Some results in equivariant homotopy theory

Graeme Segal

Introduction

In this paper I shall prove the equivariant analogues of some well-known results in homotopy theory. The proofs are almost the same as the non-equivariant ones, but some care is needed in selecting arguments which can be generalized.

Let G be a finite group.

If V is a finite dimensional real G-module let S^V be the sphere formed by compactifying V. We take ∞ as the base-point in S^V. The infinite symmetric product $SP^\infty(S^V)$ is a G-space which should be an "equivariant Eilenberg-Maclane space". That can be formulated in the following way, where $\Omega^V X$ denotes the G-space of base-point-preserving maps from S^V to a pointed G-space X.

Theorem (A) (a). There is an equivariant homotopy equivalence

$$\mathbb{Z} \to \Omega^V SP^\infty(S^V)$$

providing the fixed submodule V^G is non-zero.

(b). There is an equivariant homotopy equivalence

$$SP^\infty(W) \to \Omega^V SP^\infty(S^{V \oplus W})$$

providing $W^G \neq 0$.

The second theorem concerns the configuration space C(V) of V. This is the G-space formed by the unordered finite subsets of V. There is a well-known map $C(V) \to \Omega^V S^V$ (of [7]). One can define an embedding $a_\xi : C(V) \to C(V)$ which adds to a configuration a given configuration ξ "near infinity". Let us take ξ to consist of one representative of each orbit that occurs in $V - \{0\}$, and define $C_\infty(V)$ to be the limit of the sequence

$$C(V) \xrightarrow{a_\xi} C(V) \xrightarrow{a_\xi} C(V) \xrightarrow{a_\xi} \cdots \quad .$$

There is a corresponding map $a_\xi : \Omega^V S^V \to \Omega^V S^V$ compatible with $C(V) \to \Omega^V S^V$, and a corresponding limit $(\Omega^V S^V)_\infty$. We have

<u>Theorem (B)</u>. There is a G-homology-equivalence

$$C_\infty(V) \to (\Omega^V S^V)_\infty \quad .$$

Here a G-homology-equivalence means a G-map which induces a homology equivalence of the spaces of fixed points of every subgroup of G.

In theorem (B) we do not assume that $V^G \neq 0$, so there is no "addition" defined in $\Omega^V S^V$, and it is not clear that the connected components have the same homotopy type. But if $V^G \neq 0$ then clearly $a_\xi : \Omega^V S^V \to \Omega^V S^V$ is an equivalence, and $(\Omega^V S^V)_\infty$ can be replaced by $\Omega^V S^V$ in the theorem.

If $V^G \neq 0$ the connected compenents of $C_\infty(V)^G$ form the free abelian group A_V on the set of orbit types which occur in V. So theorem (B) implies

<u>Corollary</u> $\quad [S^V; S^V]_G \cong A_V$.

This is well-known (at least when V is large and A_V is the Burnside ring of G). But the present method provides a new and in many ways simpler proof. The two known proofs depend either ([8], [6], [2]) on equivariant transversality or [11] are by induction over orbit types.

But the present argument gives somewhat more. For the fixed-point set $C_\infty(V)^G$ is a product $\prod_H C_\infty(V_H)^G = \prod_H C_\infty(V_H/G)$, where V_H is the part of V where the isotropy group is conjugate

to H, and H runs through the conjugacy classes of subgroups of G. Abelianizing the fundamental group commutes with products, so Theorem B gives one a product decomposition of $(\Omega^V S^V)^G$. That is not altogether surprising, for if one arranges the conjugacy classes of subgroups in non-decreasing order

$$1 = H_0, H_1, \ldots, H_m = G$$

then there is a tower of fibrations

$$(\Omega^V S^V)^G = \mathrm{Map}_G(S^V; S^V) \to \mathrm{Map}_{N_{H_1}}((S^V)^{H_1}; (S^V)^{H_1})$$
$$\to \mathrm{Map}_{N_{H_2}}((S^V)^{H_2}; (S^V)^{H_2}) \to \cdots \to \mathrm{Map}_G((S^V)^G; (S^V)^G) \; .$$

Our result is that the tower is an iterated product:

<u>Corollary 2</u>. $\quad (\Omega^V S^V)^G \simeq \prod_H \mathrm{Map}_G(V_H^+ ; S^V)$.

