String-net methods for CFT correlators

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Based on work with Jürgen Fuchs and Yang Yang

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Overview

Chapter 1

Introduction: two-dimensional conformal field theory

Two-dimensional conformal field theories

- Infinite-dimensional algebra of local symmetries: vertex algebra $\mathcal{V}$
- Representation category
- Chiral conformal field theory: conformal blocks:
  - Defined on complex curves
  - Solutions of chiral Ward identities
  - Multivalued functions of position of field insertions and complex structure: monodromies
  - Factorization

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  - Multivalued functions of position of field insertions and complex structure: monodromies
  - Factorization
- Full local conformal field theory: consistent systems of correlators
  - Defined for conformal surfaces with boundaries and line defects
  - Specify field content
  - Find correlators: specific conformal blocks that are single valued and obey sewing constraints

Towards modular functors

Chiral CFT:

Vertex algebra $\mathcal{V}$ $\longrightarrow$ $\mathcal{C} := \mathrm{Rep}(\mathcal{V})$

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Rational CFT:
$\mathcal{C}$ is a modular fusion category:

- k-linear semisimple abelian category
- monoidal category
- rigid duals
- braiding, non-degenerate

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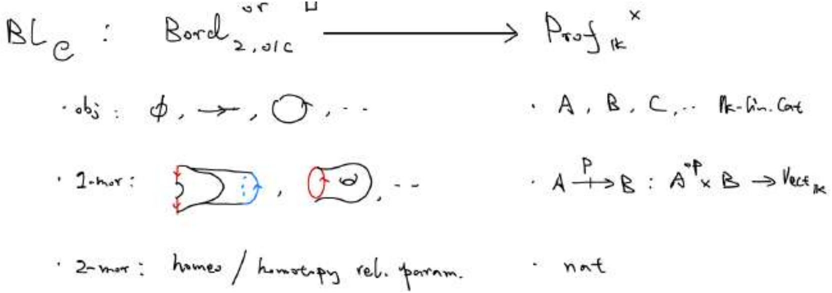
Encode monodromies of conformal blocks:
Open closed modular functor (examples will be constructed in this talk)



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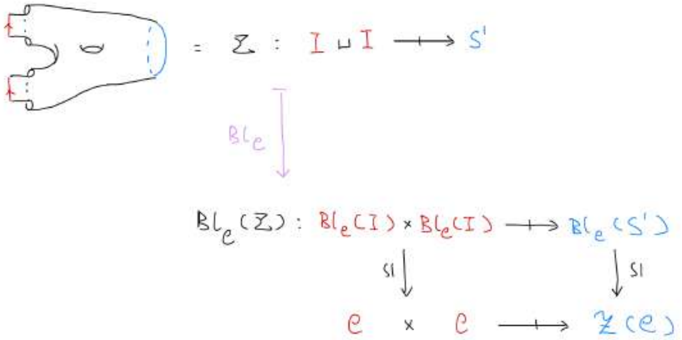
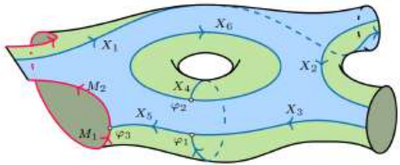
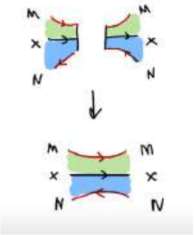


Full CFT: worldsheet $\mathcal{S}$ with topological defects

Example: world sheet with one sewing interval and two sewing circles:



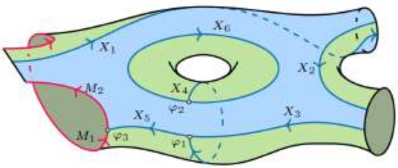


Fix modular tensor category $\mathcal{C}$. Decoration data:

- Special symmetric Frobenius algebras in $\mathcal{C}$ (here two phases, indicated in blue and green)
- Line defects: bimodules
- Point defects: bimodule morphisms

Full CFT: field objects and correlators

A world sheet $\mathcal{S}$ with one sewing interval and two sewing circles:



Expected field objects

- Interval:

$$\mathrm{Bl}_{\mathcal{C}}(I) = \mathcal{C}$$

  Boundary field: $\mathbb{F}(b_1) := \underline{\mathrm{Hom}}(M_2, X_1 \otimes_A M_1)$

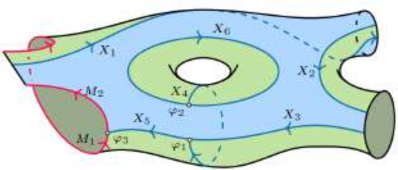
- Circle:

$$\mathrm{Bl}_{\mathcal{C}}(S^1) = \mathcal{Z}(\mathcal{C})$$

  Bulk field for lower circle: $\mathbb{F}(b_3) := \underline{\mathrm{Nat}}(G^{X_2}, G^{X_3})$
  with $G^X := X \otimes_A -$.

