
Remark. Notice that for $n \geq 1$ the nth oriental $O(\Delta^n)$ is *not* weakly equivalent to the point. As a consequence, the codescent objects in section 3.1.3 obtained from descent defined in terms of orientals are not weakly equivalent to the object on which they define descent. This is in contrast to the codescent obtained using $P_\omega([n])$. However, the difference should be negligible as long as the coefficient objects are ω-groupoids. For instance the Frobenius law in $\Pi_\omega([3])$ displayed in figure 4 follows automatically for images of $O(\Delta^{[3]})$ in a 2-groupoid:

Lemma 3.10 *The image of the third oriental in a 2-groupoid automatically satisfies the Frobenius property.*

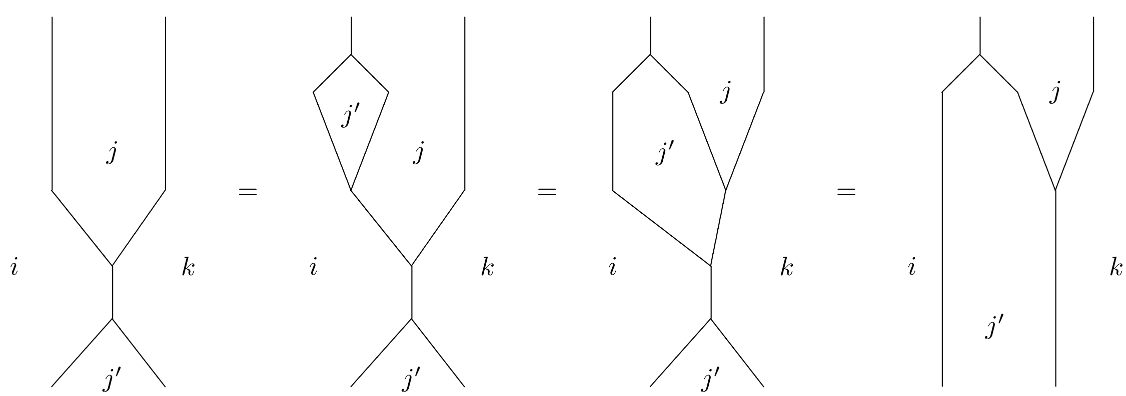
Proof. Write the image in string diagram notation (see glossary) as



Since we assume this to live in a 2-groupoid, the vertices denote invertible 2-morphisms:



Using this the Frobenius law follows:



□

3.1.2 Descent

Descent is descent of structures from "local resolutions" down to the resolved space. The archetypical example are resolutions of just points, i.e. ω-categories weakly equivalent to the *point*, the terminal ω-category. We discussed such resolutions of points in section 3.1.1 and relate them to free ω-groupoids over Street's *orientals*, which in turn are free ω-categories on single n-simplices. Both orientals and "unorientals" form cosimplicial ω-categories, in terms of which one can give a comparatively concrete description of cocycles and coboundaries in nonabelian cohomology following Street's definition of *ω-categories of descent data*. This is in section 3.1.2. The relation to the conception of cohomology in terms of ω-anafunctors as in section 2.2.5 is obtained via the notion of *codescent* in section 3.1.3.

The notions of descent and codescent can be formulated relative to any choice of cosimplicial ω-category $G(\Delta^{(-)}) : \Delta \to \omega\mathsf{Categories}$ which functions as **g***lue* and provides a **g***lobular version* of simplices. We have use of the choices

$$G = \begin{cases} O \\ U \\ \Pi_\omega \end{cases} ,$$

where the unorientals $U(\Delta^n)$ and in particular the free ω-groupoids $\Pi_\omega(\Delta^n)$ on the standard filtered n-simplex are relevant for descent and codescent themselves, while the orientals $O(\Delta^n)$ are relevant for the general relation between ω-categories and simplicial sets. So let from now on G be one of these choices, with $G = \Pi_\omega$.

Recall the enrichment of cosimplicial ω-categories by ω-categories from lemma 3.4.

Definition 3.11 (ω-category of descent data) *Given an ω-category valued presheaf $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}$ and a hypercover $\pi : Y^\bullet \longrightarrow X$ the ω-category of descent data on Y with coefficients in $\mathbf{A}$ is the end*

$$\mathrm{Desc}(Y^\bullet, C) := \int_{[n]\in\Delta} \mathrm{hom}(G(\Delta^n), \mathbf{A}(Y^{[n+1]}))) \,.$$

Remark. For $G = O$ the orientals, this formula is equivalent to the formula given by Street on p. 339 of [150] and on p. 32 of [151]. We are indebted to Dominic Verity for discussion of this reformulation of Street's descent in terms of ends.

Lemma 3.12 *There is a canonical morphism into the ω-category of descent data on $Y^\bullet \to X$ from the ω-category on X:*

$$\bar{i} : \mathbf{A}(X) \to \mathrm{Desc}(Y, A) \,.$$

Proof. The canonical morphisms

$$\mathbf{A}(X) \xrightarrow{\simeq} \mathrm{hom}(G(\Delta^0), \mathbf{A}(X)) \xrightarrow{\mathrm{hom}(i_*,\pi^*)} \mathrm{hom}(G(\Delta^n), \mathbf{A}(Y^{[n+1]}))$$

for all $n \in \mathbb{N}$ form an exceptional natural family. By the universal property of the end this yields the unique morphism $\bar{i}$ such that all diagrams

$$\begin{tikzcd}
\mathbf{A}(X) \arrow[dr, dashed, "\bar{i}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathrm{Desc}(Y, \mathbf{A}) \arrow[r] \arrow[d] & \mathrm{hom}(G(\Delta^n), \mathbf{A}(Y^{[n+1]})) \arrow[d] \\
& \mathrm{hom}(G(\Delta^m), \mathbf{A}(Y^{[m+1]})) \arrow[r] & \mathrm{hom}(G(\Delta^m), \mathbf{A}(Y^{[n+1]}))
\end{tikzcd}$$

commute. $\square$

Definition 3.13 (ω-stack) *An ω-category valued presheaf $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}$ is an $\underline{\omega\text{-stack}}$ if for all hypercovers $\pi : Y^\bullet \to X$ we have that the canonical morphism from $\bar{i} : \mathbf{A}(X) \to \mathrm{Desc}(Y^\bullet, \mathbf{A})$ from definition 3.12 is a weak equivalence:*

$$\mathbf{A} \text{ is } \omega\text{-stack} \quad \Leftrightarrow \quad \text{for all hypercovers } \pi : Y^\bullet \to X : \quad \mathbf{A}(X) \xrightarrow[i]{\simeq} \mathrm{Desc}(Y^\bullet, \mathbf{A}) \ .$$

Definition 3.14 (cohomology with coeffients in ω-category valued presheaves) *The $\underline{\text{cohomology}}$ with coefficients in $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ is*

$$H(-, \mathbf{A}) := \mathrm{colim}_Y \mathrm{Desc}(Y, \mathbf{A}) \, .$$

Remark (ω-stackification). Cohomology itself is an ω-category valued presheaf

$$H(-, \mathbf{A}) : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces}) \, .$$

Applying $H(-,-)$ makes an ω-category valued presheaf get closer and closer to being an ω-stack. This is *ω-stackification.*

[** it should be true that $H(-, \mathbf{A})$ is an ω-stack. but good proof is missing currently **]

3.1.3 Codescent

Where descent for presheaves with values in ω-categories gives rise to cohomology, dually there is a notion of *co*descent for *co*presheaves. Codescent for co-presheaves with values in $\omega\mathsf{Categories}$ gives rise to a generalized notion of *homotopy.*

Recall from section 3.1.2 that we denote by G a fixed cosimplicial ω-category.

Definition 3.15 (ω-category of codescent data) *Given an ω-category valued co-presheaf $\mathbf{A} : \mathsf{Spaces} \to \omega\mathsf{Categories}$ and a hypercover $\pi : Y^\bullet \longrightarrow X$ the $\underline{\omega\text{-category of codescent data}}$ on Y with coefficients in $\mathbf{A}$ is the coend*

$$\mathrm{Codesc}(Y^\bullet, C) := \int^{[n]\in\Delta} G(\Delta^n) \otimes \mathbf{A}(Y^{[n+1]})) \, .$$

Remark. Notice that for $G = O$ the orientals, codescent for $\mathcal{P}_0$, with $\mathcal{P}_0(X)$ the discrete ω-category on the space X, the codescent object on a simplicial set $Y^\bullet$ is the free ω-category $S(Y^\bullet)$ on that set, from proposition 3.1.

We introduce codescent as the translation from the notion of cohomology from section 3.1.2 to that in section 2.2.5.

Lemma 3.16 *There is a canonical morphism from the codescent object to the base ω-category*

$$\bar{\pi} : \mathrm{Codesc}(Y, \mathbf{A}) \to \mathbf{A}(X) \, .$$

Proof. The canonical projection

$$\mathbf{A}(Y^{n+1}) \otimes G(\Delta^n) \xrightarrow{\simeq} \mathbf{A}(Y^{n+1}) \xrightarrow{\pi_*} \mathbf{A}(X)$$

for all $n \in \mathbb{N}$ is clearly an extraordinary conatural family. By the universal property of the codescent object from proposition 3.21 this yields the unique morphism $\bar{\pi}$ such that all diagrams

$$\begin{tikzcd}
\mathbf{A}(X) & & \\
& \mathrm{Codesc}(Y, \mathbf{A}) \arrow[ul, dashed, "\bar{\pi}"'] & \mathbf{A}(Y^{[n+1]}) \otimes G(\Delta^n) \arrow[l] \arrow[ull, bend right] \\
& \mathbf{A}(Y^{[m+1]}) \otimes G(\Delta^m) \arrow[u] \arrow[uul, bend left] & \mathbf{A}(Y^{[m+1]}) \otimes G(\Delta^n) \arrow[l] \arrow[u]
\end{tikzcd}$$

commute. □

Definition 3.17 (ω-costacks) *A copresheaf $\mathbf{A}$: $\mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ is an* <u>*ω-costack*</u> *if for all hypercovers $Y^\bullet \to X$ the canonical morphism from lemma 3.16 is a weak equivalence*

$$\mathrm{Codesc}(Y, \mathbf{A}) \xrightarrow{\simeq} \mathbf{A}(X) \ .$$

Theorem 3.18 (Π_ω is an ω-costack) *The fundamental path copresheaf Π_ω : $\mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ from section 4.2 is an ω-costack.*

Proof. For $n = 1$ this is a theorem in [136] and for $n = 2$ in [138]. [** for higher n this is conjectural for the time being **] □

Remark. One can understand this as a version of the van Kampen theorem, whose interpretation in terms of groupoids and 2-groupoids is due to Ronnie Brown [34]. See also section 7 of [108].

This provides the connection between the notion of cohomology of ω-category valued presheaves on spaces, definition 3.14 and the notion of cohomology as hom-objects in the weak homnotopy category of ω-categories, definition 2.83.

Corollary 3.19 *For Π : $\mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ an ω-costack and for $C \in \omega\mathsf{Categories}(\mathsf{Spaces})$ any ω-category and $X \in \mathsf{Spaces}$ any space, there is a canonical faithful functor*

$$H(X, \mathrm{hom}(\Pi(-), C)) \hookrightarrow \mathbf{Ho}(\Pi(X), C) \, .$$

Proof. By proposition 3.21 we have $H(X, \mathrm{hom}(\Pi(-), C)) \simeq \mathrm{colim}_Y \mathrm{hom}(\mathrm{Codesc}(Y, \Pi), C)$. By the ω-costack property of Π we have that $\mathrm{hom}(\mathrm{Codesc}(Y, \Pi), C)$ is an ω-category of ω-anafunctors of the form

$$\begin{tikzcd}
\mathrm{Codesc}(Y, \Pi) \arrow[r] \arrow[d, "\simeq"] & C \\
\Pi(X) &
\end{tikzcd} .$$

□

Remark. In section 3.3.4 we discuss coefficient presheaves of the form $\mathrm{hom}(\Pi(-), C)$ with Π an ω-costack as *differential nonabelian cohomology.* For these coefficients the above corollary relates cohomology in terms of descent to cohomology in terms of ω-anafunctors, definition 2.83.

Definition 3.20 (homotopy) <u>*Homotopy*</u> *with coefficients in $\mathbf{A}$: $\mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ is*

$$\pi(-, \mathbf{A}) := \lim_Y \mathrm{Codesc}(Y, \mathbf{A}) \, .$$

Remark. For $\mathbf{A} = \Pi_\omega$ this yields a refinement of the ordinary notion of homotopy groups $\pi_n(X)$, in that these are the homology groups, definition 2.39, of the crossed complex $[\pi(X, \Pi_\omega)]$.

Proposition 3.21 *For $\mathbf{A} : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ we have*

$$\mathrm{Desc}(Y, \hom(\mathbf{A}(-), C)) \simeq \hom(\mathrm{Codesc}(Y, \mathbf{A}), C)$$

naturally in $C \in \omega\mathsf{Groupoids}(\mathsf{Spaces})$.

Proof. Using that the contravariant internal hom takes colimits to limits, lemma 6.12, and hence coends to ends: the right hand is

$$\cdots \simeq \hom\Big(\int^{[n]\in\Delta} O(\Delta^n) \otimes \mathbf{A}(Y^{[n+1]})\ , C\Big) \simeq \int_{[n]\in\Delta} \hom(O(\Delta^n) \otimes \mathbf{A}(Y^{[n+1]})\ , C)$$

and the hom-adjunction inside the internal hom (e.g. [89] (1.27)) yields

$$\cdots \simeq \int_{[n]\in\Delta} \hom(O(\Delta^n), \hom(\mathbf{A}(Y^{[n+1]}), C)) =: \mathrm{Desc}(Y, \hom(\mathbf{A}(-), C))\,.$$

Each step in this derivation is natural in C. $\square$

We now evaluate the general statements about codescent in low dimensional examples.

Proposition 3.22 (Čech groupoid is codescent for $n = 1$) *In $1\mathsf{Categories}$ the codescent object $\Pi^\pi(X)$ for $\Pi = \mathcal{P}_0$ is the familiar Čech groupoid $C(Y)$ of $\pi : Y \to X$.*

$$\Pi^\pi(X) = (\ Y \times Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y\ )\,.$$

Proof. The coend then restricts to a joint colimit over a handful of diagrams which can all be analysed separately. The diagrams

$$\begin{tikzcd} C(Y) & \mathcal{P}_0(Y) \otimes O(\Delta^0) \arrow[l] \\ \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^1) \arrow[u] & \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^0) \arrow[l, "\mathrm{Id}\otimes(\{\bullet\}\mapsto\{a\})"'] \arrow[u, "\pi_1\otimes\mathrm{Id}"'] \end{tikzcd} \qquad \begin{tikzcd} C(Y) & \mathcal{P}_0(Y) \otimes O(\Delta^0) \arrow[l] \\ \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^1) \arrow[u] & \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^0) \arrow[l, "\mathrm{Id}\otimes(\{\bullet\}\mapsto\{b\})"'] \arrow[u, "\pi_2\otimes\mathrm{Id}"'] \end{tikzcd}$$

have as top horizontal morphism the inclusion of the the objects of the Čech groupoid and as left vertical morphism the inclusion of the morphisms. Their commutativity is the source-target matching condition in the Čech groupoid.

Notice that the 2-morphisms of $O(\Delta^2) \otimes \mathcal{P}_0(Y^{[3]})$ are triangles labeled in points in $Y^{[3]}$. Hence the diagrams

$$\begin{tikzcd} C(Y) & \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^1) \arrow[l] \\ \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^2) \arrow[u] & \mathcal{P}_0(Y^{[2]}) \otimes O(\Delta^1) \arrow[l, "\mathrm{Id}\otimes(d_0^*)"'] \arrow[u, "\pi_{12}\otimes\mathrm{Id}"'] \end{tikzcd}$$

etc. encode the composition law in the Čech groupoid. □

It is helpful to draw a couple of pictures for morphisms in $O(\Delta^{(n)}) \otimes \mathcal{P}_2(Y^{[n+1]})$ to recognize all the generators discussed in [138]. For instance the 2-morphisms in $O(\Delta^{(1)}) \otimes \mathcal{P}_2(Y^{[2]}) = \{a \to b\} \otimes \mathcal{P}_2(Y^{(2)})$ coming from tensoring the interval with 1-paths $\gamma : x \to y$ in $Y^{[2]}$ are those square degree 2-generators

$$\begin{tikzcd}
(a,x) \arrow[r, "{(\gamma,a)}"] \arrow[d] & (a,y) \arrow[d] \arrow[dl, Rightarrow] \\
(b,x) \arrow[r, "{(\gamma,b)}"'] & (b,y)
\end{tikzcd}$$

Proposition 3.23 (codescent for $n = 2$ [138]) *Assuming for convenience that $Y = \sqcup_i U_i$ is a good cover by open subsets, the codescent 2-groupoid $\mathcal{P}_0^\pi(X)$ is generated from 2-morphisms*

$$\left\{ \begin{tikzcd}
& (x,j) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
(x,i) \arrow[ur] \arrow[rr] & {} & (x,k)
\end{tikzcd} \;\middle|\; (x,i,j,k) \in U_{ijk} \right\}$$

subject to the associativity relation

$$\forall (x,i,j,k,l) : \quad \begin{tikzcd}
(x,j) \arrow[r] \arrow[dr, phantom, "\Searrow"] & (x,k) \arrow[d] \\
(x,i) \arrow[u] \arrow[ur] \arrow[r] & (x,l)
\end{tikzcd}
\arrow[from=2-1, to=1-2, phantom, "\Downarrow"]
\;=\;
\begin{tikzcd}
(x,j) \arrow[r] \arrow[dr] & (x,k) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
(x,i) \arrow[u] \arrow[r] \arrow[ur, phantom, "\Downarrow"] & (x,l)
\end{tikzcd} .$$

Proof. It follows that the inverse 2-morphisms

$$\begin{tikzcd}
& (x,j) \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\
(x,i) \arrow[ur] \arrow[rr] & {} & (x,k)
\end{tikzcd}$$

satisfy a similar co-associativity relation. It also follows that the original triangles together with their inverses satisfy a mixed *Frobenius* relation. Using this one shows [138] that the canonical projection $\mathcal{P}_0^\pi(X) \to \mathcal{P}_0(X)$ is indeed a weak equivalence. □

3.2 Operations on cocycles

Natural operations on cocycles include the construction of their lifts, twisted lifts and obstructions to lifts through extensions of their coefficient object, as well transgression of cocycles to mapping spaces.

3.2.1 Lifts, twisted lifts and obstructions to lifts

A morphism $f : \mathbf{B}\hat{G} \to \mathbf{B}G$ of ω-groups naturally induces a morphism of cohomologies

$$f^* : H(-, \mathbf{B}\hat{G}) \to H(-, \mathbf{B}G) .$$

The question to which degree this morphism has a right inverse is the obstruction problem for lifts through f.

$$
\begin{tikzcd}
 & \mathbf{B}\mathbf{B}^{n-1}U(1) \arrow[d] & & \\
 & \mathbf{B}\hat{G} \arrow[d] \arrow[dr, hook, "\ker(p)"] & & \\
\mathbf{X} \arrow[ur, dotted, "\hat{g}"] \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{twLift}(g)"'] \arrow[rrr, bend right=30, "\mathrm{obstr}(g)"'] & \mathbf{B}G \arrow[r, "\simeq"] & \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G}) \arrow[r, "p"] & \mathbf{B}\mathbf{B}^{n}U(1)
\end{tikzcd}
$$

Figure 6: **Obstruction theory** for lifts of nonabelian cocycles through shifted central extensions $\mathbf{B}^{n-1}U(1) \to \hat{G} \to G$. The lift $\hat{g}$ of the G-cocycle g is obstructed by the $\mathbf{B}^nU(1)$-cocycle $p \circ \mathrm{twLift}(g)$, where $\mathrm{twLift}(g)$ is a *twisted* $\hat{G}$-cocycle, namely a $(U(1) \to \hat{G})$-cocycle, a lift to which always exists for a sufficiently fine cover $\mathbf{Y}$. Crossed arrows denote ω-anafunctors, i.e. ω-functors out of surjective equivalences (hypercovers), see section 2.2.5. The fact that the nontriviality of the horizontal composite precisely obstructs the lift of the G-cocycle g to the $\hat{G}$-cocycle $\hat{g}$ is the statement that the canonical inclusion of $\hat{G}$ into the weak quotient $\hat{G}//\mathbf{B}^{n-1}U(1) := (\mathbf{B}^{n-1}U(1) \hookrightarrow \hat{G})$ is the homotopy kernel of the projection p. See section 3.2.1.

Recall definition 2.23 of kernels and cokernels of morphisms of ω-monoids.

Definition 3.24 (well-defined obstruction problem) *We say a morphism $f : \mathbf{B}\hat{G} \to \mathbf{B}G$ induces a well-defined obstruction problem if f has a factorization $\mathbf{B}\hat{G} \overset{i}{\hookrightarrow} \mathbf{B}\tilde{G} \xrightarrow{\simeq} \mathbf{B}G$ such that the cokernel p of i exists,*

$$
\begin{tikzcd}
\mathbf{B}\hat{G} \arrow[d, "f"] \arrow[dr, hook, "i"] & & \\
\mathbf{B}G & \mathbf{B}\tilde{G} \arrow[l, "\simeq"'] \arrow[r, "p"] & \mathrm{coker}(i)
\end{tikzcd}
,
$$

and such that i is weakly equivalent to the homotopy kernel of p:

$$
\mathrm{holim}\left(
\begin{tikzcd}
 & \mathbf{B}\hat{G} \arrow[d, "f"] \\
\mathrm{pt} \arrow[r] & \mathrm{coker}(i)
\end{tikzcd}
\right) \xrightarrow{\simeq} \mathbf{B}\hat{G}\,.
$$

In this situation the following fact about homotopy limits applies. See [142] for the general issue of homotopy coherent category theory.

Lemma 3.25 *In some model category, let*

$$
\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

be an ordinary pullback which happens to be weakly equivalent to the homotopy limit

$$\mathrm{holim}\left(\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}\right) \xrightarrow{\simeq} A \times_C B\,.$$

Assume that A, B and C are fibrant and consider a cofibrant object V and a diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\Leftrightarrow"] \\ B \arrow[r] & C \end{tikzcd}$$

which commutes up to homotopy, as indicated by the double arrow. Then threre is a universal morphism $V \dashrightarrow A \times_C B$ such that the following diagram commutes up to homotopy as indicated

$$\begin{tikzcd} V \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\Leftrightarrow"] \\ & B \arrow[r] & C \end{tikzcd}$$

We are grateful to Michael Batanin for discussion of this point.

Proposition 3.26 (obstruction theorem) *A cocycle $g : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mapsto \mathbf{B}G$ has a lift through $f : \mathbf{B}\hat{G} \to \mathbf{B}G$ to a $\mathbf{B}\hat{G}$-cocycle $\hat{g}$*

$$\begin{tikzcd} & \mathbf{B}\hat{G} \arrow[d, "f"] \\ \mathbf{X} \arrow[ur, dashed, "\hat{g}"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd}$$

if and only if the composite

$$\mathbf{X} \xrightarrow{g} \mathbf{B}G \xrightarrow{\simeq} \mathbf{B}\tilde{G} \longrightarrow \mathrm{coker}(p)$$

is homotopic to the trivial cocycle (factoring through pt*), where $\mathbf{B}G \xrightarrow{\simeq} \mathbf{B}\tilde{G}$ is the weak inverse of $\mathbf{B}\tilde{G} \xrightarrow{\simeq} \mathbf{B}G$ guaranteed to exist by theory 2.81.*

Proof. Consider the diagram of ω-anafunctors

$$\begin{tikzcd} & \mathbf{B}\hat{G} \arrow[dr] & & \\ \mathbf{X} \arrow[ur, dashed, "\hat{g}"] \arrow[r, "g"] & \mathbf{B}G \arrow[r, "\simeq"] & \mathbf{B}\tilde{G} \arrow[r] & \mathrm{coker}(p) \end{tikzcd}$$

In terms of spans of ω-functors representing this we get a diagram

$$\begin{tikzcd} & \mathbf{B}\hat{G} \arrow[dr] \arrow[d, phantom, "\Updownarrow"] & & \\ \hat{\mathbf{X}} \arrow[ur, dashed] \arrow[rr] \arrow[drrr, bend right] & {} & \mathbf{B}\tilde{G} \arrow[r, "p"] \arrow[dr, phantom, "\Leftrightarrow"] & \mathrm{coker}(p) \\ & & & \mathrm{pt} \arrow[u] \end{tikzcd}$$

for $\hat{\mathbf{X}} \xrightarrow{\simeq} \mathbf{X}$ the acyclic fibration giving the ω-anafunctor. We can assume without restriction of generality that $\hat{\mathbf{X}}$ is cofibrant, for if it is not we can simply pass to a cofibrant replacement. Being ω-groups, all other objects involved are fibrant, by proposition 2.62.

Therefore, given a homotopy between the horizontal map and the map through the point, as indicated by the lower double arrow, the assumptions appearing in lemma 3.25 are met and hence the dashed morphism and the upper double arrow exist.

[** ... **] □

See also figure 6.

A special interesting case of obstruction problems comes from shifted central extensions of ω-groups.

Definition 3.27 (shifted central extension) *A shifted central extension of ω-groups is an n-group $\hat{G}$ and an abelian 1-group A yielding an extension of the form*

$$\mathbf{B}^{n-1}A \longrightarrow \hat{G} \longrightarrow G$$

such that the crossed module corresponding to $(\mathbf{B}^{n-1}A \hookrightarrow \hat{G})$ is

$$A \hookrightarrow \hat{G}_n \longrightarrow \hat{G}_{n-1} \longrightarrow \cdots \longrightarrow \hat{G}_1$$

with A central in $\hat{G}_n$.

Lemma 3.28 *This yields indeed a well-defined obstruction problem in the sense of definition 3.24 in that the canonical inclusion*

$$\mathbf{B}\hat{G} \to \mathbf{B}(\mathbf{B}U(1) \hookrightarrow \hat{G})$$

is weakly equivalent to the homotopy kernel of the canonical projection $p : \mathbf{B}(\mathbf{B}^{n-1}A \hookrightarrow \hat{G}) \to \mathbf{B}^n A$.

Proof. Expressing p in terms of crossed modules of groupoids it is immediate from definition 2.61 that p is a fibration. Since all ω-groupoids are fibrant according to proposition 2.62 the conditons in example 4.2 in [41] are met which imply that the homotopy kernel in question, which is the homotopy limit of the diagram

$$\mathrm{pt} \longrightarrow \mathbf{B}(\mathbf{B}A \hookrightarrow \hat{G}) \xleftarrow{p} \mathbf{B}\hat{G}$$

is weakly equivalent to its ordinary limit. This ordinary limit is nothing but $\mathbf{B}\hat{G}$. □

This situation is summarized in section 3.34 in terms of *twisted nonabelian cohomology.* We formulate the consequences of this in section 3.34 in terms of twisted cohomology.

3.2.2 Transgression of cocycles

Definition 3.29 (transgression) *Transgression of cocycles is the covariant inner* hom *in the weak homotopy bicategory* **Ho***: for $g : C \longrightarrow\!\!\!\!\!\!\mapsto D$ a cocycle regarded as an ω-anafunctor and for B another ω-category, the transgression of g to* $\mathrm{hom}(B, C)$ *is*

$$\tau(g) := \mathrm{hom}(B, -) : \mathrm{hom}(B, C) \longrightarrow\!\!\!\!\!\!\mapsto \mathrm{hom}(B, D) \ .$$

[** discuss inner hom of ω-anafunctors **]

Remark. Transgression of differential 2-cocycles to loop spaces by inner hom is discussed in [137, 138].

Proposition 3.30 (trangression of differential forms) *On differential cocycles representing ordinary differential forms, this notion of transgression coincides with the classical one.*

Proof. For $n = 2$ in [137]. □

3.2.3 Construction of cocycles by killing of homotopy groups

[** the following discussion eventually should be given more generally **]

In the constructions in section 4.4.1, it may happen that a construction of a cocycle $g : \Pi(X) \longrightarrow\!\!\!\!\!\!\mapsto \mathbf{B}G$ proceeds via the construction of morphism $f : \Pi(Y) \longrightarrow \mathbf{B}H$ where neither $\Pi(Y)$ is weakly equivalent to $\Pi(X)$, nor $\mathbf{B}H$ to $\mathbf{B}G$, but both become so after killing of homotopy groups. Suppose for simplicity that Y is connected and that $[\Pi(Y)]$ has all homotopy groups, definition 2.39, equal to $[\Pi(X)]$ except for the kth, $k \geq 3$, which vanishes for $[\Pi(X)]$.

Then there is an injection

$$[\mathbf{B}\mathbf{B}^{k-1}H_k([\Pi(Y)])] \longrightarrow [\Pi(Y)]$$

and we *kill* $\pi_k([\Pi(Y)])$ by forming the pushout crossed complex

$$\begin{tikzcd}
{[Y \times \mathbf{B}\mathbf{B}^{k-1}H_k([\Pi(Y)])]} \arrow[r] \arrow[d] & {[\Pi(Y)]} \arrow[d] \\
{[Y \times \mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)])]} \arrow[r] & {[\Pi(Y)] \cup [\mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)])]}
\end{tikzcd},$$

where

$$[Y \times \mathbf{B}\mathbf{b}^{k-1}H_k([\Pi(Y)])] = \Big(\bigsqcup_{y \in Y} H_k([\Pi(Y)], y) \xrightarrow{\delta_{k+1}} 0 \longrightarrow \cdots \longrightarrow 0 \rightrightarrows Y \Big)$$

and where

$$[Y \times \mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)])] = \Big(\bigsqcup_{y \in Y} H_k([\Pi(Y)], y) \xrightarrow{\delta_{k+2} = \mathrm{Id}} \bigsqcup_{y \in Y} H_k([\Pi(Y)], y) \xrightarrow{\delta_{k+1}} 0 \longrightarrow \cdots \longrightarrow 0 \rightrightarrows Y \Big).$$

This follows section 7.4.3 of [34].

By slight abuse of notation we write $\Pi(Y) \cup \mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)])$ for the ω-groupoid corresponding to the pushout crossed complex according to theorem 2.36.

If we similarly kill this group in $\mathbf{B}H$ and if $\hat{g}$ in the diagram

$$\begin{tikzcd}
\Pi(Y) \arrow[r, "f"] \arrow[d, hook] & \Pi(X) \arrow[d, hook] \\
\Pi(Y) \cup Y \times \mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)]) \arrow[r, "\hat{g}"] \arrow[d, "\simeq"] & \mathbf{B}H \cup \mathbf{B}\mathbf{E}\mathbf{B}^{k-1}H_k([\Pi(Y)]) \arrow[d, "\simeq"] \\
\Pi(X) & \mathbf{B}H
\end{tikzcd}$$

exists, then we say that f satisfied the right integrality condition to yield a cocycle.

[** ... **]

3.3 Types of cohomologies

Cohomology with coefficients in $\omega\mathsf{Categories}(\mathsf{Spaces}) \simeq \mathsf{Sheaves}(\mathsf{CartesianSpaces}, \omega\mathsf{Categories}(\mathsf{sete}))$ generalizes the classical notions of

- sheaf cohomology / Čech cohomology with coefficients in complexes of sheaves of abelian groups;
- equivariant versions of cohomology with respect to a group action;

– group cohomology (equivariant cohomology of the point) with coefficients in abelian groups;

- differential cohomologies

to more general coefficients and more general equivariance conditions on the domains. Generalizations of sheaf cohomology have been considered mostly in the context of (pre)sheaves of simplicial sets [82], which is different from but closely related to sheaves of ω-categories. In parts of the literature the term "nonabelian cohomology" is used for group cohomology with nonabelian coefficients [12], which is really just a special case of equivariant nonabelian cohomology of the point.

On the other hand, one should beware of the old terminology clash that many of these classical examples, as well as their generalizations, are *not* examples of what are called "generalized cohomology theories" in algebraic topology: functors from topological spaces to abelian groups satisfying the generalized Eilenberg-Steenrod axioms. For one, a "generalized cohomology theory" in the Eilenberg-Steenrod sense is always homotopy invariant. This axiom fails for instance manifestly for all differential cohomology theories (even those geared towards "generalized cohomology theories" [74]).

It seems that little is known about the general relation between the Eilenberg-Steenrod generalization of cohomology theories and the generalization to nonabelian cohomology theory considered here or in [82], little beyond the observation of the fact that most of the familiar "generalized cohomology theories" have cocycle representatives which do happen to be also cocycles in nonabelian cohomology: ordinary integral cohomology is represented by the "nonabelian cohomology" $H(X, \mathbf{B}^n\mathbb{Z})$ classifying higher line bundles/higher abelian gerbes, K-theory in degree 0 by $H(X, \mathbf{B}U \times \mathbb{Z})$ classifying complex vector bundles, and elliptic cohomology/TMF is expected to be geometrically represented by some kind of 2-vector bundles.

The following section 3.3.1 describes sheaf cohomology / Čech cohomology as a special case of nonabelian cohomology, and section 3.3.2 similarly considers equivariant and group cohomology. Section 3.2.1 discusses the lifting problem in nonabelian cohomology and introduces the notion of *twisted* nonabelian cohomology. Finally section 3.3.4 defines differential nonabelian cohomology and discusses non-flat differential cohomology in terms of the extension problem in nonabelian cohomology.

3.3.1 Sheaf cohomology / Čech cohomology

Theorem 3.31 (Čech cohomology with coefficients in complexes of sheaves of abelian groups) *For* $X \in \mathsf{Manifolds}$ *and* $\mathbf{A} : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Groupoids}$ *such that the presheaf of crossed complexes* $[\mathbf{A}] : \mathsf{Spaces}^{\mathrm{op}} \to \mathsf{CrossedComplexes}$ *is a presheaf of complexes of abelian groups, the (ordinary) Čech cohomology of* X *with values in* $[C]$ *coincides with the cohomology obtained from definition 3.11.*

$$H(X, [\mathbf{A}]) \simeq H(X, \mathbf{A})_\sim .$$

Proof. Since $[\mathbf{A}]$ is an ω-groupoid-valued presheaf we can use $G(\Delta^n) = \Pi_\omega(\Delta^n)$ in the definition of descent. Translating the inner hom of ω-groupoids in the integrand of $\mathrm{Desc}(Y, \mathbf{A}) := \int_{n\in\Delta} \mathrm{hom}(\Pi_\omega(\Delta^n), \mathbf{A}(Y^{[n+1]}))$ to an inner hom of crossed complexes $\mathrm{Desc}(Y, \mathbf{A}) := \int_{n\in\Delta} \mathrm{hom}([\Pi_\omega(\Delta^n)], [\mathbf{A}(Y^{[n+1]})])$ and then using the explicit description of $[\Pi_\omega(\Delta^n)]$ from the *homotopy addition lemma*, lemma 3.2, shows that the end manifestly computes the Čech double complex $\mathbf{A}(Y^{[\bullet+1]})_\bullet$. □

3.3.2 Equivariant cohomology / group cohomology

From the perspective of cohomology on ω-categories, equivariant cohomology is the generic case: for C an ω-groupoid, the cohomology of C, $H(C, D)$, can be regarded as C-equivariant cohomology on the discrete ω-category C_0 over the space of objects of C. In applications equivariant cohomology is more often understood as the special case of this situation where C is an *action groupoid* of a group acting on a space.

Definition 3.32 (equivariant cohomology with respect to group action) *For G a group and $\rho : X \times G \to X$ a G-action on $X \in$ Spaces, write $X//G := (X \times G \underset{\mathrm{pr}_1}{\overset{\rho}{\rightrightarrows}} X)$ for the corresponding action groupoid. For $D \in \omega$Categrories any coefficient object, we say that equivariant cohomology $H^G(X, D)$ of X with coefficients in D is the cohomology of $X//G$ with coefficients in D:*

$$H^G(X, D) := H(X//G, D)\,.$$

Group cohomology is the special case of equivariant cohomology for $X = \mathrm{pt}$. Notice that the action groupoid of a group acting on a point is nothing but the one-object groupoid $\mathbf{B}G$ determined by G, $\mathrm{pt}//G = \mathbf{B}G$.

Proposition 3.33 (group cohomology, [34]) *The nth group cohomoloy of a group G with coefficients in an abelian group K is*

$$H^n(G, K) := H(\mathrm{pt}//G, \mathbf{B}^n K) = H(\mathbf{B}G, \mathbf{B}^n K)\,.$$

Proof. By inspection of the boundary maps of the crossed module corresponding to the universal resolution of $\mathbf{B}G$. Details are in section 12 of [34]. □

3.3.3 Twisted cohomology

Recall the discussion of lifts, twisted lifts and obstructions of lifts from section 3.2.1.

Definition 3.34 (twisted cohomology) *We say that $H(-, \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G}))$ is the* **twisted $\hat{G}$ cohomology**. *The canonical morphism*

$$\mathrm{twLift} : H(-, \mathbf{B}G) \to H(-, \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G})$$

is the twisted lift, the canonical morphism

$$\mathrm{tw} : \; H(-, \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G})) \longrightarrow H(-, \mathbf{B}\mathbf{B}^n U(1))$$

is the projection onto the twist and their composite

$$\mathrm{obstr} : \; H(-, \mathbf{B}G) \xrightarrow{\mathrm{twLift}} H(-, \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G}) \xrightarrow{\mathrm{tw}} H(-, \mathbf{B}\mathbf{B}^n U(1))$$

is the obstruction to the lift.

The result of section 3.2.1 then says:

Corollary 3.35 (obstruction theorem for shifted central extensions) *Let $\mathbf{B}^{n-1}U(1) \to \hat{G} \to G$ a shifted central extension. Then a G-cocycle $g \in H(\mathbf{X}, \mathbf{B}G)$ has a lift to a $\hat{G}$-cocycle $\hat{g} \in H(\mathbf{X}, \mathbf{B}\hat{G})$ if and only if the cocycle $\mathrm{obstr}(g) \in H(\mathbf{X}, \mathbf{B}\mathbf{B}^n U(1))$ is equivalent to the trivial cocycle.*

Special cases.

Lemma 3.36 *For every $n \in \mathbb{Z}$ we have the shifted central extension*

$$\mathbf{B}^n\mathbb{Z} \longrightarrow \mathbf{B}^n\mathbb{R} \xrightarrow{\mathrm{mod}\mathbb{Z}} \mathbf{B}^n U(1)\;\;.$$

Corollary 3.37 *We have an equivalence*

$$H(-, \mathbf{B}^n U(1)) \xrightarrow{\simeq} H(-, \mathbf{B}^{n+1}\mathbb{Z})\;\;.$$

Proof. Use that $\mathbf{B}^n\mathbb{R} \xrightarrow{\simeq} \mathrm{pt}$ [** spell out details: need to refine above argument from classes to cocycles **] □

We are grateful to Thomas Nikolaus for discussion of this point.