This is known when $V \to \infty$. In fact then $V_H/G \simeq BW_H$, where $W_H = N_H/H$, and $C_\infty(V_H/G) \simeq \lim_{\vec{n}} E\mathfrak{S}_n \times_{\mathfrak{S}_n} BW_H^{\,n}$, whose Quillenization is $\Omega^\infty S^\infty(BW_H \amalg (\text{point}))$. (cf. [8]).

Apart from theorems (A) and (B) this papers discusses "equivariant delooping" more generally, and shows how an equivariant spectrum can be associated to an "equivariant Γ-space" in the sense of [9]. The essential point is to produce a functor which takes G-cofibrations to G-fibrations. In §4 I have explained rather sketchily how this can be done starting with a Γ-space: in fact the non-equivariant discussion in the form given in [12] applies directly.

In conclusion I should point out that neither Theorem (A) nor Theorem (B) is true if G is not finite. For example, suppose that G is the circle and V is $\mathbb{R}^3$ with G acting by rotation about an axis. Then there is a G-cofibration sequence

$$S^1 \rightarrow S^V \rightarrow S^2 (G \perp \text{point}),$$

and so for any G-space X a fibration sequence

$$\Omega^2 X \rightarrow (\Omega^V X)^G \rightarrow \Omega(X^G).$$

Taking $X = S^V$ this shows that $(\Omega^V S^V)^G \simeq \mathbb{Z} \times \Omega^2 S^3$, contradicting Theorem (B); and taking $X = SP^\infty(S^V)$ one finds that $X^G = SP^\infty(S^1) \simeq S^1$, so that $\Omega^V SP^\infty(S^V) \simeq \mathbb{Z} \times S^1$, contradicting Theorem (A).

more

§1 Preliminary remarks about G-homotopy-theory

If X, Y and Z are G-spaces with base-points a sequence of base-points preserving G-maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called a G-fibration-sequence if there is given a null-homotopy of the composite gf, and the induced map from X to the homotopic fibre of g at 0 is a G-homotopy-equivalence. (All base-points will written as 0.)

The homotopic fibre F(g,0) of g at 0 is the fibre-product $Y \times_Z P_0Z$, where P_0Z is the space of paths in Z beginning at 0. The fixed points $F(g,0)^H$ of any subgroup H of G can be identified with $F(g^H,0)$. Let us recall the theorem of [3]. It asserts that a G-map $P \to Q$ is a G-homotopy-equivalence if and only if the induced maps of fixed point sets $P^H \to Q^H$ are homotopy equivalences for all subgroups H of G, providing that P and Q are G-ANR's. Forming the homotopic fibre does not take one out of the class of G-ANR's, so we can deduce

Proposition (1.1) Providing X, Y and Z are G-ANR's a sequence

$$X \to Y \to Z$$

is a G-fibration sequence if and only if the fixed-point sequence

$$X^H \to Y^H \to Z^H$$

is a fibration sequence for all subgroups H of G.

More generally, a homotopy commutative diagram

$$\begin{array}{ccc} X' & \to & Y' \\ \downarrow & & \downarrow \\ X & \to & Y \end{array} ,$$

where the homotopy making the diagram commute is given, is called G-homotopy-cartesian if the induced map from X' to the homotopic fibre product of X and Y' over Y is a G-homotopy equivalence. The homotopic fibre-product is $X \times_Y PY \times_Y Y'$, where PY is the space of free paths in Y. For the same reason as before we can assert that if the spaces concerned are G-ANR's a diagram is G-homotopy-cartesian if and only if each of its fixed-point diagrams is homotopy cartesian.

Now suppose that one has a simplicial object $A = \{A_k\}_{k\geq 0}$ in the category of G-spaces. It has a realization as a G-space. Here I shall use the join realization, described in [9], but I shall denote it by $|A|$. It has the property that if the spaces A_k of A are G-ANR's then so is $|A|$. It also has the property that $|A|^H = |A^H|$ for all subgroups H of G.

From this point on I shall assume that all the G-spaces given are G-ANR's. The non-equivariant version of the following proposition is proved in [9], and the equivariant one follows at once.