Full CFT: field objects and correlators

A world sheet $\mathcal{S}$ with one sewing interval and two sewing circles:



Correlator for $\mathcal{S}$:

$$\mathrm{Cor}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) := \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}(b_1), \mathbb{F}(b_2) \times \mathbb{F}(b_3))$$

with $\Sigma_{\mathcal{S}}$ the underlying surface

Conditions on these elements:

- Compatible with sewing
- Invariant under action of mapping class group

$$\mathrm{MCG}(\mathcal{S}) \subset \mathrm{MCG}(\Sigma_{\mathcal{S}})$$

on $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$

Chapter 2

Graphical calculus for pivotal bicategories

Input for construction of modular functor:
"String net modular functor is a globalized version of graphical calculus"

Bicategory

Definition

A bicategory $\mathcal{B}$ is a category weakly enriched in *Cat*:

- $a, b, c \in \mathrm{Ob}(\mathcal{B})$
- for all pairs a, b a hom-category $\mathcal{B}(a, b) = \mathrm{Hom}_{\mathcal{B}}(a, b)$.
  - Objects: 1-morphisms, morphisms: 2-morphisms of $\mathcal{B}$
  - k-linear in this talk.
  - horizontal composition, units

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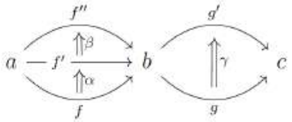
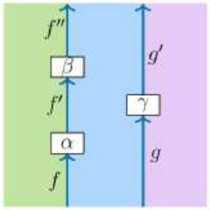
String diagram

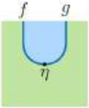
Pasting diagram



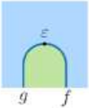


Adjoints in bicategories, pivotal bicategories

Two 1-morphisms $f \in \mathcal{B}(a, b)$ and $g \in \mathcal{B}(b, a)$, together with counit and unit



and



$\eta : 1_a \to g \cdot f$ and $\epsilon : g \cdot f \to 1_b$

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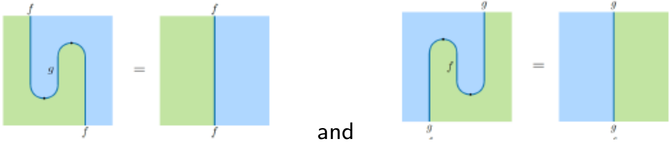
satisfying zig-zag relations:



are said to be adjoints:

$$f \dashv g \;\Leftrightarrow\; f = {}^{\vee}g \;\Leftrightarrow\; g = f^{\vee}$$

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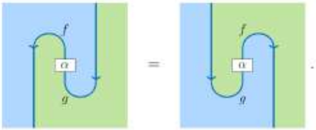
Definition

1. A pivotal structure on a bicategory $\mathcal{B}$ with fixed left and right duals is an identity component pseudonatural transformation $\mathrm{id}_{\mathcal{B}} \to (-)^{\vee\vee}$.
2. A strictly pivotal bicategory is a pivotal bicategory for which the double dual is the identity, $\mathrm{id}_{\mathcal{B}} = (-)^{\vee\vee}$.

Consequence:



Examples, graphical calculus

Examples

A pivotal tensor category $\mathcal{C}$ leads to two pivotal bicategories:

1. Delooping $B\mathcal{C}$: single object $*$ with $\mathrm{End}_{B\mathcal{C}}(*) = \mathcal{C}$.
2. $\mathcal{F}r_{\mathcal{C}}$: objects are simple special symmetric Frobenius algebras $A, B, C, \ldots$.
   Morphism categories $\mathcal{F}r_{\mathcal{C}}(A, B) = A$-mod-$B$ (bimodules)

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Remark

Graphical calculus

bicategory  2-framed, progressive

pivotal bicategory  drop progressive, drop 2-framed

Examples, graphical calculus

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Remark

Graphical calculus

bicategory  2-framed, progressive

pivotal bicategory  oriented

Formalization of graphical calculus for a pivotal bicategory

Formulate graphical calculus as a symmetric monoidal functor

$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_{k}^{\otimes}$$

Formalization of graphical calculus for a pivotal bicategory

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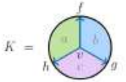
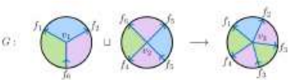
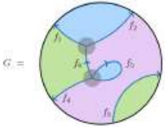
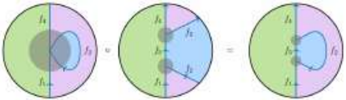
$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_k^{\otimes}$$

