Massey Products
in
Deligne-Beilinson Cohomology

Eine Dissertation zur Erlangung des akademischen Grades
Dr. rer. nat.

durch den Fachbereich Mathematik
der Universität Essen

Vorgelegt von
Alexander Schwarzhaupt
aus Worms

Contents

1. Introduction

Several cohomology theories are equipped with a multiplicative structure, called cup products. In the case of deRham cohomology of varieties over the complex numbers this cup product equals the wedge product of forms. In 1958 W.S. Massey introduced (see [Ma1]) higher order cohomology operations, called Massey products in the literature, which generalize the concept of cup products.

Massey products $M_\ell(A_1, \ldots, A_\ell)$ are partially defined up to some indeterminacy on ℓ-tuples of cohomology classes in the respective cohomology theory. The triple Massey product $M_3(A_1, A_2, A_3)$ of the cohomology classes A_1, A_2, A_3 is defined, if the cup products $A_1 \cup A_2$ and $A_2 \cup A_3$ vanish. The fourfold product $M_4(A_1, A_2, A_3, A_4)$ is defined if the triple products $M_3(A_1, A_2, A_3)$ and $M_3(A_2, A_3, A_4)$ are defined and are vanishing in some special sense which will be defined in chapter 2, definition 2.7.

Whereas the definition of the products looks somehow crucial, they can be found in different applications. W. S. Massey gave a first interpretation of the higher products in 1968 (see [Ma2]). He interprets the products as higher linking numbers. He calculates a first example, the so called Borromean rings. These three rings are configured in the three sphere in a way, such that two of them are unlinked but all the three are inseperable. Massey associates a nontrivial triple product to these rings in order to distinguish them from unlinked rings. We review this example in 2.9.

The well known theorem of Deligne, Griffiths, Morgan and Sullivan states that all the higher Massey products are vanishing in the deRham cohomology of smooth compact Kähler manifolds, or equivalently smooth projective varieties over $\mathbb{C}$ (see [DGMS], chapter 6, or [GM]). We will present this in 2.10. More general they state the vanishing of higher Massey products, if the complex which computes the cohomology is formal (for this see also [DGMS], chapter 4, and [GM]).

Our point of interest are Massey triple products in the Deligne-Beilinson cohomology of smooth projective varieties over the complex numbers. Not much is known about this topic. C. Deninger examines in [Den] Massey products in the real Deligne-Beilinson cohomology. He gets an explicit formula for the products using the resolution of the real Deligne-Beilinson complex introduced by J. I. Burgos in [Bu3]. Deninger also computes a non trivial Massey triple product on an affine curve.

T. Wenger also treats Massey products in his PhD thesis [Wen]. He generalizes the concept to absolute cohomology theories. He criterion for vanishing of the products in absolute Hodge theory, if the cohomology classes are assiociated to some invertible function f or $1 - f$. He also gets a presentation of cohomology classes in absolute Hodge cohomology.

For the case of integer valued Deligne-Beiliinson cohomology of smooth projective varieties no statements are given in the literature to our knowledge. We restrict ourselves in this thesis to the case where the cohomology classes arise from algebraic cycles. Therefore Massey products take values in $\mathrm{CH}^{\mathrm{n}}(\mathrm{X},1)$.

The obvious question is, do the products vanish. We can almost say no. Almost in the sense that up to now we were not able to find a highly nontrivial example. But we state an example (see example 3.32) which is not zero, but torsion (see definition 3.27 since the products are not uniquely defined).

We will give some criterion when the products are vanishing or are torsion via homological triviality of the algebraic cycles. This and the already mentioned example are presented in section 3.

In chapter 4 we present a relation between height pairings and Massey products. This result is quite astonishing since height pairings are a priori absolutely different defined and can be calculated in the situation of a curve as the cross-ratio of some rational functions. As a corollary we get that the difference of two special height pairings is always torsion, whereas the single height pairings can take almost every value. I am very gratefull to Stefan Müller-Stach who inspired me to look at height pairings.

Since at least to us it was not possible to construct a nontrivial example, we examine in section 5 Massey products in smooth families of algebraic varieties $\pi : X \longrightarrow S$, where S is a smooth affine curve. The natural question in this setup is: are the products constant or not. We use the concept of Griffiths infinitesimal invariant, to decide this question. The result and actually the main result is that we can construct via intersection theory an obstruction class $\widetilde{M}$ to the constance of the products which lives on the boundary $\bar{X} - X$ of the family. More precisely the class $\widetilde{M}$ lives in the singular cohomology of the boundary. We also give a way how one should be able to construct this class only using rational equivalence.

If not presented in the respective passage, technical details to hypercohomology can be found in the appendix 6.

I am very thankful to Eckart Viehweg for his guidance, helpful hints, the time he spent with me and more than everything for his patience especially with my laziness.

I thank Stefan Müller-Stach for his ideas and many explanations concerning the subjects contained in this thesis.

I also thank Hélène Esnault, whose idea it was to treat Massey products.

Without these three persons this thesis would never exist.

All in all many people had a more the less direct influence to the existence of this thesis. Thanks to all of you.

I thank my family where nobody ever knew what I did, but somehow they trusted me that it is the right thing.

I am very grateful to Manuel Blickle, Monika Dropulic, Jens, Steffi and Hanna Markus, Jörg Spritulla and Katja Stukenbrock for their very special support.

And everybody who was there when I needed them, especially: Andreas Robisch, Anja Hartmann, Ariane Liersch, Arpad Szell, Bill Gordon, Byungheup Jun, Claudia Niewerth, Claudia Czarsbon, Deana Kozsey, Enrico Sbarra, Francesco Picucci, Franziska Berg, Holger Ehrich, Ingo Kowalski, Jan Johannes, Maritha und Reinhold Stukenbrock, Nicole Liekenbröcker, Niko Naumann, Pavel Sedlak, Silke Lekaus, Stefan Meier, Steffi Krakau.

I also thank the Forschergruppe *Arithmetik und Geometrie* at the University of Essen, the Graduiertenkolleg *theoretische und experimentelle Methoden der reinen Mathemaik* and the Graduiertenkolleg *Mathematische und ingenieurwissenschaftliche Methoden für sichere Datenübertragung und Informationsvermittlung* for financial support.

Irgendwo muß man zwei Stimmen hören. Vielleicht liegen sie bloß wie stumm auf den Blättern eines Tagebuchs nebeneinander und ineinander, die dunkle, tiefe, plötzlich mit einem Sprung um sich selbst gestellte Stimme der Frau, wie die Seiten es fügen, von der weichen, weiten, gedehnten Stimme des Mannes umschlossen, von dieser verästelt, unfertig liegengebliebenen Stimme, zwischen der das, was sie noch nicht zu bedecken Zeit fand, hervorschaut. Vielleicht auch dies nicht. Vielleicht aber gibt es irgendwo in der Welt einen Punkt, wohin diese zwei, überall sonst aus der matten Verwirrung der alltäglichen Geräusche sich kaum heraushebenden Stimmen wie zwei Strahlen schießen und sich ineinander schlingen, irgendwo, vielleicht sollte man diesen Punkt suchen wollen, dessen Nähe man hier nur an einer Unruhe gewart wie die Bewegung einer Musik, die noch nicht hörbar, sich schon mit schweren unklaren Falten in dem undurchrissenen Vorhang der Ferne abdrückt....

Aus Robert Musils *Die Versuchung der stillen Veronika*

2. Massey Products

In this section we review the Definition of Massey products, state their basic properties and finally give some examples. The presentation follows the work of Kraines [Kr] for the classical Massey products and in a modified (there called linearized) version the work of C. Deninger [Den]. These definitions differ from the original one by W.S. Massey [Ma1] by a different sign convention.

Remark 2.1. A more general definition of the products can be found in [May]. Here they are called *Matric Massey products*.

2.1. **Massey products in the cohomology of Complexes of R-Modules.** We are working in the following situation. Let R be a commutative ring, and for $p \in \mathbb{N}$ let $\mathcal{C}^\bullet(p)$ be a complex of R-modules, with differential d, together with maps of complexes

$$\cup : \mathcal{C}^\bullet(p) \mathbin{\widehat{\otimes}} \mathcal{C}^\bullet(p') \longrightarrow \mathcal{C}^\bullet(p+p'),$$

where $\mathcal{C}^\bullet(p) \mathbin{\widehat{\otimes}} \mathcal{C}^\bullet(p')$ denotes the simple complex associated to the double complex $\mathcal{C}^\bullet(p) \otimes \mathcal{C}^\bullet(p')$, *i.e.* in degree n we have

$$(\mathcal{C}^\bullet(p) \mathbin{\widehat{\otimes}} \mathcal{C}^\bullet(p'))^n = \bigoplus_{q+q'=n} \mathcal{C}^q(p) \otimes \mathcal{C}^{q'}(p')$$

with differential

$$d : (\mathcal{C}^\bullet(p) \mathbin{\widehat{\otimes}} \mathcal{C}^\bullet(p'))^n \longrightarrow (\mathcal{C}^\bullet(p) \mathbin{\widehat{\otimes}} \mathcal{C}^\bullet(p'))^{n+1}$$
$$d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes \beta.$$

Furthermore we assume that $\cup$ is associative in the sense that we have

$$\cup \circ (\mathrm{id} \otimes \cup) = \cup \circ (\cup \otimes \mathrm{id}).$$

Now $\cup$ induces an associative cup product on the bigraded cohomology groups $H^*(\mathcal{C}^\bullet(*))$, where

$$H^q(\mathcal{C}^\bullet(p)) = \frac{\ker(d : \mathcal{C}^q(p) \longrightarrow \mathcal{C}^{q+1}(p))}{\mathrm{im}(d : \mathcal{C}^{q-1}(p) \longrightarrow \mathcal{C}^q(p))}$$

To be complete we give the general definition of ℓ-fold Massey products. Later we will restrict ourselves to the triple products we are concerned with.

Definition and Theorem 2.2. For $\ell \geq 2$ and integers $q_1, \dots, q_\ell$ and $p_1, \dots, p_\ell$ let define

$$q_{s,t} = \sum_{i=s}^{t} (q_i - 1) \text{ and } p_{s,t} = \sum_{i=s}^{t} p_i \text{ for } 1 \leq s \leq t \leq \ell.$$

Furthermore for $a \in \mathcal{C}^q(p)$, we denote by $\bar{a}$ the twist $(-1)^q a$.

(1) For cohomology classes $A_i \in H^{q_i}(\mathcal{C}^\bullet(p_i))$, $1 \leq i \leq \ell$ we say the ℓ-fold Massey product $M_\ell(A_1, \dots, A_\ell)$ is defined, if there exists a collection $\mathcal{M}$ of cochains

$$a_{s,t} \in \mathcal{C}^{q_{s,t}+1}(p_{s,t})$$

for $1 \le s \le t \le \ell$ and $(s,t) \neq (1,\ell)$, such that $a_i = a_{i,i}$ is a representative for A_i for $1 \le i \le \ell$ and

$$da_{s,t} = \sum_{i=s}^{t-1} \bar{a}_{s,i} \cup a_{i+1,t} \ \text{ for } \ 1 \le t-s \le \ell-2.$$

The collection $\mathcal{M}$ is called a **defining system** for ℓ-fold Massey product $M_\ell(A_1, \ldots, A_\ell)$.

(2) The associated cochain

$$c(\mathcal{M}) = \sum_{i=1}^{s-1} \bar{a}_{1,i} \cup a_{i+1,\ell} \in \mathcal{C}^{q_{1,\ell}+2}(p_{1,\ell})$$

is closed.

(3) We call the associated cohomology class $C(\mathcal{M})$ a **representative** for the ℓ-fold product

$$M_\ell(A_1, \ldots, A_\ell) \subset H^{q_{1\ell}+2}(p_{1\ell})$$

which consists of all cohomology classes $C(\mathcal{M})$, where $\mathcal{M}$ is a defining system for $M_\ell(A_1, \ldots, A_\ell)$.

(4) This definition is independent of the choice of the representatives of the A_i.

For the proof see [Kr], Theorem 3.

Before we concentrate on triple products, we list two properties of Massey products needed later.

Properties 2.3.

(1) *Scalar Multiplication:* Assume $M_\ell(A_1, \ldots, A_\ell)$ is defined, then for any $r \in R$ and any $1 \le k \le \ell$, the product $M_\ell(A_1, \ldots, rA_k, \ldots, A_\ell)$ is defined and we have

$$rM_\ell(A_1, \ldots, A_\ell) \subset M_\ell(A_1, \ldots, rA_k, \ldots, A_\ell).$$

(2) *Functoriality:* Let $\mathcal{C}^\bullet(*) \longrightarrow \mathcal{G}^\bullet(*)$ be a map of complexes of R-modules compatible with the cup product structure, then we have

$$g(M_\ell(A_1, \ldots, A_\ell)) \subset M_\ell(g(A_1), \ldots, g(A_\ell)).$$

For the proof see [Kr].

Remark 2.4. There is no strict additivity property, *i.e.* let $M_\ell(A_1, \ldots, A_k, \ldots, A_\ell)$ and $M_\ell(A_1, \ldots, A'_k, \ldots, A_\ell)$ be defined. Certainly the ℓ-fold product $M_\ell(A_1, \ldots, A_k + A'_k, \ldots, A_\ell)$ is defined. But in general

$$M_\ell(A_1, \ldots, A_k, \ldots, A_\ell) + M_\ell(A_1, \ldots, A'_k, \ldots, A_\ell) \neq M_\ell(A_1, \ldots, A_k + A'_k, \ldots, A_\ell).$$

Also there is no inclusion, neither in the one direction, nor in the other. But what we will prove later in 2.6 for the special case of triple products is, that there are common representatives for both, *i.e.* for $M_3(A_1, A_2, A_3) + M_\ell(A'_1, A_2, A_3)$ and $M_\ell(A_1 + A'_1, A_2, A_3)$ and the other two possibilities.

In the case of Massey triple products ($\ell = 3$) the above definition becomes more transparent. We simplify the notation and write $q_{s,t}$ for $q_s + q_t$ instead of $q_s + q_t - 2$ and $q_{1,2,3}$ for $q_1 + q_2 + q_3$.

Take cohomology classes $A_i \in H^{q_i}(p_i)$, such that $M_3(A_1, A_2, A_3)$ is defined. This means that we find a collection of cochains, the defining system $\mathcal{M}$,

$$a_i \in C^{q_i}(p_i) \text{ for } i = 1, 2, 3 \text{ and } a_{1,2} \in C^{q_{1,2}-1}(p_{1,2}) \ , \ a_{2,3} \in C^{q_{2,3}-1}(p_{2,3}),$$

such that the a_i are representatives for A_i and

$$da_{1,2} = \bar{a}_1 \cup a_2$$

$$da_{2,3} = \bar{a}_2 \cup a_3.$$

In particular $A_1 \cup A_2$ and $A_2 \cup A_3$ are zero.

The cochain

$$c(\mathcal{M}) = \bar{a}_1 \cup a_{2,3} + \bar{a}_{1,2} \cup a_3 = (-1)^{q_1} a_1 \cup a_{2,3} + (-1)^{q_{1,2}-1} a_{1,2} \cup a_3 \in C^{q_{1,2,3}1}(p_{1,2,3})$$

is closed.

The cochains $a_{1,2}$ and $a_{2,3}$ are well defined modulo closed cochains, *i.e.* well defined modulo the groups $H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2}))$ resp. $H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{2,3}))$. Therefore the image of $C(\mathcal{M})$ under

$$\tau : H^{q_{1,2,3}-1}(\mathcal{C}^\bullet(p_{1,2,3})) \longrightarrow \frac{H^{q_{1,2,3}-1}(\mathcal{C}^\bullet(p_{1,2,3}))}{H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2})) \cup A_3 + A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{2,3}))}$$

is well defined.

The definition of Massey products changed here from cosets to an element of a quotient group. We will denote by $M_3(A_1, A_2, A_3)$ the set of all representatives.

Remark 2.5. Keep in mind that always, when we are talking about **indeterminacy** we mean the subgroup

$$A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{2,3})) + H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2})) \cup A_3$$

of $H^{q_{1,2,3}-1}(p_{1,2,3})$.

Let us rephrase the properties of 2.3 in the case of triple products.

Properties 2.6.

(1) *Additivity 1* Let $M_3(A_1, A_2, A_3)$ and $M_3(A_1', A_2, A_3)$ be defined. Then $M_3(A_1 + A_1', A_2, A_3)$ is defined and we have

$$(M_3(A_1, A_2, A_3) + M_3(A_1', A_2, A_3)) \cap M_3(A_1 + A_1', A_2, A_3) \neq \emptyset.$$

If $M_3(A_1, A_2, A_3)$ and $M_3(A_1, A_2, A_3')$ are defined we have

$$(M_3(A_1, A_2, A_3) + M_3(A_1, A_2, A_3')) \cap M_3(A_1, A_2, A_3 + A_3') \neq \emptyset.$$

Note that we treat Massey products here as the set of all its representatives.

(2) *Additivity 2* Assume that $M_3(A_1, A_2, A_3)$ and $M_3(A_1, A_2', A_3)$ are defined, then again $M_3(A_1, A_2 + A_2', A_3)$ is defined and moreover we have

$$M_3(A_1, A_2, A_3) + M_3(A_1, A_2', A_3) = M_3(A_1, A_2 + A_2', A_3)$$

(3) *Scalar Multiplication:* Assume $M_3(A_1, A_2, A_3)$ is defined, then for any $r \in R$ the products $M_3(rA_1, A_2, A_3)$, $M_3(A_1, rA_2, A_3)$ and $M_3(A_1, A_2, rA_3)$ are defined and each contain $rM_3(A_1, A_2, A_3)$ as a subset.

(4) *Functoriality:* Let $\mathcal{C}^\bullet(*) \xrightarrow{g} \mathcal{G}^\bullet(*)$ be a map of complexes of R-modules compatible with the cup product structure, then we have

$$g(M_3(A_1, A_2, A_3)) \subset M_3(g(A_1), g(A_2), g(A_3)).$$

Proof. We just have to prove the first two points, since points (3) and (4) are just special cases of the properties listed in 2.3. Let $a_1,\ a_2,\ a_3,\ a_3'$ be representatives for $A_1,\ A_2,\ A_3$ and A_1'. By the assumptions we can find $a_{1,2},\ a_{1,2}', a_{2,3}$ with $da_{1,2} = \bar{a}_1 \cup a_2$, $da_{1,2}' = \bar{a}_1' \cup a_2$ and $da_{2,3} = \bar{a}_2 \cup a_3$. Obviously we have $d(a_{1,2} + a_{1,2}') = (\bar{a}_1 + \bar{a}_1') \cup a_2 = \overline{a_1 + a_1'} \cup a_2$. This yields representatives

$$[\bar{a}_1 \cup a_{2,3} + \bar{a}_{1,2} \cup a_3] \text{ for } M_3(A_1, A_2, A_3)$$

and

$$[\bar{a}_1' \cup a_{2,3} + \bar{a}_{1,2}' \cup a_3] \text{ for } M_3(A_1', A_2, A_3).$$

The sum of these classes is

$$[(\bar{a}_1 + \bar{a}_1') \cup a_{2,3} + (\bar{a}_{1,2} + \bar{a}_{1,2}') \cup a_3]$$

and this is obviously a representative of $M_3(A_1 + A_1', A_2, A_3)$.

The second point follows similarly. Let $\{a_1, a_2, a_3, a_{1,2}, a_{2,3}\}$ and $\{a_1, a_2', a_3, a_{1,2}', a_{2,3}'\}$ be defining systems for $M_3(A_1, A_2, A_3)$, resp. $M_3(A_1, A_2', A_3)$. Then we can construct the representative $[\bar{a}_1 \cup (a_{2,3} + a_{2,3}') + (\bar{a}_{1,2} + \bar{a}_{1,2}') \cup a_3]$ of $M_3(A_1, A_2 + A_2', A_3)$. Certainly this representative lives in the intersection

$$(M_3(A_1, A_2, A_3) + M_3(A_1, A_2', A_3)) \cap M_3(A_1, A_2 + A_2', A_3).$$

As sets we have

$$\begin{aligned} &M_3(A_1, A_2, A_3) + M_3(A_1, A_2', A_3) \\ &= [\bar{a}_1 \cup a_{2,3} + \bar{a}_{1,2} \cup a_3] + [\bar{a}_1 \cup a_{2,3}' + \bar{a}_{1,2}' \cup a_3] \\ &+ A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{1,2})) + H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{1,2})) \cup A_3 \end{aligned}$$

$$= [\bar{a}_1 \cup (a_{2,3} + a_{2,3}') + (\bar{a}_{1,2} + \bar{a}_{1,2}') \cup a_3] + A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{1,2})) + H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{1,2})) \cup A_3$$

$$= M_3(A_1, A_2 + A_2', A_3).$$

Hence equality holds. □

One point of interest and actually the main point we are concerned with is the question of vanishing of Massey triple products.

Definition and Theorem 2.7. We say that $M_3(A_1, A_2, A_3)$ for cohomology classes $A_i \in H^{q_i}(p_i)$, $i = 1, 2, 3$ vanishes if one of the following equivalent conditions holds

(1) $M_3(A_1, A_2, A_3)$ is zero viewed as an element of the quotient group
$$\frac{H^{q_{1,2,3}-1}(\mathcal{C}^\bullet(p_{1,2,3}))}{H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2})) \cup A_3 + A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{2,3}))}$$

(2) There exists a defining system M consisting of representatives $a_i \in \mathcal{C}^{q_i}(p_i)$ of A_i, for $i = 1, 2, 3$ and $a_{i,i+1} \in \mathcal{C}^{q_{ii+1}}(p_{i,i+1})$, such that
$$[(-1)^{q_1} a_1 \cup a_{2,3} + (-1)^{q_{1,2}-1} a_{1,2} \cup a_3]$$
is of the form
$$A_1 \cup A_{2,3} + A_{1,2} \cup A_3$$
for suitable $A_{i,j} \in H^{q_{i,j}-1}(p_{i,j})$. In other words we find a representative living in the indeterminacy.

(3) For $M_3(A_1, A_2, A_3)$ viewed as the set of all representatives of the triple product we have
$$M_3(A_1, A_2, A_3) = H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2}) \cup A_3 + A_1 \cup H^{q_{2,3}-1}(\mathcal{C}^\bullet(p_{2,3}).$$
Especially there is a defining system $M = \{a_1, a_2, a_3, a_{1,2}, a_{2,3}\}$ in the respective $\mathcal{C}^\bullet(\bullet)$ such that the associated cycle
$$c(M) = [(-1)^{q_1} a_1 \cup a_{2,3} + (-1)^{q_{1,2}-1} a_{1,2} \cup a_3] = [0].$$

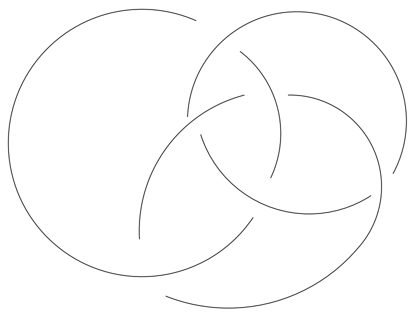
Proof. (2) follows by definition from (1). By definition 2.2 $M_3(A_1, A_2, A_3)$ is independent of the choice of representatives $a_i \in \mathcal{C}^{q_i}$. The elements $a_{1,2}$ and $a_{2,3}$ are well defined up to closed classes, *i.e.* elements representing a cohomology class in $H^{q_{1,2}-1}(\mathcal{C}^\bullet(p_{1,2})$, resp. $H^{q_{2,3}-1}(p_{2,3})$. On the other hand we can modify the chains $a_{i,j}$ by classes living in $H^{q_{i,j}-1}(p_{i,j})$, which means that we also change the classes $A_{i,j}$ by the same class to reach each class in the indeterminacy, which implies (3). The implication of (1) by (3) again is obvious. □

Remark 2.8. C. Deninger modified in [Den] the definition of general ℓ-fold Massey products to get a linearized version. The problem is that the higher products do not form homomorphisms of groups, since in the original definition the indeterminacy of the choice of a defining system does not form a subgroup of the corresponding cohomology group. However in the case of triple products both definitions agree (and as mentioned above, the indeterminacy does form a subgroup), hence we get homomorphisms
$$\mathcal{D}(M_3) \longrightarrow \mathcal{T}(M_3) = \frac{H^{q_{1,2,3}-1}(p_{1,2,3})}{H^{q_1}(p_1) \cup H^{q_{2,3}-1}(p_{2,3}) + H^{q_{1,2}-1} \cup H^{q_3}(p_3)}$$
where
$$\mathcal{D}(M_3) = \ker(\cup \otimes \mathrm{id}) \cap \ker(\mathrm{id} \otimes \cup) \subset \mathrm{H}^{q_1}(\mathrm{p}_1) \otimes \mathrm{H}^{q_2}(\mathrm{p}_2) \otimes \mathrm{H}^{q_3}(\mathrm{p}_3).$$

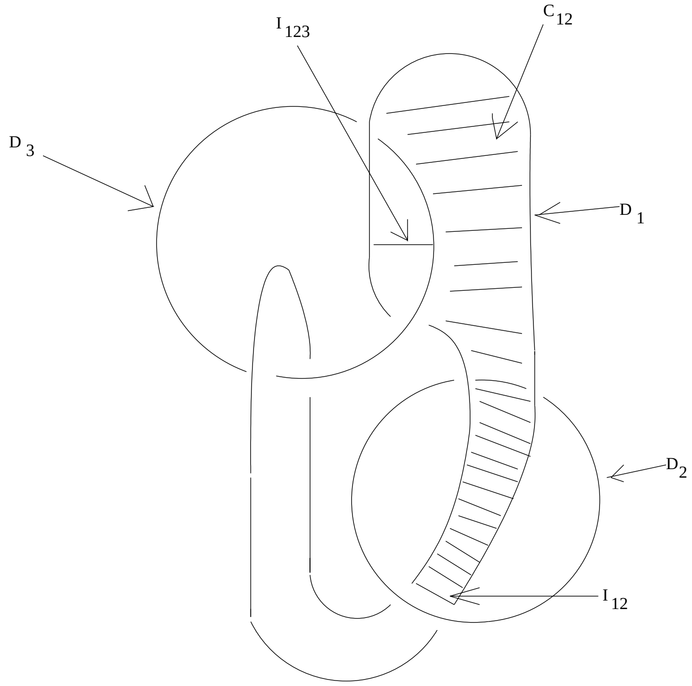
For more details the interested reader is referred to the article of C. Deninger [Den].

Let's now present some examples. The first one is the original example given by W.S. Massey in 1968 in [Ma2], where he interprets the higher products as higher linking numbers. It can be found in [GM], but we state it here to give the reader a feeling at what is going on.

Example 2.9. Consider the so called Borromean rings $\mathcal{B} \subset S^3$



Let X be the complement of $\mathcal{B}$ in S^3 and consider the rings configured as in the next picture:



As one sees in the picture, the three rings are linked, whereas two of them are unlinked. The first cohomology of X is spanned by the duals of the disks D_i,

$i = 1, 2, 3$. Let's denote them by $\widetilde{D}_i$. The fact that the rings are unlinked gives in terms of cohomology $\widetilde{D}_i \cup \widetilde{D}_j = 0$. The geometric picture is the following. Obviously $D_2 \cap D_3 = \emptyset$. On the other hand $D_1 \cap D_2 = I_{1,2} = \partial C_{1,2}$, where $C_{1,2}$ is chosen as in the picture. Therefore we can choose $a_{i,j}$, such that $a_{2,3} = 0$ and $da_{1,2} = \widetilde{D}_1 \cup \widetilde{D}_2$. This shows that the Massey product $M_3(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3)$ is defined and represented by the form $a_{1,2} \cup \widetilde{D}_3$. This form represents $C_{1,2} \cap D_3 = I_{1,2,3}$ which is an interval whose end points are living on two different circles. Hence its class in $H_1(X)$ does not vanish and the constructed representative of $M_3(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3)$ does not vanish either. What remains to check is that this representative does not live in the indeterminacy. The indeterminacy is given by $\widetilde{D}_1 \cup H^1(X) + H^1(X) \cup \widetilde{D}_3$. But two of the three circles are unlinked. As we mentioned before the classes $\widetilde{D}_i$ span the first cohomology, hence $\widetilde{D}_i \cup \widetilde{D}_j = 0$ for $i \neq j$ and $a_{1,2} \cup \widetilde{D}_3$ cannot be of the form $\alpha \widetilde{D}_1^2 + \beta \widetilde{D}_3^2$.