Proposition (1.2) If $A' \to A$ is a map of G-simplicial-spaces such that

$$\begin{array}{ccc} A'_k & \xrightarrow{\theta^*} & A'_m \\ \downarrow & & \downarrow \\ A_k & \xrightarrow{\theta^*} & A_m \end{array}$$

is G-homotopy-cartesian for each simplicial operation $\theta : [m] \to [k]$, then

$$\begin{array}{ccc} A'_0 & \to & |A'| \\ \downarrow & & \downarrow \\ A_0 & \to & |A| \end{array}$$

is G-homotopy-cartesian.

In particular suppose that X is a G-space with a composition law which is sufficiently associative for a classifying space BX to be defined. (I.e. suppose that there is a simplicial G-space $\{X_k\}$ with $X_1 = X$ and $X_k \simeq X_1{}^k$ for all $k \geq 0$.) Then we have

<u>Proposition (1.3)</u> For such an X the natural map $X \to \Omega BX$ from X to its "group-completions" is a G-homotopy-equivalence providing that $\pi_0(X^H)$ is a group for all subgroups H of G.

It seems reasonable to say that a G-map $X \to Y$ is a G-homology-equivalence if the induced map $X^H \to Y^H$ is a homology equivalence for each H. That is partly justified by the following obvious remark.

<u>Proposition (1.4)</u> A G-map $X \to Y$ is a G-homology-equivalence if and only if the induced map

$$[X; P]_G \leftarrow [Y; P]_G$$

is an isomorphism for every group-like G-space P.

Here a <u>group-like</u> G-space P means a G-space with a composition law $P \times P \to P$ which makes $\pi_0(P^H)$ into a group for all subgroups H of G.

The group-completion theorem of [5] implies the following equivariant extension. Suppose that X is a G-space with a composition law and a classifying space BX, and that

(a) $\pi_0(X^H)$ is in the centre of the Pontrjagin ring $H_*(X^H)$ for all subgroups H of G,

(b) $\pi_0(X^G)$ is a finitely generated monoid, and

(c) $\pi_0(X^G) \to \pi_0(X^H)$ is cofinal for all H, in the sense that for any $\xi \in \pi_0(X^H)$ there exists $\eta \in \pi_0(X^H)$ such that $\xi + \eta$ comes from $\pi_0(X^G)$.

Then one can choose x in X^G so that its component is cofinal in $\pi_0(X^G)$, and can form the telescope X_∞ from the sequence

$$X \xrightarrow{\times x} X \xrightarrow{\times x} X \xrightarrow{\times x} \cdots$$

<u>Proposition (1.5)</u> In the preceding situation there is a G-homology-equivalence $X_\infty \to \Omega BX_1$

§2 Infinite symmetric products

In this section I shall prove Theorem A, but I shall consider a slightly more general situation.

Suppose that F is a functor from pointed compact G-spaces to G-spaces with the properties

P1 : if X is equivariantly triangulable then F(X) is a G-ANR, and

P2 : if $Y \to X$ is a G-cofibration and Y is a G-connected then

$$F(Y) \to F(X) \to F(X/Y)$$

is a G-fibration sequence.

Here G-connected means that Y^H is connected for every subgroup H of G

Proposition (2.1) The functor $F = SP^\infty$ satisfies conditions P1 and P2.

I shall postpone the proof of this for the moment.

Let V be a real G-module.

If M is a compact G-stable subset of V let M_ε be the open ε-neighbourhood of M in V. Let U_ε be the open ball in V with centre 0 and radius ε. There is a G-map $M \times U_\varepsilon \to M_\varepsilon$ defined by $(x,y) \to x+y$. By adjunction this defines a G-map $M \to \mathrm{Map}_0(M_\varepsilon^+ ; U_\varepsilon^+)$, when $^+$ denotes one-point compactification, and Map_0 denotes the G-space of base-point-preserving maps. By functoriality, and identifying U_ε^+ with S^V, we get

$$M \to \mathrm{Map}_0(F(M_\varepsilon^+) ; F(S^V)) ,$$

and then by adjunction a base-point-preserving G-map

$$F(M_\varepsilon^+) \to \mathrm{Map}(M ; F(S^V)).$$

Proposition (2.2) If M is the unit sphere in V (and $\varepsilon \leq 1$) then the last map is a G-homotopy-equivalence.

Remark The proposition is actually true for any compact G-subset M providing M is an equivariant deformation retract of M_ε. It is essentially the assertion that M and M_ε^+ are equivariantly S-dual.