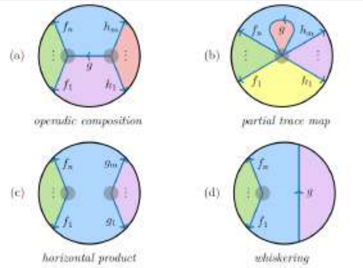
Monoidal category $\mathrm{Corollas}_{\mathcal{B}}^{\sqcup}$ of corollas:

- Objects: $\emptyset$, 
- Morphisms  given by 
- Composition 

Generation of corollas

Proposition

Any non-trivial morphism in $\mathrm{Corollas}_{\mathcal{B}}^{\sqcup}$ *can be decomposed into a finite disjoint union of partial compositions of morphisms of the following types:*



Call $\mathrm{Corollas}_{\mathcal{B}}^{\mathsf{conn}}$ the subcategory of $\mathrm{Corollas}_{\mathcal{B}}$ with the same objects, but morphisms only generated by (a) and (b).

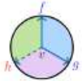
Set up the symmetric monoidal functor expressing graphical calculus:

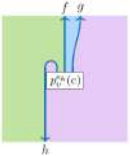
$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_{k}^{\otimes}$$

Graphical calculus on objects

To the corolla $K =$ 

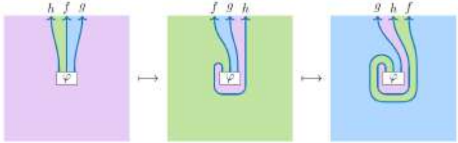
we associate a vector space H_v as follows:

Any polarization 

yields a space of morphisms = 

Pivotal structure allows to relate morphisms for different polarizations by unique isomorphism

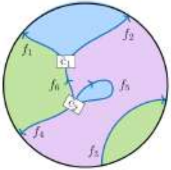
Example:



Vector space H_v assigned to corolla
$=$ the limit over this diagram of vector spaces.

Graphical calculus on morphisms

To find the linear map for the morphism of corollas



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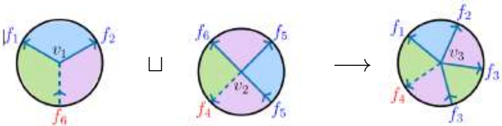
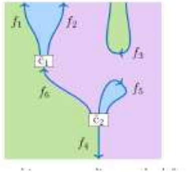
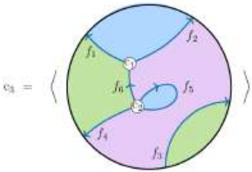


pick polarizations and consider the morphism given by the evaluation of progressive diagram



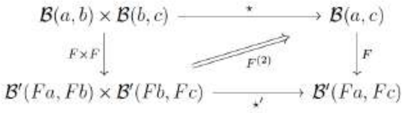
We write

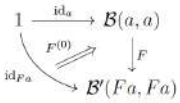


Functoriality of the graphical calculus

Lax functor $F : \mathcal{B} \to \mathcal{B}'$ between bicategories $\mathcal{B}, \mathcal{B}'$:

- On objects $F : a \mapsto Fa$
- On Hom-categories $\mathcal{B}(a, b) \mapsto \mathcal{B}'(Fa, Fb)$ with natural transformations





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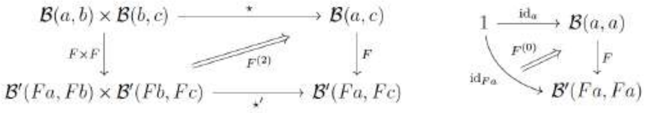

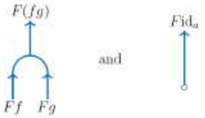

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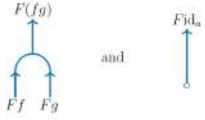

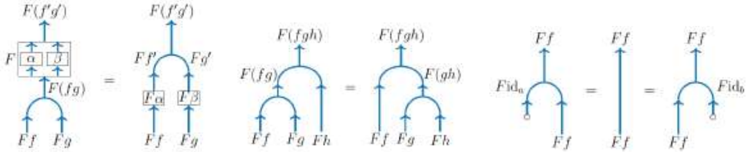

Naturality Lax Associativity Lax unity

Functoriality of the graphical calculus

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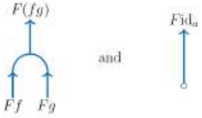
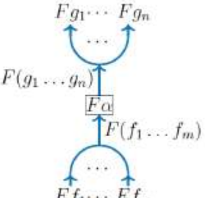


Similarly, oplax structure. For a natural transformation

$$\alpha : f_1 \star \ldots \star f_m \Rightarrow g_1 \star \ldots \star g_n$$

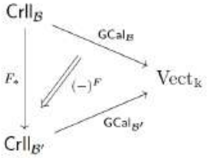
define F-conjugate:

$$\alpha^F : \; Ff_1\star\cdots\star Ff_m \xrightarrow{F^{(m)}_{f_1,\ldots,f_m}} F(f_1\star\cdots\star f_m) \xrightarrow{F\alpha} F(g_1\star\cdots\star g_n) \xrightarrow{F_{(n)\,g_1,\ldots,g_n}} Fg_1\star\cdots\star Fg_n$$