If we assume $\mathcal{B}'$ to be three unlinked circles (we can arrange them in a way, such that the spanning disks D_i are disjoint) then clearly the Massey product $M_3(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3)$ is vanishing. In particular $\mathcal{B}$ and $\mathcal{B}'$ are not isotopic.

This example of the Borromean rings was the motivation to interpret Massey products as higher linking numbers. For more results of this kind the reader is refered to [Ma2].

The following example, which can also be found in [GM], gives a statement for the triviality of the products in the deRham cohomology of compact Kähler manifolds.

Example 2.10. This example is the theorem of P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, which states the vanishing of all Massey products in the deRham cohomology of compact Kähler Manifolds (see [DGMS], chapter 6, and [GM]). There Massey products are related to the formality of Differential Graded Algebras (DGA). For more details on this subject see [DGMS], chapter 4, and [GM]. The main incredient of the proof is the *principle of two types* which we state first. We follow the presentation given in [GM].

Lemma 2.11. *Let X be a compact Kähler Manifold and $E^{p,q}(X)$ the vector space of C^∞ (p,q)-forms on X. Then the deRham cohomology of X is given by*

$$H^n_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where

$$H^{p,q}(X) = \{\phi \in E^{p,q}(X) : d\phi = 0\}/dE^{p+q-1}(X) \cap E^{p,q}(X).$$

Now suppose $\phi \in E^{p,q}(X)$, such that $\phi = d\eta$ for some C^∞-form η, then we can find forms $\eta' \in E^{p,q-1}(X)$ and $\eta'' \in E^{p-1,q}(X)$, such that

$$\phi = d\eta'$$

and

$$\phi = d\eta''.$$

Theorem 2.12. *Let X be a compact Kähler manifold and $A_i \in H^{n_i}(X, \mathbb{C})$ be deRham classes, such that $M_3(A_1, A_2, A_3)$ is defined. Then $M_3(A_1, A_2, A_3)$ is vanishing.*

Proof. The wedge product of forms induces an associative cup product in deRham cohomology. Let A_i for $i = 1, 2, 3$ be forms, such that $M_3(A_1, A_2, A_3)$ is defined. For simplicity we assume $A_i \in H^{p_i,q_i}(X)$. Let $a_i \in E^{p_i,q_i}(X)$ be some representative. By 2.10 we can find forms $\phi_{1,2} \in E^{p_{1,2}-1,q_{1,2}}(X)$, $\phi'_{1,2} \in E^{p_{1,2},q_{1,2}-1}(X)$, $\phi_{2,3} \in E^{p_{2,3}-1,q_{2,3}}$ and $\phi'_{2,3} \in E^{p_{2,3},q_{2,3}-1}(X)$, such that

$$d\phi_{1,2} = d\phi'_{1,2} = \bar{a}_1 \wedge a_2$$

and

$$d\phi_{2,3} = d\phi'_{2,3} = \bar{a}_2 \wedge a_3.$$

Now $\phi_{1,2} - \phi'_{1,2}$ is closed. Using Hodge decomposition we can vary both forms preserving their type such that the difference becomes exact. We can now construct two cohomologous representatives of $M_3(A_1, A_2, A_3)$, namely

$$M = \bar{a}_1 \wedge \phi_{2,3} + \bar{\phi}_{1,2} \wedge a_3 \in E^{q_{1,2,3}-1,q_{1,2,3}}(X)$$

and

$$M' = \bar{a}_1 \wedge \phi'_{2,3} + \bar{\phi}'_{1,2} \wedge a_3 \in E^{q_{1,2,3},q_{1,2,3}-1}(X).$$

By the direct sum decomposition and the fact that they are cohomologous, they have to be exact, which means $M_3(A_1, A_2, A_3)$ vanishes. □

The next example shows that this statement is not true anymore for general Kähler Manifolds. The example is also taken from [DGMS], chapter 4. In the language of *formality* the example shows that the deRham complex of a general Kähler Manifold must not be formal. For this see [DGMS], chapter 4. We will write the differentialforms as matrices. This differs from the presentation in [DGMS]

Example 2.13. Let $\mathcal{T}$ be the space of upper 3×3 triangular matrices over $\mathbb{C}$ with 1 on the diagonal. Let furthermore Γ be the lattice consisting of matrices of the form

$$\begin{pmatrix} 1 & \alpha + i\alpha\prime & \beta + i\beta\prime \\ 0 & 1 & \gamma + i\gamma\prime \\ 0 & 0 & 1 \end{pmatrix}$$

for $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \gamma' \in \mathbb{Z}$. Now let $X = \mathcal{T}/\Gamma$ be the Iwasawa manifold (see [GH] chapter 3.5.). By mapping

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

to the (a,c)-coordinates we see that X has the structure of a fibrebundle over a complex 2-torus, each fibre being a complex one torus. The form

$$dgg^{-1} = \begin{pmatrix} 0 & da & -cda+db \\ 0 & 0 & dc \\ 0 & 0 & 0 \end{pmatrix}$$

is a holomorphic form on X. Setting

$$\omega_1 = \begin{pmatrix} 0 & da & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \omega_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & dc \\ 0 & 0 & 0 \end{pmatrix} \ , \ \omega_3 = \begin{pmatrix} 0 & 0 & -cda+db \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One observes that

$$d\omega_3 = \omega_1 \wedge \omega_2,$$

and therefore ω_3 is not closed. Now $M_3(\omega_1, \omega_1, \omega_2)$ is defined and we have a nonexact representative

$$\omega_1 \wedge \omega_3.$$

Since

$$\omega_1 \wedge H^1(X) = H^1(X) \wedge \omega_2 = 0$$

$M_3(\omega_1, \omega_1, \omega_3)$ does not vanish.

2.2. **Massey products in the Hypercohomology of Complexes of Sheaves.**

Let X be a complex algebraic variety and as before for $p \in \mathbb{N}$ let $\mathcal{F}^\bullet(p)$ be a complex of sheaves of R-modules, $R \subset \mathbb{C}$, equipped with a an associative product

$$\cup_{\mathcal{F}} : \mathcal{F}^n(p) \otimes \mathcal{F}^m(p') \longrightarrow \mathcal{F}^{n+m}(p+p'),$$

respecting differentials.

Let furthermore $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ be the Čech resolution of the complex for some open covering $\mathcal{U}$ of X, together with the differential δ (see Appendix A).

Additionally we have the Čech product

$$\text{(2.1)} \qquad \begin{array}{rccc} \check{\cup} : & \mathcal{C}^q(\mathcal{U}, \mathcal{F}^n(p)) \otimes \mathcal{C}^{q'}(\mathcal{U}, \mathcal{F}^m(p')) & \longrightarrow & \mathcal{C}^{q+q'}(\mathcal{U}, \mathcal{F}^n(p) \otimes \mathcal{F}^m(p')) \\ & (f_{i_0 \dots i_q}) \otimes (g_{i_0 \dots i_{q'}}) & \longmapsto & (f_{i_0 \dots i_q} \otimes g_{i_q \dots i_{q+q'}}). \end{array}$$

This yields to an associative morphism of complexes

$$\text{(2.2)} \qquad \check{\cup} : \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet(p)) \,\widehat{\otimes}\, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet(p')) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet(p)) \otimes \mathcal{F}^\bullet(p')).$$

In the construction of a cup product on Čech Hypercohomology one has to be careful. Several contraction morphisms introduce signs which have to be understood to obtain a consistent theory. For a general point of view on this subject see [Del3].

The first contraction is (the notation in what follows is taken from [Den]: the indices of the complexes are labeled corresponding to the indices of the contractions, to make clear which parts are contracted):

$$s_{1,3} : (\mathcal{C}^{\bullet_1}(\mathcal{U}, \mathcal{F}^{\bullet_2}(p)) \otimes \mathcal{C}^{\bullet_3}(\mathcal{U}, \mathcal{F}^{\bullet_4}(p'))) \longrightarrow \mathcal{C}^{\bullet_{1,3}}(\mathcal{U}, \mathcal{F}^{\bullet_2}(p) \otimes \mathcal{F}^{\bullet_4}(p'))$$

which is induced by $\check{\cup}$.

After applying the contractions $s_{2,4}$ and $s_{(1,3)(2,4)}$ we get a map of simple complexes

$$s_{(1,3)(2,4)}s_{2,4}s_{1,3}(\mathcal{C}^{\bullet_1}(\mathcal{U},\mathcal{F}^{\bullet_2}(p))\otimes\mathcal{C}^{\bullet_3}(\mathcal{U},\mathcal{F}^{\bullet_4}(p'))) \xrightarrow{\check{\cup}} s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p)\otimes\mathcal{F}^{\bullet}(p'))$$

(note that $s\mathcal{C}^{\bullet}(\mathcal{F}^{\bullet})$ denotes the simple complex associated to the double complex $\mathcal{C}^{\bullet}(\mathcal{F}^{\bullet})$).

Alternatively we could walk the other way around, which means we build first the simple complexes $s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p))$ and contract the tensor product of them. In terms of the above setting this means, we build

$$s_{(1,2)(3,4)}s_{1,2}s_{3,4}(\mathcal{C}^{\bullet_1}(\mathcal{U},\mathcal{F}^{\bullet_2}(p))\otimes\mathcal{C}^{\bullet_3}(\mathcal{U},\mathcal{F}^{\bullet_4}(p'))).$$

By [Del3] there is an isomorphism

$$\begin{aligned}\tau^{\bullet}: {} & s_{(1,2)(3,4)}s_{1,2}s_{3,4}(\mathcal{C}^{\bullet_1}(\mathcal{U},\mathcal{F}^{\bullet_2}(p))\otimes\mathcal{C}^{\bullet_3}(\mathcal{U},\mathcal{F}^{\bullet_4}(p'))) \\ \longrightarrow {} & s_{(1,3)(2,4)}s_{1,3}s_{2,4}(\mathcal{C}^{\bullet_1}(\mathcal{U},\mathcal{F}^{\bullet_2}(p))\otimes\mathcal{C}^{\bullet_3}(\mathcal{U},\mathcal{F}^{\bullet_4}(p'))),\end{aligned} \tag{2.3}$$

which is multiplication by a sign. Precisely, in degree $\bar{k}=(p_1,p_2,p_3,p_4)$ we have $\tau^{\bar{k}}=(-1)^{p_2p_3}$, see [Den], or in another language [Del3].

Finally we get an associative product

$$\check{\cup}\circ\tau^{\bullet}: s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p))\otimes s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p'))\longrightarrow s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p)\,\widehat{\otimes}\,\mathcal{F}^{\bullet}(p')).$$

Setting

$$\mathcal{C}^{\bullet}(p)=\lim_{\vec{\mathcal{U}}} s\mathcal{C}^{\bullet}(\mathcal{U},\mathcal{F}^{\bullet}(p))$$

we get complexes of R-modules.

Finally we can define the desired cup product on the Čech complex

$$\cup=\cup_{\mathcal{F}}\circ\check{\cup}\circ\tau^{\bullet}:\mathcal{C}^{\bullet}(p)\,\widehat{\otimes}\,\mathcal{C}^{\bullet}(p')\longrightarrow\mathcal{C}^{\bullet}(p+p'),$$

which gives by definition 2.1 a theory of Massey products on the Hypercohomology Groups $\mathbb{H}^*(X,\mathcal{F}^{\bullet}(*))$.

3. Massey products in Deligne-Beilinson cohomology

In this section we present Deligne-Beilinson cohomology as introduced by A. Beilinson in [Be1]. We follow the presentation in *Deligne-Beilinson cohomology* by H. Esnault and E. Viehweg [EV1]. Proofs for all statements can be found there. The second part of this section is devoted to Massey triple products in Deligne-Beilinson cohomology, and first results about their vanishing. We focus on the case where the cohomology classes A_1, A_2, A_3 are the classes of algebraic cycles. We show, that in this case the only possibility to get some nontrivial example of Massey products is, if none of them is homologically equivalent to zero.

3.1. **Deligne-Beilinson Cohomology.**

Let X be an algebraic manifold and $\phi : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet$ be a morphism of complexes of sheaves on X. We define the cone of ϕ to be the complex

$$\mathrm{Cone}(\mathcal{A}^\bullet \xrightarrow{\phi} \mathcal{B}^\bullet) = \mathcal{C}^\bullet_\phi := \mathcal{A}^\bullet[1] \oplus \mathcal{B}^\bullet, \tag{3.1}$$

where $\mathcal{A}^p[1] = \mathcal{A}^{p+1}$. The differentials are given by

$$\begin{array}{rccc} d_{\mathcal{C}} : & \mathcal{A}^{p+1} \oplus \mathcal{B}^p & \longrightarrow & \mathcal{A}^{p+2} \oplus \mathcal{B}^{p+1} \\ & (a,b) & \longmapsto & (-d_{\mathcal{A}}(a), \phi(a) + d_{\mathcal{B}}(b)) \end{array} \tag{3.2}$$

The obvious embedding and projection give rise to a triangle in the derived category of complexes of sheaves

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[rr, "\phi"] & & \mathcal{B}^\bullet \arrow[dl] \\ & \mathcal{C}^\bullet_\phi \arrow[ul, "{[1]}"] & \end{tikzcd}$$

A triangle in a derived category is called distinguished, if it is of the above form or quasiisomorphic to one constructed in this way.

Applying the Hypercohomology functor (see the Appendix) we get the distinguished triangle

$$\begin{tikzcd} \mathbb{H}^*(\mathcal{A}^\bullet) \arrow[rr, "\phi"] & & \mathbb{H}^*(\mathcal{B}^\bullet) \arrow[dl] \\ & \mathbb{H}^*(\mathcal{C}^\bullet_\phi) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

In other words we get the long exact sequence of Hypercohomology

$$\ldots \to \mathbb{H}^q(\mathcal{A}^\bullet) \to \mathbb{H}^q(\mathcal{B}^\bullet) \to \mathbb{H}^q(\mathcal{C}^\bullet_\phi) \to \mathbb{H}^{q+1}(\mathcal{A}^\bullet) \to \ldots$$

A nice reference for this subject is the book of B. Iversen [Iv].

We now list some properties of this construction.

Properties 3.1. For morphisms of complexes

$$\begin{array}{rcl} \phi_1 : & \mathcal{A}_1^\bullet & \longrightarrow & \mathcal{B}^\bullet \\ \phi_2 : & \mathcal{A}_2^\bullet & \longrightarrow & \mathcal{B}^\bullet \end{array}$$

let $\mathcal{C}^\bullet$ be the complex

$$\mathcal{C}^\bullet = \mathrm{Cone}(\mathcal{A}_1^\bullet \oplus \mathcal{A}_2^\bullet \xrightarrow{\phi_1-\phi_2} \mathcal{B}^\bullet)[-1].$$

Alternatively we can construct $\mathcal{C}^\bullet$ as follows:

$$\begin{aligned} \mathcal{C}^\bullet &= \mathrm{Cone}(\mathcal{A}_1^\bullet \xrightarrow{\phi_1} \mathrm{Cone}(\mathcal{A}_2^\bullet \xrightarrow{-\phi_2} \mathcal{B}^\bullet))[-1] \\ &= \mathrm{Cone}(\mathcal{A}_2^\bullet \xrightarrow{-\phi_2} \mathrm{Cone}(\mathcal{A}_1^\bullet \xrightarrow{\phi_1} \mathcal{B}^\bullet))[-1]. \end{aligned} \tag{3.3}$$

The three different constructions imply the long exact sequences

$$\ldots \longrightarrow \mathbb{H}^q(\mathcal{C}^\bullet) \longrightarrow \mathbb{H}^q(\mathcal{A}_1^\bullet) \oplus \mathbb{H}^q(\mathcal{A}_2^\bullet) \longrightarrow \mathbb{H}^q(\mathcal{B}^\bullet) \longrightarrow \mathbb{H}^{q+1}(\mathcal{C}^\bullet) \longrightarrow \ldots \tag{3.4}$$

$$\ldots \mathbb{H}^q(\mathcal{C}^\bullet) \longrightarrow \mathbb{H}^q(\mathcal{A}_1^\bullet) \longrightarrow \mathbb{H}^q(\mathrm{Cone}(\mathcal{A}_2^\bullet \xrightarrow{-\phi_2} \mathcal{B}^\bullet)) \longrightarrow \mathbb{H}^{q+1}(\mathcal{C}^\bullet) \longrightarrow \ldots \tag{3.5}$$

$$\ldots \mathbb{H}^q(\mathcal{C}^\bullet) \longrightarrow \mathbb{H}^q(\mathcal{A}_2^\bullet) \longrightarrow \mathbb{H}^q(\mathrm{Cone}(\mathcal{A}_1^\bullet \xrightarrow{\phi_1} \mathcal{B}^\bullet)) \longrightarrow \mathbb{H}^{q+1}(\mathcal{C}^\bullet) \longrightarrow \ldots . \tag{3.6}$$

Let X be smooth of dimension n over $\mathbb{C}$, $\bar{X}$ a good compactification of X, *i.e.* a smooth compactification, such that $\bar{X} - X = Y$ is a divisor with normal crossings.

Let $\Omega^\bullet_{\bar{X}}(\log Y)$ be the deRham complex of meromorphic forms on $\bar{X}$ with logarithmic poles along Y. On $\Omega^\bullet_{\bar{X}}(\log Y)$ we have the F-filtration given by

$$F^p = F^p(\Omega^\bullet_{\bar{X}}(\log Y)) = (0 \to \Omega^p_{\bar{X}}(\log Y) \to \Omega^{p+1}_{\bar{X}}(\log Y) \to \ldots \to \Omega^n_{\bar{X}}(\log Y)) \tag{3.7}$$

where $\Omega^p_X(\log Y)$ lives in degree p. By Deligne [Del1] we have the following

Properties 3.2.

$$H^q(X, \mathbb{C}) = \mathbb{H}^q(\bar{X}, \Omega^\bullet_{\bar{X}}(\log Y))$$

The maps

$$\iota_p : \mathbb{H}^q(\bar{X}, F^{p+1}) \longrightarrow \mathbb{H}^q(\bar{X}, F^p)$$

and

$$\iota : \mathbb{H}^q(\bar{X}, F^p) \longrightarrow \mathbb{H}^q(\bar{X}, \Omega^\bullet_{\bar{X}}(\log Y))$$

are injective. Moreover $H^q(X, \mathbb{C})$ carries a mixed Hodge structure given by the weightfiltration and the maps ι. For more details see [Del1].

By GAGA we can use also algebraic differential forms for the computation of $H^q(X, \mathbb{C})$.

Now we are able to define the Deligne-Beilinson Complex and the Deligne-Beilinson cohomology groups.

Definition 3.3. Let X, $\bar{X}$ and Y be as above, $A \subset \mathbb{R}$ be a subring and $A(p) = (2\pi i)^p A \subset \mathbb{C}$. The Deligne-Beilinson Complex of the pair $(X, \bar{X})$ is constructed as follows

$$A_{\mathcal{D}}(p) = \mathrm{Cone}(Rj_*A(p) \oplus F^p \xrightarrow{\epsilon-\iota} \mathcal{R}j_*\Omega^\bullet_X)[-1] \tag{3.8}$$

where ϵ and ι are the obvious maps, and where we may choose for $\mathcal{R}j_*\Omega^\bullet_X$ any complex quasi-isomorphic to the log-forms, such that all the maps exist.

Equation 3.3 gives the quasi-isomorphisms

- $A_{\mathcal{D}}(p) \xrightarrow{quis} \mathrm{Cone}(Rj_*A(p) \xrightarrow{\epsilon} \mathrm{Cone}(F^p \xrightarrow{-\iota} \mathcal{R}j_*\Omega^\bullet_X))[-1]$ and
- $A_{\mathcal{D}}(p) \xrightarrow{quis} \mathrm{Cone}(F^p \xrightarrow{-\iota} Rj_*\mathrm{Cone}(A(p) \xrightarrow{\epsilon} \Omega^\bullet_X))[-1]$

The definition yields the following distinguished triangle

$$\begin{array}{ccc} Rj_*\mathbb{Z}(p) \oplus F^p & \longrightarrow & \mathcal{R}j_*\Omega^\bullet_X \\ & \nwarrow \quad \swarrow & \\ & A_{\mathcal{D}}(p) & \end{array} \tag{3.9}$$

Remark 3.4. Let $X = \bar{X}$. The commutative diagram

(3.10)
$$\begin{array}{ccccccccccccc} A(P)_{\mathcal{D},an} = (A(p) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \dots & \longrightarrow & \Omega^{p-2}_X & \longrightarrow & \Omega^{p-1}_X & \longrightarrow & 0 & \cdots) \\ \alpha_0 \downarrow & & \alpha_1 \downarrow & & & & \alpha_{p-1} \downarrow & & \alpha_p \downarrow & & \downarrow & \\ A_{\mathcal{D}}(p) \ = (A(p) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \dots & \longrightarrow & \Omega^{p-2}_X & \longrightarrow & \Omega^p_X \oplus \Omega^{p-1}_X & \longrightarrow & \Omega^{p+1}_X \oplus \Omega^p_X & \cdots) \end{array}$$

implies that the complex $A_{\mathcal{D}}(p)|_X$ is quasi-isomorphic to the complex $A(p)_{\mathcal{D},an}$, called the analytic Deligne Complex (see [EV], chapter 1).

We have the following maps

$$\pi_A : A_{\mathcal{D}}(p) \longrightarrow Rj_*A(p) \ \text{ and}$$
$$\pi_F : A_{\mathcal{D}}(p) :\longrightarrow F^p$$

which are the obvious projections, and the compositions

$$\epsilon_{\mathbb{A}} : A_{\mathcal{D}}(p) \xrightarrow{\pi_A} Rj_*A(p) \xrightarrow{\epsilon} \mathcal{R}j_*\Omega^\bullet_X \ \text{ and}$$
$$\epsilon_F : A_{\mathcal{D}}(p) \xrightarrow{\pi_F} F^p \xrightarrow{\iota} \mathcal{R}j_*\Omega^\bullet_X.$$

Both, ϵ_A and ϵ_F, are homotopic. Finally we denote by η the map

$$\eta : \mathcal{R}j_*\Omega^\bullet_X \longrightarrow A_{\mathcal{D}}(p).$$

Definition 3.5. Let X be a quasiprojective algebraic manifold, $\bar{X}$ a good compactification. We define the Deligne-Beilinson cohomology groups with coefficients in A by

$$H^q_{\mathcal{D}}(X, A(p)) = \mathbb{H}^q(\bar{X}, A_{\mathcal{D}}(p))$$

The quasi-isomorphisms

$$\Omega^{<p}_{\bar{X}}(\log Y) \xrightarrow{\text{quis}} \mathrm{Cone}(F^p \longrightarrow \mathcal{R}j_*\Omega^\bullet_X),$$

$$\mathcal{R}j_*\mathrm{Cone}(A(p) \longrightarrow \Omega^\bullet_X) \xrightarrow{\text{quis}} Rj_*\mathbb{C}/A(p)$$

and the long exact sequences 3.4, 3.5 and 3.6 give us

Corollary 3.6. *There are long exact sequences*

(1) $\to H^q_{\mathcal{D}}(X, A(p)) \xrightarrow{\pi_A \oplus \pi_F} H^q(X, A(p)) \oplus F^p H^q(X,\mathbb{C}) \xrightarrow{\epsilon - \iota} H^q(X,\mathbb{C})$
$\xrightarrow{\eta} H^{q+1}_{\mathcal{D}}(X, A(p)) \to$

(2) $\to H^q_{\mathcal{D}}(X, A(p)) \to H^q(X.A(p)) \to H^q(X,\mathbb{C})/F^p \to H^{q+1}_{\mathcal{D}}(X, A(p)) \to$

(3) $\to H^q_{\mathcal{D}}(X, A(p)) \to F^p H^q(X,\mathbb{C}) \to H^q(X, \mathbb{C}/A(p)) \to H^{q+1}_{\mathcal{D}}(X, A(p)) \to$

From now on we are working with integer valued Deligne-Beilinson cohomology, therefore A is replaced by $\mathbb{Z}$.

Examples 3.7.

(1) ($p = 0$) Here we have $\mathbb{Z}_{\mathcal{D}}(0) = \mathbb{Z}$ and the q-th Deligne-Beilinson cohomology Group of $\mathbb{Z}_{\mathcal{D}}(0)$ is nothing but the q-th singular cohomology.

(2) ($p = q = 1$) By the quasi-isomorphism of remark 3.4 we get

$$\mathbb{Z}(1)_{\mathcal{D}} \xrightarrow{\text{quis}} (\mathbb{Z}(1) \longrightarrow \mathcal{O}_X) \xrightarrow{qis} \mathcal{O}^*_X[-1],$$

thus

$$H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^0(X, \mathcal{O}^*_X).$$

(3) ($p = 1, q = 2$) By remark 3.4 we have a similar quasi-isomorphism, which yields the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}(1)) \longrightarrow$$

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X) \longrightarrow H^2(X; \mathbb{Z}(1)) \longrightarrow \dots$$

3.2. **Products in Deligne-Beilinson cohomology.**

The Deligne-Beilinson cohomology carries a product structure. Later in this chapter we will define a cycle map, which is compatible with the intersection of cycles.

For $\alpha \in \mathbb{R}$ we define a product

$$\cup_\alpha : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(p') \longrightarrow \mathbb{Z}_{\mathcal{D}}(p + p')$$

given on the level of local sections by

(3.11)

	$z_q \in \mathbb{Z}(q)$	$f_q \in F_{\mathcal{D}}^q$	$\omega_q \in \Omega_X^\bullet$
$z_p \in \mathbb{Z}(p)$	$z_p \cdot z_q \in \mathbb{Z}(p+q)$	0	$(1-\alpha)z_p \cdot \omega_q \in \Omega_X^\bullet$
$f_p \in F_D^p$	0	$f_p \wedge f_q \in F_{D}^{p+q}$	$(-1)^{\deg f_p} \cdot \alpha \cdot f_p \wedge \omega_q \in \Omega_X^\bullet$
ω_p	$\alpha \omega_p z_q \in \Omega_X^\bullet$	$(1-\alpha)\omega_p \wedge f_q \in \Omega_X^\bullet$	0

Verifying that $\cup_\alpha$ respects the differential, *i.e.*

$$d(\gamma \cup_\alpha \gamma') = d(\gamma) \cup_\alpha \gamma' + (-1)^\nu \gamma \cup_\alpha d(\gamma'),$$

where γ lives in degree ν, yields the following

Proposition 3.8. $\cup_\alpha$ *for* $\alpha \in \mathbb{R}$ *as defined above has the following properties:*

(1) *Let* γ, γ' *be in degree* ν, ν' *resp., then*

$$\gamma \cup_\alpha \gamma' = (-1)^{\nu\nu'} \gamma' \cup_{1-\alpha} \gamma$$

Moreover $\cup_{\frac{1}{2}}$ *is anticommutative .*

(2) $\cup_0$ *and* $\cup_1$ *are associative.*

(3) *For each* α, $\beta \in \mathbb{R}$ *the products* $\cup_\alpha$ *and* $\cup_\beta$ *are homotopic.*

(4) *The products* $\cup_\alpha$ *are compatible with the products on* $Rj_*Z(p)$ *and* F^p, *i.e.*

$$\epsilon_{\mathbb{Z}}(\gamma \cup_\alpha \gamma') = \epsilon_{\mathbb{Z}}(\gamma) \cup_{\mathbb{Z}} \epsilon_{\mathbb{Z}}(\gamma')$$

and

$$\epsilon_F(\gamma \cup_\alpha \gamma') = \epsilon_F(\gamma) \wedge \epsilon_F(\gamma').$$

For the proof see [EV1] Proposition 3.5.