3.3.4 Differential cohomology

Flat differential cohomology is equivariant cohomology with respect to the the smooth fundamental path ω-groupoid, described in section 4.2.1, hence cohomology with coefficients in $\hom(\Pi_\omega(-), \mathbf{B}G)$. Fake-flat differential cohomology is cohomology with coefficients in $\hom(\mathcal{P}_n(-), \mathbf{B}G)$. Non-flat differential cohomology is about the obstruction to lifting nonabelian cohomology to flat differential cohomology.

Definition 3.38 (trivial ω-bundles with connection) *For* $\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ *and* $C \in \mathsf{Categories}(\mathsf{Spaces})$ *we say*

$$C\mathrm{TrivBund}_\Pi := \hom(\Pi(-), C)$$

is the presheaf of trivial C-principal ω-bundles with Π-connection. In particular

- $C\mathrm{TrivBund} := C\mathrm{TrivBund}_{P_0}$ *(definition 2.20) is the presheaf of trivial C-principal ω-bundles;*
- $C\mathrm{TrivBund}_{\Pi_\omega}$ *(definition 4.13) is the presheaf of flat C-principal ω-bundles with connection;*
- $C\mathrm{TrivBund}_{\mathcal{P}_n}$ *(definition 4.13) is the presheaf of fake-flat C-principal ω-bundles with connection with curvature in degree $(n+1)$.*

Definition 3.39 (nonabelian differential cohomology) *For* $\Pi : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ *and* $C \in \mathsf{Categories}(\mathsf{Spaces})$ *nonabelian differential cohomology $H_\Pi(-, C)$ relative to Π with coefficients in A is*

$$H_\Pi(-, C) := H(-, C\mathrm{TrivBund}_\Pi(-))\,.$$

Remark. For low n the discussion of nonabelian differential cohomology classifying fake-flat smooth principal G-n-bundles ($(n-1)$-gerbes) with connection is in [11, 136, 137, 138, 107].

Proposition 3.40 (Deligne cohomology) *nth Deligne cohomology of X is $H_{\mathcal{P}_n}(X, \mathbf{B}^n U(1))$.*

Proof. One first shows that the crossed complex $[\hom(\mathcal{P}_n(-), \mathbf{B}^n U(1))]$ is the sheaf of chain complexes of abelian groups

$$C^\infty(-, \mathbb{R}/\mathbb{Z}) \xrightarrow{d} \Omega^1(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-)$$

with $C^\infty(-, \mathbb{R}/\mathbb{Z})$ in degree n and $\Omega^n(-)$ in degree 0 (the sheaf of objects). The claim then follows with theorem 3.31. $\square$

For $n \leq 2$, this is proven in [138].

Remark. This says that in the abelian case, $G = \mathbf{B}^{n-1}U(1)$, fake-flatness is automatic and no extra condition.

Definition 3.41 (ω-category of C-torsors) *Restricting the cohomology presheaf $H(-, C\mathrm{TrivBund}_\Pi(-))$ from* Spaces *to* $\mathsf{CartesianSpaces}$ *yields, via the equivalence proposition 2.17, an ω-category internal to* Spaces *which, due to the discussion in section 2.2.4, we call $C\mathrm{Tor}_\Pi$, the ω-category of C-torsors relative to Π. For $\Pi = \mathrm{P}_0$ we write just $C\mathrm{Tor} := C\mathrm{Tor}_\Pi$.*

According to section 3.1.2, $C\mathrm{Tor}$ can be regarded as the ω-stackification of $C\mathrm{TrivBund}$.

Lemma 3.42 *The canonical inclusion $C \hookrightarrow C\mathrm{Tor}$ is a local weak equivalence.*

Remark. Notice the following tautological but useful situation. Nonabelian C-cocycles $X \longrightarrow \mathbf{B}G$ were introduced as maps out of resolutions of X into $\mathbf{B}G$. With the above definition and since by Yoneda $\hom(X, C\mathrm{Tor}) \simeq H(X, C)$ we can regard them alternatively as maps into the *coresolution* $C\mathrm{Tor}$ of $\mathbf{B}G$

$$\begin{tikzcd} X \arrow[dr] & & \mathbf{B}G \arrow[dl, hook, "\simeq"] \\ & G\mathrm{Tor} & \end{tikzcd} .$$

Remark. It is familiar from differential versions of "generalized cohomology theories" [74] such as Deligne cohomology refining integral cohomology and differential K-theory refining K-theory that cocycles for differential cohomology can be realized as bundles with connection. Compare the remark at the beginning of section 3.3.

Non-flat differential cohomology and curvature. The definition of non-flat differential cohomology, its curvature and characteristic forms, proceeds via ∞-Lie integration of diagrams in $L_\infty\mathsf{Algebroids}$, discussed in section 4.

Definition 3.43 (universal ω-groupoids) *We say a smooth ω-groupoid $C \in \omega\mathsf{Groupoids}(\mathsf{Spaces})$ is universal if there is a L_∞-algebroid $\mathfrak{g}$, definition 2.91, such that C is equals the un-truncated ∞-Lie integration, definition 4.20, of $\mathfrak{g}$:*

$$C \text{ is universal} \quad \Leftrightarrow \quad C = \Pi_\omega(S(CE)(\mathfrak{g}))\,.$$

Accordingly a smooth ω-group G is universal if $\mathbf{B}G$ is a universal smooth ω-groupoid.

Remark. The idea is that every ω-groupoid internal to $\mathsf{ConcreteSpaces}$ should be the quotient of a universal smooth ω-groupoid by a discrete ω-group, in some sense.

Definition 3.44 *For G a universal smooth ω-group with $\mathfrak{g} \in L_\infty\mathsf{Algebroids}$ such that*

$$\mathbf{B}G = \Pi_\omega(S(\mathrm{CE}(\mathfrak{g})))$$

define ω-groups

$$\mathbf{B}\mathcal{E}G := \Pi_\omega(S(\mathrm{W}(\mathfrak{g})))$$

and

$$\mathbf{B}\mathcal{B}G := \Pi_\omega(S(\mathrm{W}(\mathfrak{g})_{\mathrm{basic}}))\,,$$

where $\mathrm{CE}(\mathfrak{g})$ is the Chevalley-Eilenberg algebra of $\mathfrak{g}$ from definition 2.91, $\mathrm{W}(\mathfrak{g})$ is the Weil algebra of $\mathfrak{g}$ from definition 2.92 and where $\mathrm{W}(\mathfrak{g})_{\mathrm{basic}} = \mathrm{inv}(\mathfrak{g})$ is the algebra of invariant polynomials from definition 2.93.

The ∞-Lie integration of the sequence $\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{W}(\mathfrak{g}) \longleftarrow \mathrm{W}(\mathfrak{g})_{\mathrm{basic}}$ from proposition 2.94 yields the sequence

$$\mathrm{B}G \longrightarrow \mathrm{B}\mathcal{E}G \longrightarrow \mathrm{B}\mathcal{B}G\ .$$

Definition 3.45 *Regarding the two sequences $i_X : \mathcal{P}_0(X) \to \Pi_\omega(X) \to \Pi_\omega(X)$ and $\mathrm{B}G \longrightarrow \mathrm{B}\mathcal{E}G \longrightarrow \mathrm{B}\mathcal{B}G$ as (images of) functors $i, f : I \to \omega\mathsf{Categories}(\mathsf{Spaces})$ on the double interval category $I \cup I := \{a \to b \to c\}$ we set*

$$\hom(i_X, f) := \hom\begin{pmatrix} \mathcal{P}_0(X) & \mathbf{B}G \\ \downarrow i & \downarrow \\ \Pi_\omega(X)\,, & \mathbf{B}\mathcal{E}G \\ \downarrow \mathrm{Id} & \downarrow \\ \Pi_\omega(X) & \mathbf{B}\mathcal{B}G \end{pmatrix} := \int_{a \in I \cup I} \hom(i_X(a), f(a))\,.$$

Lemma 3.46 *This construction $\hom(i_X, f)$ is contravariantly functorial in X and hence extends to a presheaf $\hom(i_-, f) : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces})$.*

Definition 3.47 (non-flat differential cohomology and curvature) *For G a universal smooth ω-group, we say that differential cohomology with curvature of G is*

$$\bar{H}(-, \mathbf{B}G) := H(-, \hom(i_-, f))$$

with $\hom(i_-, f)$ from defintion 3.45.

Proposition 3.48 *Descent with respect to the diagram morphisms $\hom(i_-, f)$ is the same as diagram morphisms with respect to descent of the component presheaves.*

Proof. Use the Fubini theorem for ends. $\square$

Remark. An object of $\bar{H}(X, \mathbf{B}G)$ is represented by a diagram of ω-anafunctors

$$\begin{tikzcd}
X \arrow[r, "g", "|"] \arrow[d] & \mathbf{B}G \arrow[d] & \text{cocycle} \\
\Pi_\omega(X) \arrow[r, "(g,\nabla)", "|"] \arrow[d] & \mathbf{B}\mathcal{E}G \arrow[d] & \text{connection and curvature} \\
\Pi_\omega(X) \arrow[r, "P"] & \mathbf{B}\mathcal{B}G & \text{characteristic forms}
\end{tikzcd} .$$

See also figure 9.

3.4 Characteristic classes

While nonabelian cohomology is powerful in its generality, abelian cohomology is of course more tractable. To each nonabelian cohomology class there is canonically associated a family of abelian cohomology classes characterizing it (entirely or partly): its *characteristic classes.*

Definition 3.49 (universal characteristic classes) *For $\mathbf{B}G \in \omega\mathsf{Groupoids}(\mathsf{Spaces})$ we say that the cohomology classes $c \in \mathbf{Ho}(\mathbf{B}G, \mathbf{B}^n U(1))$, definition 2.83, of $\mathbf{B}G$ with coefficients in $\mathbf{B}^n U(1)$ are the degree n <u>universal characteristic classes</u> of G.*

Remark. In view of section 3.3.2 this means that the universal characteristic classes of G are precisely the ω-group cohomology classes of G with coefficients in $\mathbf{B}^n U(1)$.

Definition 3.50 (characteristic classes of nonabelian cocycle) *For $g \in H(X, \mathbf{B}G)$ a G-cocycle and for $c \in \mathbf{Ho}(\mathbf{B}G, \mathbf{B}^n U(1))$ a universal characteristic class as in definition 3.49, we say that the corresponding <u>characteristic class</u> of g is the cohomology class of $c(g) := c_*(f) \in H(X, \mathbf{B}^n U(1))$, i.e the image $[c(g)] \in H(X, \mathbf{B}^n U(1))/_\sim \simeq H^{n+1}(X, \mathbb{Z})$.*

Remark. In the existing literature (universal) characteristic classes are usually defined in terms of geometric realizations of n-groupoids as topological spaces; for $n \leq 2$ in [62, 61]: for G an n-group and $|G|$ the realization of its nerve, which is a topological group with topological classifying space $B|G|$, the universal characteristic classes (with values in $\mathbb{Z}$) of G would be taken to be the ordinary singular cohomology classes $H^\bullet(B|G|, \mathbb{Z})$. Notice that, also for $n \leq 2$, it is shown in [13] that $B|G|$ is the classifying space for second nonabelian cohomology $H(-, \mathbf{B}G)$ (for G a topological group satisfying mild conditions). The above approach aims at staying within the homotopy theory of smooth ω-categories without the *need* of passing to geometric realizations in Top. Of course for explicit computations it may be convenient to do so, but the formalism itself should not depend on this.

3.4.1 Characteristic forms

A characteristic form is a differential refinement of a characteristic class. Given that we regard characteristic classes in section 3.4 as $\mathbf{B}^n U(1)$-cocycles and hence, by theorem 3.31 as $U(1)$-Čech n-cocycles, it is natural to identify characteristic forms as the curvature forms of the corresponding differential $U(1)$-cohomology in the sense of section 3.3.4, which, by theorem 3.40, is Deligne cohomology. As we show below, this naturally harmonizes with the notion in definition 3.47 of curvature in differential nonabelian cohomology.

Pseudoconnections. In the abelian case non–flat differential cocycles in the sense of definition 3.47 are related to what in parts of the literature is addressed as *pseudoconnections* [16]. This is also related to the gerbe *bimodules* of [157]. In all these cases the naive conditions on a connection are relaxed. We now show that this relaxation is due to the passage from differential cocycles of the form $\mathcal{P}_n(X) \longrightarrow \mathbf{B}G$ to those of the form

$$\begin{array}{ccc} \mathcal{P}_0(X) & \longrightarrow & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi_\omega(X) & \longrightarrow & \mathbf{B}\mathbf{E}G \end{array}$$

as in section 3.3.4.

Here, as before, $\mathcal{P}_0, \Pi_\omega : \mathsf{Spaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ are the co-presheaves which send each space to the discrete ω-category and to the fundamental ω-groupoid over it, respectively. Notice that we have the canonical inclusion $i : \mathcal{P}_0 \to \Pi_\omega$. Let now $f : \mathbf{B}G \to \mathbf{B}H$ be any morphism.

We have a hom-ω-category of diagram morphisms from i to f:

Definition 3.51 *Regarding the two morphisms $i_X : \mathcal{P}_0(X) \to \Pi_\omega(X)$ and $f : \mathbf{B}G \to \mathbf{B}H$ as (images of) functors $i, f : I \to \omega\mathsf{Categories}(\mathsf{Spaces})$ on the interval category $I := \{a \to b\}$ we set*

$$\mathrm{hom}(i_X, f) := \mathrm{hom}\left(\begin{array}{c} \mathcal{P}_0(X) \\ \downarrow i \\ \Pi_\omega(X) \end{array}, \begin{array}{c} \mathbf{B}G \\ \downarrow f \\ \mathbf{B}H \end{array}\right) := \int_{a \in I} \mathrm{hom}(i_X(a), f(a)) \,.$$

Remark. An object in $\mathrm{hom}(i_X, f)$ is a commuting diagram

$$\begin{array}{ccc} \mathcal{P}_0(X) & \longrightarrow & \mathbf{B}G \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle f} \\ \Pi_\omega(X) & \longrightarrow & \mathbf{B}H \end{array} ,$$

a morphism is a transformation of the two vertical morphisms still fitting in such a diagram, etc.

Lemma 3.52 *This construction $\mathrm{hom}(i_X, f)$ is contravariantly functorial in X and hence extends to a presheaf $\mathrm{hom}(i_-, f) : \mathsf{Spaces}^{\mathrm{op}} \to \omega\mathsf{Categories}(\mathsf{Spaces})$.*

We can therefore consider descent for this presheaf of diagram morphisms.

Proposition 3.53 *Descent with respect to the diagram morphisms $\mathrm{hom}(i_-, f)$ is the same as diagram morphisms with respect to descent of the component presheaves:*

$$\mathrm{Desc}(Y^\bullet, \mathrm{hom}\left(\begin{array}{c} \mathcal{P}_0(X) \\ \downarrow i \\ \Pi_\omega(X) \end{array}, \begin{array}{c} \mathbf{B}G \\ \downarrow f \\ \mathbf{B}H \end{array}\right)) \simeq \int_{a \in I} \mathrm{Desc}(Y^\bullet, \mathrm{hom}(i_-(a), f(a))) \,.$$

Proof. Recall that the descent ω-category itself is defined by an end

$$\mathrm{Desc}(Y^\bullet, \mathrm{hom}\left(\begin{array}{c} \mathcal{P}_0(-) \\ \downarrow i \\ \Pi_\omega(-) \end{array}, \begin{array}{c} \mathbf{B}G \\ \downarrow f \\ \mathbf{B}H \end{array}\right)) \simeq \int_{[n] \in \Delta} \mathrm{hom}(\Pi_\omega(\Delta^n), \ \mathrm{hom}\left(\begin{array}{c} \mathcal{P}_0(Y^{[n+1]}) \\ \downarrow i \\ \Pi_\omega(Y^{[n+1]}) \end{array}, \begin{array}{c} \mathbf{B}G \\ \downarrow f \\ \mathbf{B}H \end{array}\right)) \,.$$

Inserting definition 3.51 and using that the covariant hom (being a right adjoint) preserves limits we obtain the double end

$$\cdots \simeq \int_{[n]\in\Delta} \mathrm{hom}(\Pi_\omega(\Delta^n), \int_{a\in I} \mathrm{hom}(i_{Y^{[n+1]}}(a), f(a))) \simeq \int_{[n]\in\Delta}\int_{a\in I} \mathrm{hom}(\Pi_\omega(\Delta^n),\ \mathrm{hom}(i_{Y^{[n+1]}}(a), f(a)))$$

Both ends on the total integrand exist separately, so that by the Fubini theorem for ends [Kelly] they may be interchanged:

$$\cdots \simeq \int_{a\in I}\int_{[n]\in\Delta} \mathrm{hom}(\Pi_\omega(\Delta^n),\ \mathrm{hom}(i_{Y^{[n+1]}}(a), f(a))) \simeq \int_{a\in I} \mathrm{Desc}(Y^\bullet, \mathrm{hom}(i_-(a), f(a)))\,.$$

$\square$

Remark. This means that $\mathrm{Desc}(Y^\bullet, \mathrm{hom}(i_-, f))$ is the pullback of

$$\begin{tikzcd} & \mathrm{Desc}(Y^\bullet, \mathrm{hom}(\Pi_\omega(-), \mathbf{B}H)) \arrow[d, "i^*"] \\ \mathrm{Desc}(Y^\bullet, \mathrm{hom}(\mathcal{P}_0(-), \mathbf{B}G)) \arrow[r, "f_*"] & \mathrm{Desc}(Y^\bullet, \mathrm{hom}(\mathcal{P}_0(-), \mathbf{B}H)) \end{tikzcd}$$

Abelian pseudoconnections. We write $(\mathbf{B^{n-1}}\mathbb{R} \to \mathbf{B^{n-1}U(1)})$ for the $(n+1)$-group whose corresponding crossed complex is

$$|\mathbf{B}(\mathbf{B^{n-1}}\mathbb{R} \to \mathbf{B^{n-1}U(1)})| = \left(\underbrace{\mathbb{R}}_{n+1} \xrightarrow{\mathrm{mod}\mathbb{Z}} \underbrace{U(1)}_{n} \xrightarrow{0} 0 \longrightarrow \cdots \rightrightarrows \underbrace{\mathrm{pt}}_{0} \right).$$

Notice that this is a cover of $\mathbf{BEB}^{n-1}U(1)$ which is

$$|\mathbf{BEB}^{n-1}U(1)| = \left(\underbrace{U(1)}_{n+1} \xrightarrow{\mathrm{Id}} \underbrace{U(1)}_{n} \xrightarrow{0} 0 \longrightarrow \cdots \rightrightarrows \underbrace{\mathrm{pt}}_{0} \right).$$

and let f now be the canonical inclusion

$$f : \mathbf{BB}^{n-1}U(1) \to \mathbf{B}(\mathbf{B^{n-1}}\mathbb{R} \to \mathbf{B^{n-1}U(1)})\,.$$

Definition 3.54 (abelian differential cohomology with pseudoconnections) *Abelian differential cohomology with pseudoconnections is cohomology with respect to* $\mathrm{hom}(i_-, f)$ *with* f *as above:*

$$\bar{H}_{\mathrm{pseudo}}(X, \mathbf{B}^n U(1)) := H(X, \mathrm{hom}\begin{pmatrix} \mathcal{P}_0(-) & & \mathbf{BB}^{n-1}U(1) \\ \downarrow i & , & \downarrow f \\ \Pi_\omega(-) & & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{pmatrix})$$

Proposition 3.55 *Descent over a surjective submersion* $Y \to X$ *of abelian pseudoconnections is an abelian* ω*-groupoid whose underlying crossed complex is canonically isomorphic to the kernel of* $\delta|_{\Omega^0}$ *of the complex of differential* $U(1)$*-valued forms on* Y*:*

$$|\mathrm{Desc}(Y^\bullet, \mathrm{hom}\begin{pmatrix} \mathcal{P}_0(-) & & \mathbf{BB}^{n-1}U(1) \\ \downarrow i & , & \downarrow f \\ \Pi_\omega(-) & & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{pmatrix})| = \ker(\delta|_{\Omega^0} : (\Omega^\bullet(Y^{[\bullet]}, U(1)), d\pm\delta)_{n+1} \to \Omega^0(Y^\bullet, U(1)))$$

Proof. For $n \leq 2$ this follows from [S-WaldorfII], as described below. The theorem has an obvious generalization to arbitrary n, but a detailed proof still needs to be written down.

Using definition 2.11 of [S-Waldorf-II] with the main theorem there, specialized to the case of the 2-group $\mathbf{B}(\mathbb{R} \to U(1))$ we get the following characterization of $\hom(\Pi_\omega(Y), \mathbf{B}(\mathbb{R} \to U(1)))$ (here and in the following we write $d : \Omega^0(Y, U(1)) := C^\infty(Y, U(1)) \to \Omega^1(Y)$ for $d = d_{\mathrm{dR}} \circ \log$):

- objects are given by differential forms $\lambda := (\underbrace{\lambda_1}_{\in\Omega^1(Y)}, \underbrace{d\lambda_1}_{\in\Omega^2(Y)})$;
- 1-morphisms $\rho : \lambda \to \lambda'$ are forms $(\underbrace{\lambda_0}_{\in\Omega^0(Y,U(1))}, \underbrace{\lambda_1}_{\in\Omega^1(Y)})$ such that

$$(\underbrace{\lambda_1'}_{\in\Omega^1(Y)}, \underbrace{d\lambda_1'}_{\in\Omega^2(Y)}) = (\underbrace{\lambda_1 + d\rho_0 + \rho_1}_{\in\Omega^1(Y)}, \underbrace{d\lambda_1 + d\rho_1}_{\in\Omega^2(Y)});$$

- 2-morphisms $\begin{tikzcd} \lambda \arrow[r, bend left, "\rho"] \arrow[r, bend right, "\rho'"'] \arrow[r, phantom, "\Downarrow\kappa"] & \lambda' \end{tikzcd}$ are given by $(\underbrace{\kappa_0}_{\in\Omega^0(Y,U(1))})$ such that

$$(\underbrace{\lambda_0'}_{\in\Omega^0(Y,U(1))}, \underbrace{\lambda_1'}_{\in\Omega^1(Y)}) = (\underbrace{\kappa_0 \cdot \lambda_0}_{\in\Omega^0(Y,U(1))}, \underbrace{\lambda_1 + d\kappa_0}_{\in\Omega^1(Y)}).$$

$\square$

Remark. This means that the objects in the descent ω-category are sequences

$$(\underbrace{\tilde{\omega}_0}_{\in C^\infty(Y^{[n+1]},U(1))}, \underbrace{\tilde{\omega}_1}_{\in\Omega^1(Y^{[n]})}, \cdots, \underbrace{\tilde{\omega}_n}_{\in\Omega^n(Y)}) \in \Omega^\bullet(Y^{[\bullet]}, U(1))$$

satisfying $\delta\tilde{\omega}_0 = 0$. Morphisms are coboundaries between the elements $(\underbrace{\tilde{\omega}_0}_{\in\Omega^0(Y^{[n+1]})}, \underbrace{d\omega_0 \pm \delta\omega_1}_{\in\Omega^1(Y^{[n+1]})}, \cdots, \underbrace{d\omega_n}_{\in\Omega^{n+1}(Y)})$

, etc.

Definition 3.56 *We have canonical morphisms*

$$\begin{tikzcd} \bar{H}_{\mathrm{pseudo}}(X, \mathbf{B}^n U(1)) \arrow[r, "c"] \arrow[d, "\lambda"] & \bar{H}_{\mathrm{flat}}(X, \mathbf{B}^{n+1}\mathbb{R}) \\ H^{n+1}(X, \mathbb{Z}) & \end{tikzcd},$$

the <u>*curvature*</u> c *and the* <u>*class*</u> λ *of a pseudoconnection:*

- *the class $c(\omega)$ is the class of ω_0 in Čech cohomology under the equivalence with $H^{n+1}(X, \mathbb{Z})$;*
- *the curvature is the push-forward under the canonical projection*

$$\hom\left(\begin{array}{ccc} \mathcal{P}_0(-) & & \mathbf{B}\mathbf{B}^{n-1}U(1) \\ \downarrow i & & \downarrow f \\ \Pi_\omega(-) & & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{array}\right) \to \hom(\Pi_\omega(-), \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1))) \to \hom(\Pi_\omega(-), \mathbf{B}^{n+1}\mathbb{R}).$$

Remark. The curvature map takes the $(d \pm \delta)$-closure of $\tilde{\omega}$:

$$c : (\underbrace{\tilde{\omega}_0}_{\in C^\infty(Y^{[n+1]},U(1))}, \underbrace{\tilde{\omega}_1}_{\in \Omega^1(Y^{[n]})}, \cdots, \underbrace{\tilde{\omega}_n}_{\in \Omega^n(Y)}) \mapsto (\underbrace{d\tilde{\omega}_0 + \delta\tilde{\omega}_1}_{\in \Omega^1(Y^{[n+1]})}, \cdots, \underbrace{d\tilde{\omega}_{n-1} \pm \delta\tilde{\omega}_n}_{\in \Omega^n(Y^{[2]})}, \underbrace{d\tilde{\omega}_n}_{\in \Omega^n(Y)}) .$$

Using that $\mathbb{R}$ is contractible we can further map the curvature cocycle to an honest curvature form along

$$\bar{H}_{\mathrm{flat}}(X, \mathbf{B}^{n+1}\mathbb{R}) \xrightarrow{\simeq} \Omega^{n+1}_{\mathrm{flat}}(X) \; .$$

More concretely, we have

Lemma 3.57 *For* $\tilde{\omega} \in \mathrm{Desc}(Y, \mathrm{hom}\begin{pmatrix} \mathcal{P}_0(-) & & \mathbf{B}\mathbf{B}^{n-1}U(1) \\ \downarrow i & , & \downarrow f \\ \Pi_\omega(-) & & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{pmatrix})$ *as above, the curvature* $c(\tilde{\omega}) \in \mathrm{Desc}(Y, \mathrm{hom}(\Pi_\omega(-), \mathbf{B}^{n+1}\mathbb{R}))$ *is cohomologous to a closed* $(n+1)$*-form on* X *under the canonical inclusion* $\Omega^{n+1}_{\mathrm{flat}}(X) \hookrightarrow \mathrm{Desc}(Y, \mathrm{hom}(\Pi_\omega(-), \mathbf{B}^{n+1}\mathbb{R}))$.

Proof. For every Čech cocycle $\omega_0 := \tilde{\omega}_0$ we can find a Deligne cocycle lifting it, i.e. $\omega = (\underbrace{\omega_0}_{\in \Omega^0(Y^{[n+1]},U(1))}, \cdots, \underbrace{\omega_n}_{\in \Omega^n(Y)})$ with curvature $(d \pm \delta)\omega = (0, \cdots, 0, \underbrace{d\omega_n}_{\in \Omega^{n+1}(Y)}) = (0, \cdots, 0, \delta \underbrace{F_{n+1}}_{\in \Omega^{n+1}(X)})$. Using this one rewrites

$$c(\tilde{\omega}) = (\underbrace{d\tilde{\omega}_0 + \delta\tilde{\omega}_1}_{\in \Omega^1(Y^{[n+1]})}, \cdots, \underbrace{d\tilde{\omega}_{n-1} \pm \delta\tilde{\omega}_n}_{\in \Omega^n(Y^{[2]})}, \underbrace{d\tilde{\omega}_n}_{\in \Omega^n(Y)})$$

as

$$\cdots = (0, \cdots, 0, \underbrace{d\omega_n}_{\in \Omega^{n+1}(Y)}) + (d \pm \delta)(\underbrace{\tilde{\omega}_1 - \omega_1}_{\in \Omega^1(Y^{[n]})}, \cdots, \underbrace{\tilde{\omega}_n - \omega_n}_{\in \Omega^n(Y)}) ,$$

where in the first component one uses $\delta\omega_1 = -d\omega_0 = -d\tilde{\omega}_0$, the first equality sign being the cocycle condition on ω, the second being the fact that $\omega_0 = \tilde{\omega}_0$. Beware that the differential on 0-forms is $d_{\mathrm{dR}} \circ \log$. $\square$

Remark. Notice that in particular the case where $\tilde{\omega}_0 = \omega_0$ is possibly nontrivial but all higher degree forms vanish, $\tilde{\omega}_k = 0$ for $k \geq 1$, corresponding to a Čech cocycle with "vanishing pseudoconnection". Then the above statement reduces to the observation that in $\mathrm{Desc}(Y, \mathrm{hom}(\Pi_\omega(-), \mathbf{B}^{n+1}\mathbb{R}))$ the "canonical 1-form on the fibers" of a Čech cocycle, i.e. $(\underbrace{d\omega_0}_{\in \Omega^1(Y^{[n+1]})}, 0, \cdots, 0)$ is cohomologous to the curvature of any proper connection carried by the cocycle, i.e. to $(0, \cdots, 0, \underbrace{d\omega_n}_{\in \Omega^{n+1}(Y)})$:

$$(\underbrace{d\omega_0}_{\in \Omega^1(Y^{[n+1]})}, 0, \cdots, 0) = (0, \cdots, 0, \underbrace{d\omega_n}_{\in \Omega^{n+1}(Y)}) + (d \pm \delta)(\underbrace{-\omega_1}_{\in \Omega^1(Y^{[n]})}, \cdots, \underbrace{-\omega_n}_{\in \Omega^n(Y)}) .$$

At the "rationalized" level of L_∞-algebra connections the analogous statement is discussed in section 7.1.1 of [SatiSchreiberStasheff-I].

We can rephrase this in the following useful ways.

Corollary 3.58

- *The diagram*

$$\begin{tikzcd} \bar{H}_{\mathrm{pseudo}}(X, \mathbf{B}^n U(1))/_\sim \arrow[r, "c"] \arrow[d, "\lambda"'] & \Omega^{n+1}_{\mathrm{closed}}(X) \arrow[d] \\ H^{n+1}(X, \mathbb{Z}) \arrow[r] & H^{n+1}(X, \mathbb{R}) \end{tikzcd}$$

is a pullback square (in the category of abelian groups).

- *Every cocycle* $\tilde{\omega} \in \mathrm{Desc}(Y^\bullet, \mathrm{hom}\left(\begin{array}{cc} \mathcal{P}_0(-) & \mathbf{B}\mathbf{B}^{n-1}U(1) \\ \downarrow i \quad , & \downarrow f \\ \Pi_\omega(-) & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{array} \right))$ *which, by proposition* **??** *and proposition 3.53, we can represent as a a diagram*

$$\begin{tikzcd} \mathrm{Codesc}(Y, \mathcal{P}_0) \arrow[r] \arrow[d] & \mathbf{B}^n U(1) \arrow[d] \\ \mathrm{Codesc}(Y, \Pi_\omega) \arrow[r] & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \end{tikzcd}$$

is cohomologous to one which can be extended to a diagram

$$\begin{tikzcd} \mathrm{Codesc}(Y, \mathcal{P}_0) \arrow[r] \arrow[d] & \mathbf{B}^n U(1) \arrow[d] & \textit{cocycle} \\ \mathrm{Codesc}(Y, \Pi_\omega) \arrow[r] \arrow[d] & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1)) \arrow[d] & \textit{connection and curvature} \\ \Pi_\omega(X) \arrow[r] & \mathbf{B}^{n+1}\mathbb{R} & \textit{characteristic form (curvature)} \end{tikzcd}$$

.

3.4.2 The generalized Chern-Weil homomorphism

[** the following text is a placeholder **]

At the linearized level we know from [132] that the differential refinement of the cocycle is an extension to a diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\ \Omega^\bullet(Y) \arrow[u] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A, F_A)}"'] \arrow[u] \end{tikzcd}$$

which corresponds to choosing a connection A. We want to see that the invariant forms obtained from the curvature F_A of this connection reproduce in deRham cohomology the integral characteristic class just discussed. At the linearized level this comes from observing that we can extend to the larger diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \arrow[l, "\mu"'] \\ \Omega^\bullet(Y) \arrow[u] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A, F_A)}"'] \arrow[u] & \mathrm{W}(b^{n-1}\mathfrak{u}(1)) \arrow[l, "{(\mathrm{cs}, P)}"'] \arrow[u] \end{tikzcd} ,$$

where P is the invariant polynomial corresponding to μ and where cs is the correspponding transgression element ("Chern-Simons element"). So $P(A, F_A)$ here is a closed $(n+1)$-form on Y which is supposed to descend down to X where it represents the deRham image of our characteristic class. We expect (details to be described below) that we can ∞-Lie integrate this to a differential refinement in nonabelian cohomology.

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & \mathbf{B}G \arrow[r] \arrow[d] & \mathbf{B}^n U(1) \arrow[d] \\
\Pi_\omega(X) \arrow[r] & \mathbf{BE}G \arrow[r] & \mathbf{B}(\mathbf{B}^{n-1}\mathbb{R} \to \mathbf{B}^{n-1}U(1))
\end{tikzcd}
.
$$

Here the outermost square represents, and this is one of the main aspects below, a cocycle in $(n+1)$st Deligne cohomology where the shifted part sitting in $\mathbf{B}^{n+1}\mathbb{R}$ picks up the curvature $(n+1)$-form. By the general fact about Deligne cohomology this form represents our cocycle in deRham cohomology and is hence the corresponding characteristic form of our original cocycle g.

[** – **]

4 ∞-Lie theory

Ordinary Lie theory with its relation between Lie groups and Lie algebras by Sophus Lie should generalize to a relation between smooth ∞-groups and L_∞-algebras. Several aspects of this have appeared in the literature [140, 60, 68, 141].

We interpret this as saying that smooth ω-groupoids and L_∞-algebroids are naturally connected by a sequence of two adjunctions, as shown in figure 7. The first of these adjunctions, relating Spaces to L_∞Algebroids is essentially that known from rational homtopy theory (e.g. page 9 of [70]), mediated by the object $\Omega^\bullet$ of (dual) "infinitesimal paths". The second adjunction, relating Spaces to ωCategories(Spaces) is mediated instead by the object Π_ω of finite paths. Therefore both adjunctions are induced by *ambimorphic objects* and hence constitute examples of the general notion of *Stone duality* [156].

$$
\begin{tikzcd}[column sep=large]
\omega\mathsf{Groupoids} \arrow[r, shift left, "K(-)=\mathrm{Hom}(\Pi_\omega(-),-)"] \arrow[rrr, bend left, "\textbf{∞-Lie differentiation}"] & \mathsf{Spaces} \arrow[l, shift left, "\Pi_\omega(-)=\mathrm{Hom}(C^\infty(-),\Pi_\omega)"] \arrow[r, shift left, "\Omega^\bullet(-)=\mathrm{Hom}(-,\Omega^\bullet)"] \arrow[d, phantom, "\Downarrow"] & C^\infty\mathsf{qDGCAs} \arrow[l, shift left, "S(-)=\mathrm{Hom}(-,\Omega^\bullet(-))"] & L_\infty\mathsf{Algebroids} \arrow[l, "\mathrm{CE}(-)"', "\simeq"] \arrow[lll, bend left, "\textbf{∞-Lie integration}"] \\
& {} & &
\end{tikzcd}
$$

Figure 7: ∞-**Lie theory** – the relation between ∞-groupoids (ω-groupoids in our case) and L_∞-algebroids – arises from the relation between finite and infinitesimal k-paths in Spaces. For every space X there is its fundamental ω-groupoid $\Pi_\omega(X)$ whose k-morphisms are classes of certain images of the standard k-disk in X. Linear approximation to such k-paths are, dually, degree k differential forms in the deRham complex $\Omega^\bullet(X)$, which is the Chevalley-Eilenberg qDGCA of the tangent Lie algebroid TX. Conversely, every ω-groupoid C gives rise to its classifying space $|C|$ and every qDGCA A to the classifying space $S(A)$ of A-valued differential forms. Moreover, C^∞qDGCAs are precisely the Chevalley-Eilenberg algebras of L_∞-algebroids. Thus passing from right to left through the above diagram is ∞-Lie integration of L_∞-algebroids to ω-groupoids. Passing from left to right is ∞-Lie differentiation.

4.1 Infinitesimal paths

In the context of the "differential homotopy relation", spaces are related to L_∞-algebroids of infinitesimal paths in these spaces.

4.1.1 Fundamental L_∞-algebroid

Definition 4.1 (fundamental/tangent L_∞-algebroid) *For $X \in \mathsf{Spaces}$ the fundamental Lie algebroid or tangent Lie algebroid TX of X is that whose Chevalley-Eilenberg-algebra is the qDGCA of differential forms on X, from definition 2.89:*

$$\mathrm{CE}(TX) := \Omega^\bullet(X)\,.$$

Definition 4.2 (classifying space of flat $\mathfrak{g}$-valued forms) *Given any DGCA A we obtain a sheaf $S(A) \in \mathsf{Spaces}$ given by the assignment*

$$S(A) : U \mapsto \mathrm{Hom}(A, \Omega^\bullet(U))$$

for all $U \in \mathsf{Euclid}$. This extends to a contravariant functor $S : \mathrm{DGCA}s \to \mathsf{Spaces}$.

Remark. In rational homotopy theory this corresponds to the map in definition 1.22 of [70].

Definition 4.3 (flat L_∞-valued forms) *When A is the Chevalley-Eilenberg algebra of an L_∞-algebroid $\mathfrak{g}$, $A = \mathrm{CE}(\mathfrak{g})$, and for Y a space we call*

$$\Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g}) := \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))$$

the set of flat $\mathfrak{g}$-valued forms.

For more details see [132].

4.1.2 The adjunction between Spaces and L_∞Algebroids

Definition 4.4 (infinitesimal path object) *We call the sheaf $\Omega^\bullet$ given by $\Omega^\bullet : U \mapsto \Omega^\bullet(U)$ the infinitesimal path object.*

Remark. $\Omega^\bullet$ is an *ambimorphic object* [156] in that it is both a sheaf as well as a qDGCA in a compatible way. In this sense the "differential homotopy relation" is a special case of general Stone duality [83] induced by ambimorphic objects.

Theorem 4.5 ("differential homotopy relation") *The contravariant functors*

$$\mathsf{Spaces} \underset{S(\cdot)}{\overset{\Omega^\bullet(\cdot)}{\rightleftarrows}} \mathrm{DGCA}s$$

form a contravariant adjunction whose unit $\mathrm{Id}_{\mathrm{DGCA}s} \to \Omega^\bullet(S(\cdot))$ has, as component map, the canonical inclusion $A \hookrightarrow \Omega^\bullet(S(A))$ for all DGCAs A given by $(a \in A) \mapsto (\forall U \in \mathsf{Euclid} : f \in S(A)(U) \mapsto f(a))$.