When does F-conjugation preserve a graphical calculus?

in the sense that F-conjugation induces a monoidal natural transformation



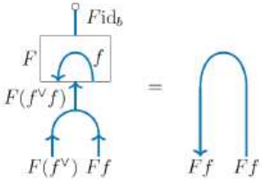
which is given on a corolla K by

$$(-)^F_K : \quad \mathsf{GCal}^{\mathrm{conn}}_{\mathcal{B}}(K) \xrightarrow{r^k_v} \widehat{h^{\mathcal{B}}_v}(k) \xrightarrow{(-)^F} \widehat{h^{\mathcal{B}'}_{v'}}(k) \xrightarrow{(r^k_{v'})^{-1}} \mathsf{GCal}^{\mathrm{conn}}_{\mathcal{B}}(F_*K)$$

When does F-conjugation preserve a graphical calculus?

Optimal case:

- F is rigid, i.e. $F(f^\vee) = F(f)^\vee$ and F preserves the counits of duals:



- And F is a pseudofunctor: $F^{(2)} = F_{(2)}^{-1}$ and $F^{(0)} = F_{(0)}^{-1}$.

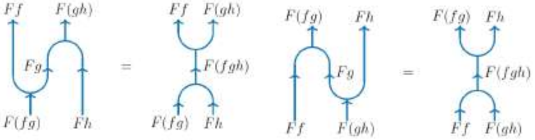
Theorem

A rigid pseudofunctor preserves the graphical calculus.

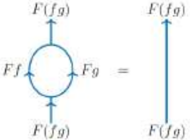
When does F-conjugation preserve a graphical calculus?

Subptimal case:

- F is rigid
- F is Frobenius

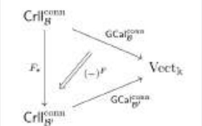


- F is separable



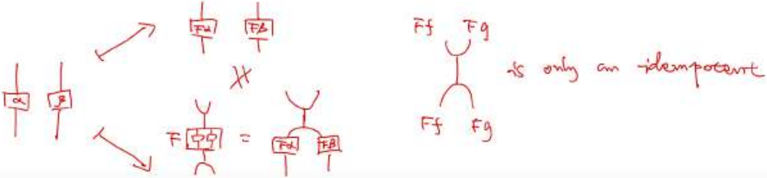
Theorem

A rigid Frobenius functor preserves the connected graphical calculus:



operadic composition and partial trace preserved, but not whiskering and horizontal product.

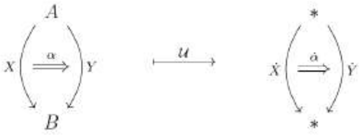
When does F-conjugation preserve a graphical calculus?



Example of a rigid separable Frobenius functor

$\mathcal{C}$ a pivotal tensor category: using separable special symmetric Frobenius algebras:

$$\mathcal{U} : \ \mathcal{F}r_{\mathcal{C}} \to B\mathcal{C}$$



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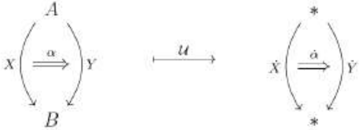
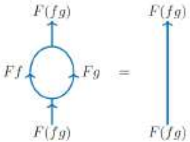
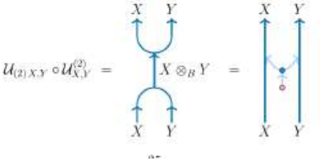


Lax monoidal and opmonoidal constraint

$$\mathcal{U}^{(2)}_{X,Y}: \quad X \otimes Y \longrightarrow X \otimes_B Y \qquad \text{and} \qquad \mathcal{U}_{(2)\,X,Y}: \quad X \otimes_B Y \longrightarrow X \otimes Y$$

defined by splitting of the idempotent





Chapter 3

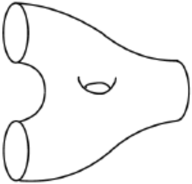
String-net models for pivotal bicategories

Will be applied to two different pivotal bicategories:

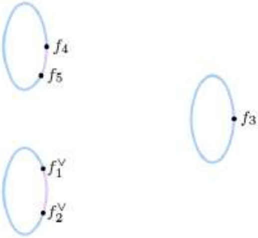
- $\mathcal{B} = B\mathcal{C}$, provides modular functor for conformal blocks
- $\mathcal{B} = \mathcal{F}r_{\mathcal{C}}$ describes world sheets with defects

String-net models for pivotal bicategories

Σ oriented surface



boundary value b, e.g.