Proof: Choose an equivariant triangulation of the sphere M. M is covered by the open stars of the simplexes of the triangulation. This is a collection $\{C_\alpha\}$ of contractible open sets of M, closed under intersection, and permuted by G. One can suppose that each set C_α either coincides with or is disjoint from its translates by elements of G.

Suppose that ε is small compared with the mesh of the triangulation. Let $\pi : M_\varepsilon \to M$ be the radial projection. For any subset X of M let $\hat{X} = \pi^{-1}(X)$, and $\check{X} = X - (M-X)_\varepsilon$. Then we have a map

$$F(\hat{X}^+) \to \mathrm{Map}(\check{X} ; F(S^V)).$$

We shall prove that this is a G-homotopy-equivalence whenever X is an G-stable union $\bigcup_{\alpha\in T} C_\alpha$ of some of the C_α. The proof is by induction on the cardinal of T. If T is a single orbit under G then $\hat{X}$ is a union of copies of V indexed by T, so $F(\hat{X}^+)$ is a corresponding product of copies of $F(V^+)$, i.e. $F(\hat{X}^+) = \mathrm{Map}(T; F(S^V))$. But clearly $\mathrm{Map}(T; F(S^V)) = \mathrm{Map}(\check{X}; F(S^V))$, so the result holds in this case.

Proposition (2.3) If X_1 and X_2 are two G-stable unions of sets C_α, and $X = X_1 \cup X_2$, $X_{12} = X_1 \cap X_2$, then

$$\begin{array}{ccc} F(\hat{X}^+) & \to & F(\hat{X}_2^+) \\ \downarrow & & \downarrow \\ F(\hat{X}_1^+) & \to & F(\hat{X}_{12}^+) \end{array}$$

is G-homotopy-cartesian.

Proof: Observe that $\hat{X}_1^+ = \hat{X}^+/(\hat{X} - \hat{X}_1)^+$, so that there is a G-cofibration sequence

$$(\hat{X} - \hat{X}_1)^+ \to \hat{X}^+ \to \hat{X}_1^+ .$$

Similarly

$$(\hat{X}_2 - \hat{X}_{12})^+ \to \hat{X}_2^+ \to \hat{X}_{12}^+$$

is a G-cofibration sequence. But $\hat{X}_2 - \hat{X}_{12} = \hat{X} - \hat{X}_1$, and the compactifications of these spaces, being suspensions, are G-connected. So the vertical maps in the diagram have the same homotopy fibre by (2.1), and that proves (2.3).

Proposition (2.3) implies Proposition (2.2) by induction, for the corresponding square of mapping-spaces

$$\begin{array}{ccc} \mathrm{Map}(\check{X};\, F(S^V)) & \to & \mathrm{Map}(\check{X}_2;\, F(S^V)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\check{X}_1;\, F(S^V)) & \to & \mathrm{Map}(\check{X}_{12};\, F(S^V)) \end{array}$$

is obviously G-homotopy-cartesian.

We shall now deduce Theorem (A) from (2.2).

Let D_r denote that closed disk of radius r in V and S_r its boundary sphere. Then in (2.2) we have $M = S_1$, and M_ε^+ is $D_\lambda/(D_\mu \cup S_\lambda)$, where $\lambda = 1+\varepsilon$ and $\mu = 1-\varepsilon$. There is a commutative diagram

$$\begin{array}{ccc} F(D_\lambda/S_\lambda) & \to & \mathrm{Map}(D_1;\, F(S^V)) \\ \downarrow & & \downarrow \\ F(D_\lambda/D_\mu \cup S_\lambda) & \to & \mathrm{Map}(S_1;\, F(S^V)) \end{array}$$

in which the bottom map is the equivalence of (2.2), and the top map is trivially an equivalence. The right hand map is a fibration with fibre $\Omega^V F(S^V)$. From the cofibration sequence

$$S^0 \simeq (D_\mu \cup S_\lambda) / S_\lambda \to D_\lambda/S_\lambda \to D_\lambda/(D_\mu \cup S_\lambda)$$

we could conclude that the homotopy fibre of the left-hand map was $F(S^0)$, and hence that $F(S^0) \simeq \Omega^V F(S^V)$, if we knew that F took <u>all</u> cofibration sequences ro fibration sequences. But in any case the cofibre of $D_\lambda/S_\lambda \to D_\lambda/(D_\mu \cup S_\lambda)$ is S^1, and $D_\lambda/S_\lambda \cong S^V$ is G-connected (as we are assuming $V^G \neq 0$), so the left-hand homotopy fibre is $\Omega F(S^1)$. If $F = SP^\infty$ then $F(S^1) \simeq S^1$, and so $\Omega F(S^1) \simeq \mathbb{Z}$. This proves the first part of Theorem A. For the second part one considers the functor F defined by $F(X) = SP^\infty(X \wedge S^W)$. This does take all cofibrations to fibrations if $W^G \neq 0$, for then $X \wedge S^W$ is always G-connected.