We are indebted to Todd Trimble for discussion of this statement.

Classifying space of flat $\mathfrak{g}$-valued forms. Using this adjunction together with definition 4.3 we find that, for $\mathfrak{g}$ an L_∞-algebroid, $S(\mathrm{CE}(\mathfrak{g}))$ is the *classifying space of flat $\mathfrak{g}$-valued forms* in the sense that maps from any space Y into it are in bijection with flat $\mathfrak{g}$-valued forms:

$$\mathrm{Hom}(Y, S(\mathrm{CE}(\mathfrak{g}))) \simeq \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y)) =: \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g})\,.$$

4.2 Finite paths

In the context of the "smooth homotopy relation" spaces are related to ω-groupoids of paths in these spaces.

4.2.1 Fundamental ω-groupoid

To every space $X \in \mathsf{Spaces}$ we assign its *smooth fundamental ω-groupoid* $\Pi_\omega(X)$ whose k-morphisms are (classes of) k-dimensional "paths" in X, namely (classes of)images of the standard k-disk in X. The equivalence relation divided out in each degree is not homotopy as in [27], but *thin homotopy* as in [11, 136, 137, 138, 106], the relation under which parallel transport on smooth spaces is invariant. The corresponding crossed complex $[\Pi_\omega(X)]$ has in degree 0 and 1 the 1-groupoid of thin homtopy classes of paths in a smooth space from [137] and in degree $k \geq 2$ a bundle of groups, over X, of classes of based $(k-1)$-dimensional spheres filled by based by based k-dimensional balls with group composition being gluing at the base point.

Definition 4.6 (path space) *Fix once and for all $\epsilon \in (0, \frac{1}{4}) \subset \mathbb{R}$. For each $t \in \mathbb{R}$, $t \geq 0$ denote by $I_t := (-\epsilon, 1+\epsilon) \subset \mathbb{R}$ the $\underline{\epsilon\text{-extended interval}}$ of length t and $C := (-\epsilon, \epsilon) \subset \mathbb{R}$ be the standard collar, equipped, for each t, with two injections* $C \underset{\mathrm{out}_t}{\overset{\mathrm{in}_t}{\rightrightarrows}} I_t$ *given by $\mathrm{in}_t : s \mapsto s$ and $\mathrm{out}_t : s \mapsto t + s$. For $X \in \mathsf{Spaces}$ and all t, we have the parameterized path space $P(t)X := \mathrm{hom}(I_t, X)$.*

To concatenate such paths a bit of overlap has to be enforced. Since in the end the parameterization of the paths will be divided out, the usual choice is to force all paths to be constant on their collars.

Definition 4.7 (paths with sitting instant/constant collars) *For $X \in \mathsf{Spaces}$ let $P_{\mathrm{sit}}(t)X$, the space of paths with sitting instants at their boundary or paths with constant collars, be the pullback*

$$\begin{tikzcd} \mathrm{hom}(\mathrm{pt}, X) \arrow[d] & P_{\mathrm{sit}}(t)X \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \mathrm{hom}(\mathrm{pt}, X) \arrow[d] \\ \mathrm{hom}(C, X) & P(t)X \arrow[l, "\mathrm{in}_t^*"'] \arrow[r, "\mathrm{out}_t^*"] & \mathrm{hom}(C, X) \end{tikzcd}.$$

Definition 4.8 (composition of paths) *For all pairs t_1, t_2 let $I_{t_1} \overset{l}{\hookrightarrow} I_{t_1+t_2} \overset{r}{\hookleftarrow} I_{t_2}$ be given by $l : s \mapsto s$ and $r : s \mapsto s + t_1$. Then the diagrams*

$$\begin{tikzcd} C \arrow[r, "\mathrm{in}"] \arrow[drr, "\mathrm{in}"'] & I_{t_1} \arrow[dr, "l"] & C \arrow[l, "\mathrm{out}"'] \arrow[r, "\mathrm{in}"] & I_{t_2} \arrow[dl, "r"'] & C \arrow[l, "\mathrm{out}"'] \arrow[dll, "\mathrm{out}"] \\ & & I_{t_1+t_2} & & \end{tikzcd}$$

commute. In particular $\{l, r\}$ is a cover of $I_{t_1+t_2}$ so that the pullback

$$\begin{tikzcd} P(t_1, t_2)X \arrow[r, "l^*"] \arrow[d, "r^*"'] & P(t_1)X \arrow[d, "\mathrm{out}^*"] \\ P(t_2)X \arrow[r, "\mathrm{in}^*"'] & \mathrm{hom}(C, X) \end{tikzcd}$$

is a subspace $P(t_1, t_2)X \hookrightarrow P(t_1 + t_2)X$. The universal property of this pullback yields the dotted morphism *in*

$$\begin{tikzcd}
 & & P_{\mathrm{sit}}(t_1)X_t \times_s P_{\mathrm{sit}}(t_2)X \arrow[dl] \arrow[dr] \arrow[ddd, dotted, bend right] & & \\
\hom(\mathrm{pt}, X) \arrow[d] & P_{\mathrm{sit}}(t_1)X \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \hom(\mathrm{pt}, X) \arrow[d] & P_{\mathrm{sit}}(t_2)X \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \hom(\mathrm{pt}, X) \arrow[d] \\
\hom(C, X) & P(t_1)X \arrow[l, "\mathrm{in}^*"'] \arrow[r, "\mathrm{out}^*"] & \hom(C, X) & P(t_2)X \arrow[l, "\mathrm{in}^*"'] \arrow[r, "\mathrm{out}^*"] & \hom(C, X) \\
 & & P(t_1, t_2)X \arrow[ul, "l^*"] \arrow[ull, "\mathrm{in}^*"] \arrow[ur, "r^*"'] \arrow[urr, "\mathrm{out}^*"'] \arrow[d, hookrightarrow] & & \\
 & & P(t_1 + t_2)X & &
\end{tikzcd}$$

which in turn yields, by the pullback property of $P_{\mathrm{sit}}(t)X$ in definition 4.7, a universal morphism

$$\mathrm{conc} : P_{\mathrm{sit}}(t_1)X_t \times sP_{\mathrm{sit}}(t_2)X \to P_{\mathrm{sit}}(t_1 + t_2)X\,.$$

This is the <u>composition of paths</u> with sitting instants.

Proposition 4.9 *This composition is associative and unital. The units are the constant paths (those that arise as pullback along maps to the point).*

Definition 4.10 *Denote by $P_1(X) \in \omega\mathsf{Categories}(\mathsf{Spaces})$ the corresponding <u>parameterized path 1-category</u>.*

To obtain a groupoid of paths and higher groupoids of paths, we concretize and divide out by equivalence relations.

Definition 4.11 (thin homotopy) *For $X \in \mathsf{ConcreteSpaces}$ define recursively the space of n-paths in X with sitting instant to be*

$$P^0X = X\,,$$

$$P^1X := P^1_{\mathrm{sit}}X$$

and

$$P^nX := \bigcup_{s,t \in P^{n-2}X} P_{\mathrm{sit}}(P^{n-1}_{s \to t}X)\,.$$

Notice that these are natually subspaces of spaces of maps from I^n to X, $P^n_{\mathrm{sit}}X \hookrightarrow \hom(I^n, X)$, and in fact naturally subspaces of spaces of maps from the n-disk D^n $P^n_{\mathrm{sit}}X \hookrightarrow \hom(D^n, X)$. On P^nX consider the equivalence relation $\sim_{\mathrm{thin}}$ which considers two n-paths $\gamma_1, \gamma_2 : I^n \to X$ as equivalent precisely if there is a concrete $(n+1)$-path $\Sigma : I^{(n+1)} \to X$ starting at γ_1 and ending at γ_2 such that all $(n+1)$-forms on X vanish when pulled back along Σ:

$$(\gamma_1 \sim_{\mathrm{thin}} \gamma_2) \iff \left(\exists\ \gamma_1 \xrightarrow{\Sigma} \gamma_2 \ : \begin{tikzcd} \Omega^{n+1}(X) \arrow[rr, "\Sigma^*"] \arrow[dr] & & \Omega^{n+1}(D^{n+1}) \\ & \mathrm{pt} \arrow[ur, "0"'] & \end{tikzcd} \right).$$

Write $P^n_{\mathrm{thtpy}}X := (P^nX)/{\sim_{\mathrm{thin}}}$ for the concrete <u>space of thin-homotopy classes of n-paths</u>.

Here $(\cdot)/\sim_{\mathrm{thin}}$ is the quotient operation on spaces from definition 2.8.

Remark. The definition of thin homotopy classes of paths in concrete spaces in terms of vanishing conditions on pulled back forms is due to [137].

Proposition 4.12 *Under the composition operations inherited from the degree-wise composition of paths, the globular space*

$$\cdots \qquad P^2_{\mathrm{thtpy}}X \underset{t}{\overset{s}{\rightrightarrows}} P^1_{\mathrm{thtpy}}X \underset{t}{\overset{s}{\rightrightarrows}} X$$

becomes an ω-groupoid $\Pi_\omega(X)$.

Proof. All compositions are associative and unital by construction. The nontrivial part is to check that all the exchange laws holds. By theorem 2.36 it is sufficient to check that the complex $[\Pi_\omega(X)]$ from definition 2.35 satisfies the axioms of a crossed complex. It is immediate that in degree 0 and 1 we have a 1-groupoid and that $[\Pi_\omega(X)]_k$ for $k \geq 2$ are bundles of groups over X and bundles of abelian groups for $k \geq 3$. It is also straightforward to to check that the action on $[\Pi_\omega(X)]_{k\geq 3}$ by $[\Pi_\omega(X)]_1$ is compatible with the δ-maps. The only nontrivial point is the compatibility of the action of $[\Pi_\omega(X)]_1$ on $[\Pi_\omega(X)]_2$. This follows using thin homotopy invariance as in [137]. □

Definition 4.13 (smooth fundamental ω-groupoid) *The ω-groupoid $\Pi_\omega(X)$ obtained this way for each $X \in \mathsf{Spaces}$ is the smooth fundamental ω-groupoid of X. Its truncation at degree n is the nth smooth path ω-groupoid $\mathcal{P}_n(X)$. The quotient by n-equivalences is the smooth fundamental n-groupoid $\Pi_n(X)$.*

For $f : X \to Y$ a morphisms of spaces we obtain an obvious ω-functor $f_* : \Pi_\omega(X) \to \Pi_\omega(Y)$. This yields an ω-groupoid valued co-presheaf $\Pi_\omega : \mathsf{Spaces} \to \omega\mathsf{Groupoids}(\mathsf{Spaces})$ which we address as the object of finite paths.

In particular

- $\Pi_1(X)$ for X a manifold is the ordinary fundamental groupoid of a manifold;
- $\mathcal{P}_0(X)$ is the discrete ω-category over (the concretization of) X;
- $\Pi_0(X)$ is the discrete ω-category over the space of connected components of (the concretization of) X;
- $\mathcal{P}_1(X)$ is the path 1-groupoid appearing in [136, 137];
- $\mathcal{P}_2(X)$ for X a manifold is the path 2-groupoid appearing in [11, 137, 138].

Remark. A closely related but different notion of a fundamental ω-groupoid of homotopy classes of paths in a filtered topological space is given in [31, 27], see the monograph [34]: there homotopy (relative vertices) is divided out, whereas here only *thin* homotopy is divided out.

Proposition 4.14 *The smooth fundamental ω-groupoid is the coend*

$$\Pi_\omega(X) = \hom(C^\infty(X), \Pi_\omega) := \int^{U \in \mathsf{CartesianSpaces}} \hom(C^\infty(X), C^\infty(U)) \cdot \Pi_\omega(U) = \int^{U \in \mathsf{CartesianSpaces}} X(U) \cdot \Pi_\omega(U)\,,$$

where the extraordinarily conatural family of morphisms

$$i_U : \; X(U) \cdot \Pi_\omega(U) \longrightarrow \Pi_\omega(X)$$

is given by the co-presheaf propery of Π_ω.

Proof. [** still needs details – we may want to turn this around and take the coend formula as the definition and then prove that it coincides with the above direct construction **] □

Realization of ωGroupoids as Spaces. The fundamental ω-groupoid of a space is obtained by mapping *into* the ambimorphic object Π_ω. The dual operation obtained by mapping *out of* Π_ω yields a notion of spatial realization of ω-groupoids.

Definition 4.15 (spatial realization of ω-categories) *For $C \in \omega\mathsf{Categories}(\mathsf{Spaces})$, the space $K(C)$ is the sheaf given by $K(C) : U \mapsto \mathrm{hom}(\Pi_\omega(U), C)$. This construction is clearly functorial*

$$K(-) : \omega\mathsf{Categories}(\mathsf{Spaces}) \to \mathsf{Spaces}\,.$$

Remark. The operation $K(-)$ is similar to but different from the familiar geometric realization $|-|$: $\omega\mathsf{Categories}(\mathsf{Sets}) \to \mathsf{TopologicalSpaces}$. For G a topological 1- or 2-group it is known [13] that

$$|\mathbf{B}G| \simeq B|G|\,,$$

where on the right we have the ordinary topological classifying space of the topological group $|G|$. On the other hand, for G a smooth 1- or 2-group we show in section 5.3 that $K(G)$ is a smooth model for a $K(G,1)$ and $K(G,1,2)$, where $K(G,1)$ is such that $\Pi_1(K(G,1)) = \mathbf{B}G$.

[** But this "$K(G,1)$" in general does have higher homotopy groups, so the notation is still not really good. What would be the best suited and most suggestive notation for the above operation $\omega\mathsf{Groupoids}(\mathsf{Spaces}) \to \mathsf{Spaces}$ **]

4.2.2 The adjunction between ωGroupoids and Spaces

Proposition 4.16 (adjunction between spaces and ω-categories) *The functors*

$$\omega\mathsf{Categories}(\mathsf{Spaces}) \underset{\Pi_\omega = \mathrm{hom}(C^\infty(-),\Pi_\omega)}{\overset{K(-)=\mathrm{hom}(\Pi_\omega(-),-)}{\rightleftarrows}} \mathsf{Spaces}$$

form an adjunction with $\Pi_\omega(-)$ left adjoint to $K(-)$, for $X \in \mathsf{Spaces}$ and $C \in \omega\mathsf{Categories}(\mathsf{Spaces})$ we naturally have

$$\mathrm{Hom}(X, K(C)) \simeq \mathrm{Hom}(\Pi_\omega(X), C)\,.$$

Proof. Using the coend characterization of $\Pi_\omega(X)$ from proposition 4.14 this amounts to a standard computation for classifying spaces, compare for instance proposition 10.4.9 in [34]: by the end-expression for natural transformations (proposition 6.22) we have

$$\mathrm{Hom}(X, K(X)) \simeq \int_{U \in \mathsf{CartesianSpaces}} \mathrm{Hom}(X(U), K(X)(U))\,.$$

Plugging in the definition of $K(X)$ and then using the Hom-adjunction this is

$$\cdots \simeq \int_{U \in \mathsf{CartesianSpaces}} \mathrm{Hom}(X(U), \mathrm{Hom}(\Pi_\omega(U), C)) \simeq \int_{U \in \mathsf{CartesianSpaces}} \mathrm{Hom}(X(U) \cdot \Pi_\omega(U), C)\,.$$

The contravariant Hom takes colimits to limits (lemma 6.12)

$$\cdots \simeq \mathrm{Hom}\left(\left(\int^{U \in \mathsf{CartesianSpaces}} X(U) \cdot \Pi_\omega(U)\right), C\right)$$

and using proposition 4.14 this is the desired result

$$\cdots \simeq \mathrm{Hom}(\Pi_\omega(X), C)\,.$$

□

Lemma 4.17 *The unit of this adjunction on $X \in \mathsf{Spaces}$ is a canonical inclusion $X \hookrightarrow K(\Pi_\omega(X))$ given by sending for each $U \in \mathsf{CartesianSpaces}$ elements $f \in X(U) \simeq \mathrm{Hom}(U, X)$ to $f_* \in K(\Pi_\omega(X))(U) = \mathrm{Hom}(\Pi_\omega(U), \Pi_\omega(X))$.*

Lemma 4.18 *For all $X, Y \in \mathsf{ConcreteSpces}$, the map $\mathrm{Hom}(\Pi_\omega(X), \Pi_\omega(Y)) \to \mathrm{Hom}(X, Y)$, obtained by restricting ω-functors to their degree 0 component, is an isomorphism. Hence $\Pi_\omega : \mathsf{CartesianSpaces} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ is faithful.*

Proof. [** roughly **] Consider an ω-functor $F : \Pi_\omega(X) \to \Pi_\omega(Y)$ with degree 0-component $F_0 : X \to Y$. Let $\Sigma : D^k \to X$ represent a k-morphism in $\Pi_\omega(X)$. The claim is that for all $\sigma \in D^k$ this already fixes $F(\Sigma)(\sigma) \in Y$. To see this consider any decomposition of Σ into a pasting diagram of k-morphisms such that σ sits on a vertex of this pasting diagram. Then the image of σ is bound to be $F_0(\sigma)$. □

[** can this be generalized to general Spaces by using that every sheaf is a colimit of representables? **]

Corollary 4.19 *Every concrete space $X \in \mathsf{ConcreteSpaces}$ is isomorphic to the spatial realization of its fundamental ω-groupoid: $X \simeq K(\Pi_\omega(X))$.*

Proof. By the definition 4.15 of the spatial realization, for $U \in \mathsf{CartesianSpaces}$

$$K(\Pi_\omega(X)) : U \mapsto \mathrm{Hom}(\Pi_\omega(U), \Pi_\omega(X))\,.$$

By lemma 4.18 the right side is $\cdots \simeq \mathrm{Hom}(U, X)$ and by Yoneda, theorem 6.26, this is $\cdots \simeq X(U)$. □

4.3 ∞-Lie integration and ∞-Lie differentiation

We combine the "smooth homotopy relation" with the "differential homotopy relation" to relate $L_\infty\mathsf{Algebroids}$ with $\omega\mathsf{Groupoids}(\mathsf{Spaces})$. See figure 7.

Definition 4.20 (∞-Lie integration and differentiation)

- *∞-Lie integration is the functor* $\mathrm{concretize} \circ \Pi_\omega \circ S \circ \mathrm{CE} : L_\infty \to \omega\mathsf{Groupoids}(\mathsf{ConcreteSpaces})$.
- *∞-Lie differentiation is the functor $\Omega^\bullet \circ |\cdot| : \omega\mathsf{Groupoids}(\mathsf{ConcreteSpaces}) \to \mathrm{DGCA}s$.*

Remark. The general idea of this perspective on Lie integration is sketched at the beginning of [140]. On the other hand, it is essentially nothing but the principle of the *Sullivan construction* in rational homotopy theory [153]. This was made explicit in [60]. In [60] and [68] this integration procedure is considered for the case of L_∞-algebras using not strict but weak ∞-categories and concentrating on the task of factoring the construction through $\mathsf{Manifolds}$ or $\mathsf{BanachSpaces}$, respectively. A prescription for ∞-Lie differentiation in this context is given in [141]. It seems that it amounts essentially to the above prescription.

Definition 4.21 (weak simplicial ∞-Lie integration) *An alternative model to the fundamental ω-groupoid of a space is the singular simplicial complex $\Pi_\infty : \mathsf{Spaces} \to \mathsf{KanComplexes}(\mathsf{Spaces})$ whose space of k-simplices is the collection of singular k-simplices in the space $(\Pi_\infty(X))^k = \mathrm{hom}(\Delta^k, X)$, where $\Delta^k \subset \mathbb{R}^k$ denotes the standard k-simplex. Using this instead of Π_ω in definition 4.20 yields the weak simplicial Lie integration*

$$\Pi_\infty \circ S \circ \mathrm{CE} : \mathsf{Spaces} \to \mathsf{KanComplexes}(\mathsf{Spaces})\,.$$

Using the Yoneda lemma, theorem 6.26, once we find that for $\mathfrak{g}$ some L_∞-algebra, the space of k-simplices of its weak simplicial integration is

$$(\Pi_\infty(S(\mathrm{CE}(\mathfrak{g}))))^k = \mathrm{hom}(\Delta^k, S(\mathrm{CE}(\mathfrak{g}))) = \mathrm{hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(\Delta^k)) = \Omega^\bullet_{\mathrm{flat}}(\Delta^k, \mathfrak{g})\,,$$

the space of flat $\mathfrak{g}$-valued forms on the standard k-simplex. This is indeed the algorithmic prescription used in [60, 68]. The bulk of [60] is concerned with factoring this general procedure through $\mathsf{Manifolds}$. The bulk of [68] is concerned with factoring this general procedure through $\mathsf{BanachSpaces}$.

Some examples.

Proposition 4.22 (∞-Lie integration of the tangent Lie algebroid) *For X a manifold, the ∞-Lie integration of its fundamental (tangent) Lie algebroid is its fundamental ω-groupoid*

$$\Pi_\omega(S(\mathrm{CE}(TX))) = \Pi_\omega(S(\Omega^\bullet(X))) = \Pi_\omega(X)\,.$$

Proof. This follows directly from the fact (for instance [116]) that the contravariant functor

$$\Omega^\bullet(-) : \mathsf{Manifolds} \to \mathrm{DGCAs}$$

is full and faithful. □

The following statement generalizes the main theorem of [137] from $n = 2$ to $n = \infty$:

Proposition 4.23 *Let G be an ω-group obtained from ∞-Lie integrating the L_∞-algebra $\mathfrak{g}$, $\mathbf{B}G = \Pi_\omega \circ S \circ \mathrm{CE}(\mathfrak{g})$. Then smooth ω-functors $\Pi_\omega(Y) \to \mathbf{B}G$ are in bijection with flat $\mathfrak{g}$-valued differential forms on X*

$$\mathrm{Hom}(\Pi_\omega(Y), \mathbf{B}G) \simeq \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g})\,.$$

Proof. By assumption $\mathrm{Hom}(\Pi_\omega(Y), \mathbf{B}G) = \mathrm{Hom}(\Pi_\omega(Y), \Pi_\omega \circ S \circ \mathrm{CE}(\mathfrak{g}))$. Then by lemma 4.18 $\cdots \simeq \mathrm{Hom}(Y, S \circ \mathrm{CE}(\mathfrak{g}))$. And finally by definition $\cdots = \Omega^\bullet_{\mathrm{flat}}(Y, \mathfrak{g})$. □

Relation of model structure under integration differentiaton. [** discussion goes here on how integration/differentiation acts on the fibrations/cofibrations/weak equivalences on both sides **]

4.4 L_∞-algebraic cocycles

The ∞-Lie integration procuedure of section 4.3 is functorial and can hence also be applied to integrate morphisms of DGCAs to ω-functors of ω-groupoids. Cohomology cocycles in the sense of section 3 are ω-functors out of the codescent object from section 3.1.3 into the structure ω-group. We can therefore look for examples of cocycles that arise from L_∞-integration of morphisms from L_∞-algebroids to some L_∞-algebra. Such L_∞-algebraic (rational) approximations to full cocycles were considered in [132] (there also generalized to *differential* cocycles, see section 3.3.4).

Definition 4.24 (vertical forms) *For $\pi : Y \to X$ a smooth map, the DGCA $\Omega^\bullet_{\mathrm{vert}}(Y)$ of vertical forms on Y with respect to π is the quotient of the full deRham DGCA $\Omega^\bullet(Y)$ by those forms that vanish when restricted in all arguments to the kernel of π_*.*

We call $\Omega^\bullet_{\mathrm{vert}}(Y)$ the Chevalley-Eilenberg algebra of the *vertical tangent Lie algebroid* of Y relative to π:

$$\Omega^\bullet_{\mathrm{vert}}(Y) =: \mathrm{CE}(T^{\mathrm{vert}}Y)\,.$$

Definition 4.25 (L_∞-algebraic cocycles) *For $Y \to X$ a surjective submersion of manifolds and $\mathfrak{g}$ an L_∞-algebra, a $\mathfrak{g}$-cocycle on X is an L_∞-morphism $T^{\mathrm{vert}}Y \to \mathfrak{g}$ i.e. a DGCA morphism*

$$\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})\ .$$

Definition 4.26 (vertical paths) *For $\pi : Y \to X$ a map of spaces the fundamental vertical path n-groupoid $\Pi_n^{\mathrm{vert}}(Y)$ of Y relative to π is the pullback*

$$\begin{tikzcd} \Pi_n^{\mathrm{vert}}(Y) \arrow[r] \arrow[d] & \Pi_n(Y) \arrow[d, "\pi_*"] \\ \mathcal{P}_0(X) \arrow[r, hook] & \Pi_n(X) \end{tikzcd}\ .$$

$\Pi_n^{\mathrm{vert}}(Y)$ is the sub-n-groupoid of $\Pi_n(Y)$ all whose n-morphisms map to constant n-paths in X under Y: the vertical n-paths run only within a given fiber of $\pi : Y \to X$.

Lemma 4.27 *Let $Y \to X$ be a fiber bundle with typical fiber F such that all homotopy groups of F up to and including the nth one vanish*

$$\forall k, 1 \leq k \leq n : \pi_n(F) = 1 .$$

Then there is a weak equivalence $\Pi_n^{\mathrm{vert}}(Y) \xrightarrow{\simeq} \mathcal{P}_0(X)$.

Proof. k-surjectivity for $1 \leq k \leq n$ requires that for any two parallel $(k-1)$-morphisms in a fiber, there is a k-morphism connecting them. This is precisely the statement that $\pi_k(F)$ vanishes. k-surjectivity at $k = (n+1)$, i.e. injectivity at $k = n$ is given since in Π_n all homotopic n-morphisms are identified by definition. □

[** warning: the following proposition is true as stated only over the test domains $U = \mathbb{R}^0$ **]

Proposition 4.28 (∞-Lie integrating vertical forms to vertical paths) *For $\pi : Y \to X$ a smooth map of manifolds, the ∞-Lie integration of the vertical tangent Lie algebroid of Y relative to π is the fundamental vertical ω-groupoid of Y:* $\Pi_\omega(S(\Omega^\bullet_{\mathrm{vert}}(Y))) = \Pi_\omega^{\mathrm{vert}}(Y)$.

Proof. A morphism $\Omega^\bullet(I^n) \xleftarrow{f} \Omega^\bullet_{\mathrm{vert}}(Y)$ is a morphism $\hat{f}$ out of all of $\Omega^\bullet(Y)$ which vanishes on forms that are zero when restricted in all arguments to the kernel of π_*

$$\begin{tikzcd} & \Omega^\bullet(Y) \arrow[dl, "\hat{f}"'] \arrow[d] \\ \Omega^\bullet(I^n) & \Omega^\bullet_{\mathrm{vert}}(Y) \arrow[l, "f"'] \end{tikzcd} .$$

As in the proof of proposition 4.22, the morphism $\hat{f}$ comes from pullback $f = \phi^*$ along a smooth map $\phi : I^n \to Y$. For that pullback to annihilate all forms which vanish when all its arguments are in the kernel of π_*, the push-forward of vectors along ϕ has to be in the kernel of π_*

$$\forall \sigma \in I^n \; : \quad \begin{tikzcd} & T_{\phi(\sigma)}Y \arrow[d, "\pi_*"] \\ T_\sigma I^n \arrow[ur, "\phi_*"] \arrow[r, "0"] & T_{\pi(\phi(\sigma))}X \end{tikzcd} .$$

This says precisely that ϕ is a vertical n-path. □

Definition 4.29 (L_∞-algebraic cocycle) *For X a space and $\mathfrak{g}$ an L_∞-algebra, an* <u>*L_∞-algebraic cocycle*</u> *on X with coefficients in $\mathfrak{g}$ is a surjection $\pi : Y \to X$ and a morphism* $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$.

4.4.1 Integrating L_∞-algebraic cocycles to nonabelian cocycles

Given a surjection $Y \to X$ and an L_∞-morphism $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ for $\mathfrak{g}$ some L_∞-algebra, the ∞-Lie integration of this morphism

$$\begin{tikzcd} \Pi_n \circ S(A_{\mathrm{vert}}) : \Pi_n^{\mathrm{vert}}(Y) \arrow[r] \arrow[d] & \mathbf{B}G \\ \mathcal{P}_0(X) \end{tikzcd} \qquad (4.1)$$

ordinary $\mathfrak{g}$-connection	attempted lift to $\mathfrak{g}_\mu$-connection	obstructing $b^n\mathfrak{u}(1)$-connection	**obstruction interpretation**
ordinary $\mathfrak{g}$-connection	twisted $\mathfrak{g}_\mu$-connection	twisting $b^n\mathfrak{u}(1)$-connection	**twisting interpretation**
ordinary $\mathfrak{g}$-connection	electric field	magnetic charge	**charge interpretation**

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] & \mathrm{W}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l, "f^{-1}"'] \arrow[u, two heads] & \mathrm{CE}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i\}"'] \arrow[u, hook] & \mathrm{inv}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^n\mathfrak{u}(1)) \arrow[l] \arrow[u, hook]
\end{tikzcd}
$$

Cartan-Ehresmann L_∞-connection

Figure 8: **Obstructing $b^n\mathfrak{u}(1)$ $(n+1)$-connections and "twisted" $\mathfrak{g}_\mu$ n-connections** are two aspects of the same mechanism: the $(n+1)$-connection is the obstruction to "untwisting" the n-connection. The n-connection is "twisted by" the $(n+1)$-connection. There may be many non-equivalent twisted n-connections corresponding to the same twisting $(n+1)$-connections.

is akin to a G-cocycle on X, but fails to be such in as much as the projection $\Pi_n^{\mathrm{vert}}(Y) \twoheadrightarrow \mathcal{P}_0(X)$ fails to be a weak equivalence. By lemma 4.27 this failure is measured by the homotopy groups of the fibers of Y. If the fibers of Y happen to be n-connected we do have surjective equivalence $\Pi_n^{\mathrm{vert}}(Y) \twoheadrightarrow \mathcal{P}_0(X)$. In this case there are choices of embeddings $\mathcal{P}_0^Y(X) \hookrightarrow \Pi_n^{\mathrm{vert}}(Y)$ of the codescent n-groupoid of Y, as in section 3.1.3, so that the span

$$
\begin{tikzcd}
\mathcal{P}_0^Y(X) \arrow[r, hook] \arrow[dr, "\simeq"] & \Pi_n^{\mathrm{vert}} \arrow[r, "\Pi_n \circ S(A_{\mathrm{vert}})"] \arrow[d, "\simeq"] & \mathbf{B}G \\
& \mathcal{P}_0(Y) \arrow[ur, "g"'] &
\end{tikzcd}
$$

defines a G-cocycle $\mathcal{P}_0^Y(X) \xrightarrow{g} \mathbf{B}G$ on X. We call this the G-cocycle integrating the original L_∞-algebraic cocycle A_{vert}.

In general the fibers of Y are not n-connected. Then A_{vert} may still be integrated to a cocycle in nonabelian cohomology if it satisfies an *integrability condition* which makes (4.1) cover a $G_\sim$-cocycle where the projection $G \to G_\sim$ is such that it sends the nontrivial periods of A_{vert} over the cycles of the fibers of Y to the identity. This is formalized in the following definition.

Definition 4.30 (∞-Lie integration of L_∞-algebraic cocycles to nonabelian cocycles) *For A_{vert} a $\mathfrak{g}$-cocycle on X as in def. 4.25 with $\mathfrak{g}$ a Lie n-algebra and $\mathbf{B}G := \Pi_n \circ S \circ \mathrm{CE}(\mathfrak{g})$ the n-group integrating it, we say that a* <u>*Lie integration of A_{vert}*</u> *is a commuting diagram*

$$
\begin{tikzcd}
& \Pi_n^{\mathrm{vert}}(Y) \arrow[r, "\Pi_n \circ S(A_{\mathrm{vert}})"] \arrow[d] & \mathbf{B}G \arrow[d] \\
\mathcal{P}_0^Y(X) \arrow[r, hook] \arrow[dr, "\simeq_w"'] & \mathbf{Y} \arrow[r, "g"] \arrow[d, "\simeq"] & \mathbf{B}G_\sim \\
& \mathcal{P}_0(X) \arrow[ur, "g"', "\vert" marking] &
\end{tikzcd}
$$

yielding a nonabelian $G_\sim$-cocycle $\mathcal{P}_0(X) \xrightarrow{g} \mathbf{B}G_\sim$ on X. The existence of such a diagram is an integrability condition *on A_{vert}.*

Here $\mathbf{Y} \xrightarrow{\simeq} \mathcal{P}_0(X)$ is a surjective equivalence in which a codescent ω-groupoid $\mathcal{P}_0^Y(X)$ as in section 3.1.3 may be injected. An important special case is that where the fibers of Y are $(n-1)$-connected. In this case we obtain $\mathbf{Y}$ by "patching in" the missing n-cells. [** merge/harmonize notation with section 3.2.3 **]

$$
\begin{tikzcd}
\mathrm{Codesc}(Y, \mathcal{P}_0) \arrow[r] \arrow[d, hook, "i"] & \mathbf{B}G \arrow[d] & G\text{-cocycle} & \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
& & \text{first Cartan-Ehresmann condition} & & \\
\mathrm{Codesc}(Y, \Pi_\omega) \arrow[r] \arrow[d, "\pi"] & \mathbf{BE}G \arrow[d] & \text{connection and curvature} & \Omega^\bullet(Y) \arrow[uu, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "{(A, F_A)}"'] \arrow[uu, two heads] \\
& & \text{second Cartan-Ehresmann condition} & & \\
\Pi_\omega(X) \arrow[r] & \mathbf{B}[\mathbf{B}G] & \text{characteristic forms} & \Omega^\bullet(X) \arrow[uu, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[uu, hook] \\
\text{nonabelian differential } G\text{-cocycle} & & & & \mathfrak{g}\text{-connection descent datum} \arrow[llll, bend left, "\vert" marking, "\boldsymbol{\infty}\textbf{-Lie integration}"]
\end{tikzcd}
$$

Figure 9: **∞-Lie integration of L_∞-connections to nonabelian (differential) cocycles.** For $\mathfrak{g}$ any L_∞-algebra and $Y \to X$ a smooth surjection, the diagram on the right encodes a generalization of a Cartan-Ehresmann connection as described in [132]. Applying the ∞-Lie integration functor $\Pi_n \circ S$ to the entire diagram yields, when certain integrability conditions are met, nonabelian cocycles and differential refinements of these, representing higher principal bundles with connection. Here we consider only the topmost horizontal morphisms which encode the bare cocycles. An outlook on the remaining parts of these diagrams is given in section 3.3.4.

Definition 4.31 (patching-in homotopy groups) *For $n \in \mathbb{N}$, $n > 1$, G an $(n-1)$-connected group in* Spaces*, i.e one whose first n homotopy groups are trivial, $\forall\ 0 \leq k < n\ :\ \pi_k(G) = 1$, and for $\pi : P \to X$ a principal G-bundle, let $P \times \mathbf{B}^{n+1}\pi_n(G) \hookrightarrow \Pi_n^{\mathrm{vert}}(P)$ be the canonical inclusion of the nth homotopy groups of the fibers into the fundamental vertical n-groupoid $\Pi_n^{\mathrm{vert}}(P)$ from definition 4.26, sending a pair consisting of $p \in P$ and an element in $\pi_n(G)$ to the vertical n-path based at p representing that element. Write*

$$\Pi_n^{\mathrm{vert}}(P)^\circ := \Pi_n^{\mathrm{vert}}(P)//(P \times \mathbf{B}^n\pi_n(G)) := (P \times \mathbf{B}^{n+1}\pi_n(G) \hookrightarrow \Pi_n^{\mathrm{vert}}(P))$$

for the weak quotient (the mapping cone over the inclusion) corresponding to the mapping cone n-groupoid for this inclusion.

Using this, we obtain the following special case of the integration procedure, definition 4.30.

Definition 4.32 (integration of L_∞-algebraic cocycles with integral n-periods) *Let $\mathfrak{g}$ be a Lie n-algebra such that the simply-connected n-group G integrating it, given by $\mathbf{B}G := \Pi_n \circ S \circ \mathrm{CE}(\mathfrak{g})$ has $\mathbf{B}^n\mathbb{Z}$ as a sub n-group $\mathbf{B}^n\mathbb{Z} \hookrightarrow G$. Let $Y \to X$ have $(n-1)$-connected fibers. Then we say that an L_∞-algebraic cocycle $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ has integral n-periods if the local horizontal morphism in*

$$\begin{tikzcd}
\Pi_n^{\mathrm{vert}} \arrow[r, "\Pi_n \circ S(A_{\mathrm{vert}})"] \arrow[d] & \mathbf{B}G \arrow[d] \\
\Pi_n^{\mathrm{vert}}(Y)^\circ \arrow[r] & \mathbf{B}(\mathbf{B}^{n-1} \hookrightarrow G)
\end{tikzcd}$$

exists. In this case $\mathcal{P}_0(X) \xrightarrow{g} \mathbf{B}(G/\mathbf{B}^{n-1}\mathbb{Z})$ in

$$\begin{tikzcd}
 & \Pi_n^{\mathrm{vert}} \arrow[r, "\Pi_n \circ S(A_{\mathrm{vert}})"] \arrow[d] & \mathbf{B}G \arrow[d] & \\
\mathcal{P}_0^Y(X) \arrow[r, hook] \arrow[dr, "\simeq"] & \Pi_n^{\mathrm{vert}}(Y)^\circ \arrow[r] \arrow[d, "\simeq"] & \mathbf{B}(\mathbf{B}^{n-1} \hookrightarrow G) \arrow[r, two heads, "\simeq"] & \mathbf{B}(G/\mathbf{B}^{n-1}\mathbb{Z}) \\
 & \mathcal{P}_0(X) \arrow[urr, "g"'] & &
\end{tikzcd}$$

is the nonabelian cocycle integrating A_{vert}.

4.4.2 Lifts, obstructions and twists of L_∞-algebraic cocycles

For $b^{n-1}\mathfrak{u}(1) \longrightarrow \mathfrak{g}_\mu \longrightarrow \mathfrak{g}$ a shifted central extension of L_∞-algebras as in [132], the L_∞-algebraic analog of the situation in section 3.2.1 is depicted in figure 8.