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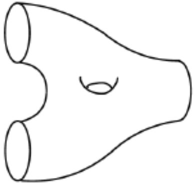
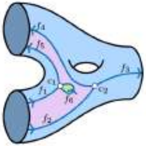


$$\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma,\mathrm{b}) := \Bbbk\mathrm{G}(\Sigma,\mathrm{b})/\mathrm{N}(\Sigma,\mathrm{b}) \,,$$

with $\mathbb{K}\mathrm{G}(\Sigma, b)$ the vector space freely generated by labelled surfaces with boundary condition b, e.g.



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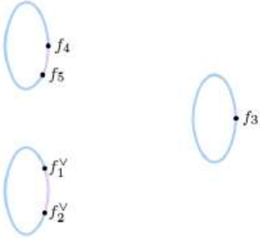


and $N(\Sigma, b)$ the subspace generated by local relations given by graphical calculus:

String-net spaces as colimits

Set up evaluation functor

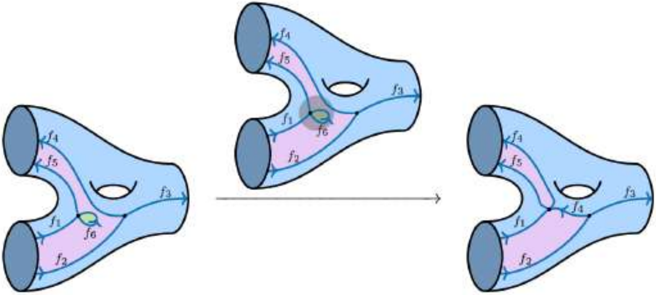
$$\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}} : \quad \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma,\mathsf{b}) \longrightarrow \mathrm{Vect}_{\Bbbk}$$

String-net spaces as colimits

Set up evaluation functor

$$\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathrm{b}} : \quad \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma,\mathrm{b}) \longrightarrow \mathrm{Vect}_{\Bbbk}$$

- Objects of $\mathcal{G}\mathrm{raphs}(\Sigma, b)$: partially colored surfaces:
  - patches → objects of $\mathcal{B}$
  - edges → 1-morphisms of $\mathcal{B}$
  - vertices are not colored
- Morphisms of $\mathcal{G}\mathrm{raphs}(\Sigma, b)$ are freely generated by e.g.



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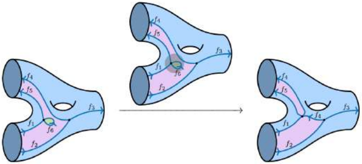


$$\mapsto \quad \bigotimes_{v\in V(\Gamma_1)} H_v^{\mathcal{B}} \longrightarrow \bigotimes_{v\in V(\Gamma_2)} H_v^{\mathcal{B}}$$

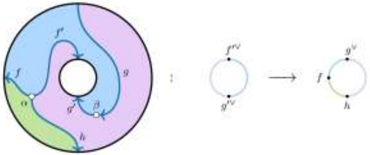
Theorem

1. *The string net space is a colimit:* $\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, b) = \mathrm{colim}\mathcal{E}_{\mathcal{B}}^{\Sigma,b}$
2. *The mapping class group* $\mathrm{MCG}(\Sigma)$ *acts on the string net space.*
3. *Sewing holds: we have a modular functor.*

Remark:
Generalization to modular functors with values in $\mathrm{Ch}_{\mathbb{K}}$ via homotopy limits.

Cylinder categories for pointed bicategories

For any closed oriented 1-manifold ℓ, define a category $\mathrm{Cyl}^{\circ}(\mathcal{B}, \ell)$.



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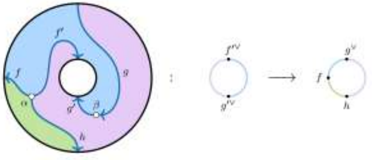

The bicategories $B\mathcal{C}$ and $\mathcal{F}r_{\mathcal{C}}$ are pointed:
Distinguished object $*_{\mathcal{B}} \in \mathcal{B}$: for $\mathcal{F}r_{\mathcal{C}}$ this is the Frobenius algebra $1 \in \mathcal{C}$.