We get the following rules:

Lemma 3.9. *For sections* γ *and* ω *of* $\mathbb{Z}_{\mathcal{D}}(p)$, *resp.* $Rj_*\Omega_X^\bullet$ *we have*

(1) $$\gamma \cup_0 \eta(\omega) = \eta(\epsilon_{\mathbb{Z}}(\gamma) \wedge \omega)$$
(2) $$\gamma \cup_1 \eta(\omega) = (-1)^{\deg \epsilon_F(\gamma)} \eta(\epsilon_F \wedge \omega)$$
(3) $$\eta(\omega) \cup_0 \gamma = \eta(\omega \wedge \epsilon_F(\gamma))$$
(4) $$\eta(\omega) \cup_1 \gamma = \eta(\omega \wedge \epsilon_{\mathbb{Z}}(\gamma))$$

The proof follows immediately from the multiplication table 3.11.

Therefore the products $\cup_\alpha$ are compatible with the morphisms in the distinguished triangle 3.9, which gives

Theorem 3.10. $\cup_\alpha$ *induces a product* $\cup$ *on* $H_{\mathcal{D}}(X) := \bigoplus_{p,q} H_{\mathcal{D}}^q(X, \mathbb{Z}(p))$, *making it into a bigraded ring with unit. Let* $\gamma \in H_{\mathcal{D}}^q(X, \mathbb{Z}(p))$ *and* $\gamma' \in H_{\mathcal{D}}^{q'}(X, \mathbb{Z}(p))$. *Then we have*

$$\gamma \cup \gamma' = (-1)^{qq'} \gamma' \cup \gamma.$$

Moreover point 4 of proposition 3.8 and lemma 3.9 translate directly into cohomology. The image of $H(X,\mathbb{C}) := \bigoplus_q H^q(X,\mathbb{C})$ under η is a square zero ideal in the bigraded ring $H_{\mathcal{D}}(X)$.

See [EV1] Theorem 3.9.

3.3. **The Cycle map in the Deligne-Beilinson Cohomology and Griffith's Intermediate Jacobian.** There is also the notion of Deligne-Beilinson cohomology with support. For generalities on cohomology with support on a subvariety see the Appendix. We denote by $\mathcal{Z}^p(X)$ the group of codimension p cycles on X.

Definition 3.11. Let first assume X to be projective. Let $A \in \mathcal{Z}^p(X)$ supported in $|A|$ and $U = X - |A|$ its complement. We define the Deligne-Beilinson cohomology Groups with support on A (see also chapters 4 and 5) as

$$H^q_{\mathcal{D}|A|}(X,\mathbb{Z}(p)) = \mathbb{H}^q_{|A|}(X,\mathbb{Z}_{\mathcal{D}}(p))$$

Similar as before, this definition fits into a long exact sequence of cohomology groups

$$0 \to H^{2p}_{\mathcal{D}|A|}(X,\mathbb{Z}(p)) \xrightarrow{(\pi_Z,\pi_F)} H^{2p}_{|A|}(X,\mathbb{Z}(p)) \oplus H^{2p}_{|A|}(X,F^p) \xrightarrow{\tau=\epsilon-\iota} H^{2p}_{|A|}(X,\mathbb{C}) \to \dots .$$

Note that $2p-1$ is smaller than the real codimension of A in X. Therefore $H^{2p-1}_{|A|}(X,\mathbb{C})$ vanishes and the map $(\pi_{\mathbb{Z}},\pi_F)$ is injective.

Let moreover denote by

$$c_{\mathbb{Z}} : \mathcal{Z}^p(X) \longrightarrow H^{2p}(X,\mathbb{Z}(p))$$

and

$$c_F : \mathcal{Z}(X) \longrightarrow H^{2p}(X,F^p)$$

the classical cycle maps (see [EV1], chapter 6). Obviously $\tau((c_{\mathbb{Z}}(A),c_F(A)) = 0$. Since (π_Z,π_F) is injective, we can assume $(c_Z(A),c_F(A))$ to be an element of $H^{2p}_{\mathcal{D}|A|}(X,\mathbb{Z}(p))$. Let us denote it by $c_{\mathcal{D}}(A)$.

Its image under the morphism

$$H^{2p}_{\mathcal{D}|A|}(X,\mathbb{Z}(p)) \longrightarrow H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

will be the cycle class of A denoted by $\gamma(A)$ from now on.

Remark 3.12. For noncompact X we work on a good compactification $\bar{X}$. We construct the cycle class of $\bar{A}$ in $H^{2p}_{\mathcal{D}}(\bar{X},\mathbb{Z}(p))$ and map it afterwards into $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$. For more details on this see [EV1] 7.2.

The next proposition asserts that the intersection product of cycles is compatible with the cup product in Deligne-Beilinson cohomology.

Moreover Deligne-Beilinson cohomology respects rational equivalence of cycles.

Proposition 3.13. *For rational equivalent cycles $A_1, A_2 \in \mathcal{Z}^p(X)$ we have $\gamma(A_1) = \gamma(A_2)$.*

See [EV1] 7.6.

Thereby the cycle map γ factors through $\mathrm{CH}^p(X)$ and we get well defined maps (also denoted by γ)

$$\gamma : \mathrm{CH}^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

The cup product in Deligne-Beilinson cohomology is compatible with the intersection in $\mathcal{Z}^\bullet(X)$.

Proposition 3.14. *Let A, A' be codimension p, resp. p' cycles on X. If they intersect proper, i.e. $A \cap A'$ is defined and a codimension $p + p'$ cycle, $\cup$ induces maps*

$$\cup : H^{2p}_{\mathcal{D}|A|}(X, \mathbb{Z}(p)) \otimes H^{2p'}_{\mathcal{D}|A'|}(X, \mathbb{Z}(p')) \longrightarrow H^{2(p+p')}_{\mathcal{D}|A \cap A'|}(X, \mathbb{Z}(p + p')).$$

We have the equalities

$$c_{\mathcal{D}}(A) \cup c_{\mathcal{D}}(A') = c_{\mathcal{D}}(A \cap A')$$

and

$$\gamma(A) \cup \gamma(A') = \gamma(A \cap A')$$

Propositions 3.13 and 3.14 yield

Proposition 3.15. *Let denote by $H^\bullet_{\mathcal{D}}(X)$ the subring*

$$H^\bullet_{\mathcal{D}}(X) = \bigoplus_p H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \subset H_{\mathcal{D}}(X) = \bigoplus_{p,q} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Then we have a homomorphism of rings

$$\gamma : \mathrm{CH}^\bullet(X) \longrightarrow H^\bullet_{\mathcal{D}}(X)$$

Moreover the map γ respects

$$f^* : \mathrm{CH}^\bullet(X) \longrightarrow \mathrm{CH}^\bullet(X')$$

for morphisms $f : X' \longrightarrow X$.

For a proof of this see [EV1], 7.7..

We recall the definition of Griffiths Intermediate Jacobian.

Definition 3.16. Let X be a projective variety defined over the complex numbers. By Deligne's description of Hodge theory [Del1] $F^p H^q(X, \mathbb{C})$ is defined as $H^q(X, F^p)$ and additionally the quotient group $H^q(X, \mathbb{C})/F^p$ is isomorphic to $\mathbb{H}^q(X, \Omega^{<p}_X)$.

The image of $H^{2p-1}(X, \mathbb{Z}(p))$ in $H^{2p-1}(X, \mathbb{C})/F^p$ is a lattice. Therefore the group

$$\mathcal{J}^p(X) = H^{2p-1}(X, \mathbb{C})/H^{2p-1}(X, \mathbb{Z}(p)) + F^p H^{2p-1}(X, \mathbb{C})$$

is a complex torus. We call $\mathcal{J}^p(X)$ the **p-th Intermediate Jacobian** of X.

Let us denote by $\mathrm{CH}^p_h(X)$ the group of cycles homologically equivalent to zero modulo rational equivalence. Then there is a map

$$\mathrm{AJ}_X : \mathrm{CH}^p_h(X) \longrightarrow \mathcal{J}^p(X)$$

called the **Abel-Jacobi map**, given by integration.

If we denote by $Hdg^p(X)$ the the group of codimension p Hodge cycles of X, *i.e.*

$$Hdg^p(X) = \ker(H^{2p}(X, \mathbb{Z}(p)) \oplus F^p H^{2p}(X, \mathbb{C}) \xrightarrow{\epsilon - \iota} H^{2p}(X, \mathbb{C})),$$

we get a commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{CH}^p_h(X) & \hookrightarrow & \mathrm{CH}^p(X) & & & & \\ & & \downarrow & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathcal{J}^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) & \longrightarrow & Hdg^p(X) & \longrightarrow 0. \end{array} \tag{3.12}$$

Let us denote by $\mathcal{J}^\bullet(X)$ the ideal $\bigoplus_p \mathcal{J}^p(X)$ of the commutative ring $H^\bullet_{\mathcal{D}}(X)$. By theorem 3.10 we get the following

Corollary 3.17. *$\mathcal{J}^\bullet(X)$ is an ideal of square zero in $H^\bullet_{\mathcal{D}}(X)$.*

If we denote by γ_0 the restriction of $\gamma : \mathrm{CH}^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ to $\mathrm{CH}^p_h(X)$ we get as a last consequence

Corollary 3.18. *The map*

$$\gamma_0 : \mathrm{CH}^p_h(X) \longrightarrow \mathcal{J}^p(X)$$

is the Abel-Jacobi map

3.4. **Massey products in Deligne-Beilinson Cohomology.**

In this section we give first properties of Massey products in Deligne-Beilinson cohomology. We will discuss some cases where the triviality of the products is obvious. Trivial means vanishing in the sense of 2.7 or torsion (which will be defined in 3.27).

In 3.32 we construct a first example of Massey triple products in Deligne Beilinson cohomology, which will be torsion, but not zero. In chapter 4 this example will be related to height pairings.

By proposition 3.8 the complex $\mathbb{Z}_{\mathcal{D}}(p)$ is equipped with maps of complexes $\cup_\alpha$ for $\alpha \in \mathbb{R}$

$$\cup_\alpha : \mathbb{Z}_{\mathcal{D}}(p) \otimes \mathbb{Z}_{\mathcal{D}}(p') \longrightarrow \mathbb{Z}_{\mathcal{D}}(p + p')$$

given by the multiplication table 3.11.

For open coverings $\mathcal{U}$ of X there is the Čech product on the double complex of $\mathbb{Z}$-modules $\mathcal{C}^\bullet(\mathcal{U}, \mathbb{Z}(\bullet))$ given by

$$\begin{array}{rccc} \check{\cup}: & \mathcal{C}^q(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p)) \otimes \mathcal{C}^{q'}(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p')) & \longrightarrow & \mathcal{C}^{q+q'}(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p+p')) \\ & (f_{i_0 \dots i_q}) \otimes (g_{j_0 \dots j_{q'}}) & \longmapsto & (f_{i_0 \dots i_q} \otimes g_{i_q \dots i_{q+q'}}) \end{array}$$

All in all the conditions of 2.2 are fulfilled and we get

Proposition 3.19. *For*

$$\mathcal{C}^q(p) = \lim_{\overrightarrow{\mathcal{U}}} s\mathcal{C}^q(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p))$$

and

$$\check{\cup}_\alpha = \cup_\alpha \circ \check{\cup} \circ \tau$$

we get a product on the bigraded complex

$$\mathcal{C} = \bigoplus_{p,q} \mathcal{C}^q(p),$$

which gives $\mathcal{C}$ the structure of a ring. Moreover there is a well defined theory of Massey products in Deligne-Beilinson cohomology.

There is a short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}^{\bullet-1}(\mathcal{U}, \Omega_X^\bullet) \xrightarrow{\eta} \mathcal{C}^\bullet(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p)) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathbb{Z}(p)) \oplus \mathcal{C}^\bullet(\mathcal{U}, F^p) \longrightarrow 0.$$

As in theorem 3.10 we get

Lemma 3.20. *The image of $\bigoplus_{p,q} \mathcal{C}^q(\mathcal{U}, \Omega_X^p)$ is an ideal of square zero in $\mathcal{C}$.*

Proof. the statement follows immediately from the multiplication table 3.11. □

As a first consequence for Massey products we get

Proposition 3.21. *Let $A_i \in H_\mathcal{D}^{q_i}(X, \mathbb{Z}(p_i))$ be classes living in the image of η. Then*

(1) *$M_3(A_1, A_2, A_3)$ is defined.*

(2) *$M_3(A_1, A_2, A_3)$ is vanishing.*

Proof. By 3.10 we have $A_1 \cup A_2 = 0$ and $A_2 \cup A_3 = 0$. Thus $M_3(A_1, A_2, A_3)$ is defined. Chose representatives $a_i \in \mathcal{C}^{q_i}(\mathcal{U}, \mathbb{Z}_\mathcal{D}(p_i))$. Then we have for each $\alpha \in \mathbb{R}$ by lemma 3.20

$$a_1 \check{\cup}_\alpha a_2 = a_2 \check{\cup}_\alpha a_3 = 0.$$

Therefore we can choose a defining system

$$\mathcal{M} = \{a_{i,j} \in \mathcal{C}^{q_{i,j}-1} | 1 \le i \le j \le 3\}$$

with $a_{1,2} = a_{2,3} = 0$ (see the definition of Massey products 2.2). By definition $M_3(A_1, A_2, A_3)$ is represented by

$$\mathcal{M} = [(-1)^{p_1} a_1 \cup 0 + (-1)^{p_{1,2}-1} 0 \cup a_3] = [0].$$

□

For the remainder of the chapter let X be smooth projective over $\mathbb{C}$. First we state a lemma, which will be frequently used in the sequel.

Lemma 3.22. *Let $q < 2p$. Then*

$$\epsilon_{\mathbb{Z}}(H^q_{\mathcal{D}}(X, \mathbb{Z}(p))) = \epsilon_F(H^q_{\mathcal{D}}(X, \mathbb{Z}(p))) = 0.$$

Proof. Remember the long exact sequence

$$\to H^{q-1}(X, \mathbb{C}) \xrightarrow{\eta} H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) \xrightarrow{\pi_{\mathbb{Z}} \oplus \pi_F} H^q(X, \mathbb{Z}(p)) \oplus F^p H^q(X, \mathbb{C}) \xrightarrow{\epsilon - \iota} H^q(X, \mathbb{C}) \to .$$

The sequence implies $\epsilon_{\mathbb{Z}}(A_i) = \epsilon_F(A_i)$ (remember $\epsilon_{\mathbb{Z}} = \epsilon \circ \pi_{\mathbb{Z}}$ and $\epsilon_F = \iota \circ \pi_F$).

On the other hand Hodge Theory gives

$$F^p H^q(X, \mathbb{C}) = \bigoplus_{k+\ell=q, k \geq p} H^\ell(X, \Omega^k_X)$$

and

$$\overline{F^p H^q(X, \mathbb{C})} = \bigoplus_{k+\ell=q, \ell \geq p} H^\ell(X, \Omega^k_X).$$

This shows that $q < 2p$ forces the intersection

$$F^p H^q(X, \mathbb{C}) \cap \overline{F^p H^q(X, \mathbb{C})}$$

to be zero.

Additionally $\overline{\epsilon_{\mathbb{Z}}(A_i)} = \epsilon_{\mathbb{Z}}(A_i)$, thus $\epsilon_{\mathbb{Z}}(A_i) = \epsilon_F(A_i) = 0$. $\square$

Proposition 3.23. *Let $A_i \in H^{q_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$ for $i = 1, 2, 3$. Assume furthermore $q_i < 2p_i$. Then there exist integers n_i for $i = 1, 2, 3$, such that*

- *$M_3(n_1 A_1, n_2 A_2, n_3 A_3)$ is defined.*
- *$M_3(n_1 A_1, n_2 A_2, n_3 A_3)$ vanishes in the sense of 2.7. Moreover there is just one representative: zero.*

Proof. By lemma 3.22 $\pi_{\mathbb{Z}}(A_i)$ lives in the kernel of

$$\epsilon : H^{q_i}(X, \mathbb{Z}(p_i)) \longrightarrow H^{q_i}(X, \mathbb{C}).$$

Hence $\pi_{\mathbb{Z}}(A_i)$ must be zero or torsion in $H^{q_i}(X, \mathbb{Z}(p_i))$. Therefore we find natural numbers n_i for $i = 1, 2, 3$, such that $n_i A_i$ lives in the image of the map

$$\eta : H^{q_i - 1}(X, \mathbb{C}) \longrightarrow H^{q_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i)).$$

Applying proposition 3.21 gives the vanishing of $M_3(A_1, A_2, A_3)$.

Moreover there is no indeterminacy. By the same argument as above we have

$$H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) \cup H^{q'}_{\mathcal{D}}(X, \mathbb{Z}(p')) = 0$$

for $q < 2p$ and $q' < 2p'$. $\square$

The proposition implies that we should focus on the case where $q_i \geq 2p_i$. Therefore we are working from now on with cohomology classes which are the cycle classes of algebraic codimension p_i cycles, *i.e.* $q_i = 2p_i$.

As a next step we allow A_2 to be a homologically non trivial Deligne-class, whereas $\pi_{\mathbb{Z}}(A_1) = \pi_{\mathbb{Z}}(A_3) = 0$. This will yield to an explicit formula, which only includes the cohomology classes A_1, A_2, A_3.

Proposition 3.24. *Let $A_i \in H^{2p_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$ for $i = 1, 2, 3$. Assume that $M_3(A_1, A_2, A_3)$ is defined and there are cohomology classes $[\alpha_1] \in H^{2p_1-1}(X, \mathbb{C})$ and $[\alpha_3] \in H^{2p_3-1}(X, \mathbb{C})$ such that*

$$\eta([\alpha_1]) = A_1 \text{ and } \eta[\alpha_3] = A_3.$$

Then the indeterminacy vanishes and the Massey product is represented by the cohomology class

$$M_3(A_1, A_2, A_3) = -\eta([\alpha_1] \wedge \iota(\epsilon_F(A_2)) \wedge [\alpha_3]).$$

In order to proof the proposition we will need the following homological

Lemma 3.25. *Let*

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[rr, "\rho"] & & \mathcal{B}^\bullet \arrow[dl, "\eta"] \\ & \mathcal{D}^\bullet \arrow[ul, "\epsilon"] & \end{tikzcd}$$

be a distinguished triangle of complexes of sheaves, where

$$\mathcal{D}^\bullet = \mathrm{Cone}(\mathcal{A}^\bullet \xrightarrow{\rho} \mathcal{B}^\bullet).$$

Take Čech resolutions $\mathcal{C}^\bullet(\mathcal{U}, \bullet)$, together with the Čech differentials δ (see the Appendix) to calculate the Hypercohomology $\mathbb{H}^\bullet(X, \bullet)$ of the respective complex. Let $[\alpha] \in \mathbb{H}^q(X, \mathcal{B}^\bullet)$ with representative $\alpha \in \mathcal{C}^q(\mathcal{U}, \mathcal{B}^\bullet)$. Assume $\eta(\alpha) = \delta\psi$ for some $\psi \in \mathcal{C}^q(\mathcal{U}, \mathcal{D}^\bullet)$, i.e. $\eta([\alpha]) = 0$.

Then

(1) *$\varphi := \epsilon(\psi)$ is closed.*

(2) *$\rho([\varphi]) = [\alpha]$*

Proof. The distinguished triangle yields the commutative diagram

$$\begin{tikzcd} o \arrow[r] & \mathcal{C}^{q-1}(\mathcal{U}, \mathcal{B}^\bullet) \arrow[r] \arrow[d, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U}, \mathcal{D}^\bullet) \arrow[r] \arrow[d, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U}, \mathcal{A}^\bullet[1]) \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & \mathcal{C}^{q}(\mathcal{U}, \mathcal{B}^\bullet) \arrow[r] & \mathcal{C}^{q}(\mathcal{U}, \mathcal{D}^\bullet) \arrow[r] & \mathcal{C}^{q}(\mathcal{U}, \mathcal{A}^\bullet[1]) & \end{tikzcd}$$

which implies

$$\begin{tikzcd} & \psi \arrow[r, "\epsilon"] \arrow[d, "\delta"'] & \varphi \arrow[r] \arrow[d, "\delta"'] & 0 \\ \alpha \arrow[r, "\eta"] & \delta\psi \arrow[r] & 0 & \end{tikzcd}$$

The statement follows from the commutativity of the diagrams.

In order to prove the second point we write $\psi = \psi_{\mathcal{B}} \oplus \psi_{\mathcal{A}}$ (in this notion $\psi_{\mathcal{A}} = \epsilon(\psi) = \phi$). The differential of the cone construction and the assumptions give

$$\eta(\alpha) = (\alpha, 0) = \delta(\psi) = (\delta(\psi_{\mathcal{B}}) - \rho(\phi), \delta(\phi)).$$

Therefore α differs from $\rho(\phi)$ by the exact cocycle $\delta(\psi_{\mathcal{B}})$. Thus

$$[\alpha] = \rho([\phi]).$$

□

We can now prove proposition 3.24.

Proof. Choose representatives

$$a_i \in \mathcal{C}^{2p_i}(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p_i)) \text{ for } A_i \ , \quad i = 1, 2, 3$$

such that

$$a_1 = \eta(\alpha_1) \text{ and } a_3 = \eta(\alpha_3)$$

for

$$\alpha_i \in \mathcal{C}^{2p_i-1}(X, \Omega_X^\bullet).$$

The product rules of lemma 3.9 translate directly to the level of Čech cochains. Therefore

$$a_1 \cup_0 a_2 = \eta(\alpha_1 \wedge \epsilon_F(a_2)) = da_{1,2}$$

and

$$a_2 \cup_0 a_3 = \eta(\epsilon_{\mathbb{Z}}(a_2) \wedge \alpha_3) = da_{2,3}.$$

By definition

$$\begin{aligned} M = [a_1 \cup_0 a_{2,3} - a_{1,2} \cup_0 a_3] &= [\eta(\alpha_1 \wedge \epsilon_F(a_{2,3}) - \epsilon_{\mathbb{Z}}(a_{1,2}) \wedge \alpha_3)] \\ &= [\eta(\alpha_1 \wedge \epsilon_F(a_{2,3}) - \epsilon_{\mathbb{Z}}(a_{1,2}) \wedge \alpha_3))] \end{aligned}$$

is a representative of $M_3(A_1, A_2, A_3)$.

The distinguished triangle

$$\begin{tikzcd} \mathbb{Z}(p)) \oplus F_D^p \arrow[rr] & & \Omega_X^\bullet \arrow[dl] \\ & \mathbb{Z}_{\mathcal{D}}(p) \arrow[ul] & \end{tikzcd}$$

induces the exact sequence of Čech complexes for some suitable open covering

$\mathcal{U}$ of X:

$$0 \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \Omega_X^\bullet) \xrightarrow{\eta} \mathcal{C}^\bullet(\mathcal{U}, \mathbb{Z}_\mathcal{D}(\bullet)) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathbb{Z}(\bullet)) \oplus \mathcal{C}^\bullet(\mathcal{U}, F^\bullet) \longrightarrow 0.$$

The cochains $a_{i,j}$ fulfill the assumptions of lemma 3.25 and we obtain

$$[(\epsilon_\mathbb{Z} - \epsilon_F)(a_{1,2})] = [\alpha_1] \wedge \epsilon_F(A_2) \tag{3.13}$$

$$[(\epsilon_\mathbb{Z} - \epsilon_F)(a_{2,3})] = \epsilon_\mathbb{Z}(A_2) \wedge [\alpha_3].$$

Since Massey products are independent of the representatives of the cohomology classes A_i, we can choose $[\alpha_1]$ and $[\alpha_3]$ after variation with elements of $F^{p_i}H^{2p_i-1}(X, \mathbb{C})$, such that $[\alpha_1] \wedge \epsilon_F(A_2)$ and $\epsilon_\mathbb{Z}(A_2) \wedge [\alpha_3]$ are integervalued, *i.e.* live in $\epsilon(H^{2p_{ij}}(X, \mathbb{Z}(p_{ij})))$. Furthermore we obtain the equalities

$$\begin{aligned} \overline{[\alpha_1] \wedge \epsilon_F(A_2)} &= [\alpha_1] \wedge \epsilon_F(A_2) \\ \overline{\epsilon_\mathbb{Z}(A_2) \wedge [\alpha_3]} &= \epsilon_\mathbb{Z}(A_2) \wedge [\alpha_3]. \end{aligned} \tag{3.14}$$

On the other hand we have by lemma 3.22

$$F^{p_{ij}}H^{2p_{ij}-1}(X, \mathbb{C}) \cap \overline{F^{p_{ij}}H^{2p_{ij}-1}(X, \mathbb{C})} = 0.$$

Thus we can assume

$$[\epsilon_F(a_{1,2})] = [\epsilon_F(a_{2,3})] = [0].$$

Equation 3.13 translates to

$$[\epsilon_\mathbb{Z}(a_{1,2})] = [\alpha_1] \wedge \epsilon_F(A_2)$$

Therefore

$$M = \eta([\alpha_1] \wedge [\epsilon_F(a_{2,3})] - [\epsilon_\mathbb{Z}(a_{1,2})] \wedge [\alpha_3]) = -\eta([\alpha_1] \wedge \epsilon_F(A_2) \wedge [\alpha_3]).$$

By lemma 3.22 and the product rules 3.9, the indeterminacy is given by

$$\eta([\alpha_1]) \cup H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) + H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) \cup \eta([\alpha_3])$$

$$= \eta([\alpha_1] \wedge \epsilon_F(H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2}))) + \epsilon_\mathbb{Z}(H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2}))) \wedge [\alpha_3]).$$

But again

$$\epsilon_F(H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})))$$

and

$$\epsilon_\mathbb{Z}(H_\mathcal{D}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})))$$

have to vanish. Hence there is no indeterminacy and we are done. □

Corollary 3.26. *The same holds true, if we assume $\pi_\mathbb{Z}(A_1)$ and $\pi_\mathbb{Z}(A_3)$ to be torsion in $H^{q_i}(X, \mathbb{Z}(p_i))$.*

We want to define what it means that Massey products are torsion. A naive definition would be the following. We find a natural number ν and a representative M of $M_3(A_1, A_2, A_3)$, such that $\nu M = 0$. But we have to be careful, since this definition is not sufficient in the general setup. But in the end it will turn out, that in our situation the naive definition is equivalent to the one we state here.

Definition and Theorem 3.27. Let $A_i \in H^{q_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$ be Deligne cohomology classes, such that $M_3(A_1, A_2, A_3)$ is defined. We say that $M_3(A_1, A_2, A_3)$ is **torsion** if one of the following two equivalent definitions holds.

(1) There exists a representative M of $M_3(A_1, A_2, A_3)$ and a natural number ν such that
$$\nu M = A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3$$
for suitable $\Psi_{i,j} \in H^{q_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$.

(2) There exists a natural number ν such that
$$\nu M_3(A_1, A_2, A_3) \subset A_1 \cup H^{2q_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3})) + H^{q_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3.$$

Proof. It is enough to proof that 1 implies 2, since the other implication is obvious.