4.5 [variation]

[** the following needs to be merged with the above **]

So far we had taken the space (sheaf) associated with an L_∞-algebroid $\mathfrak{g}$ to be the classifying space:

$$S(\mathrm{CE}(\mathfrak{g})) : U \mapsto \mathrm{Hom}_{L_\infty\mathsf{Algebroids}}(TU, \mathfrak{g}) = \mathrm{Hom}_{C^\infty\mathsf{qDGCAs}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(U))$$

of flat $\mathfrak{g}$-valued forms. ($U \in \mathsf{CartesianSpaces}$.) Then we took the ω-groupoid integrating $\mathfrak{g}$ to be the fundamental ω-groupoid $\Pi_\omega(S(\mathrm{CE}(\mathfrak{g})))$ of this space. Notice that the space of k-morphisms of this ω-groupoid is (a quotient by thin-homotopy) of the sheaf given by

$$U \mapsto \mathrm{Hom}_{L_\infty\mathsf{Algebroids}}(T(D^k \times U), \mathfrak{g})\,. \tag{4.2}$$

(With D^k denoting the standard k-disk.) This means that not only are all k-morphisms supposed to be maps from the standard k-disk D^k "parallel" in $\mathfrak{g}$, but also all U-parameterized families of these have to remain "parallel" in $\mathfrak{g}$. For L_∞-algebras this is fine, but for L_∞-algebroids one should be aware of the following subtlety:

while this produces the right *set* of k-morphisms, the smooth structure on the space of k-morphisms is restricted to "gauge orbits". Take for instance $Y \to X$ to be a surjective submersion and let $T_{\mathrm{vert}}Y$ be the vertical tangent Lie algebroid of Y. Then the integrating ω-groupoid

$$\Pi_\omega(S(\mathrm{CE}(T_{\mathrm{vert}}Y))) = \bigsqcup_{x\in X} \Pi_\omega(Y_x)$$

is the disjoint union of the fundamental ω-groupoids of the fibers: the disjoint union itself comes with the *discrete* structure, i.e. every smooth family of k-morphisms has to stay within one and the same fiber.

Dmitry Roytenberg had considered the following slight modification: instead of (4.2) he declares that the space of k-cells is

$$U \mapsto \mathrm{Hom}_{L_\infty\mathsf{Algebroids}}((TD^k) \times U, \mathfrak{g})\,, \tag{4.3}$$

(he uses the standard simplex Δ^k instead of the standard disk D^k, but that is not important for the present discussion) where now tangents are taken only on D^k, not on U. This makes the U-parameterized families more flexible. With this prescription for instance the integration of $T_{\mathrm{vert}}Y$ yields the same ω-groupoid in Sets as before, but now with the smooth structure on the k-morphisms such that a smooth U-parameterized family of k-morphisms must be such that still for each point of U it is a vertical k-path in some fiber, but the fiber may change smoothly with U now.

Clearly this is preferable at least in some cases. So I was starting to think about how to realize this conceptually, i.e. without just changing formula (4.2) "by hand", given that the interpretation of L_∞-integration as that double adjunction

$$\omega\mathsf{Groupoids}(\mathsf{Spaces}) \xleftarrow{\Pi_\omega} \mathsf{Spaces} \xleftarrow{S\circ\mathrm{CE}} L_\infty\mathsf{Algebroids}$$

which proceeds via classifying spaces is elegant and useful.

Now, I noticed that the more flexible version of (4.2) has a similar formulation, which in fact nicely fits into the context of differential cohomology: namely, I will claim now that (4.3) is obtained by taking Π_ω not of the classifying space of flat $\mathfrak{g}$-valued forms – but on the classifying space of $\mathfrak{g}$-connections in the sense of [SatiSchreiberStasheff].

Recall that while a flat $\mathfrak{g}$-valued form on $Y \to X$ is just a morphism

$$TY \longrightarrow \mathfrak{g} \quad \Leftrightarrow \quad \Omega^\bullet(Y) \longleftarrow \mathrm{CE}(\mathfrak{g})$$

while a $\mathfrak{g}$-connection (or "$\mathfrak{g}$-connection descent object", the L_∞-algebraic approximation to a differential nonabelian cocycle) is such morphism which is required to be only fiberwise flat ("first Cartan-Ehresmann condition on connections"), namely a commuting square

$$\begin{tikzcd} T_{\mathrm{vert}}Y \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ TY \arrow[r] & \mathrm{inn}(\mathfrak{g}) \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l] \\ \Omega^\bullet(Y) \arrow[u] & \mathrm{W}(\mathfrak{g}) \arrow[l] \arrow[u] \end{tikzcd}\,.$$

(In [SatiSchreiberStasheff] we also imposed a second condition coming from a second square. I could impose that in the following, too, but for the moment I'll save myself some typing by doing it just this way. This already captures the main idea).

From this one can already see how it may relate to the assignment (4.3). I'll now formulate that in detail.

∞-Lie integration using $\mathfrak{g}$-connections. I now describe an ∞-Lie theory pretty much as the one before, but with the category of generalized Spaces enlarged to a category of generalized Covers. The goal is to describe the diagram

$$\begin{tikzcd}
\omega\mathsf{Groupoids} \arrow[r, shift left, "K(-)=\mathrm{Hom}(\Pi_\omega^{\mathrm{vert}}(-),-)"] \arrow[rrr, bend left, "\infty\text{-Lie differentiation}"] & \mathsf{Covers} \arrow[l, shift left, "\Pi_\omega^{\mathrm{vert}}(-)"] \arrow[r, shift left, "\Omega^\bullet(-)=\mathrm{Hom}(-,\Omega^\bullet)"] & C^\infty\mathsf{qDGCAs} \arrow[l, shift left, "S(-)=\mathrm{Hom}(-,\Omega^\bullet(-))"] & L_\infty\mathsf{Algebroids} \arrow[l, "\mathrm{CE}(-)", "\simeq"'] \arrow[lll, bend left, "\infty\text{-Lie integration}"]
\end{tikzcd}$$

where the right and the left pair are adjunctions. This is, notice, the same diagram as before, but with Spaces replaced by Covers (and with the morphisms suitably re-interpreted, of course).

Here Covers will be defined to be the category of sheaves on the category of CartesianCovers of the form $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$. Notice that this also makes closer contact to [Ševera: *L_∞-algebras a 1-jets*] which first found the need in ∞-Lie theory to consider sheaves on the category of test covers, instead of just sheaves on the category of test spaces.

Covers

Definition 4.33 (cartesian covers) *Let* ManifoldCovers *be the category whose object ares surjective submersions $Y \to X$ of manifolds and whose morphisms are commuting diagrams of these. The category* CartesianCovers *of cartesian covers is the full subcategory of* ManifoldCovers *on objects of the form $\mathbb{R}^k \times \mathbb{R}^l \xrightarrow{p_1} \mathbb{R}^k$.*

Definition 4.34 (covers) *Let* $\mathsf{Covers} := \mathsf{Sheaves}(\mathsf{CartesianCovers})$.

We consider Covers to be tensored over Spaces by using the first embedding: for $Y \in \mathsf{Covers}$ and $X \in \mathsf{Spaces}$ let $Y \times X \in \mathsf{Covers}$ be given by

$$Y \times X : (U \times F) \mapsto Y(U \times F) \times X(U)\,.$$

It is enriched over Spaces by setting

$$\mathrm{hom}(Y, Y') : U \mapsto \mathrm{Hom}_{\mathsf{Covers}}(Y \times U, Y')\,.$$

Let $I = \{a \to b\}$ be the interval object in ωCategories.

Infinitesimal paths.

Definition 4.35 (vertical forms) *For $\pi : Y \to X$ in* CartesianCovers*, let $\Omega^\bullet_{\mathrm{vert}}(Y \xrightarrow{\pi} X)$, the C^∞*qDGCA *of*
vertical differential forms on Y, be the pushout

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y \to X) & \Omega^\bullet(Y) \arrow[l, "{:=}i^*"'] \\
C^\infty(X) \arrow[u] & \Omega^\bullet(X) \arrow[l, two heads] \arrow[u, hook, "\pi^*"']
\end{tikzcd}\,.$$

Definition 4.36 (infinitesimal path object) *Let the infinitesimal path object, for covers, be the functor* $\Omega^\bullet_- : I^{\mathrm{op}} \to \mathsf{Covers}$ *given by*

$$\Omega^\bullet_-(-) : \left(\begin{array}{c} a_1 \\ \downarrow \\ a_2 \end{array}, U \times F \to U \right) \mapsto \begin{array}{c} \Omega^\bullet_{\mathrm{vert}}(U \times F \to U) \\ \uparrow{\scriptstyle i^*} \\ \Omega^\bullet(U \times F) \end{array}$$

Definition 4.37 (universal fibration of L_∞-algebroids) *For* $\mathfrak{g}$ *an* L_∞*-algebroid,* $\mathrm{CE}(\mathfrak{g})$ *its Chevalley-Eilenberg-*C^∞ *qDGCA and* $\mathrm{W}(\mathfrak{g})$ *its Weil* C^∞ *qDGCA, let*

$$\mathrm{CE}_-(\mathfrak{g}) : I^{\mathrm{op}} \to \mathsf{C^\infty qDGCAs}$$

be the functor given by

$$\mathrm{CE}_- : \left(\begin{array}{c} a_1 \\ \downarrow \\ a_2 \end{array} \right) \mapsto \begin{array}{c} \mathrm{CE}(\mathfrak{g}) \\ \uparrow{\scriptstyle i^*} \\ \mathrm{W}(\mathfrak{g}) \end{array} .$$

Definition 4.38 *For* $U \times F \to U$ *a cartesian cover and for* $\mathfrak{g}$ *an* L_∞*-algebroid, define the* hom*-set from infinitesimal paths on* $U \times F$ *to* $\mathfrak{g}$ *by*

$$\mathrm{hom}(\mathrm{CE}_-(\mathfrak{g}), \Omega^\bullet_-(U \times F \to U)) := \int_{a \in I} \mathrm{Hom}(\mathrm{CE}_a(\mathfrak{g}), \Omega^\bullet_a(U \times F \to U)) .$$

Remark. This Hom-set is the set of horizontal morphisms in this commuting square

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(U \times F \to U) & \mathrm{CE}(\mathfrak{g}) \arrow[l] \\ \Omega^\bullet(U \times F) \arrow[u, two heads, "i^*"] & \mathrm{W}(\mathfrak{g}) \arrow[l] \arrow[u, two heads, "i^*"'] \end{tikzcd}$$

in $C^\infty\mathsf{qDGCAs}$.

This now allows us to define an adjunction $\mathsf{Covers} \underset{S}{\overset{\Omega^\bullet}{\rightleftarrows}} C^\infty\mathsf{qDGCAs}$ by homming into the ambimorphic ("schizophrenic" in [Johnstone: Stone duality]) object $\Omega^\bullet$, which can be regarded both as a C^∞qDGCA internal to Covers as well as a cover internal to $C^\infty\mathsf{qDGCAs}$.

Definition 4.39 (qDGCA of differential forms on covers) *For* $Y \in \mathsf{Covers}$ *let*

$$\Omega^\bullet(Y) := \mathrm{hom}_{\mathsf{Covers}}(Y(-), \Omega^\bullet_{a_2}(-)) ,$$

which naturally comes equipped with the structure of a $C^\infty\mathsf{qDGCA}$ *be the* C^∞ *qDGCA of differential forms on the cover* Y.

Definition 4.40 (classifying cover of L_∞-algebroids) *The functor* $S : L_\infty\mathsf{Algebroids} \to \mathsf{Covers}$, *sends* $\mathfrak{g} \in L_\infty\mathsf{Algebroids}$ *to*

$$S(\mathfrak{g}) := S(\mathrm{CE}(\mathfrak{g})) : (U \times F \to U) \mapsto \mathrm{hom}(\mathrm{CE}_-, \Omega^\bullet_-(U \times F \to U)) ,$$

the classifying cover of $\mathfrak{g}$*-connections.*

Finite paths

Definition 4.41 *Define* $\Pi_\omega : \mathsf{Covers} \to \omega\mathsf{Categories}(\mathsf{Spaces})$ *"as before", with space of k-morphisms being a quotient (thin-homotopy) of a subset (sitting instants) of*

$$\hom_{\mathrm{Covers}}(D^k, -) : \mathsf{Covers} \to \mathsf{Spaces}\,.$$

Here we use the embedding of Spaces into Covers which sends D^k to $D^k \to \mathrm{pt}$ and use the enrichment over Spaces of Covers. This means that for $Y \in \mathsf{Covers}$ the above hom-space is the sheaf

$$\hom(D^k, Y) : U \mapsto \mathrm{Hom}_{\mathsf{Covers}}(U \times D^k \to U, Y) \simeq Y(U \times D^k \to U)\,.$$

For instance for $Y = S(\mathrm{CE}(\mathfrak{g}))$ this is, by the above

$$\cdots \simeq \int\limits_{a \in I} \mathrm{Hom}(\mathrm{CE}_a(\mathfrak{g}), \Omega^\bullet_a(U \times D^k \to U))$$

which is the set of horizontal morphisms in the commuting diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(U \times D^k \to U) & \mathrm{CE}(\mathfrak{g}) \arrow[l] \\ \Omega^\bullet(U \times D^k) \arrow[u, "i^*"] & \mathrm{W}(\mathfrak{g}) \arrow[l] \arrow[u, "i^*"'] \end{tikzcd}\,.$$

(Here the tensor product is that of C^∞-algebras, hence the completed ordinary tensor product of algebras.) But this is indeed isomorphic to the set

$$\cdots \simeq \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(D^k) \otimes C^\infty(U)) \simeq \mathrm{Hom}_{L_\infty\mathrm{Algebroids}}((TD^k) \times U, \mathfrak{g})$$

as in (4.3).

5 Examples and Applications

This section lists various applications and concrete examples in the context of lifts of ω-bundles through shifted central extensions of their structure ω-groups.

5.1 ω-Groups

5.1.1 1-Groups

If G is an ordinary group, then $\mathbf{B}G = \{\, \bullet \xrightarrow{g} \bullet \,|g \in G\}$ is the 1-groupoid with the single object denoted $\bullet$ and one morphism per element of the group, with composition of morphisms being the product in the group.

5.1.2 2-Groups

For $(H \xrightarrow{t} G)$ a crossed module of groups with action $\alpha : G \to \mathrm{Aut}(H)$, the corresponding 2-group is given by the one-object 2-groupoid

$$\mathbf{B}(H \to G) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \middle|\, g_1, g_2 \in G, h \in H, t(h)g_1 = g_2 \right\}\,.$$

As described at the beginning of [126] there are four different (but isomorphic) identifications of crossed modules of groups with 2-groups, which differ in the order in which composition of morphisms relates to the product in the groups G and H. Up to this choice, horizontal composition is the product operation in the semidirect product group $H \rtimes_\alpha G$, while vertical composition is given by the product in H. For an introduction to strict 2-groups see [7]. The standard examples of 2-groups include

Definition 5.1 (automorphism 2-group) *Let H be a group. The* <u>*automorphism 2-group*</u> $\mathrm{AUT}(H)$ *of H is the 2-group*

$$\mathrm{AUT}(H) := (H \to \mathrm{Aut}(H)) .$$

corresponding to the crossed module $(H \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(H))$ *where* $\alpha = \mathrm{Id}_{\mathrm{Aut}(H)}$.

Definition 5.2 (2-groups from central extensions) *Let $H \subset G$ be a normal subgroup and $\hat{H} \to H$ a central extension of groups such that the conjugation action of G on H lifts to an automorphism action $\alpha : G \to \mathrm{Aut}(\hat{H})$ on the central extension. Then $(\hat{H} \to G)$ with this α is a crossed module.*

Examples of smooth 2-groups coming from central extensions are often obtained from cocycles on Lie groups as follows, generalizing considerations in [119]:

Proposition 5.3 (smooth 2-groups from central extensions) *Let $G \subset \Gamma$ be a simply connected normal Lie subgroup of a Lie group Γ. Write PG for the based path group of G whose elements are smooth maps $[0,1] \to G$ starting at the neutral element and whose product is given by the pointwise product in G. Consider the complex with differential $\mathrm{d} \pm \delta$ of simplicial forms on BG from definition 6.33. Let (F, a, β) be a triple where*
i. $a \in \Omega^1(G \times G)$ *such that* $\delta a = 0$*;*
ii. F *is a closed integral 2-form on* G *such that* $\delta F = da$*;*
iii. $\beta : \Gamma \to \Omega^1(G)$ *such that, for all* $\gamma, \gamma_1, \gamma_2 \in \Gamma$,

- $\gamma^* F = F + d\beta_\gamma$;
- $(\gamma_1)^* \beta_{\gamma_2} - \beta_{\gamma_1 \gamma_2} + \beta_{\gamma_1} = 0$;
- $a = \gamma^* a + \delta(\beta_\gamma)$;
- *for all based paths* $f : [0,1] \to G$, $f^* \beta_\gamma = (f, \gamma^{-1})^* a + (\gamma, f\gamma^{-1})^* a$.

1. *Then the map* $c : PG \times PG \to U(1)$ *given by* $c : (f,g) \mapsto c_{f,g} := \exp\left(2\pi i \int\limits_{0,1} (f,g)^* a\right)$ *is a group 2-cocycle leading to a central extension* $\widehat{P}G = PG \ltimes U(1)$ *with product* $(\gamma_1, x_1) \cdot (\gamma_2, x_2) = (\gamma_1 \cdot \gamma_2, x_1 x_2 c_{\gamma_1, \gamma_2})$.
2. *Since G is simply connected every loop in G bounds a disk D. There is a normal subgroup $N \subset \widehat{P}G$ consisting of pairs (γ, x) with $\gamma(1) = e$ and $x = \exp(2\pi i \int_D F)$ for any disk D in G such that $\partial D = \gamma$.*
3. *Finally, $\tilde{G} := \widehat{PG}/N$ is a central extension of G by $U(1)$ and the conjugation action of Γ on G lifts to $\tilde{G}$ by setting $\alpha(\gamma)(f,x) := (\alpha(\gamma)(f),\ x \exp(\in_f \beta_\gamma))$ such that* $\mathrm{Cent}(G, \Gamma, F, a, \beta) := (\tilde{G} \to \Gamma)$ *is a Lie crossed module and hence a strict Lie 2-group of the type in definition 5.2.*

Proof. All statements about the central extension $\hat{G}$ can be found in [119]. It remains to check that the action $\alpha : \Gamma \to \mathrm{Aut}(\tilde{G})$ satisfies the required axioms, definition **??**, of a crossed module, in particular the condition $\alpha(t(h))(h') = hh'h^{-1}$. Then we have to show that

$$\alpha(h(1))([f,z]) = [h,1][f,z] \left[h^{-1}, \exp(-\int_{(h,h^{-1})} a)\right] ,$$

where h denotes a based path in $P\mathcal{G}$, so that $[h,1]$ represents an element of $\tilde{\mathcal{G}}$. By definition of the product in $\tilde{\mathcal{G}}$, the right hand side is equal to

$$\left[hfh^{-1}, z\exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a\right)\right] .$$

This is not exactly in the form we want, since the left hand side is equal to $\left[h(1)fh(1)^{-1}, z\exp(\int_f \beta_h)\right]$. Therefore, we want to replace hfh^{-1} with the homotopic path $h(1)fh(1)^{-1}$. An explicit homotopy between these two paths is given by $H(s,t) = h((1-s)t+s)f(t)h((1-s)t+s)^{-1}$. Therefore, we have the equality

$$\begin{aligned}&\left[hfh^{-1}, z\exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a\right)\right] \\ &\qquad = \left[h(1)fh(1)^{-1}, z\exp\left(\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int H^*F\right)\right] .\end{aligned}$$

Using the relation $(F) = da$ and the fact that the pullback of F along the maps $[0,1]\times[0,1]\to G$, $(s,t)\mapsto h((1-s)t+s)$ vanish, we see that

$$\int H^*F = \int_{(f,h(1)^{-1})} a - \int_{(f,h^{-1})} a + \int_{(h,h^{-1})} a + \int_{(h(1),fh(1)^{-1})} a - \int_{(h,fh^{-1})} a .$$

Therefore the sum of integrals

$$\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int H^*F$$

can be written as

$$\int_{(h,f)} a + \int_{(hf,h^{-1})} a - \int_{(h,h^{-1})} a + \int_{(f,h(1)^{-1})} a - \int_{(f,h^{-1})} a + \int_{(h,h^{-1})} a + \int_{(h(1),fh(1)^{-1})} a - \int_{(h,fh^{-1})} a .$$

Using the condition $\delta(a) = 0$, we see that this simplifies down to $\int_{(f,h(1)^{-1})} a + \int_{(h(1),fh(1)^{-1})} a$. Therefore, a sufficient condition to have a crossed module is the equation $f^*\beta_h = (f,h(1))^*a + (h(1),fh(1)^{-1})^*a$. ☐

Proposition 5.4 *Given triples (F,a,β) and (F',a',β') as above and given $b\in\Omega^1(G)$ such that*

$$F' = F + db\,, \tag{5.1}$$

$$a' = a + \delta(b) \tag{5.2}$$

and for all $\gamma\in\Gamma$

$$\beta_\gamma + \gamma^*b = b + \beta'_\gamma\ , \tag{5.3}$$

then there is an isomorphism $\mathrm{Cent}(G,\Gamma,F,a,\beta)\simeq\mathrm{Cent}(G,\Gamma,F',a',\beta')$.

Examples. In [9] the following special case of the above general construction was considered.

Definition 5.5 ($\mathrm{String}_{\mathrm{BCSS}}(G)$ **[9]**) *Let G be a compact, simple and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Let $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ be the bilinear invariant form on $\mathfrak{g}$ normalized such that the Lie algebra 3-cocycle $\mu := \langle\cdot,[\cdot,\cdot]\rangle$ extends left invariantly to a 3-form on G which is the image in deRham cohomology of one of the two generators of $H^3(G,\mathbb{Z}) = \mathbb{Z}$. Let ΩG be the based loop group of G whose elements are* *smooth maps $\gamma : [0,1] \to G$ with $\gamma(0) = \gamma(1) = e$ and whose product is by pointwise multiplication of such maps. Define $F \in \Omega^2(\Omega G)$, $a \in \Omega^1(\Omega G \times \Omega G)$ and $\beta : \Gamma \to \Omega^1(\Omega G)$*

$$\begin{aligned} F(\gamma, X, Y) &:= \int_0^{2\pi} \langle X, Y' \rangle dt \\ a(\gamma_1, \gamma_2, X_1, X_2) &:= \int_0^{2\pi} \langle X_1, \dot{\gamma}_2 \gamma_2^{-1} \rangle dt \\ \beta(p)(\gamma, X) &:= \int_0^{2\pi} \langle p^{-1}\dot{p}, X \rangle dt \end{aligned}$$

This satisfies the axioms of proposition 5.3 and we write $\mathrm{String}_{\mathrm{BCSS}}(G) := \mathrm{Cent}(\Omega G, PG, F, \alpha, \beta)$ *for the corresponding 2-group.*

Remark. This 2-group is the termwise integration of the strict String Lie 2-algebra from theorem 5.20 as described in [9]. Here $\hat{\Omega}G$ is the Kac-Moody central extension of ΩG at level 1.

The following related construction is based on the cocycle on loop groups considered by Mickelsson [115].

Definition 5.6 ($\mathrm{String}_{\mathrm{Mick}}(G)$**)** *With all assumptions as in definition 5.5 define now*

$$\begin{aligned} F(\gamma, X, Y) &:= \frac{1}{2} \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, [X, Y] \rangle dt \\ a(\gamma_1, \gamma_2, X_1, X_2) &:= \frac{1}{2} \int_0^{2\pi} \left(\langle X_1, \dot{\gamma}_2 \gamma_2^{-1} \rangle - \langle \gamma_1^{-1}\dot{\gamma}_1, \gamma_2 X_2 \gamma_2^{-1} \rangle \right) dt \\ \beta(p)(\gamma, X) &:= \frac{1}{2} \int_0^{2\pi} \langle \gamma^{-1} p^{-1} \dot{p} \gamma + p^{-1}\dot{p}, X \rangle dt \end{aligned}$$

This satisfies the axioms of proposition 5.3 and we write $\mathrm{String}_{\mathrm{Mick}}(G) := \mathrm{Cent}(\Omega G, PG, F, \alpha, \beta)$ *for the corresponding 2-group.*

Proposition 5.7 *There is an isomorphism of 2-groups* $\mathrm{String}_{\mathrm{BCSS}}(G) \xrightarrow{\simeq} \mathrm{String}_{\mathrm{Mick}}(G)$.

Proof. We show that $b \in \Omega^1(\Omega G)$ defined by $b(\gamma, X) := \frac{1}{4\pi} \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, X \rangle dt$ satisfies the conditions of proposition 5.4 and hence defines the desired isomorphism.

- Proof of equation 5.1: We calculate the exterior derivative db. To do this we first calculate the derivative $Xb(y)$: if $\gamma_t = \gamma e^{tX}$ then to first order in t, $\gamma_t^{-1}\dot{\gamma}_t$ is equal to $\gamma^{-1}\dot{\gamma} + t[\gamma^{-1}\dot{\gamma}, X] + tX'$. Therefore

$$Xb(Y) = \frac{1}{2} \int_0^{2\pi} \left(\langle \gamma^{-1}\dot{\gamma}, [X, Y] \rangle + \langle X', Y \rangle \right) dt \, .$$

  Hence db is equal to

$$\frac{1}{2} \int_0^{2\pi} \left(\langle \gamma^{-1}\dot{\gamma}, [X, Y] \rangle + \langle X', Y \rangle + \langle \gamma^{-1}\dot{c}, [X, Y] \rangle - \langle Y', X \rangle - \langle \gamma^{-1}\dot{\gamma}, [X, Y] \rangle \right) \, ,$$

  which is easily seen to simplify down to

$$- \int_0^{2\pi} \langle X, Y \rangle dt + \frac{1}{2} \int_0^{2\pi} \langle \gamma^{-1}\dot{\gamma}, [X, Y] \rangle dt \, .$$

- Proof of equation 5.2: We get

$$\frac{1}{2}\int_0^{2\pi}\left\{\langle\gamma_2\dot\gamma_2^{-1},X_2\rangle-\langle\gamma_2^{-1}\gamma_1^{-1}\dot\gamma_1\gamma_2,\gamma_2^{-1}X_1\gamma_2\rangle-\langle\gamma_2^{-1}\gamma_1^{-1}\dot\gamma_1\gamma_2,X_2\rangle\right.$$
$$\left.-\langle\gamma_2^{-1}\dot\gamma_2,\gamma_2^{-1}X_1\gamma_2\rangle-\langle\gamma_2^{-1}\dot\gamma_2,X_2\rangle+\langle\gamma_1^{-1}\dot\gamma_1,X_1\rangle\right\}dt\ ,$$

which is equal to

$$\frac{1}{2}\int_0^{2\pi}\left\{-\langle\gamma_1^{-1}\dot\gamma_1,\gamma_2X_2\gamma_2^{-1}\rangle-\langle\dot\gamma_2\gamma_2^{-1},X_1\rangle\right\}dt\ ,$$

which in turn equals

$$\frac{1}{2}\int_0^{2\pi}\left\{\langle X_1,\dot\gamma_2\gamma_2^{-1}\rangle-\langle\gamma_1^{-1}\dot\gamma_1,\gamma_2X_2\gamma_2^{-1}\rangle\right\}dt-\frac{1}{2\pi}\int_0^{2\pi}\langle X_1,\dot\gamma_2\gamma_2^{-1}\rangle dt\ .$$

- Proof of equation 5.3: we get

$$\begin{aligned}
p^*b(\gamma;\gamma X) &= b(p\gamma p^{-1};p\gamma p^{-1}(pXp^{-1}))\\
&= \frac{1}{2}\int_0^{2\pi}\langle p\gamma p^{-1}(\dot p\gamma p^{-1}+p\dot\gamma p^{-1}-p\gamma p^{-1}\dot p p^{-1},pXp^{-1}\rangle dt\\
&= \frac{1}{2}\int_0^{2\pi}\langle p\gamma^{-1}p^{-1}\dot p\gamma p^{-1}+p\gamma^{-1}\dot\gamma p^{-1}-\dot p p^{-1},pXp^{-1}\rangle dt\\
&= \frac{1}{2}\int_0^{2\pi}\langle\gamma^{-1}p^{-1}\dot p\gamma+\gamma^{-1}\dot\gamma-p^{-1}\dot p,X\rangle dt\\
&= b(\gamma,\gamma X)+\frac{1}{2}\int_0^{2\pi}\langle\gamma^{-1}p^{-1}\dot p\gamma-p^{-1}\dot p,X\rangle dt\\
&= b(\gamma,\gamma X)+\frac{1}{2}\int_0^{2\pi}\langle\gamma^{-1}p^{-1}\dot p\gamma+p^{-1}\dot p,X\rangle dt-\frac{1}{2\pi}\int_0^{2\pi}\langle p^{-1}\dot p,X\rangle dt
\end{aligned}$$

The three conditions in proposition 5.4 are satisfied and, therefore, the desired isomorphism is established. □

Corollary 5.8 ([9]) *Let* $\mathrm{String}(G)$ *be either of* $\mathrm{String}_{\mathrm{BCSS}}(G)$ *or* $\mathrm{String}_{\mathrm{Mick}}(G)$. *This exhibits a shifted central extension in the sense of definition 3.27:* $U(1)\longrightarrow\mathrm{String}(G)\longrightarrow G$.

5.1.3 3-Groups

Lemma 5.9 *Given a 2-group* $(\hat H\to G)$ *coming from a central extension as in definition 5.2, there is a 3-group* $(U(1)\to\hat H\to G)$ *and a weak equivalence* $\mathbf{B}(U(1)\to\hat H\to G)\xrightarrow{\simeq}\mathbf{B}(H\to G)$.

5.2 ∞-Lie integration

We apply the general mechanism for ∞-Lie integrating L_∞-algebras to ω-groups described in section 4.3 to various examples of interest. In section 5.2.1 the relation to ordinary Lie theory of Lie 1-algebras and 1-groups is established.

5.2.1 Integration of Lie 1-algebras

Lie's third theorem, that every Lie algebra comes from a Lie group, is usually proven by relating everything to matrix Lie algebras using Ado's theorem[2]. That there is a more elegant and more conceptual method which identitfies the simply connected Lie group integrating a given Lie algebra with a certain quotient of based *paths* in the Lie algebra and identifies the product in the Lie group with *composition of paths*, has apparently been well known to a chosen few for a long time[3] but was certainly not widely appreciated. It received wider attention only when researchers started thinking about the more general problem of the integration of Lie algebroids to Lie groupoids. In that latter case, the more conceptual path method is the only sensible one. An exhaustive review of this theory of integration of Lie 1-algebroids can be found in [44]. In section 3.2 the reader can find a discussion of the path-method for integrating Lie algebras, which then in section 3.3 is generalized to the integration of Lie algebroids. The discussion in [44] is not formulated exactly in the language used here, but is easily translated into it as the proof of the following theorem shows.

Theorem 5.10 (integration of ordinary Lie 1-algebras and Lie 1-algebroids) *For a Lie 1-algebra* $\mathfrak{g}$ $\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) := \mathbf{B}G$ *, where G is the simply connected Lie group integrating $\mathfrak{g}$. The same result holds if $\mathfrak{g}$ is a Lie 1-algebroid.*

Proof. This is essentially the main theorem reviewed in [44]: First of all, it is well known, or otherwise easily checked (and indeed the rationale for definition 2.91), that given a manifold U then DGCA morphisms $\Omega^\bullet(U) \longleftarrow \mathrm{CE}_A(\mathfrak{g})$ are in a bijection with the Lie algebroid morphisms $T \longrightarrow (\mathfrak{g}, A)$. By the Yoneda lemma, theorem 6.26, the space of morphisms of $\Pi_1(S(\mathrm{CE}(\mathfrak{g}, A)))$ is hence precisely that of Lie algebroid morphisms $TI \to (\mathfrak{g}, A)$. The latter are the "A-paths" of [44] (see definition 2.13 and exercise 27 there) – modulo Lie algebra homotopies $T(I\times I) \to (\mathfrak{g}, A)$ – these are the "A-homotopies" of [44] (see definition 3.18 there). □

Remark (nonabelian Stokes theorem). The integration procedure of theorem 5.10 can be interpreted as follows: an element of the simply connected Lie group G is represented by a $\mathfrak{g}$-valued 1-form on the interval $A \in \Omega^1([0,1], \mathfrak{g})$. Such 1-forms are usually turned into group elements by means of their *parallel transport* (see for instance [136]):

$$A \mapsto P\exp\left(\int_{[0,1]} A\right) \in G\,,$$

the "path ordered integral" of A over the interval. Remarkably, when forming $\mathbf{B}G$ as $\Pi_1(S(\mathrm{CE}(\mathfrak{g})))$ no such integral is computed explicitly. Instead, equivalence classes of 1-forms A on the interval are formed, where two 1-forms are identified if they can be interpolated by a flat 1-form on the disk. On the other hand, the *nonabelian Stokes theorem* (for instance [136]) implies that any two 1-forms connected by a flat 1-form over the disk have the same parallel transport. Conversely, given two 1-forms on intervals with the same parallel transport, we can use lemma 5.11 below, together with the fact that G is simply connected, to deduce that they can be interpolated over the disk by a flat 1-form. This shows that while the path ordered exponential is not computed explicitly when forming $\Pi_1(S(\mathrm{CE}(\mathfrak{g})))$, the equivalence relation identifies precisely those 1-forms which would yield the same group element if integrated. One can regard this as a nonabelian instance of the principle of *integration without integration* as mentioned in a related remark after proposition 5.17 below.

Flat $\mathfrak{g}$-valued 1-forms on the n-disk. The following aspect of the path-integration method of Lie algebras, a standard fact, is crucial for the integration of shifted central extensions of ordinary Lie algebras in sections 5.2.3, 5.2.4 and 5.2.5.

[2]which states that every finite-dimensional Lie algebra over a field of characteristic zero can be viewed as a Lie algebra of square matrices under the commutator bracket.

[3]Apparently Bott taught it his students this way.

Lemma 5.11 *For $\mathfrak{g}$ a Lie algebra, G the simply conneced Lie group integrating it and $h : D^n \to PD^n$ any map that sends each point $x \in D^n$ to a path connecting it to the origin, parallel transport*

$$P\exp\left(\int_{h(-)}(-)\right) :\ \Omega^1_{\mathrm{flat}}(D^n,\mathfrak{g}) \xrightarrow{\simeq} C^\infty_*(D^n,G)$$

of flat $\mathfrak{g}$-valued forms along these paths establishes a bijection between these forms and based G-valued functions $f \in C^\infty_*(D^n,G)$ on D^n (sending the origin to the neutral element). The inverse map is pullback of the canonical $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(G,\mathfrak{g})$: $(f \in C^\infty(D^n,G)) \mapsto f^*\theta$.

Higher fundamental groupoids for Lie 1-algebras According to theorem 5.10, the simply connected Lie group G integrating the Lie algebra $\mathfrak{g}$ is the first fundamental group of the classifying space of flat $\mathfrak{g}$-valued forms, $\mathbf{B}G = \Pi_1(S(\mathrm{CE}(\mathfrak{g})))$. Recalling from section 4.2.1 that Π_1 is obtained from Π_ω by dividing out equivalences, we can also consider $\Pi_n(S(\mathrm{CE}(\mathfrak{g})))$ for higher n. When integrating shifted central extensions of Lie algebras in sections 5.2.3 and 5.2.4 these higher fundamental groupoids of $S(\mathrm{CE}(\mathfrak{g}))$ are part of the structure one finds.

Definition 5.12 (path group and group of paths) *For G a Lie group, there are two different natural group structures on the space of based paths (starting at the identity). Write PG for the space of parameterized paths $\gamma : [0,1] \to G$ with $\gamma(0) = e$ and $P'G$ for the space of thin-homotopy classes of such paths with sitting instant at the boundary. The group structure on PG is that obtained by pointwise multiplication in G. The group structure $\circ$ on $P'G$ is given on representatives by translation and concatenation:*

$$\gamma_1 \circ \gamma_2 : (\sigma \in [0,1]) \mapsto \begin{cases} \gamma_1(2\sigma) & \text{for } \sigma \leq 1/2 \\ \gamma_1(1)\gamma_2(2\sigma - 1) & \text{for } \sigma \geq 1/2 \end{cases} .$$

Write ΩG and $\Omega' G$, respectively, for the subgroups for which $\gamma(1) = e$.

Lemma 5.13 *For $\mathfrak{g}$ a Lie algebra, the second fundamental groupoid of its classifying space is*

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}))) = (\Omega' G \to P'G)\,,$$

where on the right we have the crossed module obtained by the inclusion of loops in paths, with the action of paths on loops by conjugation.

Proof. Follows directly from lemma 5.11 and the fact that π_2 of every Lie group is trivial. □

Proposition 5.14 *For $\mathfrak{g}$ any Lie algebra and G its simply connected Lie group, there are weak equivalences* $\mathbf{B}(\Omega G \to PG) \xrightarrow{\simeq} \mathbf{B}G$ *and* $\mathbf{B}(\Omega' G \to P'G) \xrightarrow{\simeq} \mathbf{B}G$.

Proof. Essentially by construction. Notice that the second statement is, by lemma 5.13, a special case of proposition 4.27. □

5.2.2 Integration of $b^{n-1}\mathfrak{u}(1)$

Recall from [132]:

Definition 5.15 (shifted $\mathfrak{u}(1)$) *For all postive $n \in \mathbb{N}$ the L_∞-algebra $b^{n-1}\mathfrak{u}(1)$ is defined to be the L_∞-algebra whose Chevalley-Eilenberg algebra is the free GCA on a single degree n-generator equipped with the trivial differential:*

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) := \left(\wedge^\bullet \underbrace{\langle b\rangle}_{\deg = n}, d = 0\right) .$$

For $n = 1$ this is the ordinary Lie algebra of $\mathfrak{u}(1)$. The classifying space $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$ of flat $b^{n-1}\mathfrak{u}(1)$-valued forms is just the classifying space of closed n-forms

$$S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) = \Omega^{n-1}_{\mathrm{closed}}\,.$$

Therefore the ∞-Lie integration of these L_∞-algebras is governed by the following lemma [** which must be a classical fact, I suppose, but I give a proof nevertheless **]

Lemma 5.16 *For all positive $n \in \mathbb{N}$ every smooth n-form $B_n \in \Omega^n(S^n)$ on the n-sphere whose integral over the n-sphere vanishes, $\int_{S^n} B_n = 0$, extends to a smooth closed n-form $\hat{B} \in \Omega^n_{\mathrm{closed}}(D^{n+1})$ on the $(n+1)$-disk with boundary the n-sphere.*

Proof. If the statement is true for some $(n-1) \in \mathbb{N}$ then it is implied for n as follows: given $B_n \in \Omega^n(S^n)$ choose any smooth surjective map $h : D^n \twoheadrightarrow S^n$ injective away from ∂D^n and consider the pullback form $h^* B_n \in \Omega^n(D^n)$. By the Poincaré lemma there is $A_{n-1} \in \Omega^{n-1}(D^n)$ such that $h^* B_n = dA_{n-1}$. We can find another choice A'_{n-1} from this with the additional property that it vanishes on the boundary of the n-disk, $A'_{n-1}|_{\partial D^n} = 0$: using that the integral of A_{n-1} over the boundary vanishes

$$\int_{\partial D^n} A_{n-1} = \int_{D^n} h^* B_n = \int_{S^n} B_n = 0\,.$$

Then applying the induction hypothesis, we find that A_{n-1} can be extended to a closed $(n-1)$-form $\hat{A}_{n-1} \in \Omega^{n-1}_{\mathrm{closed}}(D^n)$ on the n-disk. Then $A'_{n-1} := A_{n-1} - \hat{A}_{n-1}$ satisfies

$$h^* B_n = dA'_{n-1}\,; \qquad A'|_{\partial D^n} = 0\,.$$

But since A'_{n-1} vanishes on the boundary of D^n, it comes from pullback along h of an $(n-1)$-form

$$A'_{n-1} = h^* a_{n-1}\,; \qquad a_{n-1} \in \Omega^{n-1}(S^n)$$

on the n-sphere, which satisfies $da_{n-1} = B_n$.