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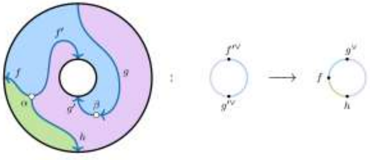

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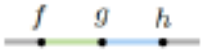
$\mathsf{b} =$ 

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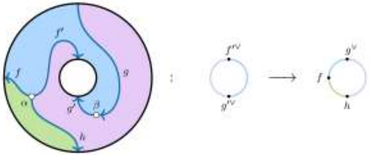

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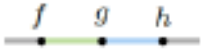
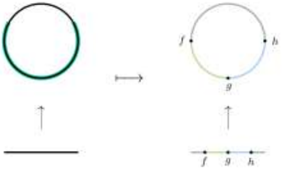
b = 



Remark

1. Functoriality under embedding of 1-manifolds: symmetric monoidal functor

$$\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -) : \quad \mathrm{Emb}_1^{\mathrm{or}} \longrightarrow \mathrm{Cat}_{\Bbbk} .$$

2. Induces on $\mathrm{Cyl}^{\circ}(\mathcal{B}, a, I)$ the monoidal structure of $\mathcal{B}(a, a)$.

Cylinder categories for pivotal fusion categories

$\mathcal{C}$ a pivotal fusion category. Karoubify:

$$\mathrm{Cyl}(\mathcal{C}, I) \cong \mathcal{C} \quad \text{and} \quad \mathrm{Cyl}(\mathcal{C}, S^1) \cong \mathcal{Z}(\mathcal{C})$$

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Geometric embedding induces left adjoint L of $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$:

$$\begin{array}{ccc} \mathrm{Cyl}(\mathcal{C}, I) & \xrightarrow{I \hookrightarrow S^1} & \mathrm{Cyl}(\mathcal{C}, S^1) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{C} & \xrightarrow{L} & \mathcal{Z}(\mathcal{C}) \end{array}$$

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String net space for surface Σ with karoubified boundary data:
for idempotent $B : b \to b$ in $\mathrm{Cyl}(\mathcal{B}, *_\mathcal{B}, \Sigma)$, consider subspace

$$\mathrm{SN}_\mathcal{B}(\Sigma, B) \;\subset\; \mathrm{SN}^0(\Sigma, b)$$

invariant under glueing with B

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invariant under glueing with B
This can be extended to a 3-2-1 TFT equivalent to the Turaev-Viro construction.

Modular functor

One can prove factorization:

Theorem 3.23. *Let* $\Sigma\colon \alpha \sqcup \beta \nrightarrow \beta \sqcup \gamma$ *be a bordism, with* $\alpha, \beta, \gamma \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,o/c}$. *Then the family*

$$\left\{ s^{\Sigma}_{-,\mathsf{b}_0,\sim} : \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \mathsf{b}_0, \mathsf{b}_0, \sim) \Longrightarrow \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim) \right\}_{\mathsf{b}_0 \in \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \tag{3.45}$$

of natural transformations whose members are given by the sewing of string nets, is dinatural and exhibits the functor $\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)$ *as the coend*

$$\int^{\mathsf{b} \in \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \mathsf{b}, \mathsf{b}, \sim) : \quad \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \gamma)\,. \tag{3.46}$$

Modular functor

and obtains a modular functor with values in profunctors:

Theorem 3.27. *Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed strictly pivotal bicategory. Then the assignments*

$$\alpha \mapsto \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \qquad \text{and} \qquad \Sigma \mapsto \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \sim) \tag{3.67}$$

extend to an open-closed modular functor, *i.e. a symmetric monoidal pseudofunctor*

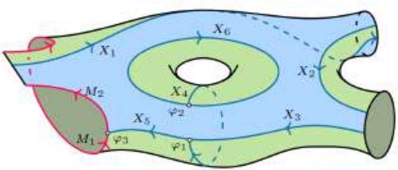
$$\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}} : \quad \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathcal{P}\mathrm{rof}_{\Bbbk} \tag{3.68}$$

from the symmetric monoidal bicategory of open-closed bordisms to the symmetric monoidal bicategory of $\Bbbk$-linear profunctors. Similarly, the Karoubified cylinder categories and string-net spaces give rise to another open-closed modular functor

$$\mathcal{S}\mathrm{N}_{\mathcal{B}} : \quad \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathcal{P}\mathrm{rof}_{\Bbbk} \,. \tag{3.69}$$

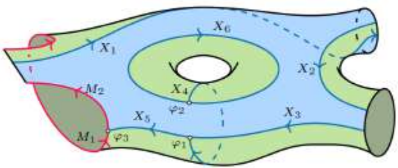
Chapter 4

Applications to RCFTs with defects: universal correlators



world sheet $\mathcal{S}$ with defects, decoration data in $\mathcal{F}r_{\mathcal{C}}$