Let M' be another representative of $M_3(A_1, A_2, A_3)$. This representative differs from M by a term of the form $A_1 \cup \Upsilon_{2,3} + \Upsilon_{1,2} \cup A_3$ for suitable $\Upsilon_{i,j} \in H^{q_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$. Therefore
$$\begin{aligned}
\nu M' &= \nu(M + A_1 \cup \Upsilon_{2,3} + \Upsilon_{1,2} \cup A_3) \\
&= \nu M + \nu A_1 \cup \Upsilon_{2,3} + \nu \Upsilon_{1,2} \cup A_3 \\
&= A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3 + A_1 \cup (\nu \Upsilon_{2,3}) + (\nu \Upsilon_{1,2}) \cup A_3 \\
&= A_1 \cup (\Psi_{2,3} + \nu \Upsilon_{2,3}) + (\Psi_{1,2} + \nu \Upsilon_{1,2}) \cup A_3,
\end{aligned}$$
which shows that for each representative M of $M_3(A_1, A_2, A_3)$ the multiple νM lives in the indeterminacy $A_1 \cup H^{q_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3})) + H^{q_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3$. Moreover we have shown that the integer ν works for all representatives of $M_3(A_1, A_2, A_3)$. $\square$

Remark 3.28. The point why the naive access $\nu M = 0$ fails is that, in general, we can not divide by ν in $H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$. But in our situation, where the A_i are the cycle classes of codimension p_i cycles on X, $\Psi_{i,j}$ of definition 3.27 lives in $H^{2p_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$. Thus some multiple of $\Psi_{i,j}$ lives in the image of $H^{2p_{i,j}-2}(X, \mathbb{C})$. After lifting this multiple to $H^{2p_{i,j}-2}(X, \mathbb{C})$ we can divide by ν and map it again to $H^{2p_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$. Remind that this procedure depends on the choice of the lifting, therefore the received class is not unique.

Let us make this more precise.

Proposition 3.29. *Let $A_i \in H^{2p_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$, such that $M_3(A_1, A_2, A_3)$ is defined. Then $M_3(A_1, A_2, A_3)$ is torsion if and only if we find a representative M of $M_3(A_1, A_2, A_3)$ and some natural number ν, such that $\nu M = 0$.*

Proof. We need only to prove that the definition of torsion 3.27 implies the existence of some M and some ν with $\nu M = 0$.

Let $a_i \in \mathcal{C}^{2p_i}(\mathbb{Z}_\mathcal{D}(p_i))$ representing A_i, $a_{1,2} \in \mathcal{C}^{2p_{1,2}-1}(\mathbb{Z}_\mathcal{D}(p_{1,2}))$ and $a_{2,3} \in \mathcal{C}^{2p_{2,3}-1}(\mathbb{Z}_\mathcal{D}(p_{2,3}))$ with

$$da_{1,2} = a_1 \cup a_2$$

$$da_{2,3} = a_2 \cup a_3.$$

By assumption we find some integer ν and $\Psi_{i,j} \in H_\mathcal{D}^{2p_{i,j}-1}(X, \mathbb{Z}(p_{i,j}))$ such that

$$[a_1 \cup a_{2,3} - a_{1,2} \cup a_3] = A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3.$$

Replacing ν and $\Psi_{i,j}$ by some suitable multiple, we can assume that $\Psi_{i,j} = \eta(\Phi_{i,j})$ for $\Phi_{i,j} \in H^{2p_{i,j}-2}(X, \mathbb{C})$. Let $\phi_{i,j} \in \mathcal{C}^{2p_{i,j}-2}(\Omega_X^\bullet)$ be corresponding cochains. The cochain

$$a_1 \cup (a_{2,3} - \eta(\frac{1}{\nu}\phi_{2,3})) - (a_{1,2} + \eta(\frac{1}{\nu}\phi_{1,2})) \cup a_3.$$

is a representative of $M_3(A_1, A_2, A_3)$. The following calculation gives the desired result

$$\nu[a_1 \cup (a_{2,3} - \eta(\frac{1}{\nu}\phi_{2,3})) - (a_{1,2} + \eta(\frac{1}{\nu}\phi_{1,2})) \cup a_3]$$

$$= \nu[a_1 \cup a_{2,3} - a_{1,2} \cup a_3] - \nu[a_1 \cup \eta(\frac{1}{\nu}\phi_{2,3}) + \eta(\frac{1}{\nu}\phi_{1,2}) \cup a_3]$$

$$= A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3 - [a_1 \cup \eta(\phi_{2,3}) + \eta(\phi_{1,2} \cup A_3]$$

$$= A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3 - (A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3) = 0.$$

□

Lemma 3.30. *Let $A \in H^k(X, \mathbb{C})$ and $Z \in H^\ell(X, \mathbb{Z})$. Assume $\tau = A \wedge Z \in H^{k+\ell}(X, \mathbb{Z})$, then there exists $Q \in H^\ell(X, \mathbb{Q})$ such that $\tau = A \wedge Z = Q \wedge Z$.*

Proof. Since $A \wedge Z = \overline{A \wedge Z} = \bar{A} \wedge Z$, we can assume $A \in H^k(X, \mathbb{R})$. Now fix a basis for $H^k(X, \mathbb{R})$, $H^\ell(X, \mathbb{Z})$ and $H^{k+\ell}(X, \mathbb{R})$, respecting the integral and rational structure. Then the map

$$H^k(X, \mathbb{R}) \xrightarrow{\phi} H^{k+\ell}(X, \mathbb{R})$$

$$X \mapsto X \wedge Z$$

is linear and can be represented in the chosen basis by an integral matrix M. Now we have $MA = \tau$, which implies that there is at least one solution for the linear equation $MX = \tau$. By elementary Gauss transformations we find a solution with $\mathbb{Q}$-coefficients. □

Corollary 3.31. *Under the assumptions made in proposition 3.24, $M_3(A_1, A_2, A_3)$ is torsion.*

Proof. By lemma 3.30 we can find classes $[q_1]$ and $[q_3]$, such that

$$[\alpha_1] \wedge \epsilon_F(A_2) = [q_1] \wedge \epsilon_F(A_2)$$

and

$$\epsilon_{\mathbb{Z}}(A_2) \wedge [\alpha_3] = \epsilon_F(A_2) \wedge [q_3].$$

By proposition 3.24 $M_3(A_1, A_2, A_3)$ is represented by

$$-\eta([\alpha_1] \wedge \epsilon_F(A_2) \wedge [\alpha_3]) = -\eta([q_1] \wedge \epsilon_F(A_2) \wedge [q_3]),$$

which is the image of a $\mathbb{Q}$-valued cohomology class $[q]$, which implies, that some multiple of $[q]$ lives in $H^{2p_{1,2,3}-1}(X, \mathbb{Z}(p_{1,2,3}))$ and therefore in the kernel of η. □

We will now present an example, which shows that the products in proposition 3.24 do not have to be zero.

Example 3.32. Let $X = E_1 \times E_2$ be the product of two elliptic curves over $\mathbb{C}$. Denote by p_1 and p_2 the projections. Let dz_1, $d\bar{z}_1$, resp. dz_2, $d\bar{z}_2$ the basis of global $\mathcal{E}^{0,1}$- and $\mathcal{E}^{1,0}$-forms and $dx_j = \frac{1}{2}(dz_j \wedge d\bar{z}_j)$ and $dy_j = \frac{1}{2i}(dz_j \wedge d\bar{z}_j)$ the real dual basis.

By construction we know, that

$$H^1(X, R(p)) =< p_1^* dx_1, p_1^* dy_1, p_2^* dx_2, p_2^* dy_2 >_{R(p)},$$

where $R(q) = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Let $[\alpha_1]$ be the cohomology class of the q-division point $\frac{1}{q} p_1^* dx_1$ and $[\alpha_3]$ the class of the q- division point $\frac{1}{q} p_1^* dy_1$ in $\mathrm{Pic}^0(X)$.

Write $A_i = \eta([\alpha_i]) \in H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$ for $i = 1, 3$ and let $A_2 \in H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$ be in the preimage of the class

$$q(p_1^*(dx_1 \wedge dy_1) + p_2^*(dx_2 \wedge dy_2)).$$

Note that the isomorphism between $\mathrm{Alb}\,(X)$ and $\mathrm{Pic}^0(X)$ is given by the map

$$\begin{array}{rccc} \tau: & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Alb}\,(X) \\ & \omega & \longmapsto & \omega \wedge (p_1^*(dx_1 \wedge dy_1) + p_2^*(dx_2 \wedge dy_2)). \end{array}$$

The construction of the A_i implies

$$A_1 \cup A_2 = A_2 \cup A_3 = 0 \in H^4_{\mathcal{D}}(X, \mathbb{Z}(2)),$$

since both classes are zero in $\mathrm{Alb}\,(X)$. Therefore $M_3(A_1, A_2, A_3)$ is defined.

By proposition 3.24 $M_3(A_1, A_2, A_3)$ is represented by the cohomology class

$$M_3(A_1, A_2, A_3) = -\eta([\alpha_1] \wedge \epsilon_F(A_2) \wedge [\alpha_3]) = -\eta(\frac{1}{q} p_1^* dx_1 \wedge p_1^* dy_1 \wedge p_2^* dx_2 \wedge p_2^* dy_2)$$

which is $\mathbb{Q}$-valued. For $q > 1$ this class is not zero in $H^5_{\mathcal{D}}(X, \mathbb{Z}(3))$.

Proposition 3.24 says that there is no indeterminacy, hence $M_3(A_1, A_2, A_3)$ does not vanish.

We can generalize the statement on torsion of Massey products in proposition 3.24:

Proposition 3.33. *Let $A_i \in H^{2p_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$ for $i = 1, 2, 3$ be Deligne classes, such that $M_3(A_1, A_2, A_3)$ is defined. Assume that $A_i = \eta([\alpha_i])$, $[\alpha_i] \in H^{2p_i-1}(X, \mathbb{C})$ for at least one of the A_i, then $M_3(A_1, A_2, A_3)$ is torsion in $H^{2p_{1,2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2,3}))$.*

Proof. Since the proof is quite technical, we give a short outline of it.

We distinguish between two cases. On the one hand A_2 is homologically equivalent to zero, and on the other hand A_1 is homologically equivalent to zero. The proof of the second case implies by symmetry the case, where A_3 is homologically equivalent to zero.

In both cases the first step is to prove that $M_3(A_1, \eta([q_2]), A_3)$, resp. $M_3(\eta([q_1]), A_2, A_3)$ is torsion for suitable chosen classes $q_i \in H^{2p_i-1}(X, \mathbb{Q}(p_i))$. In the second step we will reduce the Massey products $M_3(A_1, \eta([\alpha_2]), A_3)$ and $M_3(\eta([\alpha_1]), A_2, A_3)$ to the products $M_3(A_1, \eta([q_2]), A_3)$, resp. $M_3(\eta([q_1]), A_2, A_3)$.

Let us assume first $A_2 = \eta([\alpha_2])$ for $[\alpha_2] \in H^{2p_2-1}(X, \mathbb{C})$. Lemma 3.30 gives us classes $[q_2]$, $[\widetilde{q}_2] \in H^{2p_2-1}(X, \mathbb{Q}(p_2))$, such that

$$\begin{aligned} \epsilon_{\mathbb{Z}}(A_1) \wedge [\alpha_2] &= \epsilon_Z(A_1) \wedge [q_2] \\ [\alpha_2] \wedge \epsilon_F(A_3) &= [\widetilde{q}_2] \wedge \epsilon_F(A_3). \end{aligned} \tag{3.15}$$

We can find some natural number $n \in \mathbb{N}$, s.th.

$$n([q_2] - [\widetilde{q}_2]) \in H^{2p_2-1}(X, \mathbb{Z}(p_2)).$$

Therefore

$$n\eta(\epsilon_{\mathbb{Z}}(A_1) \wedge ([q_2] - [\widetilde{q}_2])) = n\eta(([q_2] - [\widetilde{q}_2]) \wedge \epsilon_F(A_3)) = 0,$$

which implies - after replacing A_2 by nA_2 - that we find a unique $[q_2]$ fulfilling equations 3.15 simultaneously. Note that $n \in \mathbb{Z}$, therefore we can apply point three of the properties 2.6 and we get

$$nM_3(A_1, A_2, A_3) \subset M_3(A_1, nA_2, A_3).$$

By the choice of $[q_2]$ the Massey product $M_3(A_1, \eta([q_2]), A_3)$ is defined. Furthermore we can find some number $\nu \in \mathbb{N}$, such that $\nu[q_2] \in H^{2p_2-1}(X, \mathbb{Z}(p_2))$. Since Massey products behave well under scalar multiplication (see see again point three of properties 2.6), we have

$$\nu M_3(A_1, \eta([q_2]), A_3) \subset M_3(A_1, \eta(\nu[q_2]), A_3) = M_3(A_1, [0], A_3).$$

$M_3(A_1, [0], A_3)$ vanishes, thus

$$\begin{aligned} &\nu M_3(A_1, \eta([q_2]), A_3) \subset M_3(A_1, [0], A_3) \\ &= A_1 \cup H^{2p_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3})) + H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3. \end{aligned}$$

Therefore $M_3(A_1, \eta([q_2]), A_3)$ is torsion (see definition 3.27).

Next we prove, that $M_3(A_1, \eta([\alpha_2] - [q_2]), A_3)$ vanishes.

By the choice of $[q_2]$ (see equations 3.15) we find $\phi_{1,2} \in \mathcal{C}^{p_{1,2}-2}(\mathcal{U}, \Omega_X^\bullet)$ and $\phi_{2,3} \in \mathcal{C}^{p_{2,3}-2}(\mathcal{U}, \Omega_X^\bullet)$, such that

$$\epsilon_Z(a_1) \wedge (\alpha_2 - q_2) = d\phi_{1,2}$$

and

$$(\alpha_2 - q_2) \wedge \epsilon_F(a_3) = d\phi_{2,3}.$$

In other words $M_3(\epsilon_Z(A_1), [\alpha_2] - [q_2], \epsilon_F(A_3))$ is defined in the deRham cohomology of X and the cohomology class

$$\widetilde{M} = [\epsilon_{\mathbb{Z}}(A_1) \wedge \phi_{2,3} - \phi_{1,2} \wedge \epsilon_F(A_3)] \tag{3.16}$$

is a representative of it. The theorem of Deligne, Griffiths, Morgan and Sullivan (see example 2.10) implies that $M_3(\epsilon_Z(A_1), [\alpha_2] - [q_2], \epsilon_F(A_3))$ vanishes (see definition 2.7). Thus

$$M_3(\epsilon_Z(A_1), [\alpha_2]-[q_2], \epsilon_F(A_3)) = \epsilon_Z(A_1) \wedge H^{2p_{2,3}-2}(X, \mathbb{C}) + H^{2p_{1,2}-2}(X, \mathbb{C}) \wedge \epsilon_F(A_3).$$

On the other hand we can construct a representative M of $M_3(A_1, \eta([\alpha_2] - [q_2]), A_3)$ as follows:

Choose $a_{1,2} \in \mathcal{C}^{p_{1,2}-1}(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p_{1,2}))$ and $a_{2,3} \in \mathcal{C}^{p_{2,3}-1}(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p_{2,3}))$ with

$$a_{1,2} = \eta(\phi_{1,2})$$
$$a_{2,3} = \eta(\phi_{2,3}).$$

Equation 3.16 implies

$$\begin{aligned} M = [a_1 \cup \eta(\phi_{2,3}) - \eta(\phi_{1,2}) \cup a_3)] &= \eta([\epsilon_{\mathbb{Z}}(a_1) \wedge \phi_{2,3} - \phi_{1,2} \wedge \epsilon_F(a_3)]) = \eta(\widetilde{M}) \\ &\in \eta(\epsilon_Z(A_1) \wedge H^{2p_{2,3}-2}(X, \mathbb{C}) + H^{2p_{1,2}-2}(X, \mathbb{C}) \wedge \epsilon_F(A_3)) \\ &= A_1 \wedge \eta(H^{2p_{2,3}-2}(X, \mathbb{C})) + H^{2p_{1,2}-2}(X, \mathbb{C}) \wedge A_3 \\ &\subset A_1 \cup H_{\mathcal{D}}^{2p_{2,3}-1}(X, \mathbb{Z}(p_{2,3})) + H_{\mathcal{D}}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) \wedge A_3. \end{aligned}$$

This implies in terms of definition 2.7 the vanishing of $M_3(A_1, \eta([\alpha_2] - [q_2]), A_3)$ and moreover the vanishing of $M_3(A_1, n\eta([\alpha_2] - [q_2]), A_3)$ for all numbers $n \in \mathbb{N}$.

Taking the integer ν with $\nu[q_2] \in H^{2p_2-1}(X, \mathbb{Z}(p_2))$ we obtain again by 2.3

$$\begin{aligned} M_3(A_1, \eta(\nu([\alpha_2] - [q_2])), A_3) &= M_3(A_1, \eta(\nu[\alpha_2] - \nu[q_2]), A_3) \\ &= M_3(A_1, \eta(\nu[\alpha_2]), A_3). \end{aligned}$$

Therefore

$$\nu M_3(A_1, \eta([\alpha_2]), A_3) \subset M_3(A_1, \nu\eta([\alpha_2]), A_3)$$

$$= M_3(A_1, \nu\eta([\alpha_2]-[q_2]), A_3) = A_1 \cup H_{\mathcal{D}}^{2p_{2,3}-1}(X, \mathbb{Z}(p_{2,3})) + H_{\mathcal{D}}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) \cup A_3.$$

Thus $M_3(A_1, A_2, A_3)$ is torsion by definition 3.27, which proves the first case.

In order to prove the second case let $A_1 = \eta([\alpha_1])$, for $[\alpha_1] \in H^{2p_1-1}(X, \mathbb{C})$. Note that by lemma 3.22

$$\eta([\alpha_1]) \cup H_{\mathcal{D}}^{2p_{2,3}-1}(X, \mathbb{Z}(p_{2,3})) = 0.$$

the indeterminacy is therefore given by

$$H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3,$$

By lemma 3.30 we find a class $[q_1] \in H^{2p_1-1}(X, \mathbb{Q}(p_1))$ with representative $q_1 \in \mathcal{C}^{2p_1-1}(\mathcal{U}, \Omega^\bullet_X)$, such that

$$[q_1] \wedge \epsilon_F(A_2) = [\alpha_1] \wedge \epsilon_F(A_2).$$

As in the first case we show that $M_3(\eta([q_1], A_2, A_3)$ is torsion.

We find a natural number ν, such that $\nu[q_1]$ lives in $H^{2p_1-1}(X, \mathbb{Z}(p_1))$. This implies

$$\nu M_3(\eta([q_1]), A_2, A_3) \subset M_3(\eta(\nu[q_1]), A_2, A_3) = M_3([0], A_2, A_3).$$

But $M_3([0], A_2, A_3)$ vanishes. Therefore

$$\nu M_3(\eta([q_1]), A_2, A_3) \subset H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3,$$

which shows that $M_3(\eta([q_1]), A_2, A_3)$ is torsion.

The next step is to prove the vanishing of $M_3(\eta([\alpha_1] - [q_1]), A_2, A_3)$. By the choice of q_1 we find $\phi_{1,2} \in \mathcal{C}^{2p_{1,2}-1}(\mathcal{U}, \Omega^\bullet_X)$ and $\phi_{2,3} \in \mathcal{C}^{2p_{2,3}-1}(\mathcal{U}, \Omega^\bullet_X)$, such that

$$(\alpha_1 - q_1) \wedge \epsilon_F(a_2) = d\phi_{1,2}$$

and

$$\epsilon_F(a_2) \wedge \epsilon_F(a_2) = d\phi_{2,3}.$$

Thus the Massey product $M_3([\alpha_1] - [q_1], \epsilon_F(A_2), \epsilon_F(A_3))$ is defined and the cohomology class

$$\widetilde{M} = [(\alpha_1 - q_1) \wedge \phi_{2,3} - \phi_{1,2} \wedge \epsilon_F(a_3)] \tag{3.17}$$

is a representative of it. As in the first case the theorem of Deligne, Griffiths, Morgan, Sullivan 2.10 implies the vanishing of $M_3([\alpha_1] - [q_1], A_2, A_3)$. Therefore

$$\begin{aligned}\widetilde{M} &\in M_3([\alpha_1] - [q_1], \epsilon_F(A_2), \epsilon_F(A_3)) \\ &= ([\alpha_1] - [q_1]) \wedge H^{2p_{1,2}-1}(X, \mathbb{C}) + H^{2p_{1,2}-2}(X, \mathbb{C}) \wedge \epsilon_F(A_3).\end{aligned}$$

On the other hand we construct a representative M of $M_3(\eta([\alpha_1] - [q_1]), A_2, A_3)$ as follows:

Choose

$$a_{1,2} = \eta(\phi_{1,2}) \in \mathcal{C}^{2p_{1,2}-1}(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p_{1,2}))$$

and

$$a_{2,3} \in \mathcal{C}^{2p_{2,3}-1}(\mathcal{U}, \mathbb{Z}_{\mathcal{D}}(p_{2,3})) \text{ with } \epsilon_F(a_{2,3}) = \phi_{2,3}.$$

The choice of $a_{1,2}$ and $a_{2,3}$ yields

$$\begin{aligned}M &= [\eta(\alpha_1 - q_1) \cup a_{2,3} - a_{1,2} \cup a_3] = [\eta((\alpha_1 - q_1) \wedge \epsilon_F(a_{2,3}) - \phi_{1,2} \wedge \epsilon_F(a_3)) \\ &= \eta([(\alpha_1 - q_1) \wedge \phi_{2,3} - \phi_{1,2} \wedge \epsilon_F(a_3)]) = \eta(\widetilde{M}).\end{aligned}$$

The vanishing of $\widetilde{M}$ in the deRham cohomology of X and the product rules of 3.9 imply

$$\begin{aligned}\eta(\widetilde{M}) &\in \eta(([\alpha_1]-[q_1]) \wedge H^{2p_{2,3}-1}(X,\mathbb{C}) + H^{2p_{1,2}-2}(X,\mathbb{C}) \wedge \epsilon_F(A_3) \\ &\subset \eta([\alpha_1]-[q_1]) \cup H_{\mathcal{D}}^{2p_{2,3}-1}(X,\mathbb{Z}(p_{2,3})) + H_{\mathcal{D}}^{2p_{1,2}-1}(X,\mathbb{Z}(p_{1,2})) \cup A_3.\end{aligned}$$

Therefore $M_3(\eta([\alpha_1]-[q_1]), A_2, A_3)$ vanishes and moreover $M_3(n\eta([\alpha_1]-[q_1], A_2, A_3)$ vanishes for all numbers $n \in \mathbb{N}$.

To finish the proof let ν be the integer with $\nu[q_1] \in H^{2p_1-1}(X, \mathbb{Z}(p_1))$. We get

$$\begin{aligned}M_3(\nu\eta([\alpha_1]-[q_1]), A_2, A_3) &= M_3(\nu\eta([\alpha_1]) - \nu\eta([q_1]), A_2, A_3) \\ &= M_3(\nu\eta([\alpha_1]), A_2, A_3).\end{aligned}$$

This implies

$$\nu M_3(\eta([\alpha_1]), A_2, A_3) \subset M_3(\nu\eta([\alpha_1]), A_2, A_3)) \subset H_{\mathcal{D}}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) \cup A_3,$$

which proves the second case. $\square$

We have shown up to now that the only interesting case for Massey products is the case where all the cycles A_i, for $i = 1, 2, 3$ are homologically nontrivial.

4. Massey products and Height Pairings

In this section we present a relation between Massey products and height pairings. height pairings are of particular interest in Arakelov theory. But they are also of interest in the study of algebraic cycles on complex varieties. Let X be a smooth projective variety defined over the complex numbers and $A \in \mathcal{Z}^p(X)$ and $B \in \mathcal{Z}^q(X)$ be algebraic cycles of codimension p and q. Assume furthermore that $A \cap B = \emptyset$. The classical height pairing associates to them a real number $< A, B >$, which is bilinear in its entries (see [M-S2]). We use here a refinement of the classical height pairing, which is due to S. Müller-Stach (see [M-S2]). We assume that one of the cycles, let say A, vanishes under the Abel Jacobi map, *i.e.* $\mathrm{AJ}_X(A) = 0$. We obtain a $\mathbb{C}^*$ valued height pairing. After identifying $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}(n+1)$ via the exponential map, we obtain as a result, that in special situations the Massey Product is nothing else than the difference of two height pairings. But let us first give the definition of the refined height pairing. We follow the presentation given in [M-S2]. For the various definitions of the classical height pairing, the interested reader is also referred to [M-S2] and to the article of A. Beilinson [Be2]. There A. Beilinson states some conjectures about height pairings for varieties defined over algebraic number fields.

4.1. **Definition of the Height Pairing.**

Let X be a projective variety of dimension n over $\mathbb{C}$. Let A, B be two cycles of codimension p, resp. q, such that

(1) $p + q = n + 1$
(2) $A \cap B = \emptyset$
(3) The cycle class $\gamma(A) \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ is zero
(4) B is homologically equivalent to zero.

Let $U = X - A$, then there is the long exact sequence (see 3.11)

$$\longrightarrow H^{2p-1}_{\mathcal{D}}(U, \mathbb{Z}(p)) \longrightarrow H^{2p}_{\mathcal{D}|A|}(X, \mathbb{Z}(p)) \xrightarrow{\tau} H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow$$

Denote by c_A the class of A in $H^{2p}_{\mathcal{D}|A|}(X, \mathbb{Z}(p))$. By the assumptions we know $\tau(c_A) = 0$, hence lifts to a class $\phi_A \in H^{2p-1}_{\mathcal{D}}(U, \mathbb{Z}(p))$. The cup product induces a pairing

$$<,>\colon H^{2p-1}_{\mathcal{D}}(U, \mathbb{Z}(p)) \times H^{2q}_{\mathcal{D}|B|}(X, \mathbb{Z}(q)) \longrightarrow H^{2n+1}_{\mathcal{D}|B|}(U, \mathbb{Z}(n+1))$$

$$\xrightarrow{\cong} H^{2n+1}_{\mathcal{D}|B|}(X, \mathbb{Z}(n+1)) \longrightarrow H^{2n+1}_{\mathcal{D}}(X, \mathbb{Z}(n+1)) \cong \mathbb{C}^*$$

The isomorphism $H^{2n+1}_{\mathcal{D}|B|}(U, \mathbb{Z}(n+1)) \cong H^{2n+1}_{\mathcal{D}|B|}(X, \mathbb{Z}(n+1))$ is caused by the disjointness of A and B.

The pairing $< \phi_A, c_A >$ is well defined, because the indeterminacy of the lifting ϕ_A is $H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p))$ and as in 3.23 we have, since B is homologically equivalent to zero,

$$H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p)) \cup \gamma(B) = 0.$$

We can now define the desired height pairing:

Definition 4.1. Let X, A and B be as above. Then we define the $\mathbb{C}^*$-valued **height pairing** as follows

$$< A, B > = < \phi_A, c_B > .$$

Remark 4.2. In the definition we identify $H^{2n+1}_{\mathcal{D}}(X, \mathbb{Z}(n+1)) \cong \mathbb{C}/\mathbb{Z}(n+1)$ with $\mathbb{C}^*$. To avoid confusion we use $\mathbb{C}/\mathbb{Z}(n+1)$ for calculations in the sequel.

4.2. **Massey Products as a Difference of Two Height Pairings.**

Since we want to use the concept of cohomology with support on some closed subvariety, we will calculate in this section Deligne-Beilinson cohomology for a smooth complex projective variety X via the canonical flasque resolution of Godement recalled in the Appendix. Furthermore we have to show as in the case of Čech Hypercohomology (see subsection 2.2), that the resolution is compatible with the cup product on $H^\bullet_{\mathcal{D}}(X, \mathbb{Z}(p))$.