<u>Proof of (2.1)</u>

Property P1 presents no difficulty, for if X is triangulable then so is $SP^\infty(X)$, and it is then certainly a G-ANR. As to P2, it is enough to show that

$$SP^\infty(Y)^H \to SP^\infty(X)^H \xrightarrow{\pi} SP^\infty(X/Y)^H$$

is a fibration sequence for each subgroup H of G. As each fibre of the second map is precisely homeomorphic to $SP^\infty(Y)^H$ it suffices to show that the map is a quasi-fibration. This can be done by the original argument of Dold-Thom [1].

We filter the base $B = SP^\infty(X/Y)^H$ by closed subspaces

$$B_0 \subset B_1 \subset B_2 \subset \ldots \subset B,$$

where $B_n = SP^n(X/Y)^H$. We have $\pi^{-1}(B_n - B_{n-1}) \cong (B_n - B_{n-1}) \times SP^\infty(Y)^H$. But B_{n-1} is a deformation retract of a neighbourhood U in B_n, and the retraction is covered by a corresponding deformation retraction of $\pi^{-1}(U)$ into $\pi^{-1}(B_{n-1})$. (In fact both retractions can be induced by a retraction of a neighbourhood of Y in X into Y.)

Using the results of [1] (cf. [4] (3.3)) it suffices to show that $\pi^{-1}(U) \rightarrow \pi^{-1}(B_{n-1})$ maps each fibre by a homotopy equivalence. But on any fibre the retraction can be identified with a translation in the topological monoid $SP^{\infty}(Y)^H$. This is homotopic to the identity, for $SP^{\infty}(Y)^H$ is connected because $Y^{H'}$ is connected for every subgroup H' of H.

A rather different alternative proof of (2.1) will be given in §4.

§3 Configuration spaces

In this section I shall prove theorem (B) of the introduction.

As in [4] we introduce relative configuration spaces: if X is a manifold with boundary and Y is a closed subset of X let C(X,Y) denote the quotient space of C(X), the space of finite subsets S of X, by the equivalence relation which identifies S and S' if $S \cap (X-Y) = S' \cap (X-Y)$. The analogue of (2.1) which we need is

Proposition (3.1) If Y is a compact manifold with boundary conatined in X, and of the same dimension as X, and Z is a closed subset of Y, then

$$C(Y,Z) \rightarrow C(X,Z) \rightarrow C(X,Y)$$

is a G-fibration sequence providing $C(\partial Y, Z \cap \partial Y)$ is G-connected.

The proof of this is the same as that of (2.1). We filter the base $B = C(X,Y)^H$ by $\{B_n\}$, where $B_n = \{S \in C(X,Y)^H : \text{card}(S) \leq n\}$. Over each layer $B_n - B_{n-1}$ the map $C(X,Y)^H \rightarrow C(X,Y)^H$ is a product with fibre $C(Y,Z)^H$. The desired deformation retractions can be induced by an isotopy of the identity map of X which shrinks Y into its own interior. Although C(Y,Z) is not a monoid one can still define a map $C(Y,Z) \rightarrow C(Y,Z)$ corresponding to adding configuration at the boundary of Y - i.e. the space $C(\partial Y, Z \cap \partial Y)$ "acts on" C(Y,Z). The effect of the retractions on the fibres $C(Y,Z)^H$ is given by "adding" elements of $C(\partial Y, Z \cap \partial Y)^H$; so we have a quasifibration providing $C(\partial Y, Z \cap \partial Y)$ is G-connected.