Correlators, finally

$\mathcal{S} =$ 

worldsheet

$\downarrow$

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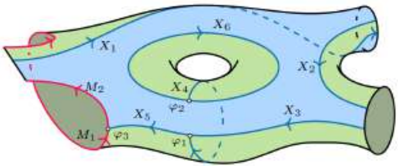
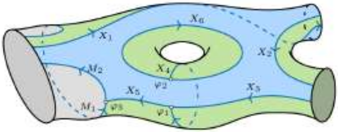


worldsheet

$\downarrow$

$\tilde{\mathcal{S}} =$



complemented worldsheet

$\downarrow$

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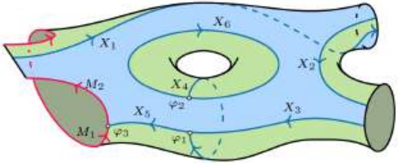

worldsheet

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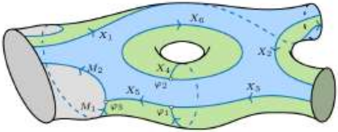
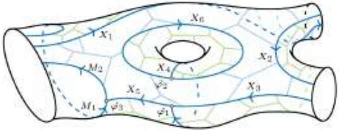


complemented worldsheet

$\downarrow$

$\mathrm{Cor}_{\mathcal{S}} :=$



$\in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}b)$

Vertices $\mathcal{U}$-conjugated, Frobenius triangulation

Field content

General scheme:

$$\mathrm{Cyl}^{\circ}(\mathcal{F}r_{\mathcal{C}},\cdot) \xrightarrow{\mathbb{F}} \mathrm{Cyl}(\mathcal{C},\cdot) \xrightarrow{\cong} \mathcal{C} \text{ or } \mathcal{Z}(\mathcal{C})$$

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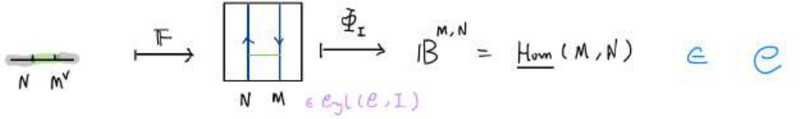
(Generalized) boundary fields:



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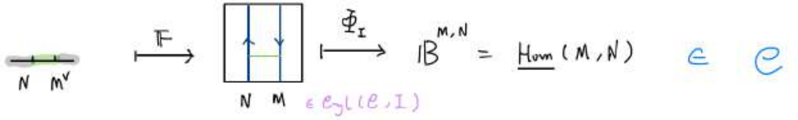
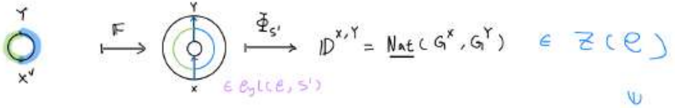
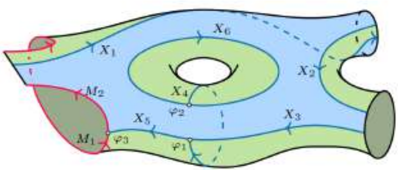




For $A = B$ and $X = Y = A$, get $\mathbb{D}^{A,A} = \underline{\mathrm{Nat}}(\mathrm{id}_{\text{mod}-A}, \mathrm{id}_{\text{mod}-A}) = Z(A)$

Quantum world sheet

$\mathcal{S} =$ 

worldsheet

$\downarrow$

Quantum world sheet

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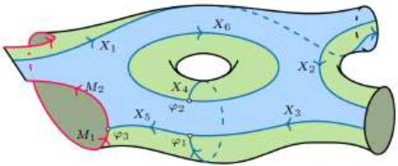
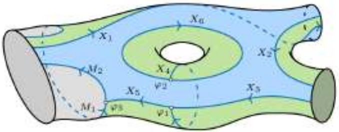


worldsheet

$\downarrow$

$[\tilde{\mathcal{S}}] =$



$\in \mathrm{SN}^0_{\mathcal{F}r_\mathcal{C}}(\Sigma_\mathcal{S}, \mathbb{F}, b_\mathcal{S})$

quantum world sheet

$\downarrow$

Quantum world sheet

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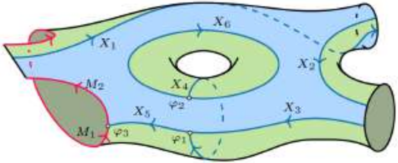

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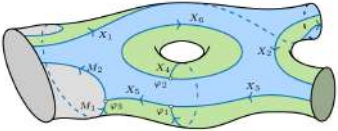

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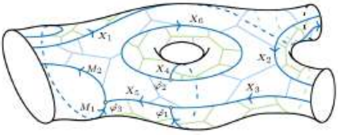
quantum world sheet

$\downarrow$

$\mathrm{Cor}_{\mathcal{S}} :=$



$\in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F} b_{\mathcal{S}})$

Universal correlators

Correlators factor through vector space of quantum world sheets:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta_{\mathcal{S}}} & \mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma) \\ {\scriptstyle \mathrm{Cor}_{\mathcal{S}}}\big\downarrow & \swarrow {\scriptstyle \mathrm{UCor}_{\Sigma}} & \\ \mathrm{SN}_{\mathcal{C}}(\Sigma) & & \end{array}$$

Correlators can be obtained as pullback of universal correlator:

$$(\delta_{\mathcal{S}})^* \mathrm{UCor}_{\Sigma} = \mathrm{Cor}_{\mathcal{S}}$$

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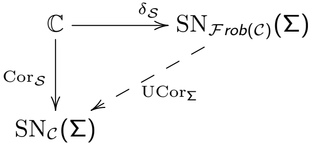

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1. Uses that $\mathcal{U} : \mathcal{F}r_{\mathcal{C}} \to B\mathcal{C}$ is a rigid separable Frobenius functor
2. Frobenius networks in $\mathrm{Cor}_{\mathcal{S}}$ compensate for the failure of preservation of horizontal products and whiskering under $\mathcal{U}$-conjugation.