For sheaves $\mathcal{F}$ and $\mathcal{H}$ denote by $\mathcal{G}^\bullet(*)$ their Godement resolutions. There is an associative map

$$\widetilde{\cup} : \mathcal{G}^p(\mathcal{F}) \otimes \mathcal{G}^q(\mathcal{H}) \longrightarrow \mathcal{G}^{p+q}(\mathcal{F} \otimes \mathcal{H})$$

given by

$$\sigma \widetilde{\cup} \tau(x_0, \ldots, x_{p+q}) = \sigma(x_0, \ldots, x_p)(x_{p+q}) \otimes \tau(x_{p+1}, \ldots, x_{p+q})$$

(see the interpretation of sections of $\mathcal{G}^\bullet(\mathcal{F}^\bullet)$ in the Appendix) This defines a morphism of complexes

$$\widetilde{\cup} : \mathcal{G}^\bullet(\mathcal{F}) \otimes \mathcal{G}^\bullet(\mathcal{H}) \longrightarrow \mathcal{G}^\bullet(\mathcal{F} \otimes \mathcal{H}).$$

As in the case of the Čech product $\check{\cup}$ this translates to morphisms of double complexes

$$\widetilde{\cup} : \mathcal{G}^\bullet(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet(\mathcal{H}^\bullet) \longrightarrow \mathcal{G}^\bullet(\mathcal{F}^\bullet \otimes \mathcal{H}^\bullet)$$

where $\mathcal{F}^\bullet$ and $\mathcal{H}^\bullet$ are complexes of sheaves.

Again we have to take care about the signs. Let as in subsection 2.2 denote by s the different contractions. We have an isomorphism

$$\begin{aligned} \tau^\bullet : \; & s_{(1,2)(3,4)} s_{1,2} s_{3,4} (\mathcal{G}^{\bullet_1}(\mathcal{F}^{\bullet_2}(p)) \otimes \mathcal{G}^{\bullet_3}(\mathcal{F}^{\bullet_4}(p'))) \\ & \longrightarrow s_{(1,3)(2,4)} s_{1,3} s_{2,4} (\mathcal{G}^{\bullet_1}(\mathcal{F}^{\bullet_2}(p)) \otimes \mathcal{G}^{\bullet_3}(\mathcal{F}^{\bullet_4}(p'))). \end{aligned} \tag{4.1}$$

Replacing the complexes $\mathcal{F}^\bullet(*)$ by the Deligne-Beilinson Complexes $\mathbb{Z}_{\mathcal{D}}(*)$ we get for $\alpha \in \mathbb{R}$ well defined products

$$\widetilde{\cup}_\alpha = \cup_\alpha \circ \widetilde{\cup} \circ \tau : \mathcal{G}^\bullet(\mathbb{Z}_{\mathcal{D}}(p)) \otimes \mathcal{G}^\bullet(\mathbb{Z}_{\mathcal{D}}(p')) \longrightarrow \mathcal{G}^\bullet(\mathbb{Z}_{\mathcal{D}}(p+p'))$$

As a consequence we get a theory of Massey products in Deligne cohomology by using this resolution, which by the functoriality of Massey products is the same as the one presented in section 3.4.

Let Z be a codimension p cycle on X and $U = X - Z$ its complement. The resolution $\mathcal{G}^\bullet(X) = \mathcal{G}^\bullet(X, \mathbb{Z}_{\mathcal{D}}(\bullet))$ gives rise to the short exact sequence of complexes

$$0 \longrightarrow \mathcal{G}^\bullet_{|Z|}(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(U) \longrightarrow 0,$$

where $\mathcal{G}^\bullet_{|Z|}(X)$ denotes the subcomplex of $\mathcal{G}^\bullet(X)$ supported on Z. This yields the long exact sequence of cohomology groups

$$\ldots \longrightarrow H^{q-1}_{\mathcal{D}}(U,\mathbb{Z}(p)) \longrightarrow H^q_{\mathcal{D}|Z|}(X,\mathbb{Z}(p)) \longrightarrow H^q_{\mathcal{D}}(X,\mathbb{Z}(p)) \longrightarrow H^q_{\mathcal{D}}(U,\mathbb{Z}(p)) \longrightarrow \ldots$$

already used in the definition of the $\mathbb{C}^*$ valued height pairing.

Let furthermore A, B be cycles on X of codimension p, resp. q and $U \subset X$ an open subset. $\widetilde{\cup}_\alpha$ gives associative maps

$$\widetilde{\cup}_\alpha : \mathcal{G}^\bullet_{|A|}(X,\mathbb{Z}_{\mathcal{D}}(*)) \otimes \mathcal{G}^\bullet_{|B|}(U,\mathbb{Z}_{\mathcal{D}}(*)) \longrightarrow \mathcal{G}^\bullet_{|A\cap B|}(U,\mathbb{Z}_{\mathcal{D}}(*)). \tag{4.2}$$

This map yields the pairing

$$\cup : H^\bullet_{\mathcal{D}|A|}(X,\mathbb{Z}(*)) \otimes H^\bullet_{\mathcal{D}|B|}(U,\mathbb{Z}(*)) \longrightarrow H^\bullet_{\mathcal{D}|A\cap B|}(U,\mathbb{Z}(*)).$$

We can now state the main theorem of this chapter.

Theorem 4.3. *Let A_i for $i = 1, 2, 3$ be algebraic cycles of codimension p_i on a n-dimensional smooth projective algebraic variety X, with cycle classes $\gamma(A_i) \in H^{2p_i}_{\mathcal{D}}(X,\mathbb{Z}(p_i))$, such that*

(1) $p_1 + p_2 + p_3 = n + 1$
(2) *A_1 and A_3 are homologically equivalent to zero*
(3) $A_1 \cap A_2 \cap A_3 = \emptyset$
(4) *$M_3(\gamma(A_1), \gamma(A_2), \gamma(A_3))$ is defined.*

Then

$$M_3(\gamma(A_1), \gamma(A_2), \gamma(A_3)) =< A_1, A_2 \cap A_3 > - < A_1 \cap A_2, A_3 >$$

Remark 4.4. To avoid confusion, we identify here $H^5_{\mathcal{D}}(X,\mathbb{Z}(3))$ with $\mathbb{C}/\mathbb{Z}(3)$. Therefore we write $M_3(A_1, A_2, A_3)$ as a difference and not as a quotient of the height pairings.

Before proving theorem 4.3 we state a lemma which is an immediate consequence of the construction of the boundary morphism.

Lemma 4.5. *Let*

$$0 \longrightarrow \mathcal{A}^\bullet \xrightarrow{\eta} \mathcal{B}^\bullet \xrightarrow{\epsilon} \mathcal{C}^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of $\mathcal{O}$-modules. The canonical flasque resolution or Čech resolution for some open covering where all the classes are living gives the short exact sequence

$$0 \longrightarrow \mathcal{G}^\bullet(\mathcal{A}^\bullet) \xrightarrow{\eta} \mathcal{G}^\bullet(\mathcal{B}^\bullet) \xrightarrow{\epsilon} \mathcal{G}^\bullet(\mathcal{C}^\bullet) \longrightarrow 0$$

which yields to the long exact sequence of hypercohomology groups

$$\ldots \longrightarrow \mathbb{H}^{p-1}(X,\mathcal{C}^\bullet) \xrightarrow{\rho} \mathbb{H}^p(X,\mathcal{A}^\bullet) \xrightarrow{\eta} \mathbb{H}^p(X,\mathcal{B}^\bullet) \xrightarrow{\epsilon} \mathbb{H}^p(X,\mathcal{C}^\bullet) \xrightarrow{\rho} \ldots$$

Given a class $[a] \in \mathbb{H}^p(X,\mathcal{A}^\bullet)$ with representative $a \in \mathcal{G}^p(\mathcal{A}^\bullet)$, such that $\eta([a]) = 0$. By assumption there exists a lifting, let's say $\phi \in \mathcal{G}^{p-1}(\mathcal{B}^\bullet)$, with $\delta(\phi) = \eta(a)$. Then

(1) *$\delta(\epsilon(\phi)) = 0$, i.e. $[\epsilon(\phi)]$ is a well defined cohomology class*

(2) $\rho([\epsilon(\phi)]) = [a]$.

In other words we can $[a]$ *to* $[\epsilon(\phi)] \in \mathbb{H}^{p-1}(X, \mathcal{C}^\bullet)$.

Proof of Lemma 4.5:

The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{p-1}(\mathcal{A}^\bullet) & \xrightarrow{\eta} & \mathcal{G}^{p-1}(\mathcal{B}^\bullet) & \xrightarrow{\epsilon} & \mathcal{G}^{p-1}(\mathcal{C}^\bullet) & \longrightarrow & 0 \\ & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & \\ 0 & \longrightarrow & \mathcal{G}^{p}(\mathcal{A}^\bullet) & \xrightarrow{\eta} & \mathcal{G}^{p}(\mathcal{B}^\bullet) & \xrightarrow{\epsilon} & \mathcal{G}^{p}(\mathcal{C}^\bullet) & \longrightarrow & 0 \end{array} \tag{4.3}$$

implies

$$\delta(\epsilon(\phi)) = \epsilon(\delta(\phi)) = \epsilon(\eta(a)) = 0,$$

i.e. $\epsilon(\phi)$ is closed.

The second point is the construction of the boundary map

$$\rho : H^{p-1}(\mathcal{C}^\bullet) \longrightarrow H^p(\mathcal{A}^\bullet),$$

which we recall: Let $c \in \mathcal{G}^{p-1}(\mathcal{C}^\bullet)$ be a closed cochain. We want to construct $\rho([c])$.

Lift c to some $b \in \mathcal{G}^{p-1}(\mathcal{B}^\bullet)$ with

$$\epsilon(b) = c.$$

Since $\delta(c) = 0$, we can find $a \in \mathcal{G}^p(\mathcal{A}^\bullet)$, such that

$$\delta(b) = \eta(a).$$

Since η is injective, a is unique and

$$\delta(a) = 0.$$

Let $b' \in \mathcal{G}^{p-1}(\mathcal{B}^\bullet)$ be another lifting of c. Then

$$b' - b = \eta(\alpha)$$

for some $\alpha \in \mathcal{G}^{p-1}(\mathcal{A}^\bullet)$.

Let $a' \in \mathcal{G}^p(\mathcal{A}^\bullet)$ be the class with

$$\delta(b') = \eta(a').$$

But

$$\delta(b') = \delta(b + \eta(\alpha')) = \eta(a + \delta(\alpha')).$$

Since η is injective

$$a' = a + \delta(\alpha')$$

and therefore

$$[a] = [a'].$$

Thus the map

$$\rho : H^{p-1}(\mathcal{C}^\bullet) \longrightarrow H^p(\mathcal{A}^\bullet)$$

given by

$$\rho([c]) = [a]$$

is well defined and gives the long exact sequence of cohomology groups.

In the situation of the lemma $\epsilon(\phi)$ certainly lifts to ϕ. By assumption we have

$$\delta(\phi) = \eta(a).$$

Therefore

$$\rho([\epsilon(\phi)]) = [a].$$

□

Let us now prove the theorem.

Proof of Theorem 4.3: We will write $\mathcal{G}^\bullet(X)$ instead of $\mathcal{G}^\bullet(X, \mathbb{Z}_\mathcal{D}(*))$ in the sequel. Let $a_i \in \mathcal{G}^{2p_i}(X)$ be representatives for $\gamma(A_i)$. We may choose a_i to be of the form

$$a_i = \eta(\gamma_i) \ \text{ for } \ \gamma_i \in \mathcal{G}^{2p_i}_{|A_i|}(X, \mathbb{Z}_{sD}(p_i)).$$

By the pairing 4.2 we find

$$\gamma_{i,j} \in \mathcal{G}^{2p_{i,j}}_{|A_i \cap A_j|}(X)$$

with

$$\gamma_{i,j} = \gamma_i \widetilde{\cup}_0 \gamma_j. \tag{4.4}$$

Therefore

$$a_i \widetilde{\cup}_0 a_j = \eta(\gamma_{i,j}).$$

By assumption $M_3(\gamma(A_1), \gamma(A_2), \gamma(A_3))$ is defined. Therefore we find $a_{1,2} \in \mathcal{G}^{2p_{1,2}-1}(X)$ and $a_{2,3} \in \mathcal{G}^{2p_{2,3}-1}(X)$ with

$$\delta(a_{1,2}) = a_1 \widetilde{\cup}_0 a_2$$

and

$$\delta(a_{2,3} = a_2 \widetilde{\cup}_0 a_3.$$

By definition

$$M = [a_1 \widetilde{\cup}_0 a_{2,3} - a_{1,2} \widetilde{\cup}_0 a_3] \in H^{2p_{1,2,3}-1}_\mathcal{D}(X, \mathbb{Z}(p_{1,2,3}))$$

is a representative of $M_3(\gamma(A_1), \gamma(A_2), \gamma(A_3))$.

Since $2p_{1,2,3} - 1 = 2n + 1$ the cochains

$$a_1 \widetilde{\cup}_0 a_{2,3} \ \text{ and } \ a_{1,2} \widetilde{\cup}_0 a_3$$

are closed.

By proposition 3.24 the indeterminacy of $M_3(\gamma(A_1), \gamma(A_2), \gamma(A_3))$ is zero. Therefore the cohomology classes

$$[a_1 \widetilde{\cup}_0 a_{2,3}] \ \text{ and } \ [a_{1,2} \widetilde{\cup}_0 a_3]$$

are defined unambigously.

We show in the sequel that

$$[a_1 \widetilde{\cup}_0 a_{2,3}] =< A_1, A_2 \cap A_3 > .$$

By assumption

$$\eta([\gamma_{2,3}]) = [a_2 \widetilde{\cup}_0 a_3] = [0].$$

Lemma 4.5 implies that

$$\delta(\epsilon(a_{2,3})) = 0$$

and

$$\rho([\epsilon(a_{2,3})]) = [\gamma_{2,3}].$$

Let $U_{2,3} = X - (A_2 \cap A_3)$, then $A_1 \subset U_{2,3}$. Thus

$$\mathcal{G}^{\bullet}_{|A_1|}(U_{2,3}) \cong \mathcal{G}^{\bullet}_{|A_1|}(X).$$

The product $\widetilde{\cup}_0$ induces the commutative diagram
(4.5)

$$\begin{array}{ccccc}
\mathcal{G}^{2p_1}_{|A_1|}(X) \otimes \mathcal{G}^{2p_{2,3}-1}(X) & \xrightarrow{\mathrm{id}\otimes\epsilon} & \mathcal{G}^{2p_1}_{|A_1|}(X) \otimes \mathcal{G}^{2p_{23}-1}(U_{23}, \mathbb{Z}_{sD}(p_{2,3})) & \xrightarrow{\tilde{\cup}} & \mathcal{G}^{2n+1}_{|A_1|}(U_{23}) \\
\tilde{\cup}\downarrow & & & & \cong\downarrow \\
\mathcal{G}^{2n+1}_{|A_1|}(X) & & \xrightarrow{\mathrm{id}} & & \mathcal{G}^{2n+1}_{|A_1|}(X) \\
\eta\downarrow & & & & \eta\downarrow \\
\mathcal{G}^{2n+1}(X) & & \xrightarrow{\mathrm{id}} & & \mathcal{G}^{2n+1}(X)
\end{array}$$

Inserting the data we have yields

$$\begin{array}{ccccc}
\gamma_1 \otimes a_{2,3} & \mapsto & \gamma_1 \otimes \epsilon(a_{2,3}) & \mapsto & \gamma_1 \widetilde{\cup}_0 \epsilon(a_{2,3}) \\
\downarrow & & & & \downarrow \\
\eta(\gamma_1)\widetilde{\cup}_0 a_{2,3} & & = & & \eta(\gamma_1 \widetilde{\cup}_0 \epsilon(a_{2,3})).
\end{array}$$

The term

$$\eta(\gamma_1)\widetilde{\cup}_0 a_{2,3} = a_1 \widetilde{\cup}_0 a_{2,3}$$

is the first term of $M = [a_1\widetilde{\cup}_0 a_{2,3} - a_{1,2}\widetilde{\cup}_0 a_3]$.

The term

$$\eta(\gamma_1 \widetilde{\cup}_0 \epsilon(a_{2,3}))$$

represents by construction $< A_1, A_{2,3} >$ (see the definition of the height pairing).

Therefore

$$[a_1\widetilde{\cup}_0 a_{2,3}] = < A_1, A_2 \cap A_3 > .$$

By symmetry the equality

$$[a_{1,2}\widetilde{\cup}_0 a_3] = < A_1 \cap A_2, A_3 >$$

holds also.

□

As a corollary we get

Corollary 4.6. *Let the situation be as in 4.3, then the height pairings $< A_1, A_2 \cap A_3 >$ and $< A_1 \cap A_2, A_3 >$ are defined. Moreover the difference (or by remark 4.4 quotient) of these height pairings is always torsion.*

We translate a special case of the example 3.32 to the situation of height pairings.

Example 4.7. Let E be an elliptic curve, given by the equation

$$y^2 = x(x-1)(x-\lambda)$$

and $X = E \times E$.

E is a covering of the projective line

$$\pi : E \longrightarrow \mathbb{P}^1$$

with ramification points

$$P_0\ ,\ \ P_1\ ,\ \ P_\lambda\ ,\ \ P_\infty.$$

The nontrivial twodivision points in $\mathrm{Pic}^0(E)$ are given by

$$P_0 - P_\infty\ ,\ \ P_1 - p_\infty\ ,\ \ P_\lambda - P_\infty.$$

For an arbitrary point $Q \in E$ we have

$$P_\infty + Q \sim_{rat} P_\infty + Q + (\frac{y + y(Q)}{x - x(Q)}) \sim_{rat} Q_1' + Q_2'.$$

Here (f) denotes the divisor associated to the rational function f and $Q,\ Q_1,\ Q_2$ are the zeroes of the function $y - y(Q)$. Prime denotes Galois conjugation.

We can write the point $P_1 - P_\infty$ in the form

$$H = P_1 + Q - Q_1' - Q_2'.$$

Let $D = P_0 - P_\infty$ be another twodivision point and let A be an ample $(1,1)$ Divisor on X. If we denote by p_1 and p_2 the two projections from X to the elliptic curve E, we see that the Massey product

$$M = M_3(\gamma(p_1^*(D)), \gamma(2A), \gamma(p_1^*(H)))$$

is defined.

By theorem 4.3 we can calculate M as

$$< p_1^*(D), 2A \cap p_1^*(H) > - < p_1^*(D) \cap 2A, p_1^*(H) > .$$

Using projection formula and the fact that A is an ample $(1,1)$ divisor we can calculate the above terms on E via

$$M =< D, 2H > - < 2D, H > .$$

Now we can write

$$2D = (x)$$

and

$$2H = \left((x-1) \frac{x^2 - 2x(Q)x + x(Q)^2}{y^2 + 2y(Q)y + y(Q)^2} \right) = (h)$$

The calculation given in [M-S2] yields

$$M = \log \left(\frac{\prod_j h(D_j)^{b_j}}{\prod_i x(H_i)^{a_i}} \right) = \log \left(\frac{x(Q_1')x(Q_2')h(0)}{x(1)x(Q)h(\infty)} \right)$$

$$= \log \left(- \frac{x(Q_1')x(Q_2)'x(Q)^2}{x(Q)y(Q)^2} \right) = \log \left(- \frac{x(Q)x(Q_1)x(Q_2)}{y(Q)^2} \right) = \log(-1).$$

We should explain some equalities. The last equality holds, since Q, Q_1, Q_2 are the solutions of the equation $y - y(Q)$.

$$h(0) = -\frac{x(Q)^2}{y(Q)^2}$$

and

$$h(\infty) = \lim_{x\to\infty} (x-1)\frac{x^2 - 2x(Q)x + x(Q)^2}{y^2 + 2y(Q)y + y(Q)^2} = 1.$$

Applying the exponential function we get

$$\exp(2M) = 1.$$

This corresponds to the result obtained in example 3.32.

Remark 4.8. As we have seen the height pairing

$$< D, 2H >= \frac{h(0)}{h(\infty)} = -\frac{x(Q)^2}{y(Q)^2}$$

depends on the choice of the point Q. But Q was chosen arbitrarily. Thus the height pairing is not invariant under rational equivalence of cycles. Theorem 4.3 implies that the difference is invariant, since $M_3(A_1, A_2, A_3)$ is invariant under rational equivalence. Moreover the difference of the two height pairings is always torsion, whereas the height pairings $< D, 2H >$ and $< 2D, H >$ are not torsion at all.

5. Infinitesimal Variations of Massey products

As mentioned in the introduction, we want to study the behaviour of Massey products in smooth families $\pi : X \longrightarrow S$. To be precise, we start again with codimension p_i cycles A_i on X, which also are of codimension p_i on the fibres $X_s = \pi^{-1}(s)$. We assume that the Massey products are defined on the fibres for all $s \in S$. By the previous chapter we know that on X_s a representative M of $M_3(A_{1s}, A_{2s}, A_{3s})$ (where $A_{is} = A_i|_{X_s}$) lives in $H_{\mathcal{D}}^{2p_{1,2,3}-1}(X_s, \mathbb{Z}(p_{1,2,3}))$ and are homologically trivial up to torsion. Therefore we can associate to a representative M of $M_3(A_1, A_2, A_3)$ a section $\mathcal{M}$ of the sheaf of intermediate Jacobians and apply the Gauss-Manin connection ∇.

The notation is as follows:

We have the diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \longleftarrow & Y \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\bar{\pi}}\big\downarrow & & {\scriptstyle\bar{\pi}}\big\downarrow \\ S & \xrightarrow{i} & \bar{S} & \longleftarrow & \Sigma \end{array}$$

where $\bar{X}$ and $\bar{S}$ are good compactifications of X resp. S and where i, j are the embeddings. Furthermore we assume π to be smooth and that Y and Σ are divisors with normal crossings.

5.1. **Gauss-Manin-Connection and Griffiths Infinitesimal Invariant.**

We will first give an algebraic construction of the Gauss-Manin-Connection on the local system defined fibrewise by $H^q(X_s, \mathbb{C})$. We follow the presentation given in the article of N.M. Katz [Ka2]. See also the article of M. Green in [GMV], [Del4], [Ka1] and [Zu].

Let $\Omega^\bullet_{\bar{X}}(\log Y)$ be the deRham complex of algebraic differential forms with logarithmic singularities on Y, and let $\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)$ be the relative deRham complex of the family $\pi : X \longrightarrow S$ with logarithmic singularities along Y.

We can filter the complex $\Omega^\bullet_{\bar{X}}(\log Y)$ in two different ways. The first filtration is given by

$$F^p(\Omega^\bullet_{\bar{X}}(\log Y))^i = \begin{cases} 0 & \text{for} \quad i < p \\ \Omega^i_{\bar{X}}(\log Y) & \text{for} \quad i \geq p \end{cases}$$

On the other hand there is the short exact sequence

$$0 \longrightarrow \bar{\pi}^*\Omega^1_{\bar{S}}(\log \Sigma) \longrightarrow \Omega^1_{\bar{X}}(\log Y) \longrightarrow \Omega^1_{\bar{X}/\bar{S}}(\log Y) \longrightarrow 0.$$

The sequence induces the second filtration $G^\bullet(\Omega^\bullet_{\bar{X}}(\log Y))$ given by

$$G^p(\Omega^\bullet_{\bar{X}}(\log Y)) = \mathrm{im}(\bar{\pi}^*\Omega^p_{\bar{S}}(\log \Sigma) \otimes_{\mathcal{O}_{\bar{X}}} \Omega^{\bullet-p}_{\bar{X}}(\log Y) \longrightarrow \Omega^\bullet_{\bar{X}}(\log Y)).$$

The associated graded pieces $\mathrm{Gr^p} = \mathrm{G^p}/\mathrm{G^{p+1}}$ are

$$\mathrm{Gr^p}(\Omega^\bullet_{\bar{\mathrm{X}}}(\log \mathrm{Y})) = \bar{\pi}^*\Omega^{\mathrm{p}}_{\bar{\mathrm{S}}}(\log \mathrm{Y}) \otimes_{\mathcal{O}_{\bar{\mathrm{X}}}} \Omega^{\bullet-\mathrm{p}}_{\bar{\mathrm{X}}/\bar{\mathrm{S}}}(\log \mathrm{Y}).$$

There is the Leray spectral sequence

$$E_1^{a,b} = \Omega^a_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^b \bar{\pi}_* \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \Rightarrow \mathcal{R}^{a+b}\bar{\pi}_* \Omega^\bullet_{\bar{X}}(\log Y)$$

whose d_1 differential is given by

$$d_1 : \Omega^a_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^b \bar{\pi}_* \Omega^{\bullet - a}_{\bar{X}/\bar{S}} \longrightarrow \Omega^{a+1}_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^b \bar{\pi}_* \Omega^{\bullet - a - 1}_{\bar{X}/\bar{S}}(\log Y).$$

Remark 5.1. For $a = 0$ the differential d_1 is the Gauss-Manin Connection

$$\nabla : \mathcal{R}^b \bar{\pi}_* \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^b \bar{\pi}_* \Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y).$$

∇ is an integrable connection. It gives rise to the complex

$$\mathcal{R}^q \bar{\pi}_* \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \xrightarrow{\nabla} \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^q \bar{\pi}_* \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \xrightarrow{\nabla} \dots.$$

By [Del4], II.6.10, the Hypercohomology of this complex computes $H^*(S, R^q\pi_*\mathbb{C})$.

We define the relative analogue of the F-filtration by

$$F^p(\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))^i = \begin{cases} 0 & \text{for } \; i < p \\ \Omega^i_{\bar{X}/\bar{S}}(\log Y) & \text{for } \; i \geq p \end{cases}$$

If we take the higher direct image sheaves and look at a fibre X_s for $s \in S$, we have

$$\mathcal{R}^q \bar{\pi}_* F^p(\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) \otimes k(s) = F^p H^q(X_s, \mathbb{C}) = \mathbb{H}^q(X_s, F^p \Omega^\bullet_{X_s}).$$

Proposition 5.2. *∇ preserves the filtration F by a shift of one, i.e.*

$$\nabla(\mathcal{R}^q \bar{\pi}_* F^p \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) \subset \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^q \bar{\pi}_* F^{p-1} \Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y).$$

Proof. By construction ∇ is the connecting morphism of the short exact sequence

$$0 \longrightarrow \mathrm{Gr}^1 \longrightarrow \mathrm{G}^0/\mathrm{G}^1 \longrightarrow \mathrm{Gr}^0 \longrightarrow 0$$

which is a consequence of the following exact sequence

$$0 \longrightarrow \bar{\pi}^* \Omega^1_{\bar{S}}(\log \Sigma) \otimes \Omega^{p-1}_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \Omega^p_{\bar{X}}(\log Y) \longrightarrow \Omega^p_{\bar{X}/\bar{S}}(\log Y) \longrightarrow 0.$$

Taking the higher direct images we get the map

$$\begin{aligned} \nabla : \mathcal{R}^q \bar{\pi}_* \Omega^p_{\bar{X}/\bar{S}}(\log Y) &\longrightarrow \mathcal{R}^{q+1} \bar{\pi}_* (\bar{\pi}^* \Omega^1_{\bar{S}}(\log \Sigma) \otimes \Omega^{p-1}_{\bar{X}/\bar{S}}(\log Y)) \\ &= \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^q \bar{\pi}_* \Omega^{p-1}_{\bar{X}/\bar{S}}(\log Y) \end{aligned}$$

which proves the proposition. □

We define the family of (p, q)-th intermediate Jacobians (see [ZZ] and [GMV]).