It is not quite true that C(X,Y) depends only on X/Y, so one cannot say that C transforms cofibrations to fibrations. But we do have an excision property: if U is an open set of X contained in Y then $C(X-U, Y-U) = C(X,Y)$. Reviewing the proof of (2.2) we find that it holds when the functor F is replaced by C in the obvious way. The only point needing care is to see that - in the notation of (2.3) - we have fibration sequences

$$C(D_\lambda - \hat{X}_1, D_\lambda - \hat{X}) \to C(D_\lambda, D_\lambda - \hat{X}) \to C(D_\lambda, D_\lambda - \hat{X}_1)$$

and

$$C(D_\lambda - \hat{X}_{12}, D_\lambda - \hat{X}_2) \to C(D_\lambda, D_\lambda - \hat{X}_2) \to C(D_\lambda, D_\lambda - \hat{X}_{12})$$

replacing the sequences

$$F((\hat{X} - \hat{X}_1)^+) \to F(\hat{X}^+) \to F(\hat{X}_1^+)$$

and

$$F((\hat{X}_2 - \hat{X}_{12})^+) \to F(\hat{X}_2^+) \to F(\hat{X}_{12}^+) .$$

But (3.1) applies to these, for, for example,

$$C(\partial(D_\lambda - \hat{X}_1), (D_\lambda - \hat{X}) \cap \partial(D_\lambda - \hat{X}_1)) = C(\partial\hat{X}_1, (\partial\hat{X}) \cap (\partial\hat{X}_1)),$$

which is of the form $C(Y \times I, Y \times \dot{I})$, and is therefore G-connected.

Thus we have

<u>Proposition (3.2)</u> $C(D_\lambda, S_\lambda \cup D_\mu) \xrightarrow{\simeq} \mathrm{Map}(S_1; C(D_\varepsilon, S_\varepsilon))$,

where $\lambda = 1+\varepsilon$ and $\mu = 1-\varepsilon$.

The following result is almost obvious.

<u>Proposition (3.3)</u> $C(D_\varepsilon, S_\varepsilon) \simeq S^V$.

For $S^V \simeq D_\varepsilon/S_\varepsilon$, and there is an inclusion $D_\varepsilon/S_\varepsilon \to C(D_\varepsilon, S_\varepsilon)$. But the multiplicative monoid $\{\lambda \in \mathbb{R}: \lambda \geq 1\}$ acts on $C(D_\varepsilon, S_\varepsilon)$ by radial expansion, and for any ξ in $C(D_\varepsilon, S_\varepsilon)$ one has $\lambda\xi \in D_\varepsilon/S_\varepsilon$ for large λ. If $C_\lambda = \{\xi \in C(D_\varepsilon, S_\varepsilon): \lambda\xi \in D_\varepsilon/S_\varepsilon\}$ then $D_\varepsilon/S_\varepsilon$ is a deformation retract of C_λ. But $\bigcup_{\lambda \geq 1} C_\lambda = C(D_\varepsilon, S_\varepsilon)$, so $D_\varepsilon/S_\varepsilon$ is also a deformation retract of $C(D_\varepsilon, S_\varepsilon)$.

If $\lambda \geq \mu \geq 0$ are real numbers let us write $A_{\mu\lambda}$ for the half-open annulus $D_\lambda - D_\mu$ in V. The configuration space of an annulus can be thought of as an H-space under juxtaposition.

To make this more precise we define

$$C(A_\star) = \bigcup_{\lambda \geq 1} C(A_{1\lambda}) ,$$

thought of as a subspace of $C(V) \times \mathbb{R}$. The obvious composition $C(A_{1\lambda}) \times C(A_{1\mu}) \to C(A_{1,\lambda\mu})$ makes $C(A_\star)$ into a monoid. Choosing $\xi \in C(A_\star)$ representing a cofinal component we define $C_\infty(A)$ as the limit of the embeddings

$$C(A_\star) \xrightarrow{\times\xi} C(A_\star) \xrightarrow{\times\xi} C(A_\star) \xrightarrow{\xi} \cdots .$$

(In other words we stabilize on the <u>outside</u> edge of the annulus.) We shall prove

<u>Proposition (3.4)</u> (a) $BC(A_\star) = C(D_\lambda, S_\lambda \cup D_\mu)$.

(b) There is a G-homotopy-equivalence $C_\infty(A) \to \Omega BC(A_\star)$.