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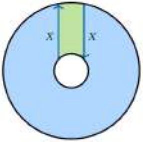
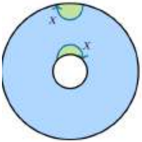
The conformal field theory cannot detect the complete world sheet geometry, only the image in the space of quantum world sheets. The latter is determined by defect data.

Mapping class groups of quantum world sheets

X is an invertible topological defect. Then the two worldsheets

$\mathbb{S}_1 :=$ 

and



$=: \mathbb{S}_2$.

have the same correlators and cannot be distinguished by the CFT.

Is the Dehn twist in the mapping class group or not?

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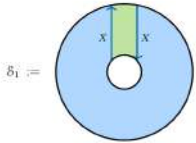
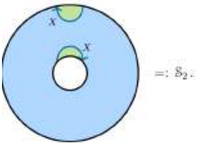


and



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Is the Dehn twist in the mapping class group or not?

$\mathrm{Map}(\Sigma)$ acts on $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ and the appropriate mapping class group for the worldsheet $\mathcal{S}$ with quantum world sheet $[\tilde{\mathcal{S}}] \in \mathrm{SN}_{\mathcal{F}r_{\mathcal{C}}}(\Sigma)$ is the stabilizer:

$$\widehat{\mathrm{Map}}(\mathcal{S}) := \mathrm{Stab}_{\mathrm{Map}(\Sigma)}([\tilde{\mathcal{S}}])$$

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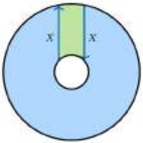
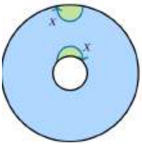
$\mathcal{S}_1 :=$ 

and



$=: \mathcal{S}_2 .$

have the same correlators and cannot be distinguished by the CFT.

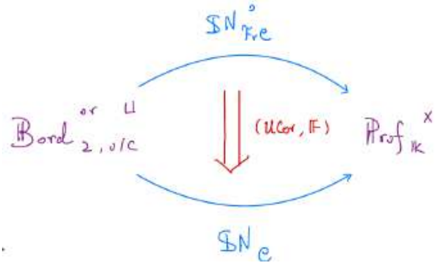
Is the Dehn twist in the mapping class group or not?

$\mathrm{Map}(\Sigma)$ acts on $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ and the appropriate mapping class group for the worldsheet $\mathcal{S}$ with quantum world sheet $[\tilde{\mathcal{S}}] \in \mathrm{SN}_{\mathcal{F}r_{\mathcal{C}}}(\Sigma)$ is the stabilizer:

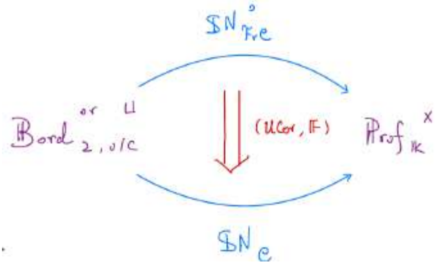
$$\widehat{\mathrm{Map}}(\mathcal{S}) := \mathrm{Stab}_{\mathrm{Map}(\Sigma)}([\tilde{\mathcal{S}}])$$

$\widehat{\mathrm{Map}}(\mathcal{S}_1)$ contains Dehn twists while $\mathrm{Map}(\mathcal{S})$ does not.
$\mathrm{Cor}_{\mathcal{S}_1} = \mathrm{Cor}_{\mathcal{S}_2}$ is invariant under $\widehat{\mathrm{Map}}(\mathcal{S}_1)$!

Schematic overview



Schematic overview



Remark

Upgrade to

- Symmetric monoidal double categories
- Symmetric monoidal double functors
- Monoidal vertical transformations

Summary and outlook

Summary

- String net constructions are a natural conceptual home for the construction of correlators
- Bicategorical string net construction capture symmetries of CFTs and the observable aspect of the world sheet.

Outlook

- Beyond semisimplicity
- Beyond rigidity (percolation)
- Construction is set up to go to homotopical versions (3dSYM/VOA correspondence)
- Higher dimensions?