Definition 5.3. For the diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \longleftarrow & Y \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \bar{\pi}}\downarrow & & {\scriptstyle \bar{\pi}}\downarrow \\ S & \xrightarrow{i} & \bar{S} & \longleftarrow & \Sigma \end{array}$$

the family of (p,q)-th intermediate Jacobians is

$$\mathcal{J}^{p,q} := R^{q-1}\bar{\pi}_* Rj_*\mathbb{C}/(R^{q-1}\bar{\pi}_* Rj_*\mathbb{Z}(p) + F^p R^{q-1}\bar{\pi}_* Rj_*\mathbb{C}).$$

The family comes together with a map

$$\pi_{p,q} : \mathcal{J}^{p,q} \longrightarrow \bar{S}$$

with

$$\begin{aligned}\pi_{p,q}^{-1}(s) &= \mathcal{J}^{p,q}(X_s) = H^{q-1}(X_s,\mathbb{C})/(H^{q-1}(X_s,\mathbb{Z}(p)) + F^p H^{q-1}(X_s,\mathbb{C})) \\ &= \ker H^q_{\mathcal{D}}(X_s,\mathbb{Z}_{\mathcal{D}}(p)) \longrightarrow H^q(X_s,\mathbb{Z}(p)) \text{ for } s \in S.\end{aligned}$$

Definition 5.4. Let $\nu \in H^0(\bar{S},\mathcal{J}^{p,q})$ be a section. We say that ν satisfies the **infinitesimal condition for normal functions**, if there exists a lifting ν to some $\widetilde{\nu} \in H^0(\bar{S}, R^{q-1}\bar{\pi}_* Rj_*\mathbb{C}) = H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))$, such that

$$\nabla(\widetilde{\nu}) \in H^0(\bar{S}, \Omega^1_{\bar{S}}(\log\Sigma) \otimes F^{p-1}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

Definition and Theorem 5.5. Let $\nu \in H^0(\bar{S},\mathcal{J}^{p,q})$ be a section. We call ν a **normal function**, if ν satisfies the infinitesimal condition for normal functions 5.4. This definition is independent of the chosen lifting $\widetilde{\nu} \in H^0(\bar{S}, R^{q-1}\bar{\pi}_* Rj_*\mathbb{C})$.

Proof. The lifting $\widetilde{\nu}$ only depends on classes

$$z \in H^0(\bar{S}, R^{q-1}\bar{\pi}_* Rj_*\mathbb{Z}(p))$$

and

$$f \in H^0(\bar{S}, F^p\mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \otimes \mathcal{O}_{\bar{S}}).$$

For those we have

$$\nabla(z) = 0$$

since integral classes are locally constant and

$$\nabla(f) \in \Omega^1_{\bar{S}}(\log\Sigma) \otimes F^{p-1}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y).$$

□

Definition 5.6. The Gauss-Manin connection induces maps

$$\Omega^k_{\bar{S}}(\log\Sigma) \otimes F^{p-k}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^{\bullet-k}_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \Omega^{k+1}_{\bar{S}}(\log\Sigma) \otimes F^{q-k-1}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^{\bullet-k-1}_{\bar{X}/\bar{S}}(\log Y).$$

Setting

$$\mathcal{K}^k_{p,q} = \Omega^k_{\bar{S}}(\log\Sigma) \otimes F^{p-k}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^{\bullet-k}_{\bar{X}/\bar{S}}(\log Y) \tag{5.1}$$

we get a complex $(\mathcal{K}^\bullet_{p,q},\nabla)$, which we call the (p,q)**-th Koszul complex** associated to the family $\pi : X \longrightarrow S$.

Let

$$\nu : \bar{S} \longrightarrow \mathcal{J}^{p,q}$$

be a normal function of $\pi : X \longrightarrow S$. Since $\nabla^2 = 0$ we get an element

$$\nabla(\widetilde{\nu}) \in H^0(\bar{S}, \Omega^1_{\bar{S}}(\log\Sigma) \otimes F^{p-1}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y)) \longrightarrow H^0(H^1(\bar{S},\mathcal{K}^\bullet_{p,q})).$$

On the other hand the lifting $\widetilde{\nu}$ only depends on elements coming from $R^{q-1}\bar{\pi}_*\mathcal{R}j_*\mathbb{Z}(p)$, which vanish under ∇, or on some $f \in F^p\mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)$, but $\nabla(f) \in \nabla(\mathcal{K}^0_{p,q})$, hence vanishes in $H^1(\mathcal{K}^\bullet_{p,q})$. Thus the following definition is well defined.

Definition 5.7. The element $\nabla(\widetilde{\nu})$ gives a well defined element

$$\delta(\nu) \in H^1(\mathcal{K}^\bullet_{p,q}).$$

We call $\delta(\nu)$ **Griffiths Infinitesimal Invariant** of the normal function ν.

Definition 5.8. A normal function is called constant, if $\delta(\nu) = 0$

In the next chapter we will associate a normal function to a Deligne cohomology class. It will allow us to formulate a criteria for Deligne cohomology classes to be constant.

5.2. **Relative Deligne-Beilinson-cohomology.**

Since we are dealing with smooth families, we need a relative version of Deligne-Beilinson-cohomology. We present first the construction of F. El Zein and S. Zucker (see [ZZ]) in a globalized setup, which is possible as stated in the remark after definition 2, chapter 3 [ZZ]. But this construction is not sufficient for what we need. Therefore we will construct the relative Deligne-Beilinson-cohomology in a different way corresponding to the construction of the Deligne-Beilinson-Cohomolgy in chapter 3.1.

Let the situation be as in 5.1 and assume furthermore, that S is a smooth curve.

In the derived category there are morphisms

$$Rj_*\mathbb{Z}(p) \longrightarrow Rj_*\mathbb{C} \xrightarrow{\cong} \Omega^\bullet_{\bar{X}}(\log Y) \longrightarrow \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \Omega^{<p}_{\bar{X}/\bar{S}}(\log Y),$$

where the Quasi Isomorphism

$$Rj_*\mathbb{C} \xrightarrow{\cong} \Omega^\bullet_{\bar{X}}(\log Y)$$

has been constructed in [Del1], chapter 3.1.

Let us denote by ϵ_p the composition map

$$\epsilon_p : Rj_*\mathbb{Z}(p) \longrightarrow \Omega^{<p}_{\bar{X}/\bar{S}}(\log Y).$$

For the realization of the morphism ϵ_p see [ZZ], chapter 3.

Definition 5.9.

(1) We call the complex

$$\mathcal{D}_{X/S}(p) := \mathrm{Cone}(Rj_*\mathbb{Z}(p) \xrightarrow{\epsilon_p} \Omega^{<p}_{\bar{X}/\bar{S}}(\log Y))$$

the **p-th relative Deligne Complex** of the family $\pi : X \longrightarrow S$.

(2) The **sheaf of (p,q)-th relative Deligne Groups** on $\bar{S}$ is defined as:

$$D^q_{X/S}(p) := \mathcal{R}^q\bar{\pi}_*\mathcal{D}_{X/S}(p).$$

Properties 5.10.

(1) Let denote by $\mathcal{Z}^p(X)$ the group of codimension p cycles on X, then there is a map

$$\nu : \mathcal{Z}^p(X) \longrightarrow H^0(\bar{S}, \mathcal{R}^{2p}\bar{\pi}_*\mathcal{D}_{X/S}(p)) \tag{5.2}$$

constructed via the natural projection

$$\mathbb{Z}_{\mathcal{D}}(p) \longrightarrow \mathcal{D}_{X/S}.$$

By construction ν respects rational equivalence, therefore we get well a defined map

$$\nu : \mathrm{CH}^p(X) \longrightarrow H^0(\bar{S}, \mathcal{R}^{2p}\bar{\pi}_*\mathcal{D}_{X/S}(p)). \tag{5.3}$$

(2) For $s \in S$ denote by $\mathcal{Z}^p(X_s)$ the subgroup of cycles $Z \in \mathcal{Z}^p(X)$, for which $T \cap X_s$ is defined. Then by slicing theory (see [Ki1], chapter 3) the map ν of 5.2 restricts to

$$\nu : \mathcal{Z}^p(X_s) \longrightarrow H^{2p}_{\mathcal{D}}(X_s, \mathbb{Z}(p)). \tag{5.4}$$

For more details on this see [ZZ], chapter 3.

Next we state the alternative construction of relative Deligne-Beilinson cohomology.

Definition 5.11.

(1) We call the complex

$$\mathbb{Z}_{\mathcal{D}/S}(p) := \mathrm{Cone}(Rj_*\mathbb{Z}(p) \oplus F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \xrightarrow{\epsilon-\iota} \mathcal{R}j_*\Omega^\bullet_{X/S})$$

the **p-th Relative Deligne-Beilinson Complex** of the family $\pi : X \longrightarrow S$.

(2) The groups

$$H^q_{\mathcal{D}/S}(X, \mathbb{Z}(p)) := \mathbb{H}^q(\bar{X}, \mathbb{Z}_{\mathcal{D}/S}(p))$$

are called the **relative Deligne-Beilinson cohomology groups**.

The complex $\mathbb{Z}_{\mathcal{D}/S}(p)$ lives in the following two short exact sequences

$$0 \longrightarrow \mathcal{N}(p) \longrightarrow \mathbb{Z}_{\mathcal{D}}(p) \longrightarrow \mathbb{Z}_{\mathcal{D}/S}(p) \longrightarrow 0$$

where $\mathcal{N}(p)$ is the complex

$$\mathcal{N}(p) := \mathrm{Cone}(\bar{\pi}^*\Omega^1_{\bar{S}}(\log \Sigma) \otimes F^{p-1}\Omega^\bullet_{\bar{X}/\bar{S}}(\log \Sigma)) \longrightarrow \mathcal{R}j_*(\pi^*\Omega^1_S \otimes \Omega^\bullet_{X/S})$$

and

$$0 \longrightarrow \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)[-1] \longrightarrow \mathbb{Z}_{\mathcal{D}/S}(p) \longrightarrow \mathcal{R}j_*\mathbb{Z}(p) \oplus F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \longrightarrow 0$$

Proposition 5.12. *The complexes $\mathbb{Z}_{\mathcal{D}/S}(p)$ and $\mathcal{D}_{X/S}(p)$ are quasi isomorphic.*

Proof. By equation 3.3 we can construct $\mathbb{Z}_{\mathcal{D}/S}(p)$ as

$$\mathrm{Cone}(\mathcal{R}j_*\mathbb{Z}(p) \longrightarrow \mathrm{Cone}(F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \mathcal{R}j_*\Omega^\bullet_{X/S})).$$

The complexes

$$\mathrm{Cone}(F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y) \longrightarrow \mathcal{R}j_*\Omega^\bullet_{X/S}) \text{ and } \Omega^{<p}_{\bar{X}/\bar{S}}(\log Y)$$

are quasi isomorphic. Therefore the proposition follows by 6.5 (see the Appendix). □

The properties of 5.10 translate directly to the complex $\mathbb{Z}_{\mathcal{D}/S}$. Putting together all the data, we get the following commutative diagram.It will be important in the sequel.

(5.5)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \bar{\pi}^*\Omega^1_{\bar{S}}(\log\Sigma)\otimes\Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y))[-1] & \rightarrow & \mathcal{N}(p) & \rightarrow & \bar{\pi}^*\Omega^1_{\bar{S}}(\log\Sigma)\otimes F^{p-1}\Omega^{\bullet-1}_{\bar{X}/\bar{S}}(\log Y) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^{\bullet}_{\bar{X}}(\log Y)[-1] & \rightarrow & \mathbb{Z}_{\mathcal{D}}(p) & \rightarrow & \mathcal{R}j_*\mathbb{Z}(p)\oplus F^p\Omega^{\bullet}_{\bar{X}}(\log Y) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^{\bullet}_{\bar{X}/\bar{S}}(\log Y)[-1] & \rightarrow & \mathbb{Z}_{\mathcal{D}/S}(p) & \rightarrow & \mathcal{R}j_*\mathbb{Z}(p)\oplus F^p\Omega^{\bullet}_{\bar{X}/\bar{S}}(\log Y) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

As a next step we construct a normal function ν_A associated to a Deligne cohomology class $A \in H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$, where A is homologically equivalent to zero on the fibres X_s. We will show, that this function satisfies the infinitesimal condition for normal functions, which enables us to calculate its infinitesimal invariant $\delta(\nu_A)$. Finally we show that we can read the infinitesimal invariant $\delta(\nu_A)$ in the cohomology class $\epsilon_F(A)$.

Let the situation be as above and let $A \in H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$ such that its class $\epsilon_F(A) \in F^pH^q(\bar{X}, \Omega^{\bullet}_{\bar{X}}(\log Y))$ vanishes on the fibres X_s, *i.e.* $\epsilon_F(A)$ maps to zero under the map

$$H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*F^p\Omega^{\bullet}_{\bar{X}}(\log Y)) \longrightarrow H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*F^p\Omega^{\bullet}_{\bar{X}/\bar{S}}(\log Y)).$$

Taking the higher direct images of the commutative diagram 5.5 we get

$$
\begin{tikzcd}[column sep=small]
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^{q-2}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))) \arrow[r] \arrow[d] & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathcal{N}(p)) \arrow[r] \arrow[d] & H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes F^{p-1}\mathcal{R}^{q-1}\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}}(\log Y)) \arrow[r] \arrow[d] & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathbb{Z}_{\mathcal{D}}(p)) \arrow[r] \arrow[d] & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathcal{R}j_*\mathbb{Z}(p)) \oplus H^0(\bar{S}, F^p\mathcal{R}^q\bar{\pi}_*\Omega^\bullet_{\bar{X}}(\log Y)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) \arrow[r] \arrow[d] & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathbb{Z}_{\mathcal{D}/S}(p)) \arrow[r] \arrow[d] & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathcal{R}j_*\mathbb{Z}(p)) \oplus H^0(\bar{S}, F^p\mathcal{R}^q\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Let us denote by λ the map

$$\lambda : H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathbb{Z}_{\mathcal{D}}(p)) \longrightarrow H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathbb{Z}_{\mathcal{D}/S(p)}).$$

Then $\lambda(A)$ lifts to a class

$$\nu_A \in H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

This lifting is unique up to elements coming from

$$H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\mathcal{R}j_*\mathbb{Z}(p)) \oplus H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

Definition and Theorem 5.13. The class ν_A is a normal function, *i.e.* it satisfies the infinitesimal condition for normal functions 5.4. We call ν_A a **normal function associated to the Deligne cohomology class** $A \in H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$.

Proof. We have the commutative diagram

$$\begin{array}{ccc} H^0(\bar{S}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) & \to & H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*\mathbb{Z}_{\mathcal{D}/S}(p)) \\ \downarrow \nabla & & \downarrow \\ H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)) & \to & H^0(\bar{S}, \mathcal{R}^{q+1}\bar{\pi}_*\mathcal{N}(p)). \end{array}$$

The class $\lambda(A)$ maps to zero in $H^0(\bar{S}, \mathcal{R}^{q+1}\bar{\pi}_*\mathcal{N}(p))$. Therefore $\nabla(\nu_A)$ maps to zero in $H^0(\bar{S}, \mathcal{R}^{q+1}\bar{\pi}_*\mathcal{N}(p))$. Thus $\nabla(\nu_A)$ lives in the image of the map

$$H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes F^{p-1}\mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))$$

$$\longrightarrow H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^{q-1}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

This is the infinitesimal condition for normal functions 5.4. □

On the other hand we can construct a class $D(A)$ associated to A as follows: $\pi_F(A)$ lives in the kernel of the map

$$H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*F^p\Omega^\bullet_{\bar{X}}(\log Y)) \longrightarrow H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

First lift $\epsilon_F(A)$ up to classes coming from

$$H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*\mathcal{R}j_*\mathbb{Z}(p)) \oplus H^0(\bar{S}, \mathcal{R}^{q-1}\bar{\pi}_*F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))$$

to a class

$$\mu_A \in H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^{q-1}\bar{\pi}_*F^{p-1}\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

Secondly map μ_A via the horizontal connecting morphism of the diagram of page ??? to the class

$$D(A) \in H^0(\bar{S}, \Omega^1_{\bar{S}}(\log \Sigma) \otimes \mathcal{R}^{q-2}\bar{\pi}_*\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y))).$$

Note that the outer vertical connecting morphisms of the diagram are given by the Gauss Manin connection ∇. This yields to the following

Proposition 5.14. *$D(A)$ equals $\nabla(\nu_A)$ modulo the image of*

$$H^0(\bar{S}, \mathcal{R}^q\bar{\pi}_*F^p\Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

This proposition is a consequence of the following

Lemma 5.15. *Let $X \xrightarrow{\pi} S$ be a flat family and*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D^\bullet & \longrightarrow & E^\bullet & \longrightarrow & F^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^\bullet & \longrightarrow & H^\bullet & \longrightarrow & I^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram of complexes of sheaves on X and $M \in H^0(S, \mathcal{R}^q\pi_*E^\bullet)$ a global section, which vanishes in $H^0(S, \mathcal{R}^q\pi_*I^\bullet)$.

Furthermore we denote the different connecting morphisms by

$$\begin{aligned}
d_1 &: H^0(S, \mathcal{R}^q\pi_*C^\bullet) \longrightarrow H^0(S, \mathcal{R}^{q+1}\pi_*A^\bullet) \\
d_2 &: H^0(S, \mathcal{R}^q\pi_*G^\bullet) \longrightarrow H^0(S, \mathcal{R}^{q+1}\pi_*A^\bullet) \\
d_3 &: H^0(S, \mathcal{R}^{q-1}\pi_*I^\bullet) \longrightarrow H^0(S, \mathcal{R}^q\pi_*C^\bullet) \\
d_4 &: H^0(S, \mathcal{R}^{q-1}\pi_*I^\bullet) \longrightarrow H^0(S, \mathcal{R}^q\pi_*I^\bullet).
\end{aligned}$$

Now map first M to a class in $H^0(S, \mathcal{R}^q\pi_*F^\bullet)$, lift this class to $H^0(S, \mathcal{R}^q\pi_*C^\bullet)$ and map this under the connecting morphism d_1 to the class $M_1 \in H^0(S, \mathcal{R}^{q+1}\pi_*A^\bullet)$. On the other side map M to a class in $H^0(S, \mathcal{R}^q\pi_*H^\bullet)$ lift it to $H^0(S, \mathcal{R}^q\pi_*G^\bullet)$ and then map it via the connecting morphism d_2 to the class $M_2 \in H^0(S, \mathcal{R}^{q+1}\pi_*A^\bullet)$. Then M_1 equals M_2 modulo

$$d_1 d_3 H^0(S, \mathcal{R}^q\pi_*I^\bullet) + d_2 d_4 H^0(S, \mathcal{R}^q\pi_*I^\bullet).$$

Proof. The lemma is an immediate consequence of the snake-lemma. □

Corollary 5.16. *By construction the class $D(A)$ lives in $H^1(\mathcal{K}^\bullet_{p,q})$ (see 5.1). There we have*

$$D(A) = \delta(\nu_A).$$

5.3. **The infinitesimal invariant of** $M_3(A_1, A_2, A_3)$**.**

In this subsection we want to define, when Massey products are called **constant**. As mentioned before we want to apply the Gauss-Manin connection on $M_3(A_1, A_2, A_3)$. But we have to be careful, since $M_3(A_1, A_2, A_3)$ is the set $M + \mathcal{I}$, where M is a representative of $M_3(A_1, A_2, A_3)$ and $\mathcal{I}$ denotes the indeterminacy

$$\mathcal{I} = A_1 \cup H_{\mathcal{D}}^{2p_{2,3}-1}(X, \mathbb{Z}(p_{2,3})) + H_{\mathcal{D}}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2})) \cup A_3.$$

Therefore we start with examining the behaviour of $\mathcal{I}$ under ∇.

Lemma 5.17. *Let $A_i \in \mathcal{Z}^{p_i}(X)$ be codimension p_i cycles on X. Denote by A_i also the corresponding cycle classes in $H_{\mathcal{D}}^{2p_i}(X, \mathbb{Z}(p_i))$. Assume that $M_3(A_1, A_2, A_3)$ is defined in the Deligne cohomology of X with representative M. Then there exists some natural number $n \in N$, such that nM is homologically trivial on the fibres and nM is a representative of $M_3(A_1, nA_2, A_3)$.*

Proof. Let us denote by $\pi_{\mathbb{Z}}$ the map

$$\pi_{\mathbb{Z}} : H_{\mathcal{D}}^{2p_{1,2,3}-1}(X_s, \mathbb{Z}(p_{1,2,3})) \longrightarrow H^{2p_{1,2,3}-1}(X_s, \mathbb{Z}(p_{1,2,3})).$$

On the fibres we have

$$F^p H^{2p_{1,2,3}-1}(X_s, \mathbb{C}) \cap H^{2p_{1,2,3}-1}(X_s, \mathbb{Z}(p_{1,2,3})) = 0.$$

Therefore the only chance for $\pi_{\mathbb{Z}}(M)$ is to be torsion in $H^{2p_{1,2,3}-1}(X_s, \mathbb{Z}(p_{1,2,3}))$. Choose n, such that $n\pi_{\mathbb{Z}}(M) = 0$.

For the second point see the properties 2.6 of Massey triple products. □

Remark 5.18. Lemma 5.17 implies that we can construct a normal function ν associated to nM and calculate its infinitesimal invariant $\delta(\nu) \in H^1(\mathcal{K}^\bullet_{2p_{1,2,3}-1, p_{1,2,3}})$. There we can divide by n.

On the other hand $\pi_F(M)$ lives in the kernel of the map

$$H^0(\bar{S}, \mathcal{R}^q \bar{\pi}_* F^p \Omega^\bullet_{\bar{X}}(\log Y)) \longrightarrow H^0(\bar{S}, \mathcal{R}^q \bar{\pi}_* F^p \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

Therefore we can construct the class $D(M)$. Since $D(nM) = \delta(nM)$ in $H^1(\mathcal{K}^\bullet_{2p_{1,2,3}-1, p_{1,2,3}})$, the class $D(M)$ has to be equal to $\delta(nM)$ divided by n.

Therefore we assume from now on the representatives M of $M_3(A_1, A_2, A_3)$ to be homologically trivial on the fibres X_s.

Proposition 5.19. *Let the situation be as in lemma 5.17. Assume that M is homologically trivial on the fibres (see remark 5.18) and let ν_M be a normal function associated to M. Furthermore denote by $\epsilon_{F/S}$ the composed map*

$$H_{\mathcal{D}}^q(X, \mathbb{Z}(p)) \longrightarrow F^p H^q(X, \mathbb{C}) \longrightarrow H^0(\bar{S}, \mathcal{R}^q \bar{\pi}_* \Omega^\bullet_{\bar{X}/\bar{S}}(\log Y)).$$

Then $\delta(\nu_M)$ is well defined modulo

$$\mathcal{I} = \epsilon_{F/S}(A_1) \wedge \delta(H_{\mathcal{D}}^{2p_{2,3}-1}(X, \mathbb{Z}(p_{2,3}))) + \delta(H_{\mathcal{D}}^{2p_{1,2}-1}(X, \mathbb{Z}(p_{1,2}))) \wedge \epsilon_{F/S}(A_3),$$

where

$$\delta(H_{\mathcal{D}}^q(X, \mathbb{Z}(p)))$$

is the set of all the possible infinitesimal invariants of normal functions associated to Deligne cohomology classes living in $\delta(H^q_{\mathcal{D}}(X, \mathbb{Z}(p)))$.

Proof. $M_3(A_1, A_2, A_3)$ is defined to be the set

$$M_3(A_1, A_2, A_3) = M + A_1 \cup H^{2p_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3})) + H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3.$$

Thus another representative of $M_3(A_1, A_2, A_3)$, let us denote it by M', differs from M by a term of the form

$$\mathcal{B} = A_1 \cup \Psi_{2,3} + \Psi_{1,2} \cup A_3.$$

with

$$\psi_{i,j} \in H^{2p_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j})).$$

By the same argument as above $\Psi_{i,j}$ and therefore $\mathcal{B}$ are homologically trivial on the fibres, hence their infinitesimal invariants are defined. What remains to prove is

$$\delta(\mathcal{B}) = \epsilon_{F/S}(A_1) \cup \delta(\Psi_{2,3}) + \delta(\Psi_{1,2}) \cup \epsilon_{F/S}(A_3)$$

In order to prove this we use first the fact that the product rules of 3.9 translate directly to the relative Deligne complex $\mathbb{Z}_{\mathcal{D}/S}(p)$. This implies that we can write $\lambda(\Psi_{i,j}) = \eta(\psi_{i,j})$, where again λ denotes the map

$$\lambda : H^0(\bar{S}, \mathcal{R}^q \bar{\pi}_* \mathbb{Z}_{\mathcal{D}}(p)) \longrightarrow H^0(\bar{S}, \mathcal{R}^q \bar{\pi}_* \mathbb{Z}_{\mathcal{D}/S}(p)).$$

We have

$$\lambda(A_1 \cup \Psi_{2,3}) = \lambda(A_1) \cup \lambda(\Psi_{2,3}) = \eta(\epsilon_F(\lambda(A_1)) \wedge \psi_{2,3}) = \eta(\epsilon_{F/S}(A_1) \wedge \psi_{2,3}).$$

Note that $\psi_{2,3}$ is a normal function associated to $\Psi_{2,3}$.

The term $\Psi_{1,2} \cup A_3$ is treated in the same way.

So far we have constructed a normal function associated to $\mathcal{B}$:

$$\nu_{\mathcal{B}} = \epsilon_{F/S}(A_1) \wedge \psi_{2,3} + \psi_{1,2} \wedge \epsilon_{F/S}(A_3).$$

On the side of the Gauss Manin connection, we have a Leibniz rule, i.e. let α be a section of $\mathcal{R}^p \bar{\pi}_* \Omega^{\bullet}_{\bar{X}/\bar{S}}(\log Y)$ and β one of $\mathcal{R}^q \bar{\pi}_* \Omega^{\bullet}_{\bar{X}/\bar{S}}(\log Y)$, then we have

$$\nabla(\alpha \wedge \beta) = \nabla(\alpha) \wedge \beta + (-1)^p \alpha \wedge \nabla(\beta).$$

Applying this to $\mathcal{B}$ we get

$$\begin{aligned}\nabla(\mathcal{B}) &= \nabla(\epsilon_{F/S}(A_1)) \wedge \psi_{2,3} + \epsilon_{F/S}(A_1) \wedge \nabla(\psi_{2,3}) \\ &+ \nabla(\psi_{1,2}) \wedge \epsilon_{F/S}(A_3) - \psi_{1,2} \wedge \nabla(\epsilon_{F/S}(A_3))\end{aligned}$$

$\epsilon_{F/S}(A_i)$ is an integral class, therefore

$$\nabla(\epsilon_{F/S}(A_i)) = 0.$$

This yields to

$$\nabla(\mathcal{B}) = \epsilon_{F/S}(A_1) \wedge \nabla(\psi_{2,3}) + \nabla(\psi_{1,2}) \wedge \epsilon_{F/S}(A_3)$$

and therefore

$$\delta(\nu_{\mathcal{B}}) = \epsilon_{F/S}(A_1) \wedge \delta(\nu_{\Psi_{2,3}}) + \delta(\nu_{\Psi_{1,2}}) \wedge \epsilon_{F/S}(A_3).$$

Since the $\Psi_{i,j}$ were chosen arbitrarily, we are done. $\square$

Now we are ready to define the infinitesimal invariant of $M_3(A_1, A_2, A_3)$ and when we call $M_3(A_1, A_2, A_3)$ to be constant.