To extend B_n to the $n-1$-disk it is now sufficient to extend a_{n-1}. To explicitly do this let $f : [0,1] \to [0,1]$ be a smoothing function, i.e a smooth orientation preserving diffeomorphism of the interval onto itself which is constant in a neighborhood of the boundary of the interval. For r the standard radial coordinate of D^{n+1} for unit radius set $\hat{a}_{n-1} := f \wedge a_{n-1} \in \Omega^{n-1}(D^{n+1})$. Then

$$\hat{B}_n := d\hat{a}_{n-1} \in \Omega^n_{\mathrm{closed}}(D^{n+1})$$

is an extension of the original B_n to a closed n-form on the $(n+1)$-ball.

It remains to show that the induction hypothesis is true for $n = 1$. In that case let $(0 < r \leq 1, 0 \leq s < 2\pi)$ be the standard polar coordinates on D^2 away from the origin, notice that $g(s) := \int_0^s B_1$ is a well-defined function on the circle, because $\int_{S^1} B_1 = 0$, and set $\hat{B}_1 := f \wedge B_1 + f' g \wedge dr$. □

Using this lemma the ∞-Lie integration of $b^{n-1}\mathfrak{u}(1)$ is now immediate:

Proposition 5.17 *For all positive $n \in \mathbb{N}$, the fundamental n-groupoid of the classifying space $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$ of flat n-forms is the $(n-1)$-fold shifted copy of the group $\mathbb{R}$: $\Pi_n(S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))) \simeq \mathbf{B}^n\mathbb{R}$.*

Proof. First it is clear that all spaces of $(k < n)$-disks consist of a single point $P^k_{\mathrm{sit}} S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) = \{0\}$, namely the unique n-form on the $(k < n)$-disk: the 0-form. On the other hand, the space of n-disks with sitting instant in $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$ is the space

$$P^n_{\mathrm{sit}} S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) = \Omega^n(D^n)$$

of smooth n-forms on the n-disk vanishing in a neighborhood of the boundary. The homotopies between two such n-disks

$$P^{n+1}_{\mathrm{sit}} S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) = \Omega^n_{\mathrm{closed}}(D^{n+1})$$

are closed n-forms on the $(n+1)$-ball interpolating between the n-forms on the two bounding hemi-n-spheres.

Lemma 5.16 implies that the map which sends an n-form on D^n to its integral $\int_{D^n} : \Omega^n(D^n) \to \mathbb{R}$ becomes a bijection on the quotient of homotopy classes:

$$\int_{D^n} \; : \; P^n_{\mathrm{htpy}} S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) \xrightarrow{\simeq} \mathbb{R} \; .$$

(Because by Stokes' theorem every two homotopic n-disks are given by n-forms with the same integral and by lemma 5.16 every two n-disks coming from n-forms with the same integral are homotopic in $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$).

It remains to check that the n different compositions of n-paths all correspond to the addition operation in $\mathbb{R}$. This follows simply from the additivity of integration. $\square$

Remark. Notice how this process of forming homotopy classes of n-paths in the classifying space of n-forms amounts to doing *integration without integration* [88]: instead of actually integrating an n-form one sends it to the equivalence class of n-forms that would yield the same integral, if integrated.

For the integration of shifted central extensions in section 5.2.3, 5.2.4 and 5.2.5 it is crucial to notice that

Proposition 5.18 *For $n \geq 2$ all homotopies of n-paths in $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$ are thin.*

Proof. For $n \geq 2$, all differential forms on $S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1)))$ are linear combinations of wedge powers of the canonical n-form ω_n

$$\begin{tikzcd} S(\mathrm{CE}(b^{n-1}\mathfrak{u}(1))) \arrow[rr, "\omega_n"] \arrow[dr, "\simeq"'] & & \Omega^\bullet \\ & \Omega^n_{\mathrm{closed}} \arrow[ur, hook] & \end{tikzcd} .$$

So all forms of degree $d \geq n+1$ actually have degree $d > n+1$ and hence vanish on the $(n+1)$-disk. $\square$

5.2.3 Integration of $\mathfrak{string}(n)$

Recall the definition of the String Lie 2-algebra, the archetypical special case of definition **??**. This appeared originally in [8] and was then used in [9, 68]. See [132] for the context and notation used here.

Definition 5.19 (String Lie 2-algebra) *For $\mathfrak{g}$ an ordinary semisimple Lie algebra with invariant bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathbb{R}$, let μ_3 be the canonical Lie algebra 3-cocycle $\mu_3 := \langle\cdot,[\cdot,\cdot]\rangle$* normalized *such that its left-invariant extension to a 3-form $\hat{\mu}_3 \in \Omega^3_{\mathrm{closed}}(G)$ on the simply connected Lie group G integrating $\mathfrak{g}$ is the image in deRham cohomology of the generator of $H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$. Then the skeletal version of the String Lie 2-algebra of $\mathfrak{g}$ is the Lie 2-algebra denoted $\mathfrak{g}_{\mu_3}$ defined by its CE-algebra as*

$$\mathrm{CE}(\mathfrak{g}_{\mu_3}) := \left(\wedge^\bullet(\underbrace{\mathfrak{g}^*}_{\mathrm{deg}=1} \oplus \underbrace{\langle b \rangle}_{\mathrm{deg}=2}), d \right)$$

with $d|_{\mathfrak{g}^*} = d_{\mathfrak{g}}$ and $d : b \mapsto \mu_3 \in \mathfrak{g}^* \wedge \mathfrak{g}^* \wedge \mathfrak{g}^*$.

Remark. For every $\lambda \in \mathbb{R}$ the rescaled cocycle $\lambda\mu_3$ is still a cocycle and still defines a Lie 2-algebra $\mathfrak{g}_{\lambda\mu_3}$. The condition that μ_3 be normalized such that it generates $H^3(G,\mathbb{Z})$ is an integrability condition in the sense of definition 4.30, as will become clear in the following.

Theorem 5.20 ([9]) *The skeletal Lie 2-algebra $\mathfrak{g}_\mu$ is equivalent, $\mathfrak{g}_{\mu_3} \simeq \mathfrak{string}(\mathfrak{g})$, to the strict Lie 2-algebra $\mathfrak{string}(\mathfrak{g}) := (\hat{\Omega}\mathfrak{g} \to P\mathfrak{g})$, where $\hat{\Omega}\mathfrak{g}$ is the Kac-Moody central extension of the loop Lie algebra of $\mathfrak{g}$ at level 1.*

This result was obtained in [9] by guessing the form of $\mathfrak{string}(\mathfrak{g})$ and then constructing an explicit equivalence. Strict Lie n-algebras, being crossed complexes of Lie algebras, can be integrated simply by integrating them termwise to crossed complexes of Lie groups. The result of this termwise integration from [9] is the 2-group $\mathrm{String}(G)_{\mathrm{BCSS}}$ from definition 5.5.

We now integrate $\mathfrak{g}_{\mu_3}$ in the more systematic way by forming the second fundamental 2-groupoid of the classifying space of flat $\mathfrak{g}_{\mu_3}$-valued forms.

Lemma 5.21 *For $n \in \mathbb{N}$ maps from the n-ball into the classifying space $S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))$ are in bijection with pairs (f, B) consisting of smooth maps $f : D^n \to G$ and 2-forms $B \in \Omega^2(D^n)$ trivializing the pullback of μ_3 along f, $dB = f^*\mu_3$.*

Proof. By lemma 5.11. □

Definition 5.22 ($\mathrm{String}'(G)$) *We write* $\mathrm{String}'(G)$ *for the strict 2-group defined by*

$$\mathbf{B}\mathrm{String}'(G) := \Pi_2(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) .$$

This is essentially the procedure described in [68], only that we form the strict fundamental 2-groupoid instead of a weak fundamental ∞-groupoid of definition 4.21.

To see what $\mathrm{String}(G)$ is like, first consider this for $\mathfrak{g}$ instead of $\mathfrak{g}_{\mu_3}$:

Lemma 5.23 *For $\mathfrak{g}$ a Lie 1-algebra and G the simply connected Lie group integrating it, the second fundamental 2-goupoid $\Pi_2(S(\mathrm{CE}(\mathfrak{g})))$ of the classifying space of flat $\mathfrak{g}$-valued forms is*

$$\Pi_2(S(\mathrm{CE}(\mathfrak{g}))) = (\Omega' G \to P' G) .$$

Proof. This is a direct consequence of lemma 5.11. □

Proposition 5.24 *The strict 2-group from definition 5.22 comes from the crossed module $\hat{\Omega}' G \xrightarrow{h'} P' G$, where*

- *$P'G$ is the group whose elements are* thin-homotopy *classes of based smooth paths in G and whose composition is obtained by translating one path so that its basepoint matches the other path's endpoint and then concatenating;*

- *$\Omega' G$ is the group whose elements are equivalence classes of pairs (d, x) consisting of* thin homotopy *classes of disks $d : D^2 \to G$ in G with sitting instant at a chosen point on the boundary which is sent to the neutral element. Also $x \in \mathbb{R}/\mathbb{Z}$. Composition is by* gluing *of disks at the baseboint. Two disks are taken to be equivalent if their boundary has the same thin homotopy classes and if the labels x and x' differ, in $\mathbb{R}/\mathbb{Z}$ by the integral $\int_{D^3} f^*\mu_3$ over any 3-ball $f : D^3 \to G$ cobounding the two disks.*

Proof. The 1-morphisms are thin-homotopy classes of 1-paths in $S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))$, which are $\mathfrak{g}$-valued 1-forms on the inerval modulo thin homotopy. By lemma 5.11 this are based thin-homotopy classes of paths in the simply connection Lie group G integrating $\mathfrak{g}$

$$P_{\mathrm{thtpy}} S(\mathrm{CE}(\mathfrak{g}_{\mu_3})) = P_{\mathrm{thtpy}} S(\mathrm{CE}(\mathfrak{g})) = (P_{\mathrm{thtpy}})_* G .$$

Composition of paths corresponds to gluing intervals with their 1-forms, which corresponds to the composition of paths in G as stated. The 2-morphisms are homotopy classes of 2-paths in $S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))$. First consider thin-homotopy classes of such 2-paths: Representatives of these are pairs consisting of a flat $\mathfrak{g}$-valued 1-form and a 2-form $B \in \Omega^2(D^2)$ on the disk, the latter being the image of $b \in \mathrm{CE}(\mathfrak{g}_{\mu_3})$. A thin homotopy between two such pairs is an extension of these tuples to 3-disks interpolating between two such 2-disks. Thinness requires all 3-forms to vanish on this 3-ball and hence the extension of B to the 3-ball to be flat. By lemma 5.16 and proposition 5.17 this means that of the 2-form B precisely its integral $\int_{D^2} B$ survives in thin homotopy equivalence classes. So again with lemma 5.11 we find that thin-homotopy classes of 2-paths are given by pairs (Σ, r) consisting of thin-homotopy classes Σ of disks in G together with a real number r.

Still using lemma 5.11, a homotopy between pairs (Σ_i, r_i) is a 3-disk $g : D^3 \to G$ in G with $\Sigma_{1,2}$ the two hemispheres of its boundary, such that the 2-form representatives $B_{1,2}$ are interpolated by $\hat{B} \in \Omega^2(D^3)$ satisfying the equation

$$d\hat{B} = g^*\mu(\theta) = g^*\langle \theta \wedge [\theta \wedge \theta] \rangle \,,$$

which is the image of the equation defining the differential in $\mathrm{CE}(\mathfrak{g}_{\mu_3})$ in definition 5.19. This means that r_2 and r_1 are in the same equivalence class if

$$r_2 - r_1 = \int_{D^2_{\mathrm{in}}} g^* B_1 - \int_{D^2_{\mathrm{out}}} g^* B_2 = \int_{D^3} g^*\mu$$

for all g^*. We need to show that, conversely, for all pairs B_1, B_2 satisfying this condition there is a $\hat{B} \in \Omega^2(D^3)$ interpolating between them representing a 3-morphism in $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3})))$: the 3-ball $g : D^3 \to G$ is to be thought of as a map $g : [0,1]^3 \to G$ with sitting instants on $([0,\epsilon) \cup (1-\epsilon, 1)) \times [0,1]^2$

$$\begin{tikzcd}
& (0,1,0) \arrow[r, dash] & (\epsilon,1,0) \arrow[rr, dash] \arrow[dl, dash, dashed] & & (1-\epsilon,1,0) \arrow[r, dash] \arrow[dl, dash, dashed] & (1,1,0) \arrow[dl, dash] \arrow[dd, dash] \\
(0,0,0) \arrow[ur, dash, "s^2"] \arrow[r, "s^1"] \arrow[dd, "s^3"'] & (\epsilon,0,0) \arrow[rr, dash] \arrow[dd, dash, dashed] & & (1-\epsilon,0,0) \arrow[r, dash] \arrow[dd, dash, dashed] & (1,0,0) \arrow[dd, dash] & \\
& & & & & (1,1,1) \arrow[dl, dash] \\
(0,0,1) \arrow[r, dash] & (\epsilon,0,1) \arrow[rr, dash] & & (1-\epsilon,0,1) \arrow[r, dash] & (1,0,1) &
\end{tikzcd}
.$$

This requires that on $([0,\epsilon) \cup (1-\epsilon,1)) \times [0,1]^2$ $\hat{B}$ vanishes . Since g is constant on $([0,\epsilon) \cup (1-\epsilon,1)) \times [0,1]^2$ this is achieved by setting

$$\hat{B}(s_1,s_2,s_3)(\vec{v}_1,\vec{v}_2) := \int_0^{s^3} g^*\mu(s^1,s^2,\sigma)(\vec{v}_1,\vec{v}_2,\frac{\partial}{\partial s^3}) + B_1(s^1,s^2)(p_*\vec{v}_1, p_*\vec{v}_2)\,,$$

where $p : [0,1]^3 \to [0,1]^2$ is the projection $(s^1,s^2,s^3) \mapsto (s^1,s^2)$.

By the integrality of μ, for fixed Σ_i this difference is unique modulo $\mathbb{Z}$. And all values in $\mathbb{Z}$ appear for some choice of g because there is always the horizonatal composite of the 3-morphism g by a 3-morphism with source and target the constant 2-path representing an element in $\pi_3(G) = \mathbb{Z}$. This means that r_i represent elements of $U(1)$. In terms of such, our equivalence relation which equates disks in G labeled by elements in $U(1)$ coincides precisely with that defining the Kac-Moody central extension $\hat{\Omega}G$ of loops in G. $\square$

We show below that these two strict models of the String 2-group, $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$ and $\mathrm{String}_{\mathrm{BCSS}}(G)$ are ana-equivalent. In the process of this proof we naturally encounter the third strict model of the String 2-group, $\mathrm{String}_{\mathrm{Mick}(G)}$ from definition 5.6.

The relation between the various ways to integrate $\mathfrak{g}_{\mu_3}$ is depicted in figure 10.

Recall the definition 2.78 of ana-ω-functors.

$$
\begin{tikzcd}
\mathsf{StrictLie}n\mathsf{Groups} & \mathsf{StrictLie}n\mathsf{Groups} & \mathsf{CrossedComplexes} \arrow[l, "\simeq"'] \\
\mathsf{Lie}n\mathsf{Algebras} \arrow[u, "\Pi_n \circ S \circ \mathrm{CE}"] & \mathsf{StrictLie}n\mathsf{Algebras} \arrow[l, hook'] \arrow[u, dashed] \arrow[r, "\simeq"] & \mathsf{DifferentialCrossedComplexes} \arrow[u, "\text{termwise ordinary 1-Lie integration}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3}) \arrow[r, "\text{prop. 5.25}", "\simeq"'] & \mathrm{String}_{\mathrm{Mick}}(G) \arrow[r, mapsto, "\text{prop. 5.7}", "\simeq"'] & \mathrm{String}_{\mathrm{BCSS}}(G) & (\hat{\Omega} G \to PG) \arrow[l] \\
\mathfrak{g}_{\mu_3} \arrow[u, "\text{prop. 5.24}"] \arrow[urr, "\text{integration in [9]}"] \arrow[rr, "\simeq"] & & \mathfrak{string}(\mathfrak{g}) \arrow[u, dashed] \arrow[r, mapsto] & (\hat{\Omega}\mathfrak{g} \to P\mathfrak{g}) \arrow[u]
\end{tikzcd}
$$

Figure 10: **Integration of the String Lie 2-algebra.** Strict Lie n-algebras are equivalent to crossed complexes of ordinary Lie algebras. Applying ordinary 1-Lie integration these integrate termwise to crossed complexes of Lie groups. These, in turn, are equivalent to strict Lie n-groups. Using this method, a weak Lie n-algebra can be integrated if one has an equivalence of L_∞-algebras with a strict Lie n-algebra. This way the String Lie 2-algebra $\mathfrak{g}_{\mu_3}$ was integrated in [9]. In contrast to that, [68] integrated $\mathfrak{g}_{\mu_3}$ by computing the *weak* fundamental ∞-groupoid $\Pi_\infty(S(\mathrm{CE}(\mathfrak{g}_{\mu_3})))$ of the classifying space of flat $\mathfrak{g}_{\mu_3}$-valued forms. Here we consider something in between by computing the *strict* fundamental 2-groupouid $\Pi_2(S(\mathrm{CE}(\mathfrak{g}_{\mu_3})))$. This in fact the strict 2-group which is *implicit* in [35], as discussed in section 5.8.2. We construct a weak ana-equivalence to the strict 2-group $\mathrm{String}_{\mathrm{BCSS}}(G)$ from [9]. In doing so we find yet another different but weakly equivalent strict model of the String 2-group, denoted $\mathrm{String}_{\mathrm{Mick}}(G)$, which is built not using the Kac-Moody cocycle but Mickelsson's cocycle.

Proposition 5.25 *The strict 2-group* $\mathrm{String}_{\mathrm{Mick}}(G)$ *from definition 5.6 is ana-equivalent to the model* $\mathrm{String}'(G)$ *from definition 5.22:*

$$
\mathrm{String}_{\mathrm{Mick}}(G) \xrightarrow{\simeq} \Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})
$$

Proof. We define a weak 2-functor $F : \mathrm{String}_{\mathrm{Mick}}(G) \to \Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$, which by proposition 2.60 corresponds to a strict ana-2-functor. Its action on 1- and 2-morphisms is obvious: it sends parameterized paths $\gamma : [0,1] \to G$ to their thin-homotopy equivalence class

$$
F : \gamma \mapsto [\gamma]
$$

and similarly for parameterized disks. On the $\mathbb{R}/\mathbb{Z}$-labels of these disks it acts as the identity.

The subtle part is the compositor measuring the coherent failure of this assignment to respect composition: Define the components of this compositor for any two parameterized based paths $\gamma_1, \gamma_2 : [0,1] \to G$ with pointwise product $(\gamma_1 \cdot \gamma_2) : [0,1] \to G$ and images $[\gamma_1], [\gamma_2], [\gamma_1 \cdot \gamma_2]$ in thin homotopy classes to be represented by a parameterized disk in G

$$
\begin{tikzcd}
& \cdot \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\Downarrow d_{\gamma_1,\gamma_2}"] & \\
\cdot \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_1 \cdot \gamma_2"'] & {} & \cdot
\end{tikzcd}
$$

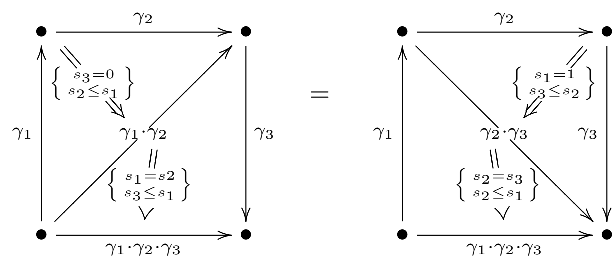
equipped with a label $x_{\gamma_1,\gamma_2} \in \mathbb{R}/\mathbb{Z}$ to be determined. Notice that this triangle is a diagram in $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$ so that composition of 1-morphisms is concatenation $\gamma_1 \circ \gamma_2$ of paths as in definition 5.12. A suitable disk in G is obtained via the map

$$D^2 \xrightarrow{a} [0,1]^2 \xrightarrow{(s_1,s_2)\mapsto \gamma_1(s_1)\cdot\gamma_2(s_2)} G \;,$$

where a is a smooth surjection onto the triangle $\{(s_1,s_2)|s_2 \leq s_1\} \subset [0,1]^2$ such that the lower semi-circle of $\partial D^2 = S^1$ maps to the hypotenuse of this triangle. The coherence law for this compositor for all triples of parameterized paths $\gamma_1, \gamma_2, \gamma_3 : [0,1] \to G$ amounts to the following: consider the map

$$D^3 \xrightarrow{a} [0,1]^3 \xrightarrow{(s_1,s_2,s_3)\mapsto \gamma_1(s_1)\cdot\gamma_2(s_2)\cdot\gamma_3(s_3)} G \;,$$

where the map a is a smooth surjection onto the tetrahedron $\{(s_3 \leq s_2 \leq s_1)\} \subset [0,1]^3$. Then the coherence condition



requires that the integral of the canonical 3-form on G pulled back to the 3-ball along these maps accounts for the difference in the chosen labels of the disks involved:

$$\int_{D^3} (b\circ a)^*\mu = \int_{s_3\leq s_2\leq s_1} (\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = x_{\gamma_1,\gamma_2} + x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_1,\gamma_2\cdot\gamma_3} - x_{\gamma_2,\gamma_3} \quad \in \mathbb{R}/\mathbb{Z}\,.$$

(Notice that there is no further twist on the right hand side because whiskering in $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$ does not affect the labels of the disks.) To solve this condition, we need a 2-form to integrate over the triangles. This is provided by the degree 2 component of the simplicial realization $(\mu,\nu) \in \Omega^3(G)\times\Omega^2(G\times G)$ of the first Pontryagin form on BG as described in proposition 6.35. So, define the label assigned by our compositor to the disks considered above by

$$x_{\gamma_1,\gamma_2} := \int_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu\,.$$

To show that this assignment satisfies the above condition, use the closedness of (μ,ν) in the complex of simplicial forms on BG, recalled in definition 6.33: $\delta\mu = d\nu$ and $\delta\nu = 0$. From this one obtains

$$(\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = -d(\gamma_1\cdot\gamma_2,\gamma_3)^*\nu = -d(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu$$

and

$$(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu = (\gamma_1\cdot\gamma_2,\gamma_3)^*\nu + (\gamma_1,\gamma_2)^*\nu - (\gamma_2,\gamma_3)^*\nu\,.$$

Now we compute as follows: Stokes' theorem gives

$$\int\limits_{s_3\leq s_2\leq s1} (\gamma_1\cdot\gamma_2\cdot\gamma_3)^*\mu = \left(\int\limits_{s_3=0,s_2\leq s_1} + \int\limits_{s_1=s_2,s_3\leq s_1} - \int\limits_{s_1=1,s_3\leq s_2} - \int\limits_{s_2=s_3,s_2\leq s_1}\right)(\gamma_1,\gamma_2\cdot\gamma_3)^*\nu\,.$$

The first integral is manifestly equal to x_{γ_1,γ_2}. The last integral is manifestly equal to $-x_{\gamma_1,\gamma_2\cdot\gamma_3}$. For the remaining two integrals we rewrite

$$\cdots = x_{\gamma_1,\gamma_2} - x_{\gamma_1,\gamma_2\cdot\gamma_3} + \left(\int\limits_{s_1=s_2,s_3\leq s_1} - \int\limits_{s_1=1,s_3\leq s_2}\right)\left((\gamma_1\cdot\gamma_2,\gamma_3)^*\nu + (\gamma_1,\gamma_2)^*\nu - (\gamma_2,\gamma_3)^*\nu\right)\,.$$

The first term in the integrand now manifestly yields $x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_2,\gamma_3}$. The second integrand vanishes on the integration domain. The third integrand, finally, gives the same contribution under both integrals and thus drops out due to the relative sign. So in total what remains is indeed

$$\cdots = x_{\gamma_1,\gamma_2} - x_{\gamma_1,\gamma_2\cdot\gamma_3} + x_{\gamma_1\cdot\gamma_2,\gamma_3} - x_{\gamma_2,\gamma_3}\,.$$

This establishes the coherence condition for the compositor.

Finally we need to show that the compositor is compatible with the horizontal composition of 2-morphisms. We consider this in two steps, first for the horizontal composition of two 2-morphisms both starting at the identity 1-morphism in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)$ – this is the product in the loop group $\hat{\Omega} G$ centrally extended using Mickelsson's cocycle – then for the horizontal composition of an identity 2-morphism in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)$ with a 2-morphism starting at the identity 1-morphisms – this is the action of PG on $\hat{\Omega} G$. These two cases then imply the general case.

- Let (d_1, x_1) and (d_2, x_2) represent two 2-morphisms in $\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)$ starting at the identity 1-morphisms. So

  $$d_i : [0,1] \to \Omega G$$

  is a based path in loops in G and $x_i \in U(1)$. We need to show that

  $$\begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Downarrow (d_1{,}x_1)"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow (d_2{,}x_2)"] \arrow[d, phantom, "\Downarrow (d_{\gamma_1,\gamma_2}{,}x_{\gamma_1,\gamma_2})"] & \bullet \\ & {} & \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow (d_1\cdot d_2{,}x_1+x_2+\rho(d_1{,}d_2))"] & \bullet \\ & {} & \end{tikzcd}$$

  as a pasting diagram equation in $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$. Here on the left we have gluing of disks in G along their boundaries and addition of their labels, while on the right we have the pointwise product from definition 5.6 of labeled disks as representing the product of elements $\hat{\Omega} G$.

  There is an obvious 3-ball interpolating between the disk on the left and on the right of the above equation:

  $$(\{s_2 \leq s_1\} \subset [0,1]^3) \to G$$
  $$(s_1, s_2, t) \mapsto (d_1(t, s_1) \cdot d_2(t, s_2))$$

  $$\begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Downarrow \{s_2=0\}"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow \{s_1=0\}"] \arrow[d, phantom, "\Downarrow \{t=1\}"] & \bullet \\ & {} & \end{tikzcd} , \quad \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id"] \arrow[d, phantom, "\Downarrow \{s_1=s_2\}"] & \bullet \\ & {} & \end{tikzcd} .$$

  The compositor property demands that the integral of the canonical 3-form over this ball accounts for the difference between x_{γ_1,γ_2} and $\rho(\gamma_1, \gamma_2)$

  $$\rho(d_1, d_2) = \int\limits_{\substack{s_2 \leq s_1 \\ 0 \leq t \leq 1}} (d_1 \cdot d_2)^* \mu + \int\limits_{s_2 \leq s_1} (\gamma_1, \gamma_2)^* \nu\,.$$

  Now use again the relation between μ and $d\nu$ to rewrite this as

  $$\cdots = \int\limits_{\substack{s_2 \leq s_1 \\ 0 \leq t \leq 1}} ((d_1)^* \mu + (d_2)^* \mu - d(d_1, d_2)^* \nu) + \int\limits_{s_2 \leq s_1} (\gamma_1, \gamma_2)^* \nu\,.$$

The first two integrands vanish. The third one leads to boundary integrals

$$\cdots = -\left(\int\limits_{s_2=0} + \int\limits_{s_1=0} \right) (d_1, d_2)^*\nu - \int\limits_{\substack{t=1\\ s_2\leq s_1}} (d_1, d_2)^*\nu + \int\limits_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu + \int\limits_{\substack{0\leq t\leq 1\\ s_1=s_2}} (d_1, d_2)^*\nu \,.$$

The first two integrands vanish on their integration domain. The third integral cancels with the fourth one. The remaining fifth one is indeed the 2-cocycle on $P\Omega G$ which from definition 5.22.

- The second case is entirely analogous: for γ_1 a path and (d_2, x_2) a centrally extended loop we need to show that

$$\begin{tikzcd}[column sep=large, row sep=large]
\bullet \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow (d_2,x_2)"] \arrow[d, phantom, "\Downarrow (d_{\gamma_1,\gamma_2},x_{\gamma_1,\gamma_2})"] & \bullet \\
 & {} &
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=large]
\bullet \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow (\gamma_1\cdot d_2,x_1+x_2+\lambda(\gamma_1,d_2))"] & \bullet \\
 & {} &
\end{tikzcd}$$

as a pasting diagram equation in $\mathbf{B}\mathrm{String}'(G)$.

There is an obvious 3-ball interpolating between the disk on the left and on the right of the above equation:

$$(\{s_2 \leq s_1\} \subset [0,1]^3) \to G$$

$$(s_1, s_2, t) \mapsto (\gamma_1(s_1)\cdot d_2(t, s_2))$$

$$\begin{tikzcd}[column sep=large, row sep=large]
\bullet \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_1"'] \arrow[r, phantom, "\Downarrow \{s_2=0\}"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow \{s_1=0\}"] \arrow[d, phantom, "\Downarrow \{t=1\}"] & \bullet \\
 & {} &
\end{tikzcd}
\,,\quad
\begin{tikzcd}[column sep=large, row sep=large]
\bullet \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend right=70, "\gamma_1\cdot\gamma_2"'] & \bullet \arrow[r, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow \{s_1=s_2\}"] & \bullet \\
 & {} &
\end{tikzcd}\,.$$

The compositor property demands that the integral of the canonical 3-form over this ball accounts for the difference between x_{γ_1,γ_2} and $\lambda(\gamma_1,\gamma_2)$

$$\lambda(\gamma_1, d_2) = \int\limits_{\substack{s_2\leq s_1\\ 0\leq t\leq 1}} (d_1\cdot d_2)^*\mu + \int\limits_{s_2\leq s_1} (\gamma_1,\gamma_2)^*\nu \,.$$

This is essentially the same computation as before, so that the result is

$$\cdots = \int\limits_{\substack{0\leq t\leq 1\\ s_1=s_2}} (\gamma_1, d_2)^*\nu \,.$$

This is indeed the quantity from definition 5.22.

$\square$

Remark (strict models of the String 2-group as multiplicative bundle gerbes). As plain groupoids, forgetting their monoidal structure, the three 2-groups $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$, $\mathrm{String}_{\mathrm{BCSS}}(G)$ and $\mathrm{String}_{\mathrm{Mick}}(G)$ are the *tautological bundle gerbe* on G as defined in [117], the only difference being that for $\mathrm{String}_{\mathrm{Mick}}(G)$ and $\mathrm{String}_{\mathrm{Mick}}(G)$ the bundle gerbe is defined with respect to the surjective submersion given by the space of parameterized paths, whereas for $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$ the surjective submersion is the quotient of this by thin homotopy of paths. There are obvious ("stable") isomorphism between these 2-groups as bundle gerbes. By [** ... a list of known results to be named it should apparently be true that... **] for G compact simple and simply connected, multiplicative bundle gerbes on G are equivalent as multiplicative bundle gerbes already if they are equivalent as plain bundle gerbes. This yields an alternative way to understand the above situation.

String 1-group. With the 1-groupoid *String*(G) defined by $\mathbf{B}\mathrm{String}(G) := \Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_\mu)$, the spatial realization $|\mathrm{String}(G)|$, according to definition 4.15 should be a 1-group internal to Spaces.

[** the following 1-group should be related to $|\mathrm{String}(G)|$ **]

Definition 5.26 (String 1-group) *For G, $\mathfrak{g}$ and μ_3 as in definition 5.19, define a group G_{μ_3} internal to Spaces as follows. Consider the simplicial forms $(\mu_3, \nu_2) \in \Omega^3(G) \times \Omega^2(G \times G)$ representing the first Pontryagin 4-form on BG, as in proposition 6.35. Then the set of plots of G_{μ_3} over any test domain U is defined to be*

$$G_{\mu_3} : U \mapsto \{(f, B) | f \in C^\infty(U, G),\ B \in \Omega^2(U),\ f^*\mu_3 = dB\}\ ,$$

with the pullback operation being the obvious one. The product morphism $G_\mu \times G_\mu \to G_\mu$ is given over test domain U by $((f_1, B_1), (f_2, B_2)) \mapsto (f_1 \cdot f_2, B_1 + B_2 - (f_1, f_2)^*\nu)$. The identity element $\mathrm{pt} \to G_\mu$ is given by the plot $(e, 0)$. The inverse map $G_\mu \to G_\mu$ is given over U by $(f, B) \mapsto (f^{-1}, -B)$.

Lemma 5.27 *The object G_μ defined this way is indeed a group internal to Spaces.*

Proof. The product is well defined by $\delta\mu_3 = d\nu$, which implies that $d(B_1 + dB_2 + (f_1, f_2)^*\nu_2) = f_1^*\mu_3 + f_2^*\mu_3 - (f_1, f_2)^* d\nu_2$ can be rewritten as $\cdots = (f_1 \cdot f_2)^*\mu_3$. Associativity of this product requires that for all $f_1, f_2, f_3 \in C^\infty(U, G)$ we have $(f_1, f_2)^*\nu_2 + (f_1 \cdot f_2, f_3)^*\nu_2 = (f_2, f_3)^*\nu_2 + (f_1, f_2 \cdot f_3)^*\nu_2$. This follows from $\delta\nu_2 = 0$. $\square$

Proposition 5.28 *G_μ fits into a short exact sequence of groups internal to Spaces:*

$$1 \to S(\mathrm{CE}(b\mathfrak{u}(1))) \to G_\mu \to G \to 1$$

Lemma 5.29 *Let $[\mu_3]$ denote the generator in $H^3(G, \mathbb{Z})$ that μ_3 is the deRham image of. Think of this as a homotopy class of maps $G \to K(\mathbb{Z}, 3)$ being an element in $[G, K(\mathbb{Z}, 3)]$. By postcomposition this yields a map*

$$[\mu_3] : [S^3, G] \to [S^3, K(\mathbb{Z}, 3)] \simeq H^3(S^3, \mathbb{Z})\,.$$

This is an isomorphism.

Proof. [...] $\square$

Proposition 5.30 *The fundamental groups $\pi_0(G_\mu)$ and $\pi_1(G_\mu)$ are trivial. For $2 \leq k \leq 3$, maps from k-spheres to G which factor through G_μ homotopically trivial.*

Proof. Follows from lemma 5.29. $\square$

Corollary 5.31 *The 3-group $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3})))$ is surjectively equivalent to $\mathbf{B}\mathrm{Sting}'(G)$:*

$$\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) \xrightarrow{\simeq} \mathbf{B}\mathrm{String}'(G)\ .$$

Proof. The 1-morphisms of $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3})))$ are the same as those of $\mathbf{B}\mathrm{String}'(G)$, namely thin-homotopy classes of paths in G starting at the identity, 2-morphisms are thin-homotopy classes of disks cobounding these paths and labeled with an element $r \in \mathbb{R}$, and 3-morphisms are *homotopy* classes $[V]$ of 3-balls $V : D^3 \to G$ such that $V^*\mu_3$ is exact and cobounding disks the difference of whose lables r_1, r_2 is $r_2 - r_1 = \int_{D^3} V^*\mu_3$.

The functor $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) \xrightarrow{\simeq} \mathbf{B}\mathrm{String}'(G)$ just divides out 3-morphisms. This ω-functor is manifestly k-surjective for $0 \leq k \leq 3$. As a direct consequence of lemma 5.29 every 3-morphism is parallel only to itself and hence the ω-functor is injective in degree 3 and hence also 4-surjective. □

5.2.4 Integration of $\mathfrak{fivebrane}(n)$

Definition 5.32 *For $\mathfrak{g} = \mathfrak{so}(n)$ and $G = \mathrm{Spin}(n)$ we abbreviate $\mathfrak{string}(n) := \mathfrak{string}(\mathfrak{so}(n))$ and* $\mathrm{String}(n) := \mathrm{String}(\mathrm{Spin}(n))$.

Definition 5.33 (The $\mathfrak{fivebrane}(n)$ Lie 6-algebra) *We define $\mathfrak{fivebrane}(n) := (\mathfrak{so}(n)_{\mu_3})_{\mu_7}$.*

Theorem 5.34 (Integration of $\mathfrak{fivebrane}(n)$ to the $\mathrm{Fivebrane}(n)$-6-group) *The 6-group*

$$\mathbf{B}\mathrm{Fivebrane}(n) := \Pi_6 \circ S \circ \mathrm{CE}(\mathfrak{fivebrane}(n))$$

is as follows: ...

5.2.5 Integration of $\mathfrak{sugra}(11)$

In [132] the super Lie 3-algebra $\mathfrak{sugra}(11)$ was described, whose Chevalley-Eilenberg algebra is used in [] for the description of eleven-dimensional supergravity. Integrating that yields...

5.3 ∞-Lie differentiation

According to section 4.3 ∞-Lie differentiation sends ω-groupoids to the DGCA of differential forms on their classifying spaces. See figure 7.

5.3.1 L_∞-Differentiation of Lie 1-groups

Recall the operation of spatial realization $|\cdot| : \omega\mathsf{Categories}(\mathsf{Spaces}) \to \mathsf{Spaces}$ from 4.2.1.

Theorem 5.35 ([136]) *For $\mathfrak{g}$ an ordinary Lie algebra and G a Lie group integrating it we have*

$$|\mathbf{B}G| \simeq S(\mathrm{CE}(\mathfrak{g}))\,.$$

In particular this means that $|\mathbf{B}G|$ always produces the realization corresponding to the simply connected cover of G.