Definition 5.20. Let $A_i \in H^{2p_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$ be Deligne cohomology classes, such that $M_3(A_1, A_2, A_3)$ is defined. Let M be a representative of $M_3(A_1, A_2, A_3)$ and denote by $\mathcal{I}$ the set

$$\mathcal{I} = \epsilon_{F/S}(A_1) \wedge \delta(H^{2p_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3}))) + \delta(H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3}))) \wedge \epsilon_{F/S}(A_3).$$

(1) We call the set

$$\delta(M_3(A_1, A_2, A_3)) = \delta(\nu_M) + \mathcal{I}$$

the **infinitesimal invariant of** $M_3(A_1, A_2, A_3)$

(2) $M_3(A_1, A_2, A_3)$ is said to be **constant** if

$$\delta(\nu_M) \in \mathcal{I}$$

in other words

$$\delta M_3(A_1, A_2, A_3) = \mathcal{I}$$

or equivalently

$$\delta(M_3(A_1, A_2, A_3)) = 0$$

in the quotient

$$H^1(\mathcal{K}^\bullet_{2p_{1,2,3}-1, p_{1,2,3}})/\mathcal{I}$$

Remark 5.21. Proposition 5.19 implies that $\delta(M_3(A_1, A_2, A_3))$ does not depend on the choice of the representative M.

Secondly 5.19 gives in the case of rigidity, that the infinitesimal invariants of all the representatives M of $M_3(A_1, A_2, A_3)$ are of the form

$$\delta(\nu_M) = \epsilon_{F/S}(A_1) \wedge \delta(\nu_{\Psi_{2,3}}) + \delta(\nu_{\Psi_{1,2}}) \wedge \epsilon_{F/S}(A_3)$$

for suitable $\Psi_{i,j} \in H^{p_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$.

The third implication is, that we really can find a representative M of $M_3(A_1, A_2, A_3)$ such that

$$\delta(M) = 0,$$

since all the elements of

$$\epsilon_{F/S}(A_1) \wedge \delta(H^{2p_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3}))) + \delta(H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3}))) \wedge \epsilon_{F/S}(A_3)$$

are the infinitesimal invariants of some representative of $M_3(A_1, A_2, A_3)$.

5.4. **An alternative approach to** $\delta(M_3(A_1, A_2, A_3))$**.**

Let us begin with an example.

Assume $X = \Delta^* \times Y$, where Δ^* is the punctured disc and Y is smooth projective. Let $M \in H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p))$, then $\epsilon_F(M)$ maps to zero under the map

$$F^p H^{2p-1}(\bar{X}, \Omega^\bullet_{\bar{X}}(\log Y)) \longrightarrow H^0(\Delta, \mathcal{R}^{2p-1}\bar{\pi}_* F^p \Omega^\bullet_{\bar{X}/\Delta}(\log Y)).$$

Here Δ is the disc and $\bar{\pi}$ the projection from $\bar{X}$ to Δ.

As shown in section 5.2 the infinitesimal invariant $\delta(\nu_{\mathbb{M}})$ can be calculated via lifting $\epsilon_F(M)$ to the class

$$\widetilde{M} \in H^0(\Delta, \Omega^1_\Delta(\log 0) \otimes \mathcal{R}^{2p-2}\bar{\pi}_* F^{p-1}\Omega^\bullet_{\bar{X}/\Delta}(\log Y)).$$

But this object is just isomorphic to

$$H^0(\Delta, \Omega^1_\Delta(\log 0)) \otimes H^{2p-2}(\bar{X}, F^{p-1}\Omega^\bullet_{\bar{X}/\Delta}(\log Y))$$

$$= H^0(\Delta, \Omega^1_\Delta(\log 0)) \otimes H^{2p-2}(Y, F^{p-1}\Omega^\bullet_Y).$$

In other words we lift $\epsilon_F(M)$ to a class in $H^{2p-2}(Y, \mathbb{C})$ tensorized with a differential form coming from Δ with logarithmic poles at zero.

On $\bar{X}$ this class corresponds to a class supported on the boundary, i.e. we have a class living in

$$H^{2p-1}_{|Y|}(\bar{X}, \mathbb{C}).$$

The aim of the section is to construct a class $\widetilde{\mathcal{M}}$ supported on the boundary $Y = \bar{X} - X$ and to compare this class with the infinitesimal invariant $\delta(\nu_M)$. For constructing the class we will use intersection theory, which is possible since we assume that our cohomology classes A_i are the cycle classes of some codimension p_i cycles on X.

<u>Construction of $\widetilde{\mathcal{M}}$</u>

Let $A_i \in \mathcal{Z}^{p_i}(X)$ for $i = 1, 2, 3$ be algebraic cycles of codimension p_i. We will also write A_i for the corresponding cycle class in $H^{2p_i}_{\mathcal{D}}(X, \mathbb{Z}(p_i))$. We denote by $\bar{A}_i$ their compactifications in $\bar{X}$. Furthermore by intersection theory we can assume that all the cycles intersect proper also with the boundary Y. We write $A_{i,j}$ for the intersection of A_i with A_j and $\bar{A}_{i,j}$ for the compactification of $A_{i,j}$ (and **not** for $\bar{A}_i \cap \bar{A}_j$).

Let's assume that $M_3(A_1, A_2, A_3)$ is defined in the Deligne cohomology of X. By assumption there exist cycles $B_{i,j}$ of codimension $p_{i,j}$ and support on Y with

$$\bar{A}_{1,2} \sim_{\mathcal{D}} B_{1,2}$$

$$\bar{A}_{2,3} \sim_{\mathcal{D}} B_{2,3},$$

where $\sim_{\mathcal{D}}$ stands for Deligne equivalence, *i.e.* they have the same cycle class in Deligne Cohomology. This implies certainly that they are also homologically equivalent.

Assume that we have chosen the $B_{i,j}$ in a way that they again intersect properly with A_i. We construct a cycle of codimension $p_{1,2,3}$, namely

$$\widetilde{\mathcal{M}} = \bar{A}_1 \cap B_{2,3} - B_{1,2} \cap \bar{A}_3.$$

$\widetilde{\mathcal{M}}$ has support on Y and on $\bar{X}$ we have

$$\bar{A}_1 \cap B_{2,3} \sim_{hom} \bar{A}_{1,2,3}$$

as well as

$$B_{1,2} \cap \bar{A}_3 \sim_{hom} \bar{A}_{1,2,3}.$$

Let's turn now to cohomology

Lemma 5.22. *The cycle class of $\widetilde{\mathcal{M}}$ in $H^{2p_{1,2,3}}(\bar{X}, \mathbb{C})$ vanishes.*

Proof. Since both summands of $\widetilde{\mathcal{M}}$ are homologically equivalent to $\bar{A}_{1,2,3}$, $\widetilde{\mathcal{M}}$ must be homologically trivial on $\bar{X}$. $\square$

Now we have the exact sequence of cohomology with support

$$\ldots \longrightarrow H^{2p_{1,2,3}-1}(X, \mathbb{C}) \xrightarrow{\rho} H^{2p_{1,2,3}}_{|Y|}(\bar{X}, \mathbb{C}) \longrightarrow H^{2p_{1,2,3}}(\bar{X}, \mathbb{C}) \longrightarrow \ldots.$$

By this we can lift the cycle class of $\widetilde{\mathcal{M}}$ in $H^{2p_{1,2,3}}_{|Y|}(\bar{X}, \mathbb{C})$ to a class

$$\mathcal{M} \in H^{2p_{1,2,3}-1}(X, \mathbb{C}).$$

Proposition 5.23. *The class $\mathcal{M}$ can be chosen to be $\epsilon_F(M)$, where M is a representative of $M_3(A_1, A_2, A_3)$.*

Proof. What we did not mention up to now is the fact, that the cohomology class $\widetilde{\mathcal{M}}$ lives already in the image of

$$H^{2p_{1,2,3}}_{\mathcal{D},|Y|}(\bar{X}, \mathbb{Z}(p_{1,2,3}))$$

since on $\bar{X}$ it is the difference of two Hodge classes supported on Y. We denote this class by

$$\widetilde{M} \in H^{2p_{1,2,3}}_{\mathcal{D},|Y|}(\bar{X}, \mathbb{Z}(p_{1,2,3})).$$

Obviously $\widetilde{M}$ is mapping to zero in $H^{2p_{1,2,3}}_{\mathcal{D}}(\bar{X}, \mathbb{Z}(p_{1,2,3}))$ and we have

$$\epsilon_F(\widetilde{M}) = \widetilde{\mathcal{M}}.$$

Here we denoted by ϵ_F the corresponding map.

We claim that we can lift $\widetilde{M}$ to a class $M \in H^{2p_{1,2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2,3})$ where M is a representative of $M_3(A_1, A_2, A_3)$.

Denote by $\mathcal{S}^\bullet(p)$ the complex

$$\mathcal{S}^\bullet(p) = \mathrm{Cone}(\mathbb{Z}_{\mathcal{D},\bar{X}}(p) \xrightarrow{f} \mathbb{Z}_{\mathcal{D},X}(p)).$$

We get a distinguished triangle

$$\begin{tikzcd} \mathbb{Z}_{\mathcal{D},\bar{X}}(p) \arrow[rr, "f"] & & \mathbb{Z}_{\mathcal{D},X}(p) \arrow[dl, "\rho"] \\ & \mathcal{S}^\bullet(p) \arrow[ul, "g"] & \end{tikzcd}$$

By the 5-lemma $\mathcal{S}^\bullet(p)$ computes the Deligne Cohomology with support on Y

$$H^\bullet_{\mathcal{D},|Y|}(\bar{X},\mathbb{Z}(p)).$$

Let $\cup_0$ be the product of section 3.2 on $\mathbb{Z}_{\mathcal{D},\bar{X}}(p)$ as well as on $\mathbb{Z}_{\mathcal{D},X}(p)$.

The natural pairing

$$\cup : H^q_{\mathcal{D},|Y|}(\bar{X},\mathbb{Z}(p)) \times H^{q'}_{\mathcal{D}}(\bar{X},\mathbb{Z}(p')) \longrightarrow H^{q+q'}_{\mathcal{D},|Y|}(\bar{X},\mathbb{Z}(p+p'))$$

is induced on the level of sections by $\cup_0$ in the following manner.

Remember that we can realize $\mathcal{S}^\bullet(p)$ as the direct sum

$$\mathcal{S}^\bullet(p) = \mathbb{Z}_{\mathcal{D},\bar{X}}(p) \oplus \mathbb{Z}_{\mathcal{D},X}(p)[1]$$

together with the differential

$$d_{\mathcal{S}} = (d_{\mathbb{Z}_{\mathcal{D},\bar{X}}}, d_{\mathbb{Z}_{\mathcal{D},X}} - f).$$

We can define the pairing

$$\cup_0 : \mathcal{S}^\bullet(p) \times \mathbb{Z}_{\mathcal{D},\bar{X}}(p') \longrightarrow \mathcal{S}^\bullet(p+p')$$
$$(a,b) \times c \longrightarrow (a \cup_0 c, b \cup_0 f(c)).$$

Let $\mathcal{C}^\bullet(*)$ denote the Čech resolutions of the complexes.

Choose

$$\bar{\alpha}_i \in \mathcal{C}^{2p_i}(\mathbb{Z}_{\mathcal{D},\bar{X}}(p_i))$$

representing the Deligne classes $\bar{A}_i$.

$$\alpha_i = f(\bar{\alpha}_i) \in \mathcal{C}^{2p_i}(\mathbb{Z}_{\mathcal{D},X}(p_i))$$

represents the cycle class

$$A_i \in H^{2p_i}_{\mathcal{D}}(X,\mathbb{Z}(p_i)).$$

Furthermore let

$$\beta_{i,j} = (\omega_{i,j}, \gamma_{i,j}) \in \mathcal{C}^{2p_{i,j}}(\mathcal{S}^\bullet(p_{i,j}))$$

be a representative of $B_{i,j}$.

By construction we have

$$g(B_{i,j}) = \bar{A}_{i,j},$$

in other words $g(\beta_{i,j}) = \omega_{i,j}$ differs from $\bar{\alpha}_{i,j}$ by some exact cocycle. Therefore we can assume $\omega_{i,j} = \bar{\alpha}_{i,j}$.

On the other hand we know since $M_3(A_1,A_2,A_3)$ is defined in Deligne cohomology that

$$f(\bar{\alpha}_{i,j}) = \alpha_{i,j} = d\phi_{i,j} \text{ for some } \phi_{i,j} \in \mathcal{C}^{2p_{i,j}-1}(\mathbb{Z}_{\mathcal{D},X}(p_{i,j})).$$

Hence the cocycle

$$\psi_{i,j} = (\bar{\alpha}_{i,j}, \phi_{i,j}) \in \mathcal{C}^{2p_{i,j}}(\mathcal{S}^\bullet(p_{i,j}))$$

represents another lifting of $\bar{A}_{i,j}$ to $H^{2p_{i,j}}_{\mathcal{D},|Y|}(\bar{X}, \mathbb{Z}(p_{i,j}))$.

$\psi_{i,j}$ differs from $\beta_{i,j}$ by some closed cocycle of the form $\rho(\varphi_{i,j})$ for some $\varphi \in \mathcal{C}^{2p_{i,j}-1}(\mathbb{Z}_{\mathcal{D},X}(p_{i,j}))$. This is the indeterminacy of Massey products as defined in 2.5. Thus we can choose

$$\phi_{i,j} = \gamma_{i,j}.$$

The class $\widetilde{M}$ is now constructed as follows:

$$\widetilde{\mathcal{M}} = [\bar{\alpha}_1 \cup_0 \psi_{2,3} - \psi_{1,2} \cup_0 \bar{\alpha}_3].$$

The following simple calculation finishes the proof.

$$\begin{aligned}
\widetilde{\mathcal{M}} &= [\bar{\alpha}_1 \cup_0 \psi_{2,3} - \psi_{1,2} \cup_0 \bar{\alpha}_3] \\
&= [\bar{\alpha}_1 \cup_0 (\bar{\alpha}_{2,3}, \phi_{2,3}) - (\bar{\alpha}_{1,2}, \phi_{1,2}) \cup_0 \bar{\alpha}_3] \\
&= [(\bar{\alpha}_1 \cup_0 \bar{\alpha}_{2,3}, f(\bar{\alpha}_1) \cup_0 \phi_{2,3}) - (\bar{\alpha}_{1,2} \cup_0 \bar{\alpha}_3, \phi_{1,2} \cup_0 f(\bar{\alpha}_3))] \\
&= [(\bar{\alpha}_{1,2,3} - \bar{\alpha}_{1,2,3}, \alpha_1 \cup_0 \phi_{2,3} - \phi_{1,2} \cup_0 \alpha_3)] \\
&= [g(\alpha_1 \cup_0 \phi_{2,3} - \phi_{1,2} \cup_0 \alpha_3)].
\end{aligned}$$

$[\alpha_1 \cup_0 \phi_{2,3} - \phi_{1,2} \cup_0 \alpha_3)]$ is by construction a representative M of $M_3(A_1, A_2, A_3)$. $\square$

Definition 5.24. We say that the cycle class of $\widetilde{\mathcal{M}}$ vanishes in $H^{2p_{1,2,3}}_{|Y|}(\bar{X}, \mathbb{C})$ if we can write it as

$$\widetilde{\mathcal{M}} = \epsilon_F(A_1) \wedge \rho(\epsilon_F(\Psi_{2,3})) + \rho(\epsilon_F(\Psi_{1,2})) \wedge \epsilon_F(A_3)$$

for classes

$$\Psi_{i,j} \in H^{2p_{i,j}}_{\mathcal{D}}(X, (p_{i,j})).$$

Lemma 5.25. *The vanishing of $\widetilde{\mathcal{M}}$ is equivalent to*

$$\epsilon_F(M_3(A_1, A_2, A_3)) = \epsilon_F(A_1 \cup H^{2p_{2,3}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{2,3})) - H^{2p_{1,2}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{1,2})) \cup A_3),$$

i.e. the image of each representative of $M_3(A_1, A_2, A_3)$ lies in the image of the indeterminacy of $M_3(A_1, A_2, A_3)$.

Proof. By definition

$$\widetilde{\mathcal{M}} = \epsilon_F(A_1) \wedge \rho(\epsilon_F(\Psi_{2,3})) + \rho(\epsilon_F(\Psi_{1,2})) \wedge \epsilon_F(A_3).$$

for suitable $\psi_{i,j} \in H^{2p_{i,j}-1}_{\mathcal{D}}(X, \mathbb{Z}(p_{i,j}))$. Let M be the lifting of proposition 5.23. The class

$$M' = M - A_1 \cup \Psi_{2,3} - \Psi_{1,2} \cup A_3$$

is another representative of $M_3(A_1, A_2, A_3)$ with the property

$$\rho(\epsilon_F(M')) = 0$$

Thus $\epsilon_F(M')$ lifts to a class in $H^{2p_{1,2,3}-1}(\bar{X}, \mathbb{C})$. But this cannot be, since $\epsilon_F(M')$ lives in the image of Deligne cohomology and is therefore integer valued, thus has only weights bigger or equal one, hence cannot come from $H^{2p_{1,2,3}-1}(\bar{X}, \mathbb{C})$. $\square$

Now we can state the main result of the section

Theorem 5.26. $\widetilde{\mathcal{M}}$ *vanishes if and only if* $M_3(A_1, A_2, A_3)$ *is constant.*

Proof. The proof follows from the construction of $D(M)$. $\square$

As an immediate corollary we get

Corollary 5.27. *If the family*

$$\bar{X} \xrightarrow{\bar{\pi}} \bar{S}$$

is smooth (i.e. $Y = \emptyset$*), then* $M_3(A_1, A_2, A_3)$ *is constant.*

Proof. This result is evident, since we cannot construct any $\widetilde{\mathcal{M}}$. $\square$

5.5. **some informal discussion.**

We try to develop the necessary conditions for constructing a nontrivial example.

First we will assume that the cycle $\bar{A}_2$ has proper intersection with the boundary Y. The next point is, that we choose in the rational equivalence classes of A_1, A_3, cycles which also intersect $\bar{A}_2 \cap Y$ properly. In the notation of the previous section this means

$$\bar{A}_i \cap \bar{A}_j = \bar{A}_{i,j}$$

Let us assume

$$A_{1,2} = A_1 \cup A_2 \sim_{rat} = 0$$
$$A_{2,3} = A_2 \cup A_3 \sim_{rat} = 0$$

on X.

We can write

$$A_{i,j} = \sum_{\ell} \mathrm{Div}(r^{\ell}_{i,j})$$

where the $r^{\ell}_{i,j} \in K(D^{\ell}_{i,j})$ are rational functions on suitable $(p_{i,j}-1)$-codimensional subvarieties $D^{\ell}_{i,j}$.

If we look at the rational functions $r^{\ell}_{i,j}$ as rational functions $\bar{r}^{\ell}_{i,j}$ on $\bar{X}$ (possible since $K(D^{\ell}_{i,j}) = K(\bar{D}^{\ell}_{i,j})$) we can construct on $\bar{X}$ the cycles

$$B_{i,j} = \bar{A}_{i,j} - \sum_{\ell} \mathrm{Div}(\bar{r}^{\ell}_{i,j}).$$

Obviously

$$B_{i,j} \sim_{rat} \bar{A}_{i,j}$$

and moreover since both summands equals $A_{i,j}$ on X, we have

$$B_{i,j} \subset Y.$$

Note that the $B_{i,j}$ are the liftings of the previous section.

We can now construct our class

$$\begin{aligned} \widetilde{\mathcal{M}} &= \bar{A}_1 \cap B_{2,3} - B_{1,2} \cap \bar{A}_3 \\ &= -\bar{A}_1 \cap \sum_{\ell} \mathrm{Div}(\bar{r}^{\ell}_{2,3}) + \sum_{k} \mathrm{Div}(\bar{r}^{k}_{1,2}) \cap \bar{A}_3, \end{aligned}$$

since the other both terms equal $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$. But there is more what vanishes, namely all the expressions coming from the closure of some relevant cycle on X. What is left is the expression

$$-\bar{A}_1 \cap \sum_{\ell} R^{\ell}_{2,3} + \sum_{k} R^{k}_{1,2} \cap \bar{A}_3,$$

where $R^{\ell}_{i,j}$ denotes the irreducible components of $\mathrm{Div}(\bar{r}^{\ell}_{i,j})$ with support on Y. Now by dimensional reasons they must be equal to some irreducible components of $Y \cap D^{\ell}_{i,j}$.

Let us give some example to make the picture a little bit more clear.

Assume that $A_{1,2} = \mathrm{Div}(f)$ for some rational function $f \in K(D)$ where D is an irreducible divisor on X and $\bar{D}$ has proper intersection with Y. Assume furthermore that $\bar{A}_{2,3} \sim_{rat} 0$.

What is left is

$$\mathcal{M} = F \cap \bar{A}_3$$

where F contains the irreducible components of $\mathrm{Div}(\bar{f})$ which are contained in $Y = \bar{\pi}^{-1}(0)$. Since we assumed $\bar{D}$ and Y to have proper intersection and $\bar{D}$ irreducible, we know that

$$F = \bar{D} \cap Y.$$

All in all we get

$$\mathcal{M} = \bar{D} \cap Y \cap \bar{A}_3.$$

Let us consider the case of a constant family $Z \times S$, S a curve X smooth projective and $\bar{S} - S = \{0\}$, given by the local coordinate z. Now on D we can write

$$f = \frac{g}{h}$$

where g, h are homogenous polynomials of the same degree on $\bar{D}$. Since F equals $Y \cap D$ we know that locally either z divides g or z divides h.

Let D be locally given on Y by the set (U_α, f_α) for some open covering U_α and regular functions $f_\alpha \in \mathcal{O}(U_\alpha)$. Assume additionally, that $\bar{A}_3$ itself is an effective irreducible divisor given on Y by the set (V_α, g_α). Let $V_\alpha = U_\alpha$, then we get that locally on U_α our class $\mathcal{M}$ is given by the zero-set of the regular functions f_α, g_α.

6. Appendix: Hypercohomology

In this appendix we explain the concept of Hypercohomology and quasiisomorphisms of complexes of sheaves and present the two methods of calculating Hypercohomology used in this thesis. We follow the presentation given in [EV2].

Throughout this appendix let X be a variety over a commutative ring k, $\mathcal{O}$ a sheaf of commutative rings on X (*i.e.* $\mathcal{O} = k$, $\mathcal{O} = \mathbb{Z}$, $\mathcal{O} = \mathcal{O}_X$) and $\mathcal{F}^\bullet$ a complex of $\mathcal{O}$-modules with differential $d_{\mathcal{F}}$.

6.1. General Definition of Hypercohomology.

We begin with several definitions of the objects we are concerned with.

Definition 6.1. We call the sheaf $\mathcal{H}^i(\mathcal{F}^\bullet)$ which is the sheaf associated to the presheaf given by

$$\mathcal{H}^i(\mathcal{F}^\bullet) : U \mapsto \frac{\ker(\Gamma(U,\mathcal{F}^i) \to \Gamma(U,\mathcal{F}^{i+1}))}{\operatorname{im}(\Gamma(U,\mathcal{F}^{i-1}) \to \Gamma(U,\mathcal{F}^i))} =: \mathcal{H}^i(U,\mathcal{F}^\bullet)$$

for open sets U in X, the **i'th cohomology sheaf** of the complex $\mathcal{F}^\bullet$.

We can now define Quasi-isomorphisms of complexes of sheaves of $\mathcal{O}$-modules.

Definition 6.2. Let $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ be a morphism of complexes of $\mathcal{O}$-modules. We call σ a **Quasi Isomorphism** if and only if for all i the induced maps on the i'th cohomology sheaf

$$\mathcal{H}^i(\sigma) : \mathcal{H}^i(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^i(\mathcal{G}^\bullet)$$

are isomorphisms of sheaves, *i.e.* for all open sets $U \subset X$ the maps

$$\mathcal{H}^i(\sigma) : \mathcal{H}^i(U,\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^i(U,\mathcal{G}^\bullet)$$

are isomorphisms.

Example 6.3. Let X be a smooth quasiprojective variety over $\mathbb{C}$ with its deRham complex of holomorphic forms $\Omega^\bullet_X$. By the Poincaré lemma, which states that locally on open disks Δ closed forms are exact and $\ker(\Gamma(\Delta,\mathcal{O}_X) \xrightarrow{d} \Gamma(\Delta,\Omega^1_X)) = \mathbb{C}$, we have

$$\mathcal{H}^i(\Omega^\bullet_X) = 0 \text{ for } i \geq 1$$

and

$$\mathcal{H}^0(\Omega^\bullet_X) = \mathbb{C}.$$

Therefore the map

$$\mathbb{C} \longrightarrow \Omega^\bullet_X = (\mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^n_X)$$

is a quasi-isomorphism.

Definition 6.4. Let X be an algebraic manifold and

$$\phi : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet$$

be a map of complexes of sheaves on X. We define the cone of ϕ to be

$$\operatorname{Cone}(\mathcal{A}^\bullet \xrightarrow{\phi} \mathcal{B}^\bullet) = \mathcal{C}^\bullet_\phi := \mathcal{A}^\bullet[1] \oplus \mathcal{B}^\bullet$$

(note $\mathcal{A}[1]^n = \mathcal{A}^{n+1}$) with differential

$$\begin{array}{rccc} d_{\mathcal{C}}: & \mathcal{C}^n = \mathcal{A}^{n+1} \oplus \mathcal{B}^n & \longrightarrow & \mathcal{C}^{n+1} = \mathcal{A}^{n+2} \oplus \mathcal{B}^{n+1} \\ & (a, b) & \mapsto & (-d_{\mathcal{A}}(a), \phi(a) + d_{\mathcal{B}}(b)). \end{array}$$

This construction is compatible with Quasi Isomorphims in the following sense:

Proposition 6.5. *Let*

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[r, "\phi"] \arrow[d, "\alpha"'] & \mathcal{B}^\bullet \arrow[d, "\beta"'] \\ \mathcal{F}^\bullet \arrow[r, "\psi"] & \mathcal{G}^\bullet \end{tikzcd}$$

be a commutative diagram of complexes of sheaves on an algebraic manifold X. Assume that α and β are Quasi Isomorphisms. Then the map

$$\eta : \mathcal{C}_\phi \longrightarrow \mathcal{C}_\psi$$

given by

$$\eta((a, b)) = (\alpha(a), \beta(b))$$

is a Quasi Isomorphism.

Proof. Let U be an open subset of X. For all i we get commutative diagrams

$$\begin{tikzcd} \mathcal{H}^i(U, \mathcal{A}^\bullet) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{H}^i(U, \mathcal{B}^\bullet) \arrow[r] \arrow[d, "\beta"'] & \mathcal{H}^i(U, \mathcal{C}_\phi) \arrow[r] \arrow[d, "\eta"'] & \mathcal{H}^{i+1}(U, \mathcal{A}^\bullet) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{H}^{i+1}(U, \mathcal{B}^\bullet) \arrow[d, "\beta"'] \\ \mathcal{H}^i(U, \mathcal{F}^\bullet) \arrow[r] & \mathcal{H}^i(U, \mathcal{G}^\bullet) \arrow[r] & \mathcal{H}^i(U, \mathcal{C}_\psi) \arrow[r] & \mathcal{H}^{i+1}(U, \mathcal{A}^\bullet) \arrow[r] & \mathcal{H}^{i+1}(U, \mathcal{G}^\bullet). \end{tikzcd}$$

By assumption the maps α and β are isomorphisms. Therefore the proposition follows by the five lemma (see [Iv], lemma I.1.7). □

For the sake of simplicity we assume from now on the complexes $\mathcal{F}^\bullet$ are bounded below, *i.e.* $\mathcal{F}^i = 0$ for $i << 0$.

Definition 6.6.

(1) We call an $\mathcal{O}$- module $\mathcal{I}$ injective, if the contravariant functor $\mathcal{H}om_{\mathcal{O}}(\bullet, \mathcal{I})$ from the category of sheaves of $\mathcal{O}$-modules to itself is right exact, *i.e.* for each injective map of $\mathcal{O}$-modules

$$\mathcal{A} \longrightarrow \mathcal{B}$$

the induced map

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{B}, \mathcal{I}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \mathcal{I})$$

is surjective.