5.3.2 L_∞-Differentiation of Lie 2-groups

In [137] it was proven that

Theorem 5.36 ([137]) *For $\mathfrak{g}_2 = (\mathfrak{h} \to \mathfrak{g})$ a strict finite-dimensional Lie 2-algebra and $G_2 = (H \to G)$ a strict Lie 2-group integrating it, we have $|\mathbf{B}G_2| \simeq S(\mathrm{CE}(\mathfrak{g}_2))$.*

This can now be seen as a consequence of the combination of 5.10 and corollary 4.19. Using the results of section 5.2.3 we obtain generalizations of this statement involving weak Lie 2-algebras:

Corollary 5.37 *Let $\mathfrak{g}_{\mu_3} \simeq \mathrm{string}(\mathfrak{g})$ be as in definition 5.19 and theorem 5.20, respectively and let $\Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3}) \simeq \mathrm{String}_{\mathrm{Mick}}(G)$ be as in definition 5.22 and definition 5.6. Then*

$$|\mathbf{B}\mathrm{String}_{\mathrm{Mick}}(G)| = S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))\,.$$

5.4 Principal ω-bundles

5.4.1 Principal 1-bundles

The following example spells out the familiar description of ordinary principal G-bundles for G an ordinary group in the language of nonabelian cohomology with coefficients in ω-category valued presheaves. If G is an ordinary (1-)group then the 1-groupoid valued constant presheaf (definition **??**) $\mathrm{TrivBund}_G(X) = \mathbf{B}C^\infty(X,G)$ has as morphisms the continuous maps from X to G, with composition of morphisms the pointwise product of such maps. Given a principal G-bundle $P \to X$ we can locally trivialize it on a good cover $Y := \bigsqcup_i U_i$ of X by open subsets $\{U_i \subset X\}$, i.e. by identifying the pullback $\pi^* P$ along the obvious projection map $\pi : Y \to X$ (the restriction of P to the subsets in the cover) with the unique trivial G-bundle $\mathrm{triv} := Y \times G$ on Y, which we identify with the unique object of $\mathrm{TrivBund}_G(Y)$

$$\mathrm{triv} \in \mathrm{Obj}(\mathrm{TrivBund}_G(Y))\,.$$

The particular choice of identification leads to a gauge transformation between the two copies of this trivial bundle over double overlaps, coming from a continuous function $g : (Y \times_X Y = \bigsqcup_{i,j} U_i \cap U_j) \to G$, whose restriction to each double overlap is written $g_{ij} := g|_{U_i \cap U_j}$. By the above we can identify this with a morphism

$$(\ \pi_1^*\mathrm{triv} \xrightarrow{\ g\ } \pi_2^*\mathrm{triv}\ ) \ \in\ \mathrm{Mor}(\mathrm{TrivBund}_G(Y \times_X Y))\,,$$

where $Y \times_X Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y$ are the two projections from double intersections to elements of the cover. If we write $(x,i) \in U_i$ for a point $x \in X$ regarded as an element of $U_i \subset X$ and $(x,i,j) \in U_i \cap U_j$ for the same point regarded as an element of the double intersection $U_i \cap U_j$ then these projections are simply given by $\pi_1 : (x,i,j) \mapsto (x,i)$ and $\pi_2 : (x,i,j) \mapsto (x,j)$. Finally, the function $g : Y \times_X Y \to G$ will satisfy the *cocycle condition* $\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g$, where now the π_{nm} are the three possible projections

$$\begin{tikzcd} Y \times_X Y \times_X Y \arrow[r, shift left=2, "\pi_1"] \arrow[r, "\pi_2" description] \arrow[r, shift right=2, "\pi_3"'] & Y \times_X Y \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \end{tikzcd}$$

from triple overlaps to double overlaps. In terms of morphisms in $\mathrm{TrivBund}_G(\cdot)$ the cocycle condition says that the triangle

$$\begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^* g"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & & \pi_3^*\mathrm{triv} \end{tikzcd}$$

commutes in $\mathrm{TrivBund}_G(Y \times_X Y \times_X Y)$, i.e. that it is filled by a (necessarily identity) 2-morphism. In terms of the component maps of the transformations $\pi_{nm}^* g$ this says that all the triangles

$$\forall (x,i,j,k) \in \bigsqcup_{i,j,k} U_i \cap U_j \cap U_k : \quad \begin{tikzcd} & \bullet \arrow[dr, "g_{jk}(x)"] & \\ \bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] & & \bullet \end{tikzcd}$$

commute in $\mathbf{B}G$. We summarize all this by saying that the tuple

$$\left(\begin{array}{ll} \mathrm{triv} & \in \ 0\mathrm{Mor}(\mathrm{TrivBund}_G(Y))\,, \\ \left(\pi_1^*\mathrm{triv} \xrightarrow{g} \pi_2^*\mathrm{triv} \right) & \in \ 1\mathrm{Mor}(\mathrm{TrivBund}_G(Y \times_X Y))\,, \\ \left(\begin{tikzcd} & \pi_2^*\mathrm{triv} \arrow[dr, "\pi_{23}^* g"] \arrow[d, phantom, "\|"] & \\ \pi_1^*\mathrm{triv} \arrow[ur, "\pi_{12}^* g"] \arrow[rr, "\pi_{13}^* g"'] & {} & \pi_3^*\mathrm{triv} \end{tikzcd} \right) & \in \ 2\mathrm{Mor}(\mathrm{TrivBund}_G(Y \times_X Y \times_X Y))\,, \end{array} \right) \in \mathrm{Obj}(\mathrm{Desc}(Y^\bullet, \mathrm{TrivBund}_G(\cdot)))$$

is an object in the *descent category* $\mathrm{Desc}(Y^\bullet, \mathrm{TrivBund}_G(\cdot))$ of G-bundles relative to Y. The terminology indicates that this data ensures that a trivial G-bundle on Y "descends" down from Y to a G-bundle P on X

$$\begin{tikzcd} Y \arrow[d, "\pi"] \\ X \end{tikzcd} .$$

The morphisms in the descent category are gauge transformations of such cocycle data given by functions $h : Y \to G$. Such a gauge transformation relates the transition function $g = \{g_{ij}\}$ with another transition function $g = \{g'_{ij}\}$ if $g_{ij} \cdot h_j = h_i \cdot g'_{ij}$. Diagrammatically, this means that the tuple

$$\left(\begin{array}{ll} \left(\mathrm{triv} \xrightarrow{h} \mathrm{triv}' \right) & \in \ 1\mathrm{Mor}(\mathrm{TrivBund}_G(Y))\,, \\ \left(\begin{tikzcd} \pi_1^*\mathrm{triv} \arrow[r, "g"] \arrow[d, "\pi_1^* h"'] & \pi_2^*\mathrm{triv} \arrow[d, "\pi_2^* h"] \arrow[dl, phantom, "="] \\ \pi_1^*\mathrm{triv}' \arrow[r, "g'"'] & \pi_2^*\mathrm{triv}' \end{tikzcd} \right) & \in \ 2\mathrm{Mor}(\mathrm{TrivBund}_G(Y \times_X Y)) \end{array} \right) \in 1\mathrm{Mor}(\mathrm{Desc}(Y^\bullet, \mathrm{TrivBund}_G(\cdot)))$$

is a morphism in the descent category relative to Y from the cocycle g to the cocycle g'.

The descent category $\mathrm{Desc}(Y^\bullet, \mathrm{TrivBund}_G(\cdot))$ thus defined knows everything about principal G-bundles on X which can be locally trivialized with respect to the chosen cover Y. To get rid of the dependence on the irrelevant choice of cover, one can form the directed limit (the colimit over all possible covers Y) to get the category

$$H(X, \mathbf{B}G) \ := \ \mathrm{colim}_Y \mathrm{Desc}(Y^\bullet, \mathrm{TrivBund}_G(\cdot))\,.$$

The category $H(X, \mathbf{B}G)$ thus obtained is the categorified version of the nonabelian G-cohomology of X: its objects are the G-cocycles on X and its morphisms the G-coboundaries. Cohomology classes are the isomorphism classes of objects in $H(X, \mathbf{B}G)$. Hence the standard fact about principal bundles now reads

Theorem 5.38 *The category of principal G-bundles on X is equivalent to the nonabelian cohomology of X with values in G:* $\mathrm{Bund}_G(X) \simeq H(X, \mathbf{B}G)$ *.*

Generalization to higher n. From just looking at the above example it is essentially clear what the right definition of descent of trivial n-bundles with structure n-group G is: a descent datum relative to a cover $Y \to X$ should be a tuple of:

- an *object* in $\mathrm{TrivBund}_G(Y)$;
- a *morphism* in $\mathrm{TrivBund}_G(Y \times_X Y)$ between the two pullbacks of this object;
- a *triangle* in $\mathrm{TrivBund}_G(Y \times_X Y \times_X Y)$ between the three pullbacks of this morphism;

- a *tetrahedron* in $\mathrm{TrivBund}_G(Y \times_X Y \times_X Y \times_X Y)$ between the four pullbacks of this triangle;
- and so on: an n-simplex in $\mathrm{TrivBund}_G(Y^{\times_X n+1})$ between the $n+1$ pullbacks of the previous $(n-1)$-simplex.

A morphism of descent data is a "prism homotopy" between the corresponding simplices, and so on.

5.4.2 Principal 2-bundles

Definition 5.39 (principal 2-bundles) *For G a strict 2-group, a principal G-2-bundle over X is a groupoid P equipped with a functor $p : P \to \mathcal{P}_0(X)$ and equipped with a strict right G-action $\rho : P \times G \to P$ such that there exists a cover $\pi : Y \to X$ and a (possibly weak) equivalence*

$$t : \ \pi^* P \xrightarrow{\simeq} \Pi_0(Y) \times G$$

of groupoids with right G-action (meaning that t is an equivalence of categories which is strictly G-equivariant). Principal G-2-bundles over X form a 2-category $2\mathrm{Bund}_G(X)$ whose morphisms are strictly G-equivariant functors $P \to P'$ leaving X invariant and whose 2-morphisms are transformation between these.

Principal G-2-bundles were introduced as such in [15] and [14]. See also [160]. Then we have

Theorem 5.40 *Principal G-2-bundles are classified by nonabelian G-cohomology: $2\mathrm{Bund}_G(X) \simeq H(X, \mathbf{B}G))$.*

Proof. Given a principal G-2-bundle $P \to X$ and picking a local trivialization $t : \pi^* P \to Y \times G$ over a good cover $\pi : (Y := \sqcup_i U_i) \to X$ yields the G cocycle $\{g_{ij}, h_{ijk}\}$ defined by

$$\begin{tikzcd} & P|_{U_{ij}} \arrow[dr, "t_j"] & \\ U_{ij} \times G \arrow[ur, "\bar{t}_i"] \arrow[rr, "g_{ij}"'] & & U_{ij} \times G \end{tikzcd}$$

and

$$\begin{tikzcd} & U_{ijk} \times G \arrow[dr, "g_{jk}"] \arrow[d, Rightarrow, "h_{ijk}"] & \\ U_{ijk} \times G \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & {} & U_{ijk} \times G \end{tikzcd} \quad := \quad \begin{tikzcd} & U_{ijk} \times G \arrow[dr, "g_{jk}"] \arrow[d, bend left, "\bar{t}_j"] \arrow[d, phantom, "\Downarrow"] & \\ & P|_{U_{ijk}} \arrow[u, bend left, "t_j"] \arrow[dr, "t_k"] & \\ U_{ijk} \times \mathcal{G} \arrow[uur, "g_{ij}"] \arrow[ur, "\bar{t}_i"] \arrow[rr, "g_{ik}"'] & & U_{ijk} \times G \end{tikzcd} .$$

One checks that this respects equivalences on both sides. Conversely, given a G-cocycle regarded as a 2-functor $g : \mathcal{P}_0^Y(X) \to \mathbf{B}G$ out of the codescent 2-groupoid of Y, one gets the pullback

$$\begin{tikzcd} g^* \mathbf{E}G \arrow[r] \arrow[d] & \mathbf{E}G \arrow[d] \\ \mathcal{P}_0^Y(X) \arrow[r, "g"] & \mathbf{B}G \end{tikzcd}$$

as described in section 7 of [126] and in our definition **??**. Quotienting out 2-isomorphisms yields the smooth principal G-2-bundle $g^* \mathbf{E}G/_\sim$. One checks that picking a local trivialization of this reproduces the cocycle

g up to equivalence. $\square$

In particular, for $G = \mathbf{B}U(1) = (U(1) \to 1)$ principal G-2-bundles are equivalent to $U(1)$-bundle gerbes on X

$$H(X, \mathbf{BB}U(1))_\sim = H^3(X, \mathbb{Z})\,.$$

5.4.3 Principal 3-bundles

5.5 Characteristic classes

We apply the general theory of characteristic classes of ω-bundles, described in section 3.4, to special examples.

5.5.1 Characteristic classes of principal 1-bundles

Let G be a compact, simple and simply connected Lie group with Lie algebra $\mathfrak{g}$. It is well known that its third integral cohomology is $H^3(G, \mathbb{Z}) = \mathbb{Z}$, as is the fourth integral cohomology of the classifying space $H^4(BG, \mathbb{Z}) = \mathbb{Z}$. In terms of characteristic classes in the sense of cohomology of ω-groupoids, as described in section 3.4, this has the following geometric interpretation, in view of the constructions in section 5.2.1:

according to proposition 5.13 there is a weak equivalence $\Pi_2(S(\mathrm{CE}(\mathfrak{g}))) \xrightarrow{\simeq} \mathbf{B}G$, due to the fact that $\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G$ and $\pi_2(G) = 0$. Accordingly $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))$ fails to be weakly equivalent due to the existence of nontrivial endomorphism 3-cells, i.e due to the nontriviality of $\pi_3(G)$. Following section 3.2.3 this can be remedied again by throwing in suitable 4-cells that *kill* these nontrivial 3-endomorphisms by connecting them to the identity endomorphism:

Lemma 5.41 *The ω-groupoid $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ$ corresponding to the pushout crossed complex*

$$\begin{tikzcd}
{[\mathbf{B}^3\pi_3(G)]} \arrow[r] \arrow[d] & {[\Pi_3(S(\mathrm{CE}(\mathfrak{g})))]} \arrow[d] \\
{[\mathbf{BEB}^2\pi_3(G)]} \arrow[r] & {[\Pi_3(S(\mathrm{CE}(\mathfrak{g})))] \cup [\mathbf{BEB}^2\pi_3(G)]}
\end{tikzcd}$$

is surjectively equivalent to $\mathbf{B}G$, $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ \xrightarrow{\simeq} \mathbf{B}G$.

Proof. The 1-morphisms of $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ$ are thin-homotopy classes of paths in G, starting at the identity, the 2-morphisms are thin-homotopy classes of disks cobounding such paths and the 3-morphisms are *homotopy* classes of 3-balls cobounding such surfaces. From each identity 3-morphism on an identity 2-morphisms on the identity 1-morphisms originates one 4-morphism per element $k \in \pi_3(G)$, connecting that identity 3-morphisms to the 3-morphism being the homotopy class of 3-spheres in G representing that element.

The functor $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ \xrightarrow{\simeq} \mathbf{B}G$ is given by endpoint evaluation on paths. This is clearly k-sujective for $0 \leq k \leq 4$. 5-surjectivity is due to the fact that there is, by the killing construction, a unique 4-morphism connecting any two parallel 3-morphisms. $\square$

By definition 2.83 a cocycle on $\mathbf{B}G$, hence a cocycle on $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ$, is an ω-anafunctor out of a a replacement. The universal choice is the universal free resolution

$$\Pi_3(\widehat{S(\mathrm{CE}(\mathfrak{g}))})^\circ := \mathrm{Codesc}(N^\bullet(\mathbf{B}(\Omega G \to PG), \mathcal{P}_0) = \int^{[n]\in\Delta} \Pi_\omega(\Delta^n) \otimes \mathcal{P}_0(N^n(\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ))$$

from definition 3.1 and definition 3.15.

While this may look complicated, it has a simple geometric interpretation: the 3-simplices in $\mathbf{B}\Pi_3(\widehat{S(\mathrm{CE}(\mathfrak{g}))})^\circ$ are (thin-homotopy classes of) oriented tetrahedra in G and the 4-simplices are in bijection with their boundaries consisting of five such tetrahedra, as in figures 5 and 11. Its 3-morphisms are generated from tetrahedra in G.

Using this surjectively equivalent model for $\mathbf{B}G$ there is a direct geometric way to see that every normalized Lie algebra 3-cocycle on $\mathfrak{g}$ yields a universal characteristic class on $\mathbf{B}G$:

Lemma 5.42 *Let $\mu_3 \in \mathrm{CE}(\mathfrak{g})$ be a Lie algebra 3-cocycle, i.e. $d_{\mathrm{CE}(\mathfrak{g})}\mu_3 = 0$, which is normalized, in that its left-invariant continuation to a closed left-invariant 3-form $\mu_3 \in \Omega^3(G)$ is integral. Then there is an ω-functor*

$$\int \mu_3/\mathbb{Z} : \Pi_3(\widehat{S(\mathrm{CE}(\mathfrak{g}))})^\circ \to \mathbf{B}^3U(1)$$

which sends a 3-cell, given by a thin-homotopy class $[V]$ of a tetrahadron $V : \Delta^3 \to G$ in G to the integral

$$[V] \mapsto \int_{\Delta^n} V^*\mu_3 \bmod \mathbb{Z}\,.$$

Proof. This kind of construction is precisely the one appearing in the integration of the String Lie 2-algebra in section 5.2.3, which in turn is the kind of construction appearing in [35]: the integrality of μ_3 implies that its integral over five tetrahedra which form the boundary of a 4-simplex vanishes in $\mathbb{R}/\mathbb{Z}$, which in turn ensures that $\int \mu_3/\mathbb{Z}$ is indeed an ω-functor sending the 4-cells of $\Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ$ to the identity 4-morphisms in $\mathbf{B}^3U(1)$. □

5.5.2 Characteristic classes of String(n)-principal bundles

The cohomology of the 2-group String(G) from section 5.2.3 has been analyzed in [68] and [9], and the cohomology of String(n)-principal bundles in [13]. Using the description of the third universal characteristic classes on $\mathbf{B}G$ in $\mathbf{Ho}(\mathbf{B}G, \mathbf{B}^3U(1))$ from section 5.5.1, we can now describe these universal characteristic classes on $\mathbf{B}\mathrm{String}(G)$ conveniently using the surjectively equivalent model $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) \twoheadrightarrow \mathbf{B}\mathrm{String}'(G)$ from corollary 5.31. In terms of these surjectively equivalent resolutions, the canonical ω-functor $\mathbf{B}\mathrm{String}(G) \twoheadrightarrow \mathbf{B}G$ is realized as an ω-functor $\Pi_3(S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))) \twoheadrightarrow \Pi_3(S(\mathrm{CE}(\mathfrak{g})))^\circ$ which simply sends labeled k-disks in G to unlabeled k-disks.

Proposition 5.43 *The pullback of the universal characteristic class $[\int \mu_3/\mathbb{Z}] \in \mathbf{Ho}(\mathbf{B}G, \mathbf{B}^3U(1))$ from lemma 5.42 along $p :$ $\mathbf{B}\mathrm{String}(G) \twoheadrightarrow \mathbf{B}G$ is trivial:*

$$[p^* \int \mu_3/\mathbb{Z}] = 0 \in \mathbf{Ho}(\mathbf{B}\mathrm{String}(G), \mathbf{B}^3U(1))\,.$$

Proof. The pulled back cocycle $p^* \int \mu_3/\mathbb{Z}$ is an ω-anafunctor

$$\begin{tikzcd}
\Pi_3(\widehat{S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))}) \arrow[r, two heads, "\hat{p}"] \arrow[d, two heads, "\simeq"] \arrow[rr, bend left, "0"] & \Pi_3(\widehat{S(\mathrm{CE}(\mathfrak{g}_{\mu_3}))})^\circ \arrow[r, "\int \mu_3/\mathbb{Z}"] \arrow[d, two heads, "\simeq"] \arrow[u, phantom, "\lambda \Downarrow \simeq"] & \mathbf{B}^3U(1) \\
\mathbf{B}\mathrm{String}'(G) \arrow[r, "p"] & \mathbf{B}G &
\end{tikzcd}$$

which sends a 3-morphism in $\Pi_3(S(\widehat{\mathrm{CE}(\mathfrak{g}_{\mu_3})}))$, given by a tetrahedron in G whose faces carry labels in $\mathbb{R}$, to the 3-morphism in $\mathbf{B}^3U(1)$, given by the element in $U(1)$ obtained as the integral of μ_3 over the tetrahedron, modulo $\mathbb{Z}$. By corollary 5.31 this element equals the oriented sum of the labels of the faces of the tetrahedron, modulo $\mathbb{Z}$. Therefore these faces tautologically provide a coboundary λ, given by the transformation which sends each labeled triangle in G to its label modulo $\mathbb{Z}$. □

Corollary 5.44 (Pontryagin class of $\mathrm{String}(G)$**-bundles is trivial)** *Let $\hat{g} \in H(X, \mathbf{B}\mathrm{String}(G))$ be the cocycle representing a* $\mathrm{String}(G)$*-principal bundle, then the characteristic class $[\hat{g}^* c]$ of this cocycle corresponding to the characteristic class $c = \int \mu_3/\mathbb{Z}$ from above vanishes, $[\hat{g}^*c] = 0$.*

5.6 Chern-Simons ω-bundles

Definition 5.45 (Chern-Simons cocycles) *For $\mathbf{B}^{n-1}U(1) \to \hat{G} \to G$ a shifted central extension of ω-groups, definition 3.27, we say a differential Chern-Simons cocycle with respect to $\hat{G}$ is a $\mathbf{B}\mathbf{B}^nU(1)$-cocycle in the image of the obstruction map, definition 3.34, for differential cohomology, definition 3.47:*

$$\hat{G}\mathsf{ChernSimons}(-) = \mathrm{im}(\ \bar{H}(-, \mathbf{B}G) \xrightarrow{\mathrm{twLift}} \bar{H}(-, \mathbf{B}(\mathbf{B}^{n-1}U(1) \to \hat{G})) \xrightarrow{\mathrm{twist}} \bar{H}(-, \mathbf{B}(\mathbf{B}^nU(1))\ )\,.$$

5.6.1 Chern-Simons 3-bundles

We describe abelian 3-bundles arising as obstructions to lifts through the shifted abelian String-extension from corollary 5.8 and identify them with *Chern-Simons 3-bundles* classified by the first Pontryagin class of the underlying ordinary principal bundle.

Lemma 5.46 *Let $G = \mathrm{Spin}(n)$, which is compact, simple and simply connected with Lie algebra $\mathfrak{so}(n)$ such that $p : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a double cover. Let μ_3 be the 3-form on $\mathrm{Spin}(n)$ as in definition 5.5. Let $\mu_3' \in \Omega^3(\mathrm{SO}(n))$ be the 3-cocycle corresponding to the first Pontryagin form. Then we have*

$$\mu_3 = \frac{1}{2} p^* \mu_3'\,.$$

Definition 5.47 (Chern-Simons 3-bundles) *For $P \to X$ a principal* $\mathrm{Spin}(n)$ *bundle with first Pontryagin class $\frac{1}{2}p_1 \in H^4(X, \mathbb{Z})$, we say that a cocycle in $H(X, \mathbf{B}^3U(1))$ represents the Chern-Simons 3-bundle or Chern-Simons 2-gerbe [40, 158] of P if its image under the isomorphism $H(X, \mathbf{B}^3U(1))/\sim\ \xrightarrow{\simeq} H^4(X, \mathbb{Z})$ is $\frac{1}{2}p_1$.*

An explicit way to construct Chern-Simons cocycles from principal bundles was given in [35]. We review this construction with slight technical modifications adapted to our context (for instance we restrict attention to the simply connected case, replace formal addition of chains with gluing of chains along common boundaries and work in $H(X, \mathbf{B}^3U(1))$ instead of the isomorphic $H(X, \mathbf{B}^4\mathbb{Z})$).

Definition 5.48 (Brylinski-McLaughlin's geometric construction of p_1**)** *Given a principal* $\mathrm{Spin}(n)$*-bundle P, construct an object in $H(X, \mathbf{B}^3U(1))$ as follows:*

1. *choose a cocycle in $H(X, \mathbf{B}G)$ representing P, given by a good cover $Y = \sqcup_i U_i$ of X and a transition function $g : Y^{[2]} \to G$;*
2. *choose a smooth lift $\hat{g}$ of g to the based group of paths $P'G$ from definition 5.12 – recall that elements are thin-homotopy classes of paths with sitting instants at their boundaries and that composition is by concatenation, not by pointwise multiplication;*
3. *choose a map $\sigma : Y^{[3]} \to \mathrm{Maps}(D^2, G)$ cobounding the triangles formed by the pullback of $\hat{g}$;*
4. *choose a map $T : Y^{[4]} \to \mathrm{Maps}(D^3, G)$ cobounding the tetrahedra formed by the pullback of h;*
5. *form the map $\kappa : Y^{[4]} \to \mathbb{R}/\mathbb{Z}$ given by $\kappa(y) := \int_{D^3} T(y)^*\mu_3$.*

Remark. Notice that all these choices of lifts are guaranteed to exist because the first non-vanishing homotopy group of G is $\pi_3(G) = \mathbb{Z}$. It then follows from the integrality of μ_3 that the κ defined this way indeed satisfies the cocycle condition: for $(\delta\kappa) : Y^{[5]} \to \mathbb{R}/\mathbb{Z}$ is given by the integral of μ_3 over 3-sphere in G:

$$\begin{aligned}
\beta_{ijklm}(x) &:= (\delta\kappa)_{ijklm}(x) \\
&= \int_{S^3} f^*_{ijklm}\, \mu_3 \\
&= \int_{D^3} T_{ijkl}(x)^*\mu_3 - \int_{D^3} T_{ijkm}(x)^*\mu_3 + \int_{D^3} T_{ijlm}(x)^*\mu_3 - \\
&\qquad - \int_{D^3} T_{iklm}(x)^*\mu_3 + \int_{D^3} (g_{ij}(x)\cdot T_{iklm}(x))^*\mu_3 \\
&= 0 \in \mathbb{R}/\mathbb{Z}\,,
\end{aligned}$$

where $f_{ijklm} : Y^{[5]} \to \mathrm{Maps}(S^3, G)$ is the 3-sphere obtained by gluing the solid tetrahedra $T(y)$ at their common boundaries. This is an integer and hence vanishes in $\mathbb{R}/\mathbb{Z}$.

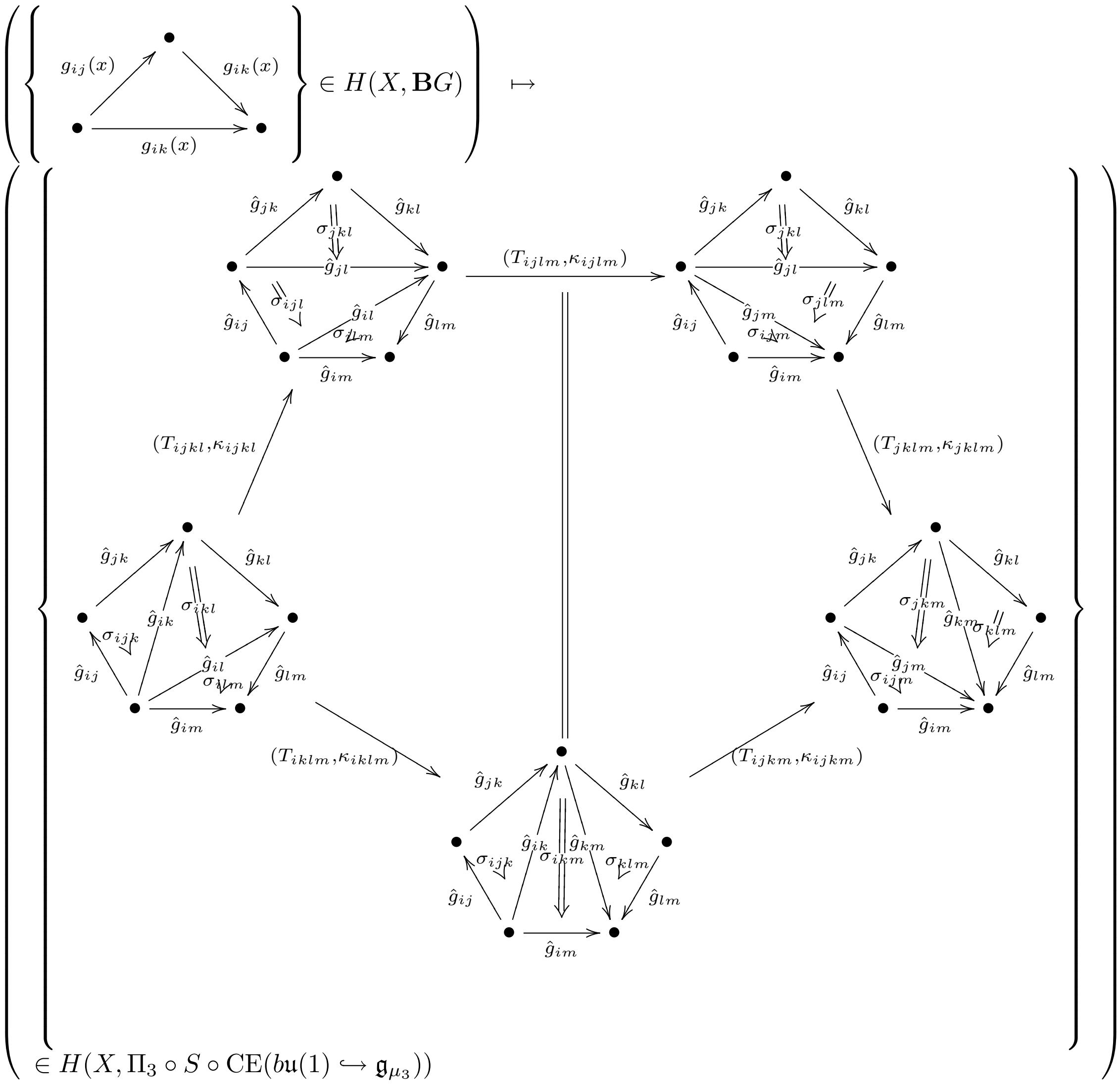


Figure 11: **Lifting a G-cocycle to a twisted** String(G)**-cocycle.** The diagram illustrates the construction of an abelian cocycle κ representing the first Pontryagin class of a principal G-bundle due to [35], reviewed in definition 5.48. As noticed in section 5.7.2, also the lower-dimensional data $\{\hat{g}, \sigma, T\}$ appearing here has a cocyclic interpretation, but in *nonabelian* cohomology: the diagram really illustrates the lift of a G-cocycle to a *twisted* String(G)-2-cocycle, namely to a $(\mathbf{B}U(1) \hookrightarrow \mathrm{String}(G))$-cocycle in the sense of section **??**. The abelian component κ in top degree is only the twist itself.

Theorem 5.49 ([35, 36]) *The κ constructed above is a cocycle in $H(X, \mathbf{B}^3 U(1))$ and under the isomorphism*

$$H(X, \mathbf{B}^3 U(1))/\sim \;\xrightarrow{\simeq} H^4(X, \mathbb{Z})$$

it maps to the first Pontryagin class of P: $g \mapsto \frac{1}{2} p_1$.

Remark. The proof proceeds by noticing that κ is indeed the top degree component of a cocycle in *differential* cohomology with curvature the Ponryagin 4-form. We find this in our context from a discussion of lifts in nonabelian differential cohomology as indicated in section 3.3.4 which should be given elsewhere.

Using the notion of lifts in nonabelian cohomology from section **??**, and using the nature of the String 2-group $\mathrm{String}(G) := \Pi_2 \circ S \circ \mathrm{CE}(\mathfrak{g}_{\mu_3})$ from proposition 5.24 we can interpret the construction in definition 5.48 as computing the obstruction to lifting a principal G-bundle to a $\mathrm{String}(G)$-2-bundle by lifting the G-cocycle $g : \mathcal{P}_0^Y(X) \to \mathbf{B}G$ to a twisted $\mathrm{String}(G)$-cocycle g_{tw}

$$\begin{tikzcd} \mathcal{P}_0^Y(X) \arrow[r, "g"] \arrow[rr, bend right, "g_{\mathrm{tw}}"'] & \mathbf{B}G & \mathbf{B}(\mathbf{B}U(1) \to \mathrm{String}(G)) \arrow[l, two heads, "\simeq"'] \end{tikzcd} .$$

The above algorithm then reads as follows, (see figure 11):

1. start with a G-cocycle $g : \mathcal{P}_0^Y(X) \to \mathbf{B}G$ representing P;
2. define g_{tw} on 1-morphisms by lifting the 1-morphisms in the image of g from $1\mathrm{Mor}(\mathbf{B}G)$ to $1\mathrm{Mor}(\mathbf{B}\mathrm{String}(G))$: the latter are paths in G with endpoint the original point in G;
3. on triple intersections define g_{tw} by choosing suitable 2-morphisms in $\mathbf{B}\mathrm{String}(G)$: these are represented by triangles in G labeled by an element $x \in \mathbb{R}/\mathbb{Z}$. Choose $x = 0$;
4. on quadruple intersections this fails to be a $\mathrm{String}(G)$-cocycle by the integral of μ_3 over any 3-ball filling the corresponding tetrahedra $f : D^3 \to G$. So $\int_{D^3} f^*\mu_3 \in \mathbb{R}/\mathbb{Z}$ gives the unique 3-morphism in the weak quotient 3-group $(\mathbf{B}U(1) \to \mathrm{String}(G))$ measuring the failure of the lift to be a lift to $\mathrm{String}(G)$.

This way we reinterpret the construction in [35] as

Theorem 5.50 (first Pontryagin class obstructs the lift through $\mathrm{String}(G) \to G$**)** *Write* $\mathrm{String}(G)$ *for any of the three ana-equivalent strict 2-groups* $\mathrm{String}_{\mathrm{BCSS}}(G)$ *(definition 5.5),* $\mathrm{String}_{\mathrm{Mick}}(G)$ *(definition 5.6) or* $\Pi_2(S\mathrm{CE}(\mathfrak{g}_{\mu_3}))$ *(section 5.2.3) and recall from proposition 5.8 that we have a shifted central extension* $\mathbf{B}U(1) \to \mathrm{String}(G) \to G$ *. Then the obstruction*

$$\mathrm{obstr}(\mathrm{String}(G) \to G) : H(-, \mathbf{B}G) \to H(-, \mathbf{B}^3 U(1))$$

from corollary **??** *to lifting a G-1-bundle to a* $\mathrm{String}(G)$*-2-bundle is a* $\mathbf{B}^3 U(1)$*-3-bundles whose class is the first Pontryagin class of the original G-bundle:*

$$[\mathrm{obstr}(\mathrm{String}(G) \to G)]/\sim = p_1 : H(-, \mathbf{B}G)/\sim \;\longrightarrow H^4(-, \mathbb{Z})\,.$$

The theorem implies that the obstruction to lifting a $\mathrm{Spin}(n)$-1-bundle coming to a $\mathrm{String}(n)$-2-bundle is a circle 3-bundle with class $\frac{1}{2} p_1(P)$.

$$\begin{tikzcd} H(-, \mathbf{B}\mathrm{Spin}(n)) \arrow[rr, "\mathrm{obstr}(\mathrm{String}(n) \to \mathrm{Spin}(n))"] \arrow[d, equal] & & H(-, \mathbf{B}^3 U(1)) \arrow[d] \\ H(-, \mathbf{B}\mathrm{SO}(n)) \arrow[r, "p_1"'] & H^4(-, \mathbb{Z}) & H^4(-, \mathbb{Z}) \arrow[l, "\cdot 2"'] \end{tikzcd}$$

Historically this fact was first understood in terms of topological string groups:

Definition 5.51 (String structure) *A principal G-bundle $P \to X$ is said to have String structure if, as a topological bundle, its structure group lifts to the 3-connected cover $\hat{G} \to G$, $\pi_3(\hat{G}) = 1$, given by the $K(\mathbb{Z}, 2)$-bundle over G whose class is a generator of $H^3(G, \mathbb{Z}) = \mathbb{Z}$.*

The existence of such a String structure is also obstructed by half of $p_1(P)$. From the above point of view this can be understood from the main result in [13]

Theorem 5.52 ([13]) *Nonabelian cohomology with values in the String 2-group is isomorphic to the non-abelian cohomology of $\hat{G}$*

$$H(X, \mathbf{String}(G))/\sim \quad \simeq H(X, \mathbf{B}\hat{G}) .$$

Geometric interpretation of String-lifts. Using the above considerations, we obtain the following geometric interpretation of lifts of structure groups through the String-extension, which is the crucial starting point for the discussion of further lifts through the Fivebrane-extension in section 5.6.2: Suppose that a lift $T : Y^{[4]} \to \mathrm{Maps}(D^3, G)$ of the original $\mathrm{Spin}(n)$-cocycle to solid tetrahedra in $\mathrm{Spin}(n)$ existed such that the 3-spheres formed by these solid tetrahedra as above all represented the trivial element in $\pi_3(G)$. Then the corresponding cocycle κ is trivial (in fact vanishes identically, $\kappa = 0$). The following asserts that also the converse is true:

Proposition 5.53 *If κ as above is a coboundary, $\kappa = \delta\rho$, then there exists a choice T' of lifts such that the image of $f : Y^{[5]} \to \mathrm{Maps}(S^3, G)$ always represents the trivial element in $\pi_3(G)$.*

Proof. Under the isomorphism $H(X, \mathbf{B}^3 U(1)) \simeq H(X, \mathbf{B}^4\mathbb{Z})$ the $U(1)$-3-cocycle κ corresponds equivalently to a $\mathbb{Z}$-4-cocycle β as in [35]. This being a coboundary implies that there is a $\mathbb{Z}$-valued cochain ρ satisfying $\delta\rho = \beta$, i.e.

$$\beta_{ijklm}(x) = \rho_{ijkl}(x) - \rho_{ijkm}(x) + \rho_{ijlm}(x) - \rho_{iklm}(x) + \rho_{jklm}(x) .$$

Being integer-valued, ρ_{ijkl} is necessarily independent of x. Picking $x \in U_{ijkl}$ we can find a 3-morphisms in $\Pi_3(\mathrm{CE}(\mathfrak{g}))$ starting and ending at the identity 2-morphism on the 0-cell source of the 3-morphisms $T_{ijkl}(x)$ and given by a smooth 3-sphere $s_{ijkl}(x) : S^3 \to G$ whose image in $\pi_3(G)$ is $-\rho_{ijkl}(x) \in \mathbb{Z}$. By left-translating this with the group elements that measure the difference between the 0-cell sources of T_{ijkl} as x varies, we obtain a smooth family s_{ijkl} of 3-morphisms represented by 3-spheres. Define the new family T'_{ijkl} to be the horizontal composite of 3-morphisms

$$T'_{ijkl}(x) := s_{ijkl} \cdot T_{ijkl}(x) .$$

By construction, the β' corresponding to these T', being the integral of μ_3 over the 3-spheres given by gluing the T' vanishes identically:

$$\beta'_{ijkl}(x) := \int_{S^3} f'_{ijkl}(x)^* \mu_3 = 0 .$$

By lemma 5.16 this implies that $f'_{ijkl}(x)^*\mu_3$ is exact. By lemma 5.29 this implies that the 3-sphere $f'_{ijkl}(x)$ in G is homotopic to the constant map for all x. $\square$

Remark. In words this says that the existence of a String-structure on a principal G-bundle implies that the structure-functions of the G-bundle can be lifted to topologically trivial 3-cells in the fibers. Such a lift can then serve as a starting point for lifts to even higher cells, such as Fivebrane lifts, section 5.6.2.

5.6.2 Chern-Simons 7-bundles

We obtain circle 7-bundles given by cocycles in $H(X, \mathbf{B}^7 U(1))$ which obstruct lifts through the shifted central extension

$$\mathbf{B}^6 U(1) \longrightarrow \mathrm{Fivebrane}(n) \longrightarrow \mathrm{String}(n)$$

as the image of

$$\mathrm{obstr}(\mathrm{Fivebrane}(n) \to \mathrm{String}(n)) : \ H(-, \mathbf{B}\mathrm{Fivebrane}(n)) \to H(-, \mathbf{B}^7 U(1))$$

and show that under

$$H(-, \mathbf{B}^7 U(1)) \to H(-, \mathbf{B}^7 U(1))/\sim \simeq H^7(-, U(1)) \simeq H^8(-, \mathbb{Z})$$

these correspond to half the second Pontryagin class of the $\mathrm{Spin}(n)$-cocycle underlying the original $\mathrm{String}(G)$-cocycle. Eventually we conclude that

Theorem 5.54 *The obstruction to lifting a* $\mathrm{String}(n)$*-2-bundle coming from a principal* $\mathrm{Spin}(n)$*-1-bundle* P *to a* $\mathrm{Fivebrane}(n)$*-6-bundle is a circle 7-bundle with class* $\frac{1}{6} p_2(P)$*.*

$$\begin{tikzcd}
H(-, \mathbf{B}\mathrm{String}(n)) \arrow[rr, "\mathrm{obstr}(\mathrm{Fivebrane}(n) \to \mathrm{String}(n))"] \arrow[d] & & H(-, \mathbf{B}^7 U(1)) \arrow[d] \\
H(-, \mathbf{B}\mathrm{SO}(n)) \arrow[r, "p_2"'] & H^8(-, \mathbb{Z}) & H^8(-, \mathbb{Z}) \arrow[l, "\cdot 6"']
\end{tikzcd}$$

5.6.3 Chern-Simons 11-bundles

We obtain circle 11-bundles given by cocycles in $H(X, \mathbf{B}^{11} U(1))$ which obstruct lifts through the shifted central extension

$$\mathbf{B}^{10} U(1) \longrightarrow \mathrm{Ninebrane}(G) \longrightarrow \mathrm{Fivebrane}(G)$$

for G a compact, simple and simply connected Lie group, as the image of

$$\mathrm{obstr}(\mathrm{Ninebrane}(n) \to \mathrm{Fivebrane}(n)) : \ H(-, \mathbf{B}\mathrm{Fivebrane}(n)) \to H(-, \mathbf{B}^{11} U(1))$$

and show that under

$$H(-, \mathbf{B}^{11} U(1)) \to H(-, \mathbf{B}^{11} U(1))/\sim \ \ \simeq H^{11}(-, U(1)) \simeq H^{12}(-, \mathbb{Z})$$

these correspond to the fractional third Pontryagin class of the $\mathrm{Spin}(n)$-cocycle underlying the original $\mathrm{Fivebrane}(n)$-cocycle. Eventually we conclude that

Theorem 5.55 *The obstruction to lifting a* $\mathrm{Fivebrane}(n)$*-6-bundle coming from a principal* $\mathrm{Spin}(n)$*-1-bundle* P *to a* $\mathrm{Ninebrane}(n)$*-10-bundle is a circle 11-bundle with class* $\frac{1}{240} p_3(P)$*.*

$$\begin{tikzcd}
H(-, \mathbf{B}\mathrm{Fivebrane}(n)) \arrow[rr, "\mathrm{obstr}(\mathrm{Ninebrane}(n) \to \mathrm{Fivebrane}(n))"] \arrow[d] & & H(-, \mathbf{B}^{11} U(1)) \arrow[d] \\
H(-, \mathbf{B}\mathrm{SO}(n)) \arrow[r, "p_3"'] & H^{12}(-, \mathbb{Z}) & H^{12}(-, \mathbb{Z}) \arrow[l, "\cdot k"']
\end{tikzcd}$$

5.7 Twisted ω-bundles

5.7.1 Twisted 1-bundles

To set the scene for the discussion of twisted bundles, let $1 \to U(1) \to \hat{G} \to G \to 1$ be an ordinary central extension of groups. We could use other abelian groups instead of $U(1)$, but in all our concrete examples the extension will be by $U(1)$ – often twisted bundles are discussed exclusively in the context of the extension $U(1) \to U(H) \to PU(H)$, for H a separable Hilbert space – and consider a G-cocycle on a smooth space X relative to surjective submersion $Y \to X$ given by a functor

$$\Pi_0^Y(X) \xrightarrow{g} \mathbf{B}G \qquad g\ : \begin{tikzcd} & (x,j) \arrow[dr] & \\ (x,i) \arrow[ur] \arrow[rr] & & (x,k) \end{tikzcd} \mapsto \begin{tikzcd} & \bullet \arrow[dr, "g_{jk}(x)"] & \\ \bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] & & \bullet \end{tikzcd}$$

from the codescent groupoid $\Pi_0^Y(X)$ corresponding to a surjective submersion $Y \to X$ of manifolds. This is the same as a function $g : Y \times_X Y \to G$ satisfying the cocycle condition $\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g$. It represents (the descent data of) a principal G-bundle on X. As indicated, the reader can think of Y as being the disjoint union of open subsets U_i of a good cover of X: $Y = \bigsqcup_i U_i$. Then the function g decomposes into a collection of functions $\{g_{ij} : U_i \cap U_j \to G\}$ and the cocycle condition takes the possibly more familiar form $g_{ij} \cdot g_{jk} = g_{ik}$ for all i, j, k.

We ask if we can lift this to a $\hat{G}$-cocycle $\hat{g}$ through the extension of groups

$$\begin{tikzcd} & \mathbf{B}U(1) \arrow[d] \\ & \mathbf{B}\hat{G} \arrow[d] \\ \Pi_0^Y(X) \arrow[r, "g"] \arrow[ur, dotted, "\hat{g}"] & \mathbf{B}G \end{tikzcd} .$$

In general this is not possible. But we can form the crossed module of groups $(U(1) \to \hat{G})$, regarded as a strict 2-group, and consider the corresponding 1-object 2-groupoid $\mathbf{B}(U(1) \to \hat{G})$. This has a canonical projection

$$\begin{tikzcd} & \mathbf{B}U(1) \arrow[d] & \\ & \mathbf{B}\hat{G} \arrow[d] & \\ \Pi_0^Y(X) \arrow[r, "g"] & \mathbf{B}G & \mathbf{B}(U(1) \to \hat{G}) \arrow[l, "\simeq"'] \end{tikzcd}$$

down to $\mathbf{B}G$, which is a weak equivalence. We can invert this locally and by refining our cover Y sufficiently we can always extend the original cocycle to a $(U(1) \to \hat{G})$-cocycle g_{tw}

$$
\begin{tikzcd}
 & \mathbf{B}U(1) \arrow[d] & \\
 & \mathbf{B}\hat{G} \arrow[d] & \\
\Pi_0^Y(X) \arrow[r, "g"] \arrow[rr, bend right, "g_{\mathrm{tw}}"'] & \mathbf{B}G & \mathbf{B}(U(1) \to \hat{G}) \arrow[l, "\simeq"']
\end{tikzcd}
\qquad
g_{\mathrm{tw}} :
\begin{tikzcd}
 & (x,j) \arrow[dr] & \\
(x,i) \arrow[ur] \arrow[rr] & & (x,k)
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & \bullet \arrow[dr, "\hat{g}_{jk}(x)"] \arrow[d, phantom, "\Downarrow c_{ijk}(x)"] & \\
\bullet \arrow[ur, "\hat{g}_{ij}(x)"] \arrow[rr, "\hat{g}_{ik}(x)"'] & {} & \bullet
\end{tikzcd}
$$

by locally choosing lifts $g_{ij} \mapsto \hat{g}_{ij}$. If the choice of lifts is bad (either because it was badly chosen or because there is in principle no good choice), then the lifted functions $\hat{g}_{ij}$ will satisfy the cocycle equation only up to a correction term c_{ijk}. But more systematically, we realize that the $\hat{g}_{ij}$ and the c_{ijk} together form a $(U(1) \to \hat{G})$-cocycle which we call a cocycle of a *twisted* $\hat{G}$-bundle. It is the same as a function $\hat{g} : Y \times_X Y \to \hat{G}$ which lifts the original cocycle function g and a function $c : Y \times_X Y \times_X Y \to U(1)$ satisfying $\pi_{12}^*\hat{g} \cdot \pi_{23}^*\hat{g} = c^{-1} \cdot \pi_{13}^*\hat{g}$. If Y is a good cover this reads $\hat{g}_{ij}(x)\hat{g}_{jk}(x) = c_{ijk}^{-1}(x) \cdot \hat{g}_{ik}(x)$. We would have a proper $\hat{G}$-cocycle if the c could be gauged away.

This is formalized by noticing that there is a canonical projection $p : (U(1) \to \hat{G}) \to (U(1) \to 1) = \mathbf{B}U(1)$, composing with which

$$
\begin{tikzcd}
 & \mathbf{B}U(1) \arrow[d] & & \\
 & \mathbf{B}\hat{G} \arrow[d] & & \\
\Pi_0^Y(X) \arrow[r, "g"] \arrow[rr, bend right, "g_{\mathrm{tw}}"'] & \mathbf{B}G & \mathbf{B}(U(1) \to \hat{G}) \arrow[l, "\simeq"'] \arrow[r, "p"] & \mathbf{B}\mathbf{B}U(1)
\end{tikzcd}
\qquad
p \circ g_{\mathrm{tw}} :
\begin{tikzcd}
 & (x,j) \arrow[dr] & \\
(x,i) \arrow[ur] \arrow[rr] & & (x,k)
\end{tikzcd}
\mapsto
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow c_{ijk}(x)"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"'] & {} & \bullet
\end{tikzcd}
$$

yields a $\mathbf{B}U(1)$-cocycle $p \circ g_{\mathrm{tw}}$ given by the function c from above. This represents a line 2-bundle or equivalently an abelian gerbe, known as the "lifting gerbe" of the original G-bundle. If this has a trivial class there is a gauge in which the c cocycle trivializes and the lift to a $\hat{G}$ cocycle $\hat{g}$ does exist. Formally this follows from the fact that the canonical inclusion $\hat{G} \hookrightarrow (U(1) \to \hat{G})$ is the kernel of the projection $(U(1) \to \hat{G}) \to BU(1)$. By the universal property of the kernel this implies that if the morphism $p \circ g_{\mathrm{tw}}$ in the diagram

$$
\begin{tikzcd}
 & \mathbf{B}U(1) \arrow[d] & & \\
 & \mathbf{B}\hat{G} \arrow[d] \arrow[dr, hook] & & \\
\Pi_0^Y(X) \arrow[ur, dotted, "\hat{g}"] \arrow[r, "g"] \arrow[rr, bend right, "g_{\mathrm{tw}}"'] & \mathbf{B}G & \mathbf{B}(U(1) \to \hat{G}) \arrow[l, "\simeq"'] \arrow[r, "p"] & \mathbf{B}\mathbf{B}U(1)
\end{tikzcd}
$$

is trivial, then it factors through $\mathbf{B}\hat{G}$, via our lift $\hat{g}$.

There are alternative perspectives on the same phenomenon which are useful for illustrating the general situation with which we are dealing. One of them is further described in section 5.7.1. Another one, closely related but not needing the notion of 2-vector spaces, is this one: From the central extension of groups that we started with we can also form the strict 2-group $(\hat{G} \to G)$. There are canonical injections of both $\mathbf{B}U(1)$ as well as G into $(\hat{G} \to G)$. One finds that as $(\hat{G} \to G)$-2-bundles a G bundle is equivalent to the lifting $\mathbf{B}U(1)$-2-bundle that obstructs its lift to a $\hat{G}$-bundle. The equivalence itself

$$g \xrightarrow[\simeq]{g_{\mathrm{tw}}} c$$

in this sense "is" again the twisted bundle, in that in components it is again given by the twisted cocycle relation, which in $\mathbf{B}(\hat{G} \to G)$ looks like a prism one of whose triangular sides is degenerate.

$$\begin{tikzcd}
 & \mathbf{B}(1 \to G) \arrow[dr, hook] \arrow[dd, phantom, "\Downarrow g_{\mathrm{tw}}"] & \\
\Pi_0^Y(X) \arrow[ur, "g"] \arrow[dr, "c"'] & & \mathbf{B}(\hat{G} \to G) \\
 & \mathbf{B}(U(1) \to 1) \arrow[ur, hook] &
\end{tikzcd}
\quad \Leftrightarrow \quad \forall y \in Y \times_X Y \times_X Y : \quad
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Downarrow c(y)"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & {} & \bullet \arrow[d, "\mathrm{Id}"] \arrow[dll, phantom, "\Swarrow \pi_{13} g_{\mathrm{tw}}(y)"] \\
\bullet \arrow[rr, "\pi_{13}^* g"'] & & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \bullet \arrow[dr, "\mathrm{Id}"] \arrow[dd, "\mathrm{Id}"] & \\
\bullet \arrow[ur, "\mathrm{Id}"] \arrow[dd, "\mathrm{Id}"'] \arrow[r, phantom, "\Swarrow \pi_{12}^* g(y)_{\mathrm{tw}}"] & {} \arrow[r, phantom, "\Swarrow \pi_{23}^* g(y)_{\mathrm{tw}}"] & \bullet \arrow[dd, "\mathrm{Id}"] \\
 & \bullet \arrow[dr, "\pi_{23}^* g(y)"] \arrow[d, phantom, "\Vert"] & \\
\bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & {} & \bullet
\end{tikzcd} .$$

This aspect of the twisting bundle as a morphism between 2-bundles becomes more amplified still when we pass from principal 2-bundles to 2-vector bundles.

2-vector bundles. We now explain the following: there is a way to understand the $\mathbf{B}U(1)$-cocycles c from above, which represented bundle gerbes and measured the obstruction to lifting G-bundles to $\hat{G}$-bundles, as inducing (cocycles for) associated rank 1 2-vector bundles ("line 2-bundles"). If we denote this 2-vector bundle still by c, then we have the following

Fact. *The vector bundles g_{tw} twisted by a line 2-bundle c are precisely the morphisms $1 \xrightarrow{g_{\mathrm{tw}}} c$ of 2-vector bundles from the trivial line 2-bundle into the twisting one. More generally, for c and c' two line 2-bundles the morphisms $c \xrightarrow{g_{\mathrm{tw}}} c'$ are the vector bundles twisted by the class $[c'] - [c]$.*

The reader will notice at this point the relation to the general situation discussed in the introduction. In order to describe this in more detail, we now develop the necessary concepts of 2-vector spaces and associated 2-bundles.

2-Vector spaces. Depending on the precise application there is some flexibility in what one may want to understand as a 2-vector space. But usually one will want to take 2-vector spaces to be abelian module categories over a given monoidal category. Two important classes of examples are these:

1. For k the ground field and $\mathrm{Disc}(k)$ the discrete monoidal category over it, the 2-category of $\mathsf{Disc}(k)$-module categories
$$2\mathsf{Vect}_{\mathsf{Disc}(k)} := \mathsf{Disk}(k) - \mathrm{Mod} \simeq \mathsf{Categories}(\mathsf{Vect}_k)$$
is the 2-category of categories internal to k-vector spaces. These "Baez-Crans 2-vector spaces" [8] are the right flavor of 2-vector spaces for 2-Lie theory. In general ∞-vector spaces of this kind are strict ∞-categories internal to vector spaces, which by the Dold-Kan theorem are equivalent to non-positively graded cochain complexes of vector spaces. These are the kinds of ∞-vector spaces which we consider in section **??**.

2. The other main example is module categories over the monoidal category Vect

$$2\mathsf{Vect}_{\mathsf{Vect}_k} := \mathsf{Vect} - \mathrm{Mod}\,.$$

In its totality this is rather unwieldy, but it contains the sub-2-categories Bimod of algebras and bimodules [143] and $\mathsf{KV2Vect}$ of Kapranov-Voevodsky 2-vector spaces [86]:

$$\begin{tikzcd} \mathsf{KV2Vect} \arrow[r, hook] \arrow[d, "\simeq"] & 2\mathsf{Vect}_{\mathsf{Vect},\mathrm{bas}} \arrow[r] \arrow[d, ":="] & 2\mathsf{Vect}_{\mathsf{Vect}} \arrow[d, ":="] \\ k^{\oplus\bullet}\mathsf{Bimod} \arrow[r, hook] & \mathsf{Bimod} \arrow[r, "i_{\mathsf{bimod}}"] & \mathsf{Vect} - \mathrm{Mod} \end{tikzcd} .$$

Any ordinary algebra A canonically specifies a Vect-module category, namely the ordinary category Mod_A of modules over the algebra: each right A-module can be tensored from the left by a vector space to produce another A-module. The 2-functor from the category Bimod of algebras, bimodules and bimodule homomorphisms to 2-vector spaces is

$$i_{\mathrm{bimod}} \;:\; \begin{tikzcd} A \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\phi"] & B \end{tikzcd} \;\mapsto\; \begin{tikzcd} \mathrm{Mod}_A \arrow[r, bend left, "(\cdot)\otimes_A N"] \arrow[r, bend right, "(\cdot)\otimes_A N'"'] \arrow[r, phantom, "\Downarrow(\cdot)\otimes_A\phi"] & \mathrm{Mod}_B \end{tikzcd} .$$

Under this inclusion Bimod behaves like the sub-2-category of 2Vect consisting of those 2-vector spaces with a *basis*: regarding an algebra A as a one-object Vect-enriched category $\mathcal{B}A$, we find the category of A-modules as the category of Vect-functors from $\mathcal{B}A$ to Vect:

$$\mathrm{Mod}_A \simeq \mathrm{Hom}(\mathcal{B}A, \mathsf{Vect})\,.$$

(More generally, we could replace $\mathcal{B}A$ by any Vect-enriched category here, i.e. by an *algebroid*.) This is analogous to how a set S is a basis for a k-vector space V if $V \simeq \mathrm{Hom}(S,k)$. Inside all of Bimod we have the full sub-2-category on those algebras that are direct sums, $A = k^{\oplus n}$, of the ground field algebra, for all $n \in \mathbb{N}$. Under the above maps these algebras map to 2-vector spaces of the form Vect^n. 2-vector spaces of this form have originally been considered by Kapranov and Voevodsky [86]. $k^{\oplus n} - k^{\oplus m}$-bimodules are $n \times m$ matrices whose entries are k-vector spaces.

The canonical 2-representation Every automorphism 2-group $\mathcal{G} = \mathrm{AUT}(H)$ has a canonical representation on 2-vector spaces obtained from the canonical composite

$$\mathcal{B}\mathrm{AUT}(H) \xrightarrow{=} \mathcal{B}\mathrm{Aut}_{\mathsf{Groups}}(\mathcal{B}H) \hookrightarrow \mathsf{Groups} \xrightarrow{k[-]} \mathsf{Algebras} \longrightarrow \mathsf{Bimod} \xrightarrow{i_{\mathsf{bimod}}} 2\mathsf{Vect}\,.$$

Here Groups and $\mathsf{Algebras}$ denote the 2-categories obtained by regarding groups as one-object groupoids and algebras as one-object Vect-enriched categories. The 2-functor $k[-]$ is that obtained by forming for each group the algebra which is the group algebra for finite groups and the group's convolution algebra for Lie groups. For $(H \to G)$ any other crossed module we can pull back this representation along the canonical 2-functor

$$\mathcal{B}(H \to G) \to \mathcal{B}AUT(H)$$

to get the induced 2-representation for any strict 2-group $(H \to G)$. More generally, for every ordinary linear representation ρ_0 of the group H such that the representing endomorphisms are linearly independent over the ground field, we get a 2-functor

$$\mathcal{B}(H \to G) \to \mathsf{Algebras}$$

based on the algebra $A = \langle \rho_0(H) | h \in H\rangle$, generated by the representation endomorphisms ρ_0. The 2-representation

$$\rho : \mathcal{B}(H \to G) \to \mathsf{Bimod} \to \mathsf{2Vect}$$

is given by

$$\begin{tikzcd}[column sep=large]
\bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, phantom, "\mapsto"] & \langle\rho(H)\rangle \arrow[r, bend left=50, "{}_{\alpha(g)}\langle\rho(H)\rangle"] \arrow[r, bend right=50, "{}_{\alpha(g')}\langle\rho(H)\rangle"'] \arrow[r, phantom, "\Downarrow (\cdot)\cdot h"] & \langle\rho(H)\rangle \arrow[r, phantom, "\mapsto"] & \mathrm{Mod}_{\langle\rho(H)\rangle} \arrow[r, bend left=50, "(\cdot)\otimes_{\langle\rho(H)\rangle} {}_{\alpha(g)}\langle\rho(H)\rangle"] \arrow[r, bend right=50, "(\cdot)\otimes_{\langle\rho(H)\rangle} {}_{\alpha(g')}\langle\rho(H)\rangle"'] \arrow[r, phantom, "\Downarrow (\cdot)\cdot h"] & \mathrm{Mod}_{\langle\rho(H)\rangle}
\end{tikzcd}$$

for all $g \in G, h \in H$.

Important examples are the 2-representation of $\mathcal{B}U(1) = (U(1) \to 1)$ induced from the standard rep of $U(1)$ on $\mathbb{C}$ as well as the 2-representation of $\mathrm{String}(G) = (\hat{\Omega} G \to PG)$ [9] induced from a positive energy representation of the centrally extended loop group $\hat{\Omega}G$ of some simple, simply connected compact Lie group G.

The standard 2-representation of $\mathcal{B}U(1)$. A very simple but useful example is the standard 2-representation of $\mathcal{B}U(1)$ induced from the defining representation of $U(1)$ on $\mathbb{C}$

$$\rho_0 : \mathcal{B}U(1) \to \mathrm{Vect}_{\mathbb{C}} \,.$$

In this case the 2-representation 2-functor acts simply as

$$\begin{tikzcd}[column sep=large]
\bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow c"] & \bullet \arrow[r, phantom, "\mapsto"] & \mathbb{C} \arrow[r, bend left=50, "\mathbb{C}"] \arrow[r, bend right=50, "\mathbb{C}"'] \arrow[r, phantom, "\Downarrow (\cdot)\cdot c"] & \mathbb{C} \arrow[r, phantom, "\mapsto"] & \mathrm{Vect}_{\mathbb{C}} \arrow[r, bend left=50, "(\cdot)\otimes\mathbb{C}"] \arrow[r, bend right=50, "(\cdot)\otimes\mathbb{C}"'] \arrow[r, phantom, "\Downarrow (\cdot)\cdot c"] & \mathrm{Vect}_{\mathbb{C}}
\end{tikzcd}$$

for all $c \in U(1)$. Notice that $\mathrm{Vect}_{\mathbb{C}}$ is the canonical 1-dimensional 2-vector space in the same sense in that $\mathbb{C}$ is the canonical 1-dimensional complex 1-vector space. Therefore, 2-vector bundles with local $\mathcal{B}U(1)$-structure under the above 2-representation deserve to be called *line 2-bundles*: their typical fiber is the "complex 2-line" in the above sense. Given that 2-functors with local $\mathcal{B}U(1)$-structure correspond to line bundle gerbes according to [138], this gives a genuine 2-vector bundle interpretation of line bundle gerbes.

The standard 2-representation of $\mathrm{String}(G)$**.** The infinite-dimensional loop group $\hat{\Omega}G$ does not have sensible representations on finite dimensional vector spaces. Instead the right substitute for the 2-category of finite dimensional algebras and bimodules is the 2-category $\mathsf{Bimod}_{\mathrm{vN}}$ of vonNeumann algebras and Hilbert bimodules between these, whose composition as 1-morphisms is not the algebraic tensor product but the Connes fusion tensor product [148].

Despite the difference in the technical details, the above construction of the 2-representation of the crossed module $\mathrm{String}(G) = (\hat{\Omega}G \to PG)$ from a representation of $\hat{\Omega}G$ should go through as in the finite dimensional case, since the Connes fusion product still respects the composition of twisting algebra homomorphisms: for A a von Neumann algebra and ${}_gA$ the bimodule structure on it induced from twisting the left action by an algebra automorphism g, we have

$${}_gH \otimes {}_{g'}H \simeq {}_{g' \circ g}H$$

under the Connes fusion tensor product. Therefore, by the above general principle, a positive energy representation of $\hat{\Omega}G$ induces a 2-representation ρ of the String 2-group on the von Neumann algebra generated by that representation. A ρ-associated String(G) 2-vector transport functor hence assigns a von Neumann algebra to each point and a vonNeumann bimodule to each path. In conjunction with the result [13, 4] that String(G)-2-bundles have the same classification as topological 1-bundles with structure group the topological String-group $|\mathrm{String}(G)|$, this says that ρ-associated String(G)-2-vector transport reproduces essentially the notion of 2-connections on String-bundles already appearing in [148].

Twisted vector bundles. Now with this understanding of 2-vector spaces and 2-representations, we can come back to twisted 2-vector bundles. This situation of twisted bundles becomes more manifestly an example of an n-functorial twist in the above sense by passing to associated 2-vector bundles. Use the canonical 2-representation of $\mathcal{B}U(1)$ on Bimod to pass from the $\mathcal{B}U(1)$-cocycle c to the associated Bimod-valued cocycle

$$\rho_* c : \; \Pi_0^Y(X) \longrightarrow \mathcal{B}U(1) \xrightarrow{\;\rho\;} \mathsf{Bimod} \; .$$

Then transformations into this 2-functor from the tensor unit

$$\begin{tikzcd}
\Pi_0^Y(X) \arrow[r, "c"] \arrow[rr, bend left, "1"] & \mathcal{B}U(1) \arrow[r, "\rho"] \arrow[u, phantom, "\Downarrow E"] & \mathsf{Bimod}
\end{tikzcd}$$

correspond precisely to twisted vector bundles, i.e. to twisted bundles for the central extension $U(n) \to PU(n)$. Again this is manifest from the naturality prism diagram

$$\begin{tikzcd}
1 \arrow[d, "g_{\mathrm{tw}}"] \\
c
\end{tikzcd}
\quad \Leftrightarrow \quad
\forall y \in Y \times_X Y \times_X Y : \quad
\begin{tikzcd}
& \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, phantom, "\Downarrow \mathrm{Id}"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] \arrow[d, "(\pi_1^* E)_y"'] & {} & \mathbb{C} \arrow[d, "\pi_3^* E_y"] \arrow[dll, phantom, "\swarrow \pi_{13}^* g_{\mathrm{tw}}(y)"] \\
\mathbb{C} \arrow[rr, "\mathbb{C}"] & & \mathbb{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[dd, "(\pi_2^* E)_y"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[ddd, "\mathbb{C}"'] \arrow[r, phantom, "\swarrow \pi_{12}^* g_{\mathrm{tw}}(y)"] & {} & \mathbb{C} \arrow[ddd, "(\pi_1^* E_y)"] \arrow[l, phantom, "\swarrow \pi_{23}^* g_{\mathrm{tw}}(y)"] \\
& \mathbb{C} \arrow[ddr, "\mathbb{C}"] \arrow[dd, phantom, "\Downarrow \rho_* c(y)"] & \\
& & \\
\mathbb{C} \arrow[uur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] & {} & \mathbb{C}
\end{tikzcd}
\; .$$

Here the fibers of E are $\mathbb{C}$–$\mathbb{C}$ bimodules, hence simply vector spaces.

Proposition 5.56 i. *For $\rho_* c : \Pi_0(X) \to \mathcal{B}\mathcal{B}U(1) \to \mathsf{Bimod}$ the transport 2-functor of a ρ-associated line 2-bundle, the transformations from the tensor unit into which are the vector bundles twisted by c:*

$$\mathsf{TwVectBund}_c(X) \simeq \mathrm{Hom}_{\mathrm{Trans}}(1, \rho_* c) \, .$$

ii. *For $\rho_* c : \mathcal{P}_2(X) \to \mathcal{B}\mathcal{B}U(1) \to \mathsf{Bimod}$ the transport 2-functor of a ρ-associated line 2-bundle with connection, the transformations from the tensor unit into which are the vector bundles twisted by c equipped with projectively flat connection:*

$$\mathsf{TwVectBund}_c^{\mathrm{proj.flat}}(X) \simeq \mathrm{Hom}_{\mathrm{Trans}}(1, \rho_* c) \, .$$

5.7.2 Twisted 2-bundles

An entirely analogous discussion applies to twisted String bundles. The strict String 2-group is $(\hat{\Omega} G \to PG)$ [9]. It was shown in [11] that String 2-bundles are equivalent to the String 1-bundles from [148]. The situation above now is

$$\begin{tikzcd}
 & \mathbf{B}\mathbf{B}U(1) \arrow[d] & & \\
 & \mathbf{B}(\hat\Omega G \to PG) \arrow[d] \arrow[dr, hook] & & \\
\Pi_0^Y(X) \arrow[ur, dotted] \arrow[r, "g"] \arrow[rr, bend right, "g_{\mathrm{tw}}"'] & \mathbf{B}(1 \to G) & \mathbf{B}(U(1) \to \hat\Omega G \to PG) \arrow[l, "\simeq"'] \arrow[r] & \mathbf{B}\mathbf{B}\mathbf{B}U(1)
\end{tikzcd}$$

with g_{tw} the cocycle for a twisted String bundle, which is twisted by the element $\frac{1}{2}p_1(X) \in H^4(X,\mathbb{Z})$ given by the $\mathbf{B}^3U(1)$-Chern-Simons cocycle.

∞-Lie integration of cocycles for twisted String 2-bundles Recall from [132] the details of the L_∞-algebra morphism which we need to integrate according to section 4.4.2 in order to lift (cocycles for) principal G-1-bundles to (cocycles for) (twisted) String 2-bundles: For $\mathfrak{g}$ a semisimple Lie algebra and $\mathfrak{g}_{\mu_3}$ its Lie 2-algebra from definition 5.19 the morphism

$$\mathrm{CE}(\mathfrak{g}) \longleftarrow \mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_{\mu_3}) \ : q$$

from section 4.4.2 is the following: we have

$$\mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_{\mu_3}) = \left(\wedge^\bullet(\underbrace{\mathfrak{g}^*}_{1} \oplus \underbrace{\langle b\rangle}_{2} \oplus \underbrace{\langle c\rangle}_{3}), d|_{\mathfrak{g}^*} = d_{\mathfrak{g}} \ ; db = \mu_3 + c \ ; dc = 0 \right)$$

and the morphism acts as

$$\begin{aligned}
q|_{\mathfrak{g}^*} &= \mathrm{Id}_{\mathfrak{g}^*} \\
q &: b \mapsto 0 \\
q &: c \mapsto -\mu_3 \in \wedge^3\mathfrak{g}^* \,.
\end{aligned}$$

Now for G the simply connected Lie group integrating $\mathfrak{g}$ and $\pi : P \to X$ the principal G-bundle with canonical vertical 1-form $\Omega^\bullet_{\mathrm{vert}}(P) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ as in definition 5.58 which we want to lift to a String 2-bundle, the composite DGCA morphism

$$\Omega^\bullet_{\mathrm{vert}}(P) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \xleftarrow{q} \mathrm{CE}(b\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu_3})$$

acts as

$$\begin{aligned}
A_{\mathrm{vert}}(q)|_{\mathfrak{g}^*} &= A_{\mathrm{vert}} \\
A_{\mathrm{vert}}(q) &: b \mapsto 0 \\
A_{\mathrm{vert}}(q) &: c \mapsto \mu_3(A_{\mathrm{vert}}) = \langle A_{\mathrm{vert}}[A_{\mathrm{vert}} \wedge A_{\mathrm{vert}}]\rangle \,.
\end{aligned}$$

Proposition 5.57 *The integrated twisted* $\mathrm{String}'(G)$ *cocycle*

$$\begin{tikzcd}
 & \Pi_3^{\mathrm{vert}}(P) \arrow[r, "\Pi_2 \circ S(A_{\mathrm{vert}(q)})"] \arrow[d, two heads] & \mathbf{B}(\mathbf{B}\mathbb{R} \to \mathrm{String}'(G)) \arrow[d, two heads] \\
\mathcal{P}_0^Y(X) \arrow[r, hook] \arrow[dr, "\simeq"'] & (\mathbf{B}^3\mathbb{Z} \to \Pi_3^{\mathrm{vert}}(P)) \arrow[r, "g"] \arrow[d, "\simeq"'] & \mathbf{B}(\mathbf{B}U(1) \to \mathrm{String}'(G)) \\
 & \mathcal{P}_0(X) &
\end{tikzcd}$$

*is given by... [** lift of G-cocycle to $(\Omega' G \to PG)$-cocycle together with the Brylinski-MacLaughlin data in degree 3 **])*

Untwisting twisted String 2-cocycles. [** if the twist is trivializable, we gauge it away to turn the $(\mathbf{B}U(1) \to \mathrm{String}'(G))$-bundle into a proper String′(G)-2-bundle **]

5.7.3 Twisted 6-bundles

[** twisted Fivebrane lifts go here **]
[...]

5.8 L_∞-integration of L_∞-cocycles to nonabelian cocycles

5.8.1 Principal 1-bundles

Here we exhibit the method of ∞-Lie integration of L_∞-algebraic n-cocycles to nonabelian n-cocycles classifying principal n-bundles from section 4.4.1 for the simple special case $n = 1$. This is to illustrate the method in a familiar context but also serves to establish a repository of some facts and notation that reappear in the more interesting examples.

The canonical vertical 1-form on a principal bundle. Recall the DGCA description of the canonical vertical 1-form on a principal bundle from [132].

Definition 5.58 *Let $\mathfrak{g}$ be a Lie algebra, G some Lie group integrating it and $\pi : P \to X$ a principal G-bundle over a manifold X. We write $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{vert}}(P, gg)$ for the canonical flat $\mathfrak{g}$-valued vertical 1-form which can be expressed in terms of a DGCA morphisms as*

$$\Omega^\bullet_{\mathrm{vert}}(P) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \ .$$

The classical way to think of A_{vert} is to choose any Cartan-Ehresmann connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ on the total space P. Its image under the quotient map $\Omega^\bullet(P) \twoheadrightarrow \Omega^\bullet_{\mathrm{vert}}(P)$ is A_{vert}. (See definition 4.24 for $\Omega^\bullet_{\mathrm{vert}}(P)$.)

Notice that every principal G-bundle canonically trivializes over itself, which in our context reads as follows:

Definition 5.59 *For G a Lie group and $\pi : P \to X$ a principal G-bundle on X, let $g : \mathcal{P}_0(X) \to \mathbf{B}G$ be the canonical 2-functor from the codescent groupoid $\mathcal{P}_0^P(X)$ of P with the property that*

$$\forall y, y' \in P \times_X P : \ \ y' = y g(y \to y') \, .$$

Simply connected structure groups. Before considering the general case of a principal 1-bundles we take G to be the simply connected Lie group integrating the Lie algebra $\mathfrak{g}$.

Proposition 5.60 *Let $\pi : P \to X$ be a principal G bundle on a manifold X for G the simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{vert}}(P, \mathfrak{g})$ be the canonical vertical 1-form from definition 5.58. Acting with the contravariant integration functor from definition 4.20*

$$\Pi_1 \circ S : \mathrm{DGCAs} \to \mathsf{1Groupoids}(\mathsf{Spaces})$$

from DGCAs to categories internal to Spaces *on the morphism* $\Omega^\bullet_{\mathrm{vert}}(P) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ *yields the canonical cocycle of P from definition 5.59:*

$$\Pi_1 \circ S \left(\Omega^\bullet_{\mathrm{vert}}(P) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g}) \right) = \mathcal{P}_0^P(X) \xrightarrow{g} \mathbf{B}G \ .$$

Proof. First consider the integration of the objects: First of all $\Pi_1(S(\mathrm{CE}(\mathfrak{g}))) = \mathbf{B}G$ is just proposition 5.10. By proposition 4.28 we have $\Pi_1(S(\Omega^\bullet_{\mathrm{vert}}(P))) = \Pi_1^{\mathrm{vert}}(P)$. Then observe that by the assumption that G, and hence the fibers of P, are simply connected, the vertical fundamental path groupoid happens to be canonically isomorphic $\Pi_1^{\mathrm{vert}}(Y) \simeq \mathcal{P}_0^Y(X)$ to the codescent groupoid

$$\mathcal{P}_0^Y(X) := \left(Y \times_X Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y \right)$$

(section 3.1.3) of the surjective submersion $\pi : P \to X$: there is a unique homotopy class of paths between any two points in the same fiber of P.