(2) Let
$$\iota : \mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$$
be a map of complexes of $\mathcal{O}$-modules bounded below. We call $\mathcal{I}^\bullet$ an **injective resolution** of the complex $\mathcal{F}^\bullet$ if
  (a) ι is a quasi-isomorphism
  (b) $\mathcal{I}^i$ is an injective $\mathcal{O}$-module for all i.

Remark 6.7. In the case where k is a field and $\mathcal{O} = k$, or all the $\mathcal{F}^i$ are coherent sheaves, we now that injective resolutions always exist. (See [Ha], chapter III)

We are now able to define the Hypercohomology of a complex of $\mathcal{O}$-modules.

Definition and Theorem 6.8. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules bounded below and
$$\iota : \mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$$
an injective resolution. We define the i-th Hypercohomology group $\mathbb{H}^i(X, \mathcal{F}^\bullet)$ of the complex $\mathcal{F}^\bullet$ to be
$$\mathbb{H}^i(X, \mathcal{F}^\bullet) := \frac{\ker(\Gamma(X, \mathcal{I}^i) \longrightarrow \Gamma(X, \mathcal{I}^{i+1}))}{\operatorname{im}(\Gamma(X, \mathcal{I}^{i-1}) \longrightarrow \Gamma(X, \mathcal{I}^i))}$$
This definition is independent of the chosen injective resolution and moreover for quasi-isomorphisms
$$\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$$
the induced maps
$$\mathbb{H}^i(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^i(X, \mathcal{G}^\bullet)$$
are isomorphisms.

Proof. The independency can be found in [Ha]. The second statement follows from the fact, that an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{G}^\bullet$ is also an injective resolution of $\mathcal{F}^\bullet$. □

Definition 6.9.
(1) We call an $\mathcal{O}$-module $\mathcal{A}$ **acyclic**, if $H^i(X, \mathcal{A}) = 0$ for all $i \geq 1$, where $H^i(X, \bullet)$ denotes sheaf-cohomology, (see [Ha]).
(2) We call a map of complexes of $\mathcal{O}$-modules $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{A}^\bullet$ an acyclic resolution of the complex $\mathcal{F}^\bullet$ if
  (a) σ is a quasi-isomorphism.
  (b) $\mathcal{A}^i$ is acyclic for all i.

Proposition 6.10. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules and $\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{A}^\bullet$ an acyclic resolution of it. Then*
$$\mathbb{H}^i(X, \mathcal{F}^\bullet) \cong \frac{\ker(\Gamma(X, \mathcal{A}^i) \longrightarrow \Gamma(X, \mathcal{A}^{i+1}))}{\operatorname{im}(\Gamma(X, \mathcal{A}^{i-1}) \longrightarrow \Gamma(X, \mathcal{A}^i))}.$$

For the proof see [Iv].
Similar to sheaf cohomology we get the next

Proposition 6.11. *Let*

$$0 \longrightarrow \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet \longrightarrow \mathcal{C}^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of bounded below $\mathcal{O}$-modules. Then we obtain the long exact sequence

$$\ldots \to \mathbb{H}^i(X, \mathcal{A}^\bullet) \to \mathbb{H}^i(X, \mathcal{B}^\bullet) \to \mathbb{H}^i(X, \mathcal{C}^\bullet) \to \mathbb{H}^{i+1}(X, \mathcal{A}^\bullet) \to \ldots$$

Now as we have presented the generalities on Hypercohomology, we give the announced methods of calculating it. The first one is

6.2. **Čech Hypercohomology.**
What follows is a generalization of Čech cohomology of sheaves of $\mathcal{O}$-modules.

Definition 6.12.

(1) Let $\mathcal{U} = \{U_\alpha, \alpha \in A \subset \mathbb{N}\}$ be an open covering of X. We define for a bounded below complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$ its associated Čech-complex

$$\check{\mathcal{C}}^i(\mathcal{U}, \mathcal{F}^\bullet) = \bigoplus_{p+q=i} \mathcal{C}^p(\mathcal{U}, \mathcal{F}^q)$$

where

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}^q) = \prod_{\alpha_0 < \alpha_1 < \ldots < \alpha_p} \rho_* \mathcal{F}^q|_{U_{\alpha_0 \cdots \alpha_p}}$$

with the notation

$$U_{\alpha_1 \ldots \alpha_p} = U_{\alpha_0} \cap \ldots \cap \ldots \alpha_p$$

and ρ the corresponding embedding.

(2) We construct the Čech differential $\check{\delta}$ of $\check{\mathcal{C}}^i(\mathcal{U}, \mathcal{F}^\bullet)$ as follows: On the one side we construct a map

$$\delta : \mathcal{C}^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}^q)$$

with

$$\delta(s)_{\alpha_0, \ldots, \alpha_{p+1}} = \sum_{k=0}^{p+1} s_{\alpha_0, \ldots, \hat{\alpha}_k, \ldots, \alpha_{p+1}}|_{U_{\alpha_0, \ldots, \alpha_{p+1}}},$$

where $\hat{\alpha}_k$ means that we leave out the index α_k. Together with the differential $d_{\mathcal{F}}$ of the complex $\mathcal{F}^\bullet$ we construct the desired map

$$\check{\delta} : \mathcal{C}^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}^q) \oplus \mathcal{C}^p(\mathcal{U}, \mathcal{F}^{q+1})$$

$$\check{\delta}(s) = (-1)^{p+q} \delta(s) + d_{\mathcal{F}}(s)$$

for $s \in \mathcal{C}^p(\mathcal{U}, \mathcal{F}^q)$.

Proposition 6.13. *The map $\check{\delta}$ gives $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ the structure of a bounded below complex of $\mathcal{O}$-modules.*

We can now define the Čech cohomology groups associated to the open covering $\mathcal{U}$ by

$$\check{\mathbb{H}}^i(\mathcal{U},\mathcal{F}^\bullet) := \frac{\ker(\Gamma(X,\check{\mathcal{C}}^i(\mathcal{U},\mathcal{F}^\bullet)) \xrightarrow{\check{\delta}} \Gamma(X,\check{\mathcal{C}}^{i+1}(\mathcal{U},\mathcal{F}^\bullet)))}{\operatorname{im}(\Gamma(\check{\mathcal{C}}^{i-1}(\mathcal{U},\mathcal{F}^\bullet)) \xrightarrow{\check{\delta}} \Gamma(\check{\mathcal{C}}^i(\mathcal{U},\mathcal{F}^\bullet)))}.$$

Remark that these groups depend on the chosen open covering $\mathcal{U}$. Taking the direct limit gives us the well defined Čech cohomology groups of the complex $\mathcal{F}^\bullet$.

Definition 6.14. We call

$$\check{\mathbb{H}}^i(X,\mathcal{F}^\bullet) := \lim_{\mathcal{U}} \check{\mathbb{H}}^i(\mathcal{U},\mathcal{F}^\bullet)$$

the **i'th Čech cohomology group** of the complex $\mathcal{F}^\bullet$.

As in the case of sheaf cohomology (see [Ha] ex. III.4.4.) we get the following

Proposition 6.15. *The natural map*

$$\check{\mathbb{H}}^i(X,\mathcal{F}^\bullet) \longrightarrow \mathbb{H}^i(X,\mathcal{F}^\bullet)$$

is an isomorphism.

See [EV2].

The last proposition justifies that we can use Čech Hypercohomology for calculations and may also denote by $\mathbb{H}^i(X,\mathcal{F}^\bullet)$ the Čech Hypercohomology.

6.3. **Godement resolution.**

Since in chapter 4 and 5 we are working with cohomology with support on a closed subvariety, we give now the concept of the flasque resolution of sheaves given by Godement in [Go]. For proofs we refer the reader to this work and alternatively the book of Kultze [Ku].

Let throughout this section $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We define inductively sheaves $\mathcal{C}^i(\mathcal{F})$ by

$$\Gamma(U,\mathcal{C}^0(\mathcal{F})) = \prod_{x\in U} \mathcal{F}_x$$

and

$$\Gamma(U,\mathcal{C}^i(\mathcal{F})) := \Gamma(U,\mathcal{C}^0(\mathcal{C}^{i-1}(\mathcal{F}))) = \prod_{x\in U} \mathcal{C}^{i-1}(\mathcal{F})_x$$

for open subsets U in X.

The sheaves $\mathcal{C}^i(\mathcal{F})$ have one important property which allows us to use them for calculating the cohomology of $\mathcal{F}$.

Definition and Theorem 6.16.

(1) We call a sheaf $\mathcal{F}$ **flasque**, if for all inclusions $U \subset V$ of open subsets of X the restrictionmaps

$$\rho_{VU} : \Gamma(V,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F})$$

are surjective.

(2) The sheaves $\mathcal{C}^i(\mathcal{F})$ are flasque.

(3) We call a bounded below complex (for simplicity let's say $\mathcal{G}^i = 0$ for $i < 0$) $\mathcal{G}^\bullet$ with differential d a flasque resolution of the sheaf $\mathcal{F}$, if all the $\mathcal{G}^i$ are flasque and the complex

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \xrightarrow{d} \mathcal{G}^1 \xrightarrow{d} \dots$$

is exact.

(4) If $\mathcal{G}^\bullet$ is a flasque resolution of the sheaf $\mathcal{F}$ we have

$$H^p(X, \mathcal{F}) = \frac{\ker(\Gamma(X, \mathcal{G}^p) \xrightarrow{d} \Gamma(X, \mathcal{G}^{p+1}))}{\operatorname{im}(\Gamma(X, \mathcal{G}^{p-1}) \xrightarrow{d} \Gamma(X, \mathcal{G}^p))}$$

Before we go on in how and why the sheaves $\mathcal{C}^i(\mathcal{F})$ compute the cohomology of $\mathcal{F}$, we give some interpretation of the sections of $\mathcal{C}^i(\mathcal{F})$.

Let s be a section of $\mathcal{C}^i(\mathcal{F})$ over some open subset $U \subset X$, *i.e.* s is a map

$$s : U \longrightarrow \Gamma(U, \mathcal{C}^i(\mathcal{F})) = \prod_{x \in U} \mathcal{C}^{i-1}(\mathcal{F})_x$$

with

$$x_0 \longmapsto s(x_0) \in \mathcal{C}^{i_1}(\mathcal{F})_{x_0}$$

which is not necessarily continuous.

Since by definition $\mathcal{C}^{i-1}(\mathcal{F})_{x_0} = \lim_{x_0 \in V}(\Gamma(V, \mathcal{C}^{i-1}(\mathcal{F})))$ we find some open subset $U(x_0)$ such that

$$s(x_0) \in \Gamma(U(x_0), \mathcal{C}^{i-1}(\mathcal{F})).$$

Again we can interpret $s(x_0)$ as a function

$$s(x_0) : U(x_0) \longrightarrow \Gamma(U(x_0), \mathcal{C}^{i-1}(\mathcal{F})) = \prod_{x \in U(x_0)} \mathcal{C}^{i-2}(\mathcal{F})_x$$

$$x_1 \longmapsto s(x_0, x_1) \in \mathcal{C}^{i-2}(sF)_{x_1}.$$

Repeating this procedure gives in the end a function

$$s(x_0, \dots, x_i) : U \times U(x_0) \times \dots \times U(x_i) \longrightarrow \mathcal{F}_{x_i}$$

where $x_i \in U(x_i)$.

Since all the sheaves $\mathcal{C}^j(\mathcal{F})$ are flasque, we can extend this function to U^{i+1}. On the other side each such function gives rise to a section of $\mathcal{C}^i(\mathcal{F})$.

Now we construct a differential $\widetilde{\delta}$ making $\mathcal{C}^\bullet$ into a complex of sheaves.

Let s be a section of $\mathcal{C}^i(\mathcal{F})$ over some open subset $U \subset X$. Now by the interpretation just given we can interpret s as a function

$$s : U^{i+1} \longrightarrow \prod_{x \in U} \mathcal{F}_x$$

$$s(x_0, \dots, x_i) \in \mathcal{F}_{x_i}.$$

We can now define $\widetilde{\delta}$ to be

$$\widetilde{\delta}(s)(x_0, \ldots, x_{i+1}) = \sum_{j=0}^{i} (-1)^j s(x_0, \ldots, \hat{x_j}, \ldots, x_{i+1}) +$$

$$(-1)^{i+1} s(x_0, \ldots, x_i)(x_{i+1}),$$

where again $\hat{x_j}$ means to forget the element x_j and the last summand should be interpreted in the following way: The element $s(x_0, \ldots, x_i) \in \mathcal{F}_{x_i}$ gives rise to a continuous function

$$s : U \longrightarrow \Gamma(U, \mathcal{F})$$

with

$$s(x_0, \ldots, x_i)(y) \in \mathcal{F}_y.$$

Now we regard the last summand to be the value of this function in the point x_{i+1}.

Similar to Čzech cohomology (one verifies that the choice of signs is the same) it can be shown that $\widetilde{\delta}^2 = 0$. This gives us the following

Proposition 6.17. *The sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{C}^0(\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{F}) \longrightarrow \mathcal{C}^2(\mathcal{F}) \longrightarrow \ldots \tag{6.1}$$

where ι is the obvious embedding, is exact. Hence the complex $\mathcal{C}^\bullet(\mathcal{F})$ is a flasque resolution of $\mathcal{F}$ and by this we can compute the cohomology of $\mathcal{F}$ via

$$H^p(X, \mathcal{F}) = \frac{\ker(\Gamma(X, \mathcal{C}^p(\mathcal{F})) \longrightarrow \Gamma(X, \mathcal{C}^{p+1}(\mathcal{F})))}{\operatorname{im}(\Gamma(X, \mathcal{C}^{p-1}(\mathcal{F})) \longrightarrow \Gamma(X, \mathcal{C}^p(\mathcal{F})))}$$

We now want to extend this somehow canonical flasque resolutions of sheaves to a flasque resolution of a complex of sheaves. This will be done in the same way as we did it for the Čech Hypercohomology in the previous section.

But first we define what a flasque resolution of a complex $\mathcal{F}^\bullet$ is and state that this resolution computes the Hypercohomology of the complex $\mathcal{F}^\bullet$.

Definition and Theorem 6.18.

(1) Let

$$\sigma : \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$$

be a map of bounded below complexes of $\mathcal{O}$-modules. We call $\mathcal{G}^\bullet$ a **flasque resolution** of the complex $\mathcal{F}^\bullet$, if

(a) All the $\mathcal{G}^i$ are flasque.

(b) σ is a quasi-isomorphism.

(2) If $\mathcal{G}^\bullet$ is a flasque resolution of the complex $\mathcal{F}^\bullet$, we have

$$\mathbb{H}^p(X, \mathcal{F}^\bullet) = \frac{\ker(\Gamma(X, \mathcal{G}^p) \xrightarrow{d_\mathcal{G}} \Gamma(X, \mathcal{G}^{p+1}))}{\operatorname{im}(\Gamma(X, \mathcal{G}^{p-1}) \xrightarrow{d_\mathcal{G}} \Gamma(X, \mathcal{G}^p))}.$$

Let $\mathcal{F}^\bullet$ be a complex of sheaves. We define the flasque sheaves

$$\mathcal{G}^{p,q}(\mathcal{F}^\bullet) := \mathcal{C}^p(\mathcal{F}^q).$$

The differential

$$\begin{aligned} \delta : \mathcal{G}^{p,q}(\mathcal{F}^\bullet) &\longrightarrow \mathcal{G}^{p+1,q}(\mathcal{F}^\bullet) \oplus \mathcal{G}^{p,q+1}(\mathcal{F}^\bullet) \\ s &\longmapsto (-1)^{p+q}\widetilde{\delta}(s) \oplus d^\bullet_{\mathcal{F}}(s). \end{aligned}$$

Obviously $\delta^2 = 0$. Therefore δ gives $\mathcal{G}^\bullet(\mathcal{F}^\bullet)$,where

$$\mathcal{G}^\ell(\mathcal{F}^\bullet) = \bigoplus_{p+q=\ell} \mathcal{G}^{p,q}(\mathcal{F}^\bullet),$$

the structure of a complex of $\mathcal{O}$-modules.

Definition and Theorem 6.19. We call $\mathcal{G}^\ell(\mathcal{F}^\bullet)$ the **canonical flasque resolution** of the complex $\mathcal{F}^\bullet$, hence computes the cohomology of $\mathcal{F}^\bullet$.

To be precise we should proof, that the embedding

$$\mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet(\mathcal{F}^\bullet)$$

is indeed a quasi isomorphism. This follows by the fact, that the complexes $\mathcal{G}^{\bullet,q}(\mathcal{F}^\bullet)$ are flasque resolutions of the sheaves $\mathcal{F}^q$.

References

[AO] D. Arapura, K. Oh, *On the Abel-Jacobi Map for Non-Compact Varieties*, Osaka J. Math. **34** (1997), pp 769-781

[Be1] A. Beilinson, *Higher Regulators and Values of L-Functions*, J. Soviet Math. **30** (1985), pp. 2036-2070.

[Be2] A. Beilinson, *Height Pairing between algebraic cycles*, Contemporary Mathematics **67** (1987) pp. 1-24.

[Bl1] S. Bloch, *Semi-Regularity and deRham Cohomology*, Inventiones Mathematicae **17** (1972), pp. 51-66.

[Bl2] S. Bloch, *Algebraic Cycles and Higher K-Theory*, Advances in Math. **61** (1986), pp. 267-304.

[Bu1] J. I. Burgos, *Anillos de Chow Aritmeticos*, PhD Thesis, Departamento de Algebra y Geometria, Universidad de Barcelona, 1994.

[Bu2] J. I. Burgos, *A C^∞ Logarithmic Dolbeault Complex*, Compositio Math. **92** (1994) No. 1, pp. 61-86.

[Bu3] J. I. Burgos, *Arithmetic Chow Rings and Deligne-Beilinson Cohomology*, J. Algebraic Geometry **6** (1997), pp. 335-377.

[BST] J. Buhler, Ch. Schoen, J. Top, *Cycles, L-Functions and Triple Products of Elliptic Curves*, J. reine angew. Math. **492** (1997), pp.93-133

[dR] G. de Rham, *Variétés différentiables*, Publications de l'Institut de Mathématique de l'Université de Nancago **III**, Hermann, Paris 1973.

[Cle] C.H. Clemens, *Degeneration of Kähler Manifolds*, Duke Math. Journal **44** No. 2 (1977), pp.215-290.

[CDK] E. Cattani, P. Deligne, A. Kaplan, *On the Locus of Hodge Classes*, J. Amer. Math. Soc. **8**, No. 2 (1995), pp. 483-506.

[Del1] P. Deligne, *Théorie de Hodge II*, Pub. Math. IHES **40** (1972), pp. 5-57.

[Del2] P. Deligne, *Théorie de Hodge III*, Pub. Math. IHES **44** (1974), pp. 5-78.

[Del3] P. Deligne, *Cohomologie a supports propres*, in *SGA IV, 3*, Lecture Notes in Mathematics **305**, Springer Verlag 1973.

[Del4] P. Deligne, *Equations Différenrielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics **163**, Springer Verlag 1970.

[Den] C. Deninger, Higher Order Operations in Deligne Cohomology, Inventiones Mathematicae**120** (1995), pp. 289-315.

[DGMS] P.Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real Homotopy Theory of Kähler Manifolds*, Inventiones Mathematicae **29** (1975) No. 3, pp. 245-274.

[EL] H. Esnault, M. Levine, *Surjectivity of Cycle Maps*,in *Journées de Géométrie Algébrique d'Orsay*, Astérisque **218** (1993), pp. 203-226.

[EV1] H. Esnault, E. Viehweg, *Deligne-Beilinson cohomology*, in *Beilinson's Conjectures on special values of L-functions*, Perspectives in Math. **4**, pp. 43-92, Academic Press, New York 1988.

[EV2] H. Esnault, E. Viehweg, *Lectures on Vanishing Theorems*, Birkhäuser Verlag 1992.

[Fu] W. Fulton, *Intersection Theory*, Ergebnisse der Mthematik und Ihrer Grenzgebiete, 3. Folge **2**, Springer Verlag 1984.

[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, John Wiley & Sons 1978.

[Gi] H. Gillet, *Deligne Homology and Abel-Jacobi Maps*, Bull. Amer. Math. Soc. **10** No. 2 (1984), pp. 285-288.

[GM] P. A. Griffiths, J. W. Morgan, *Rational Homotopy Theory and Differential Forms*, Progress in Mathematics **16**, Birkhäuser Verlag 1981.

[GMS] M. Green, S. Müller-Stach, *Algebraic Cycles on a General Complete Intersection of High Multi-Degree of a Smooth Projective Variety*, Compositio Math. **100** (1996), pp. 305-309.

[GMV] M. Green, J. Murre, C. Voisin, *Algebraic Cycles and Hodge Theory*, Lecture Notes in Mathematics **1594**, Springer Verlag 1993.

[Go] R. Godement, *Théorie des Faisceaux*, Publications de l'Institut de Mathématique de l'Université de Strasbourg **XIII**, Hermann, Paris 1964.

[Gri1] J. Carlson, M. Green, P. Griffiths, J. Harris, *Infinitesimal Variations of Hodge Structure (I)*, Compositio Math. **50** (1983), pp. 109-205.

[Gri2] P. Griffiths, J. Harris, *Infinitesimal Variations of Hodge Structure (II): An Infinitesimal Invariant of Hodge Classes*, Compositio Math. **50** (1983), pp. 207-265.

[Gri3] P. Griffiths, *Infinitesimal Variations of Hodge Structure (III): Determinantal Varieties and the Infinitesimal Invariant of Normal Functions*, Compositio Math. **50** (1983), pp. 267-324.

[Gr] M.L. Green, *Griffiths' Infinitesimal Invariant And the Abel-Jacobi Map*, J. Differential Geometry **29** (1989), pp 545-555.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer Verlag 1977.

[Iv] B. Iversen, *Cohomology of Sheaves*, Universitext, Springer Verlag 1986.

[Ja] U. Jannsen, *Deligne Homology, Hodge-D-Conjecture and Motives*, in *Beilinson's Conjectures on Special Values of L-Functions*, Perspectives in Math. **4**, pp. 305-373, Academic Press, New York 1988.

[Ka1] N. M. Katz, *On the Differential Equations Satisfied by Period Matrices*, Pub. Math. IHES **35** (1968), pp. 223-258.

[Ka2] N. M. Katz, *Algebraic Solutions of Differential Equations (p-curvature and the Hodge Filtration)*, Inventiones Math. **18** (1972), pp.1-118.

[Ki1] J.R. King, *The Currents defined by Analytic Varieties*, Acta Mathematica 127, No. 3-4 (1985), pp.185-220.

[Ki2] J. R. King, *Global Residues and Intersections on a Complex Manifold*, Trans. Amer. Math. Soc. **192** (1974), pp.163-199.

[Ki3] J. R. King, *Log Complexes of Currents and Functorial Properties of the Abel-Jacobi Map*, Duke Math. Journal **50**, No. 1 (1983), pp. 1-53.

[Kr] D. Kraines, *Massey Higher Products*, Trans. Amer. Math. Soc. **124** (1966), pp. 431-449.

[KS1] K. Kodaira, D.C. Spencer, *On Deformations of Complex Analytic Structures, I*, Annals of Mathematics **67**, No. 2 (1958), pp. 328-401.

[KS2] K. Kodaira, D.C. Spencer, *On Deformations of Complex Analytic Structures, II*, Annals of Mathematics **67**, No. 2 (1958), pp. 403-466.

[Ku] R. Kultze, *Garbentheorie*, B. G. Teubner Stuttgart, 1970.

[Lev] M. Levine, *Bloch's Higher Chow Groups Revisited*, in *K-Theory (Strasbourg 1992)*, Astérisque **226** (1994), pp. 235-320.

[Lew] J. D. Lewis, *A Survey of the Hodge Conjecture*, Les Publications **CRM**, Montréal 1991.

[Ma1] W.S. Massey, *Some Higher Order Cohomology Operations*, in *Symposium internacional de topologia algebraica*, pp. 145-154, Mexico City 1958.

[Ma2] W.S. Massey, *Higher Order Linking Numbers*, in *Conference on Algebraic Topology (University of Ullinois at Chicago Circle)*, pp. 174-205, Chicago 1968.

[May] J. P. May, Matric Massey Products, Journal of Algebra **12** (1969), pp. 533-568.

[MS] S. Müller-Stach, *Constructing Indecomposable Motivic Cohomology Classes on Algebraic Surfaces*, J. Algebraic Geometry **6** (1997), pp. 513-543.

[M-S2] S.Müller-Stach, A Remark on Height Pairings, in *Algebraic Cycles and Hodge Theory*, Lecture Notes in Mathematics **1594**, Springer Verlag 1993.

[Na] J. Nagel, The Abel-Jacobi Map for Complete Intersections, Indag. Mathem., N.S. **8**(1) (1997), pp. 95-113.

[Qu] D. Quillen, *Rational Homotopy Theory*, Annals of Math. (2) **90** (1969), pp. 205-295.

[Ro] J. Rosenberg, *Algebraic K-Theory and its Applications*, Graduate Texts in Mathematics **147**, Springer Verlag 1996.

[Sch1] Ch. Schoen, *Algebraic Cycles on Certain Desingularized Nodal Hypersurfaces*, Math. Annalen **270** (1985), pp.17-27.

[Sch2] Ch. Schoen, *Complex Multiplication Cycles on Elliptic Modular Threefolds*, Duke Math. Journal **53** (1986) No. 3, pp.771-794.

[Sch3] Ch. Schoen, *On Fiber Products of Rational Elliptic Surfaces with Section*, Math. Zeitschrift **197** (1988) No. 2, 177-199.

[Sch4] Ch. Schoen, *Cyclic Covers of* $\mathbb{P}^\nu$ *Branched along* $\nu+2$ *Hyperplanes and the Generalized Hodge Conjecture for Certain Abelian Varieties*, in *Arrithmetic of Complex Manifolds (Erlangen 1988)*, Lecture Notes in Mathematics **1399**, pp. 137-154, Springer Verlag 1989.

[SZ] R. Stöcker, H. Zieschang, *Algebraische Topologie*, Mathmatische Leitfäeden, B.G. Teubner Stuttgart 1994.

[SZu1] J. Steenbrink, S. Zucker, *Variations of mixed Hodge Structure I*, Inventiones Mathematicae **80** (1985), pp. 489-542.

[Vo1] C.Voisin, *Sur l'Application d'Abel-Jacobi des Variétés de Calabi-Yau de Dimension Trois*, Ann. scient. Éc. Norm. Sup. **4/27** (1994), pp. 209-226.

[Vo2] C. Voisin, *Variations of Hodge Structure and Algebraic Cycles*, in *Proceedings of the ICM, Zürich, Switzerland 1994*, pp.706- 715, Birkhäuser Verlag Basel 1995.

[Vo3] C. Voisin, *Some Results on Green's higher Abel-Jacobi map*, Annals of Mathematics **149** (1999), pp. 451-473.

[We] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge studies in advanced mathematics **38**, Camebridge University Press 1994.

[Wen] T. Wenger, *Massey Products in Deligne-Cohomology*, Preprintreihe des SFB 478 *Geometrische Strukturen in der Mathemaik*, Heft 107, Münster 2000.

[Wh] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics **61**, Springer Verlag 1978.

[ZZ] F. El Zein and S. Zucker, *Extendability of Normal Functions Associated to Algebraic Cycles* in *Topics in transcendental algebraic geometry*, Princeton University Press 1981.

[Zu] S. Zucker, *Hodge Theory with degenerating coefficients:* L_2 *cohomology in the Poincaré metric*, Ann. of Mathematics (2) **109** (1979) No. 3, pp 415-476.