Finally, to see that $\Pi_1(S(A_{\mathrm{vert}})) = g$ notice that on each fiber $\simeq G$ A_{vert} restricts to the canonical $\mathfrak{g}$-valued 1-form on G. This has the property that its parallel transport over any path in G is the group element relating the starting point to the endpoint of that path. Again by lemma 5.10 (see the remark below) this is precisely the definition of the canonical cocycle g. $\square$

Remark. Due to its relevance for the following constructions, the main mechanism at work in this proof deserves further amplification: The crucial aspect to notice here is that it is the *flatness* of A_{vert} which allows the interpretation of its parallel transport as a cocycle. Namely the integration process indicated is effectively regarding the ordinary cocycle condition for a principal G-bundle $Y := P \to X$

$$g : \mathcal{P}_0^Y(X) \quad \to \quad \mathbf{B}G$$

$$\begin{tikzcd} & \pi_2(y) \arrow[dr] & \\ \pi_1(y) \arrow[ur] \arrow[rr] & & \pi_3(y) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g(y)"] & \\ \bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & & \bullet \end{tikzcd}$$

as the flat parallel transport around a closed loop:

$$\begin{tikzcd} & \pi_2(y) \arrow[dr, "\gamma_2"] & \\ \pi_1(y) \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_3"'] & & \pi_3(y) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \bullet \arrow[dr, "P\exp(\int_{\gamma_2} A_{\mathrm{vert}})"] & \\ \bullet \arrow[ur, "P\exp(\int_{\gamma_1} A_{\mathrm{vert}})"] \arrow[rr, "P\exp(\int_{\gamma_3} A_{\mathrm{vert}})"'] & & \bullet \end{tikzcd} .$$

flatness of A_{vert}	$\longleftrightarrow$	cocycle condition for $g := \Pi_1 \circ S(A_{\mathrm{vert}})$
$\begin{tikzcd} & \bullet \arrow[dr, "P\exp(\int_{\gamma_2} A_{\mathrm{vert}})"] & \\ \bullet \arrow[ur, "P\exp(\int_{\gamma_1} A_{\mathrm{vert}})"] \arrow[rr, "P\exp(\int_{\gamma_3} A_{\mathrm{vert}})"'] & & \bullet \end{tikzcd}$		$\begin{tikzcd} & \bullet \arrow[dr, "\pi_{23}^* g(y)"] & \\ \bullet \arrow[ur, "\pi_{12}^* g(y)"] \arrow[rr, "\pi_{13}^* g(y)"'] & & \bullet \end{tikzcd}$

Table 3: For a $\mathfrak{g}$-connection descent datum with respect to a surjection $Y \to X$ with sufficiently high connected fibers, the integration (the parallel n-transport) of the vertical part $\Omega^\bullet(Y) \xleftarrow{A_{\mathrm{vert}}} \mathrm{CE}(\mathfrak{g})$ over singular simplices in the fibers produces a G-cocycle, for G a quotient of the ω-group integrating $\mathfrak{g}$. The quotient is by the vertical holonomy ω-group of A_{vert}.

General structure groups. If the structure group G, and hence the fibers of a principal G-bundle $P \to X$, are not simply connected, the above procedure requires an additional step in which the cells are added into the fundamental vertical groupoid that "patch" the nontrivial homotopies using definition 4.31.

Proposition 5.61 *For G a Lie group as above and $\pi : P \to X$ a principal G-bundle, there is a weak equivalence from the patched fundamental vertical n-groupoid of definition 4.31 to the discrete ω-groupoid over the base X: $\Pi_n^{\mathrm{vert}}(P) \xrightarrow{\simeq} \mathcal{P}_0(X)$.*

Proposition 5.62 *For $\mathfrak{g}$ a Lie algebra, G its simply connected Lie group, A a discrete abelian normal subgroup of G, G/A the corresponding quotient Lie group, let $\pi : P \to X$ be a principal G/A-bundle with canonical vertical 1-from A_{vert}, then*

$$\begin{tikzcd}
 & \Pi_1^{\mathrm{vert}}(P) \arrow[r, "\Pi_1 \circ S(A_{\mathrm{vert}})"] \arrow[d] & \mathbf{B}G \arrow[d] \\
\mathcal{P}_0^P(X) \arrow[r, hook] \arrow[dr, "\simeq"'] & \Pi_1^{\mathrm{vert}}(P)^\circ \arrow[r, "g"] \arrow[d, "\simeq"'] & \mathbf{B}(G/A) \\
 & \mathcal{P}_0(X) &
\end{tikzcd}$$

is a G/A-cocycle representing P.

Here $\Pi_1^{\mathrm{vert}}(P)^\circ$ is the *patched* fundamental vertical 1-groupoid from definition 4.31.

5.8.2 Chern-Simons n-bundles

Proposition 5.63 *For G an n-connected Lie group with Lie algebra $\mathfrak{g}$ and $H^{n+1}(G, \mathbb{Z}) = \mathbb{Z}$, for μ_{n+1} a Lie algebra $(n+1)$-cocycle on G such that its left-invariant extension to $H^{n+1}_{\mathrm{dR}}(G)$ is the image of a generator of $H^{n+1}(G, \mathbb{Z})$, for $\pi : (Y = P) \to X$ a principal G bundle and for $\mu(A_{\mathrm{vert}})$ the corresponding vertical form from figure 12, we have that*

$$\begin{tikzcd}
 & \Pi_n^{\mathrm{vert}} \arrow[r, "\Pi_n \circ S(\mu(A_{\mathrm{vert}}))"] \arrow[d] & \mathbf{B}^n\mathbb{R} \arrow[d] \\
\mathcal{P}_0^P(X) \arrow[r, hook] \arrow[dr, "\simeq"'] & \Pi_n^{\mathrm{vert}}(P)^\circ \arrow[r, "g"] \arrow[d, "\simeq"'] & \mathbf{B}^n U(1) \\
 & \mathcal{P}_0(X) &
\end{tikzcd}$$

is the Čech cocycle representing the corresponding characteristic class in $H^{n+1}(X, \mathbb{Z})$ as constructed in [35, 36]

[...]

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with bilinear invariant form $P = \langle \cdot, \cdot \rangle$, normalized such that the canonical 3-cocycle $\mu = \langle \cdot, [\cdot, \cdot] \rangle \in \wedge^3 \mathfrak{g}^*$ extends left-invariantly to the image in deRham cohomology of the generator – either one of the two – of $H^3(G, \mathbb{Z})$, where G is the simply connected compact semisimple Lie group integrating $\mathfrak{g}$.

Let $\pi : P \to X$ be a principal G-bundle with Cartan-Ehresmann connection $A \in \Omega^1(P, \mathfrak{g})$, which we read as a $\mathfrak{g}$-connection descent datum. By the discussion in [132], there is a $b^2\mathfrak{u}(1)$-connection descent datum obstructing the lift of the $\mathfrak{g}$-connection through the String-extension $0 \to b\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{g} \to 0$,

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[lll, bend left=15, "\mu(A_{\mathrm{vert}})"] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] & \mathrm{W}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, two heads] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \arrow[lll, bend left=15, "(\mathrm{cs}(A,F_A), P(F_A))"] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook] & \mathrm{inv}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, hook] \arrow[lll, bend left=15, "P(F_A)"]
\end{tikzcd}
$$

Figure 12: **Obstructions and twisted lifts** for lifts through String-like extensions at the level of L_∞-algebraic cocycles.

- whose diagram is the canonically constructed $b^2\mathfrak{u}(1)$-connection

$$
\begin{tikzcd}
\Omega^\bullet_{\mathrm{vert}}(Y) & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] & \mathrm{CE}(b\mathfrak{u}(1) \to \mathfrak{g}_\mu) \arrow[l] & \mathrm{CE}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[lll, bend left=15, "\mu(A_{\mathrm{vert}})"] \\
\Omega^\bullet(Y) \arrow[u, two heads] & \mathrm{W}(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u, two heads] & \mathrm{W}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, two heads] & \mathrm{W}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, two heads] \arrow[lll, bend left=15, "(\mathrm{cs}(A,F_A), P(F_A))"] \\
\Omega^\bullet(X) \arrow[u, hook] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "\{P_i(F_A)\}"'] \arrow[u, hook] & \mathrm{inv}(b\mathfrak{u}(1) \to \mathfrak{g}) \arrow[l] \arrow[u, hook] & \mathrm{inv}(b^2\mathfrak{u}(1)) \arrow[l] \arrow[u, hook] \arrow[lll, bend left=15, "P(F_A)"]
\end{tikzcd}
$$

• whose connection 3-form on $Y := P$ is the Chern-Simons 3-forms with respect to P of the original connection 1-form A,

• and whose vertical connection 3-form is, therefore $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{\mu(A_{\mathrm{vert}})} \mathrm{CE}(b^{\mathfrak{u}}(1))$.

We will now apply $\Pi_3 \circ S : \mathrm{DGCA}s \to S3\mathsf{Categories}$ to the vertical part $\mu(A_{\mathrm{vert}})$ of the Chern-Simons 3-connection obtained above, making use of lemma 5.11. Let $\Pi_0^Y(X)$ denote the strict Čech 3-groupoid of $Y \to X$:

- objects are points in Y,
- morphisms are sequences of jumps between points in the same fiber,
- 2-morphisms are free pasting diagrams of 2-simplices with boundary such jumps,
- 3-morphimss are pasting diagrams of 3-simplices with boundary such 2-simplices, freely generated modulo the relation that all boundaries of 4-simplices they form 3-commute.

Similar to the situation for $U(1)$-bundles above, but now in higher categorical dimension, we see that this Čech 3-groupoid is covered by the vertical fundamental 3-groupoid $\Pi_3^{\mathrm{vert}}(Y)$ of Y. More precisely, The Čech 3-groupoid is covered by its Kan-complex simplicial version, where $(k \leq 2)$-simplices are thin homotopy classes of maps from the standard k-simplex (as opposed to the standrad k-disk as for the globular version) into a fiber of Y, and where 3-simplices are full homotopy classes of maps from the standard 3-simplex:

$$
\begin{tikzcd}
\Pi_3^{\mathrm{vert}}(Y) \arrow[r, "\hat{g} := \Pi_3 \circ S(\mu(A_{\mathrm{vert}}))"] \arrow[d, two heads] & \mathbf{B}^3\mathbb{R} \arrow[d, two heads] \\
\mathcal{P}_0^Y(X) \arrow[r, "g"] & \mathbf{B}^3 U(1)
\end{tikzcd} .
$$

By applying our integration procedure, $\Pi_3 \circ S : \mathrm{DGCAs} \to S3\mathsf{Categories}$, to $\Omega^\bullet_{\mathrm{vert}}(Y) \xleftarrow{\mu(A_{\mathrm{vert}})} \mathrm{CE}(b^2\mathfrak{u}(1))$ we thereby find a cocycle $g : \Pi_0^Y(X) \to \mathbf{B}^3U(1)$, which

- colors jumps between two point in the fiber by chosen (thin homotopy classes of) paths equipped with a map to G (coming from the flat 1-form on that path and choosing the starting point of the path as the basepoint) – these paths always exist since G is connected;
- colors triangles of jumps in the fiber with surfaces bounded by the corresponding paths and again equipped with a map to G – these surfaces always exists sice G is simply connected;
- colors tetrahedra of jumps in the fiber with volumes fillings these and equipped with a map $f : F \to G$ – this exists because G is necessarily also 2-connected;
- finally assigns to each such tetrahedron T the real number obtained by integrating $\mu(A_{\mathrm{vert}})$ over the tetrahedron, which is the same as the integral $\int_T f^*\mu$, but taking this number only modulo the holonomy of $\mu(A_{\mathrm{vert}})$ over closed 3-dimensional volumes, hence, by assumption of the integrality of μ, modulo $\mathbb{Z}$.

It is again the flatness of the vertical connection 3-form which ensures that the construction indeed yields a 3-cocycle for a line 3-bundle: the Chern-Simons 3-bundle whose existence obstructs the lift of the original G-bundle to a String(G)-2-bundle.

One can see that the construction just sketched – the systematic procedure of integrating L_∞-connection descent data to nonabelian cocycles by hitting the Cartan-Ehresmann diagram with $\Pi_n \circ S$ – reproduces in the case we have described precisely the prscription which Brylinski and McLaughlin have described in [36]. They have a general such prescription for all higher Pontrjagin and Euler classes [35]. This involves passing from the principal G-bundle $P \to X$ first to an associated bundle (with fiber certain Stiefel manifolds) and then proceeding essentially as above. This step can be understood, from our point of view, as an integrability condition on the regular epimorphism $Y \to X$ appearing in the L_∞-connection descent datum: that needs to have sufficiently highly connected fibers, or else needs to have torsion cohomology groups, such that the higher holonomies of the vertical connection form have a chance of covering all required higher morphisms in the Čech groupoid.

5.9 Transgression to mapping spaces: σ-models

5.9.1 Chern-Simons and Dijkgraaf-Witten σ-model

For both Dijkgraaf-Witten and Chern-Simons theory the background field is a $\mathbf{B}^3U(1)$-cocyle.

In Dijkgraaf-Witten target space is $\mathbf{B}G$ for G a finite group and the background field

$$\nabla : \ \mathbf{B}G \longrightarrow \mathbf{B}^3U(1)$$

is any such ω-anafunctor, hence, by proposition 3.33, any $U(1)$-valued group-cocycle on G.

For Chern-Simons theory target space is $\mathcal{P}_3(X)$ and $\nabla : \ \mathcal{P}_3(X) \longrightarrow \mathbf{B}^3U(1)$ is a Chern-Simons 3-bundle with connection, i.e. in the image of

$$\mathrm{obstr} : \ H_{\mathcal{P}_3}(X, \mathbf{B}G) \xrightarrow{\mathrm{twLift}} H_{\mathcal{P}_3}(X, \mathbf{B}(\mathbf{B}U(1) \to \mathrm{String}(G))) \xrightarrow{\mathrm{twist}} H_{\mathcal{P}_3}(X, \mathbf{B}(\mathbf{B}^3U(1))) \ .$$

5.9.2 BF- and Yetter-Martins-Porter σ-model

Now the target is $\mathbf{B}G$ for G a strict 2-group.

6 Glossary

Higher algebraic structures appear in quantum field theory, notably in String theory. Put the other way round: quantum field theory and String theory is a source of examples of higher algebraic structures. At the time of this writing a language barrier inhibits interaction among practicioners on both sides. The following glossary is meant to provide a minimum of joint background equipped with further pointers to the literature.

6.1 Category theory

[** concerning size issues, recall lore about accessible categories, as recalled in section 5.4 in [100] **]

Definition 6.1 (category) *A category C is a set of objects $\mathrm{Obj}(C)$ and for every pair $a, b \in \mathrm{Obj}(C)$ a set $C(a, b)$ of morphisms from a to b and for every triple $a, b, c \in \mathrm{Obj}(C)$ a composition map $\circ = \circ_{a,b,c} : C(a, b) \times C(b, c) \to C(a, c)$ which is associative in the obvious sense. In addition, for every object a there is a special element $i_a \in C(a, a)$, the identity morphism on a such that $i_b \circ f = f \circ i_a$ for any $f \in C(a, b)$.*

Elements $f \in C(a, b)$ are denoted by arrows $f : a \to b$ or $a \xrightarrow{f} b$ and composition is denoted by juxtaposition $a \xrightarrow{g\circ f} c \;=\; a \xrightarrow{f} b \xrightarrow{g} c$.

Definition 6.2 (functor) *A functor $F : C \to D$ from a category C to a category D is a map $F_0 : \mathrm{Obj}(C) \to \mathrm{Obj}(D)$ and for all $a, b \in \mathrm{Obj}(C)$ a map $F_{a,b} : C(a, b) \to D(f(a), f(b))$ which respects composition on C and D.*

Definition 6.3 (natural transformation) *A natural transformation from a functor $F : C \to D$ to a functor $G : C \to D$, denoted* $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D \end{tikzcd}$ *, is for each $a \in \mathrm{Obj}(C)$ an element $\eta(a) \in D(F(a), G(a))$ such that for all ($a \xrightarrow{f} b$) $\in C(a, b)$ we have*

$$\begin{tikzcd} F(a) \arrow[r, "\eta(a)"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta(b)"'] \arrow[ur, phantom, "="] & G(b) \end{tikzcd} .$$

Natural transformations can be composed by composing their components. Thus functors $C \to D$ and natural transformations between them form the functor category denoted $\mathsf{Functors}(C, D)$ or D^C.

Definition 6.4 (initial and terminal object) *An object in a category C is terminal if there is a unique morphism from every other object to it. It is initial if there is a unique morphism from it to every other object.*

Lemma 6.5 (uniqueness of initial and terminal objects) *If it exists in a category C, the initial object is unique up to isomorphism: any two initial objects are isomorphic. Similarly for terminal objects.*

Definition 6.6 (limit and colimit) *For $F : C \to D$ a functor the category $\mathsf{Cones}(F)$ has as objects natural transformations of the form*

$$\begin{tikzcd} & \mathrm{pt} \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow c"] & \\ C \arrow[ur] \arrow[rr, "F"'] & & D \end{tikzcd}$$

and as morphisms $f : (c, T) \to (c', T')$ natural transformations

$$
\begin{tikzcd}
 & \mathrm{pt} \arrow[dr, "T"] \arrow[dd, Rightarrow, "f"] & \\
C \arrow[ur] \arrow[dr] & & D \\
 & \mathrm{pt} \arrow[ur, "T'"'] &
\end{tikzcd}
$$

such that $c = c' \circ f$. The <u>*limit*</u> *of F over C, denoted $\lim_C F$, is, if it exists, the initial object of $\mathsf{Cones}(F)$.*

The category $\mathsf{CoCones}(F)$ is defined as that of cones, but with the direction of the natural transformations reversed. The <u>*colimit*</u> *of F over C, denoted $\mathrm{colim}_C F$, is, if it exists, the terminal object of $\mathsf{CoCones}(F)$.*

Definition 6.7 (pullback and pushout) *A* <u>*pullback*</u> *is a limit over the category $\{\, a \longrightarrow c \longleftarrow b \,\}$. A* <u>*pushout*</u> *is a colimit over the category $\{\, a \longleftarrow c \longrightarrow b \,\}$.*

Definition 6.8 (category internal to K) *For K any category with pullback, a* <u>*category*</u> *C* <u>*internal to K*</u> *is*

- *two objects $C_1, C_0 \in \mathrm{Obj}(K)$ and three morphisms $C_1C_1C_1 \begin{smallmatrix}\xrightarrow{s}\\ \xleftarrow{i}\\ \xrightarrow[t]{}\end{smallmatrix} C_0$ in K as well as a morphism $\circ : C_{1t} \times_s C_1 \to C_1$, where*

$$
\begin{tikzcd}
C_{1t} \times_s C_1 \arrow[r] \arrow[d] & C_1 \arrow[d, "s"] \\
C_1 \arrow[r, "s"] & C_0
\end{tikzcd}
$$

  is a pullback diagram;

- *such that*
  - *(*<u>*associativity*</u>*)*

$$
\begin{tikzcd}
C_{1t} \times_s C_{1t} \times_s C_1 \arrow[r, "\mathrm{Id}\times\circ"] \arrow[d, "\circ\times\mathrm{Id}"'] & C_{1t} \times_s C_1 \arrow[d, "\circ"] \\
C_{1t} \times_s C_1 \arrow[r, "\circ"] & C_1
\end{tikzcd}
$$

  - *(*<u>*unity*</u>*) $s \circ i = \mathrm{Id}$, $t \circ i = \mathrm{Id}$;*

$$
\begin{tikzcd}
C_0 \times_s C_1 \arrow[r, "i\times\mathrm{Id}"] \arrow[dr, "\mathrm{pr}_2"'] & C_{1t} \times_s C_1 \arrow[d, "\circ"] \\
 & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_{1t} \times C_0 \arrow[r, "\mathrm{Id}\times i"] \arrow[dr, "\mathrm{pr}_1"'] & C_{1t} \times_s C_1 \arrow[d, "\circ"] \\
 & C_1
\end{tikzcd}
$$

Remark. So a category as in definition 6.1 is a category internal to Sets.

Definition 6.9 (cartesian product of categories) *Given categories C and D, their* <u>*cartesian product*</u> *$C \times D$ is the category with $\mathrm{Obj}(C \times D) := \mathrm{Obj}(C) \times \mathrm{Obj}(D)$ and for all $(a, a'), (b, b') \in \mathrm{Obj}(C \times D)$ $(C \times D)((a, a'), (b, b')) := C(a, b) \times D(a', b')$ with the obvious composition and units.*

Definition 6.10 (monoidal category) *A monoidal category is a triple $(C, \otimes, I)$ consisting of a category C, a functor $\otimes : C \times C \to C$ called the tensor product functor and an object $I \in C$ such that $\otimes$ is associative up to coherent isomorphism [...] and such that I is the unit under this product up to coherent isomorphism [...].*

Definition 6.11 (closed category) *A monoidal category is left closed if the functor $A \otimes - : C \to C$ has a right adjoint naturally in A, the left internal hom, $\hom_l(A, -) : C \to C$ and right closed if $- \otimes A : C \to C$ has a right adjoint, the right internal hom $\hom_r(A, -) : C \to C$. It is biclosed if it is both left and right closed.*

Lemma 6.12 *For $\mathcal{C}$ a monoidal biclosed category, the contravariant left and right internal homs $\hom_{l,r}(-, A)$ send colimits to limits*

Proof. For $A, B, C \in \mathcal{C}$ the equivalence $\mathcal{C}(A, \hom_l(B, C)) \simeq \mathcal{C}(B, \hom_r(A, C))$, can be rewritten as

$$\mathcal{C}^{\mathrm{op}}(\hom_l(B, C), A) \simeq \mathcal{C}(B, \hom_r(A, C)) \,,$$

which shows that $\hom_l(-, C) : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a left adjoint and hence sends colimits in $\mathcal{C}$ to colimits in $\mathcal{C}^{\mathrm{op}}$, hence to limits in $\mathcal{C}$. Analogously for $\hom_r$. $\square$

We are grateful to Robin Houston for discussion of this point.

Definition 6.13 (simplicial objects) *[...]*

Definition 6.14 (2-category) *A ("strict") 2-category C internal to K is a diagram*

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "s_1"] \arrow[r, shift right=2, "t_1"'] & C_1 \arrow[l, "i_1" description] \arrow[r, shift left=2, "s_0"] \arrow[r, shift right=2, "t_0"'] & C_0 \arrow[l, "i_0" description] \end{tikzcd}$$

and morphisms

$$\circ_0 : C_2 \times_{t_0, s_0} C_2 \to C_2$$
$$\circ_1 : C_2 \times_{t_1, s_1} C_2 \to C_2$$

in K such that

$$\left(\begin{tikzcd} C_2 \arrow[r, shift left=2, "s_0 \circ s_1"] \arrow[r, shift right=2, "t_0 \circ t_1"'] & C_0 \arrow[l, "i_1 \circ i_0" description] \end{tikzcd} , \circ_0 \right)$$

and

$$\left(\begin{tikzcd} C_2 \arrow[r, shift left=2, "s_1"] \arrow[r, shift right=2, "t_1"'] & C_1 \arrow[l, "i_1" description] \end{tikzcd} , \circ_1 \right)$$

are categories in K and such that the exchange law

$$(\circ_1) \circ (\circ_0 \times \circ_0) = (\circ_0) \circ (\circ_1 \times \circ_1)$$

holds.

Definition 6.15 (string diagrams) *[...]*

Definition 6.16 (essentially surjective, full, faithful functors) *A functor $F : C \to D$ is*

- *essentially surjective if it is surjective on equivalence classes of objects of D;*
- *full if for all $c_1, c_2 \in \mathrm{Obj}(C)$ the map $F_{c_1,c_2} : \mathrm{Hom}_C(c_1, c_2) \to \mathrm{Hom}_D(F(c_1), F(c_2))$ is surjective;*
- *faithful if for all $c_1, c_2 \in \mathrm{Obj}(C)$ the map $F_{c_1,c_2} : \mathrm{Hom}_C(c_1, c_2) \to \mathrm{Hom}_D(F(c_1), F(c_2))$ is injective.*

Theorem 6.17 *A functor $F : C \to D$ is an equivalence of categories if and only if it is essentially surjective, full and faithful.*

Remark. The relevance of this theorem is that it characterizes equivalences of categories, which in general involve the existence of higher morphisms, namely natural transformations between functors, just in terms of properties of 1-morphisms.

6.2 Enriched category theory

Definition 6.18 (enriched category) *An enriched category C, enriched over a monoidal category $(\mathcal{V}, \otimes, I)$ a monoidal category C is a set of objects $\mathrm{Obj}(C)$ and for every pair $a, b \in \mathrm{Obj}(C)$ an object $C(a,b) \in \mathrm{Obj}(\mathcal{V})$ of morphisms from a to b and for every triple $a, b, c \in \mathrm{Obj}(C)$ a composition morphism $\circ_{a,b,c} : C(a,b) \otimes C(b,c) \to C(a,c)$ in $\mathcal{V}$ which is associative in the obvious sense. In addition, for every object a there is a morphisms $i_a : I \to C(a,a)$ in $\mathcal{V}$, which acts as an identity under composition in the obvious sense.*

Extraordinary naturality. Recall from enriched category theory [89] the notion of extraordinary natural families and *ends*: For $\mathcal{V}$ a closed monoidal category with a faithful functor $\mathsf{Sets} \hookrightarrow \mathcal{V}$ (for us: $\mathcal{V} = \omega\mathsf{Categories}$), for C an ordinary (hence Sets-eriched and therefore, by the above assumption, $\mathcal{V}$-enriched) category and for $F : C^{\mathrm{op}} \times C \to \mathcal{V}$ a $\mathcal{V}$-functor, an extraordinary natural family for F is a familiy of morphisms $\{ K \xrightarrow{\lambda_c} F(c,c) \mid c \in \mathrm{Obj}(C)\}$ from some object K in $\mathcal{V}$, such that for all morphisms $f : a \longrightarrow b$ in C the diagram

$$\begin{tikzcd}
K \arrow[r, "\lambda_a"] \arrow[d, "\lambda_b"'] & F(a,a) \arrow[d, "f_*"] \\
F(b,b) \arrow[r, "f^*"] & F(a,b)
\end{tikzcd}$$

commutes.

End. By definition, the end of F, denoted $\int_{c \in C} F(c,c) \in \mathcal{V}$ is the domain of the *universal* extraordinary family, in that there is a universal family $\{ \int_{c\in C} F(c,c) \longrightarrow F(c,c) \mid c \in \mathrm{Obj}(C)\}$ for F and every other extraordinary family for F uniquely factors through this one.

$$\begin{tikzcd}
K \arrow[dr, dashed] \arrow[drr, bend left, "\lambda_a"] \arrow[ddr, bend right, "\lambda_b"'] & & \\
& \int_{c\in C} F(c,c) \arrow[r] \arrow[d] & F(a,a) \arrow[d, "f_*"] \\
& F(b,b) \arrow[r, "f^*"] & F(a,b)
\end{tikzcd}$$

In other words: there is a bijection between extraordinary universal families $\{ K \xrightarrow{\lambda_c} F(c,c) \mid c \in \mathrm{Obj}(C)\}$ for F and morphisms $K \to \int_{c\in C} F(c,c) \in \mathcal{V}$.

Coend. The coend $\int^c F(c,c)$ is defined entirely analogously, with all morphisms reversed.

Proposition 6.19 (coend form of the Yoneda lemma) *For any $F : C^{\mathrm{op}} \to \mathcal{V}$ we have*

$$F(-) \simeq \int^c F(c) \otimes \hom(-,c)\,.$$

Definition 6.20 (Day convolution) *When C is monoidal, the preshaf category $\mathcal{V}^{C^{\mathrm{op}}}$ naturally inherits a biclosed monoidal structure with tensor product being the Day convolution product defined by*

$$(F \star G)(-) : \int^{c,d \in C} F(c) \otimes G(d) \otimes \hom_C(-, c \otimes d)\,.$$

The tensor unit it $I = \hom_C(-, I)$

Remark. The Day convolution product generalizes the ordinary convolution product of functions on groups: let C be the discrete category over a monoidal set, a group for instance, and with $\mathcal{V} = \mathsf{Sets}$ regard presheaves $F, G : C^{\mathrm{op}} \to \mathsf{Sets}$ as categorified $\mathbb{N}$-valued functions on C. Then the Day convolution product reduces to $(F \star G)(c) = \bigoplus_{d \cdot e = c} F(d) \times G(e)$.

Definition 6.21 (enriched functor category) *For C and D $\mathcal{V}$-enriched categories, the $\mathcal{V}$-enriched functor category $[C, D]$ has as objects the morphisms $F : C \to D$ of $\mathcal{V}$-enriched categories and the $\mathcal{V}$-object of morphisms between $F, G : C \to D$ is the end*

$$[C, D](F, G) := \int_{c \in C} D(F(c), G(c))\,.$$

Proposition 6.22 *In the case $\mathcal{V} = \mathsf{Sets}$ the $\mathcal{V}$-enriched functor category coincides with the category of functors and natural transformations from definition* **??***: $[C, D] = \mathsf{Functors}(C, D)$.*

See section 2.2 of [89].

6.3 Sheaf theory

Definition 6.23 (presheaf) *A presheaf on a category C as such is nothing but a contravariant functor from C with values in* Sets

$$F : C^{\mathrm{op}} \to \mathsf{Sets}\,.$$

Being functors, presheaves naturally form a category $\mathsf{Sets}^{C^{\mathrm{op}}} = \mathsf{Functors}(C^{\mathrm{op}}, \mathsf{Sets})$. Replacing Sets *by other* $\mathsf{Objects}$ *such as for instance* $\mathsf{AbelianGroups}$*, one obtains the corresponding presheaves of* $\mathsf{Objects}$ *forming the category $\mathsf{Objects}^{C^{\mathrm{op}}}$.*

Remark. By itself the concept of presheaf adds nothing but jargon to the concept of functor. One speaks of presheaves instead of functors when i) one is using the Yoneda embedding, definition 6.25 below, and ii) when a sheaf condition is to be imposed, definition 6.30 below.

Definition 6.24 (representable presheaves) *If the category C is enriched over* $\mathsf{Objects}$ *every object $c \in C$ yields a presheaf $Y(c) := C(-, c)$. These are the representable presheaves or representables, represented by the object c. One often directly writes c instead of $Y(c)$ for representable presheaves, if the context is clear.*

Definition 6.25 (Yoneda embedding) *Sending objects of C to representable presheaves yields a functor*

$$Y := \mathrm{Hom}(-_2, -_1) : C \to \mathsf{Sets}^{C^{\mathrm{op}}}$$

called the Yoneda embedding.

A central statement in category theory is the Yoneda lemma.

Theorem 6.26 (Yoneda lemma) *For $c \in C$ and $F \in \mathsf{Sets}^{C^{\mathrm{op}}}$ morphisms of presheaves from the representable $Y(c)$ into F are in bijection with the value of F on c:*

$$\mathrm{Hom}_{\mathsf{Sets}^{C^{\mathrm{op}}}}(Y(c), F) \simeq F(c)\,.$$

In particular, setting $F = Y(d)$ for d any object in C, we obtain $\mathrm{Hom}_{\mathsf{Sets}^{C^{\mathrm{op}}}}(Y(c), Y(d)) \simeq Y(d)(c) = \mathrm{Hom}_C(c, d)$. that is, the Yoneda embedding Y (from 6.25)is full and faithful functor.

Proposition 6.27 *Every presheaf is the colimit of representables.*

Proof. This is just proposition 6.19:

$$F(-) = \int^{c} F(x) \times \hom(-, c)\,.$$

$\square$

Definition 6.28 (closed monoidal structure on presheaves) *The cartesian tensor product on presheaves is given objectwise by*

$$F_1 \times F_2 : c \mapsto F_1(c) \times F_2(c)\,,$$

where on the right we have the cartesian product of sets. The internal hom is given by

$$\hom(F_1, F_2) : c \mapsto \mathrm{Hom}_{\mathsf{Sets}^{C^{\mathrm{op}}}}(F_1 \times c, F_2)\,.$$

As the name suggests, sheaves are the presheaves with an extra property; to make sense of the property the domain category has to have an additional structure.

Definition 6.29 (site) *A Grothendieck (pre)topology τ on a category C with pullbacks is a choice for every object c in C of a collection of distinguished families of morphisms, called covers (of c), with target c, so that*

(i) $c \xrightarrow{\mathrm{id}} c$ is a cover;

(ii) (stability) If $\{f_\alpha : c_\alpha \to c\}_{\alpha \in A}$ is a cover of c, then for any morphism $g : d \to c$, the family of pullbacks $\{c_\alpha \times_c d \to d\}_{\alpha \in A}$ is a cover of d;

(iii) (transitivity) If $\{f_\alpha : c_\alpha \to c\}_{\alpha \in A}$ is a cover of c, and for every α family $\{g_{\alpha\beta} : b_{\alpha\beta} \to b_\alpha\}_{\alpha \in A, \beta \in B_\alpha}$ is a cover of c_α, then the family of compositions $\{g_{\alpha\beta} \circ f_\alpha : b_{\alpha\beta} \to c\}_{\alpha \in A, \beta \in B_\alpha}$ is a cover of c.

A **site** *$(\mathcal{C}, \tau)$ is a category equipped with a Grothendieck topology.*

Definition 6.30 (sheaf) *A sheaf is a presheaf such that[...]*

6.4 Homotopy theory

Definition 6.31 (model category) *A (closed) model category is a category A equipped with 3 classes of distinguished morphisms called fibrations, cofibrations and weak equivalences satisfying some axioms: [...]*

Definition 6.32 (homotopy category) *[...]*

6.5 Differential geometry

Definition 6.33 (simplicial differential forms on BG) *For G a Lie group, the complex $\Omega^\bullet_{\mathrm{simp}}(BG)$ of simplicial differential forms on BG is the total complex of the double complex $\bigoplus_{k,l} = \Omega^k(G^{\times l})$ with differentials $d : \Omega^k(G^l) \to \Omega^{k+1}(G^l)$ the de Rham differential and $\delta : \Omega^k(G^l) \to \Omega^k(G^{l+1})$*

$$\Omega^\bullet(G) \xrightarrow{\delta} \Omega^\bullet(G \times G) \xrightarrow{\delta} \Omega^\bullet(G \times G \times G) \longrightarrow \cdots$$

be given by alternating sumps of pullback along face maps, $\delta = d_0^* - d_1^* + d_2^* + \cdots$.

Theorem 6.34 (Chern-Weil map) *There is an injection of DGCAs*

$$w : \mathrm{inv}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{simp}}(BG)\,.$$

Proof. The concrete realization given by [53] is as follows [...]. $\square$

Proposition 6.35 (simplicial version of first Pontryagin 4-form) *For $\mathfrak{g}$ a semisimple Lie algebra, the image of the normalized invariant bilinear polynomial $\langle \cdot, \cdot \rangle$ under the Chern-Weil map from definition 6.34 is $(\mu_3, \nu_2) \in \Omega^3(G) \times \Omega^2(G \times G)$ with*

$$\mu_3 := \langle \theta \wedge [\theta \wedge \theta] \rangle$$

and

$$\nu_2 := \langle \theta_1 \wedge \bar{\theta}_2 \rangle \,,$$

where θ is the left-invariant canonical $\mathfrak{g}$-valued 1-form on G and $\bar{\theta}$ the right-invariant one.

6.6 Cohomology theory

Cocycles for Hopf algebras The classification of extensions of Hopf algebras is well understood only when the Hopf algebras are either commutative or cocommutative. Similarly, S. Majid [104] has written down formulas for nonabelian cochain spaces and coboundary operators for bialgebras which generalize various abelian cases like group cocycles, Lie algebra cocycles, and also important low-dimensional nonabelian cocycles like Drinfeld twists and the Drinfeld associator. Thus, this looks like a right cohomology theory for bialgebras. However, the cohomology *classes* are not defined for $n > 2$ for general bialgebras, and we do not know what should replace them. [To refine this...]

6.7 Higher bundles in string theory

We briefly indicate how examples of higher bundles with connection arise in string theory.

Fundamental $(n-1)$-brane. Given a model nCob of the n-category of n-dimensional cobordisms and a version nVect of topological vector spaces, suitably well-behaved functors

$$n\mathrm{Cob} \to n\mathrm{Vect}$$

are called backgrounds for the fundamental $(n-1)$-brane. Their value on objects is the space of states of the $(n-1)$-brane and their value on n-morphisms is the correlator. Such functors have recently found a refined formulation in [73] extending ideas presented in [6].

In many concrete cases such functors are constructed by an intermediate step involving functors on cobordisms with extra structure, notably Lorentzian, Riemannian or conformal structure. A good understanding of the conformal 2-dimensional case is by now available.

Σ-models. Large classes of examples of such functors are thought to arise from path integral functionals on spaces of maps from n-dimensional cobordisms to specified spaces, called target spaces, equipped with various extra structures, see [159] for a review. These are called Σ-models or geometric backgrounds. In some special cases this has been made precise – see for instance [99] for a review of examples – but in most cases physicists rely on a body of well-tested but heuristic methods.

Perturbative String theory. Perturbative String theory is the study of the stringy perturbation series, which is the formal series in the surface genus of the correlators of a given functor on 2-dimensional conformal cobordisms over the Hom-spaces of $2\mathrm{Cob}_{\mathrm{conf}}$. A precise formulation is available at the moment only after passing to rational approximations of these Hom-spaces, see [43], which follows ideas by Kontsevich.

String backgrounds with higher connections. It turns out that target spaces for conformal 2-dimensional Σ-models generically are spaces equipped with the structure of various higher bundles with connection. A precise identification and formulation of these structures is achieved in [56] in the language of (abelian) differential cohomology as developed in [74]. Further discussion [51] is in preparation at time of this writing.

The following lists the main higher bundles with connection appearing on target spaces for the fundamental 1-brane. We list the corresponding physics terminology together with the interpretation in differential cohomology following [56], refined here in terms of nonabelian differential cohomology following [132, 133, 134]. Compare the notation on lifts, twisted lifts and obstructions to lifts from section **??**.

- The Neveu-Schwarz B-field on 10-dimensional spacetime X is an object $B_2 \in \hat{H}(X, \mathbf{B}^2 U(1))$. Restricted to submanifolds $W \hookrightarrow X$ called D-branes this field is required [57] to be, in our notation, in the image of
$$\mathrm{obstr}(U(H) \to PU(H)) : H(W, \mathbf{B}PU(H)) \to H(W, \mathbf{B}^2 U(1))\,.$$
The corresponding objects in the image of
$$\mathrm{twLift}(U(H) \to PU(H)) : H(W, \mathbf{B}PU(H)) \to H(W, \mathbf{B}(U(1) \to U(H)))$$
are the Chan-Paton bundles on the D-brane.

- The supergravity C-field on 11-dimensional spacetime Y is an object $C_3 \in \hat{H}(Y, \mathbf{B}^3 U(1))$. Restricted to submanifolds called end-of-the-world M9-branes M this field is required [75] to be in the image of
$$\mathrm{obstr}(\mathrm{String}(G) \to G) : H(M, \mathbf{B}G) \to H(M, \mathbf{B}^3 U(1))\,.$$
This condition is a manifestation of the Green-Schwarz mechanism [66].

- The dual supergravity C-field on Y is an object $C_7 \in \hat{H}(Y, \mathbf{B}^7 U(1))$. In the duality-symmetric situation described in [133, 134] this field, when restricted to M, is in the image of
$$\mathrm{obstr}(\mathrm{Fivebrane}(G) \to \mathrm{String}(G)) : H(-, \mathbf{B}\mathrm{String}(G)) \to H(-, \mathbf{B}^7 U(1))\,.$$
This condition is a manifestation dual Green-Schwarz mechanism [131, 59].

- The RR-fields on X are objects in $\hat{H}(X, \mathbf{B}U \times \mathbb{Z})$, [56].

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