The theorem we want to prove follows from this. By analogy with $C(A_\star)$ let us define $C(D_\star) = \bigcup_{\lambda \geq 1} C(D_\lambda) \subset C(V) \times \mathbb{R}$. This admits an action of the monoid $C(A_\star)$, induced by $C(D_\lambda) \times C(A_{1\mu}) \to C(D_{\lambda\mu})$. The space $C_\infty(V)$ of the introduction can be identified with $C_\infty(D)$, the limit of

$$C(D_\star) \xrightarrow{\times\xi} C(D_\star) \xrightarrow{\times\xi} \cdots .$$

Now combining (3.4) and (3.2) gives a G-homology-equivalence $C_\infty(A) \to \Omega \mathrm{Map}(S_1; S^V) = \mathrm{Map}_0(D_\lambda/(S_\lambda \cup D_\mu); S^V)$. But $\Omega^V S^V = \mathrm{Map}_0(D_\lambda/S_\lambda; S^V)$ fits into a G-fibration sequence

$$\mathrm{Map}_0(D_\lambda/(S_\lambda \cup D_\mu); S^V) \to \mathrm{Map}_0(D_\lambda/S_\lambda; S^V) \to \mathrm{Map}(D_\mu; S^V).$$

We can compare this with the G-homology-fibration sequence

$$C_\infty(A) \to C_\infty(D) \to C(D_1, S_1)$$

which arises as the limit of the sequences

$$C(A_{1\lambda}) \to C(D_\lambda) \to C(D_1, S_1) .$$

(The last sequence is not a homotopy fibration. Each fibre is a copy of $C(A_{1\lambda})$ but when one filters the base in the usual way the attaching maps of the fibres when one passes from one layer to the next are given by adding a configuration on the inside of the annulus. Left translations in the monoid $C(A_\star)$ are not homology equivalences. But after stabilization the fibres are $C_\infty(A)$. The attaching maps are given by the left action of $C(A_\star)$ on $C_\infty(A)$, and <u>are</u> homology equivalences. I shall retrun to this point below.)

The maps we use to compare the sequences arises from the commutative diagram

$$\begin{array}{ccc} C(A_{1\lambda}) & \to & Map_0(D_{\lambda+\varepsilon}/(S_{\lambda+\varepsilon} \cup D_{1-\varepsilon}); C) \\ \downarrow & & \downarrow \\ C(D_\lambda) & \to & Map_0(D_{\lambda+\varepsilon}/S_{\lambda+\varepsilon}; C) \\ \downarrow & & \downarrow \\ C(D_1, S_1) & \to & Map(D_{1-\varepsilon}; C) \ , \end{array}$$

where $C = C(D_\varepsilon, S_\varepsilon)$, by stabilizing.

It remains to prove (3.4). Assertion (b) is just the group-completion theorem (1.5). Unfortunately it is not clear that the Pontrjagin ring $H_\star(C(A_\star)^H)$ is commutative. But all that we need (cf. [5]) is that the localization $H_\star(C(A_\star)^H)[\pi^{-1}]$, where $\pi = \pi_0(C(A_\star)^H)$, can be formed by right fractions, i.e. that when one forms the limit $H_\star(C_\infty(A))$ of

$$H_\star(C(A_\star)) \overset{\times\xi}{\to} H_\star(C(A_\star)) \overset{\times\xi}{\to} \dots ,$$

where ξ generates π, using the <u>right</u> action of π, the <u>left</u> action of π on the limit is by automorphisms. This is the same as the assertion above that when $C_\infty(A)$ is formed by stabilizing on the outside of the annulus the operation of adding a configuration on the inside of the annulus is a G-homology-equivalence. It is true because the homology of the n-particle configuration space $C_n(A)^H$ becomes independent of n when n is large. (In fact $C_n(A)^H = \prod_{K<H} C_n(A_K/H)$, as we saw in the introduction, and the stability of ordinary configuration spaces is proved in [4].)

Turning to (3.4)(a), the monoid $C(A_*)$ acts on $C(D_*, D_\mu)$, where $0 < \mu < 1$, so one can form a space $C(D_*, D_\mu)//C(A_*)$. (cf. [10]. Here I am using the notation $X//M$ for the space written X_M in [5].) This has a projection to $BC(A_*)$, which is a G-homotopy-equivalence because $C(D_*, D_\mu)$ is contractible. Now consider

$$C(A_*) \to C(D_*, D_\mu) \to C(D_1, S_1 \cup D_\mu) .$$

This is not a fibration sequence, for although each fibre is homeomorphic to $C(A_*)$ the attaching maps when one passes from one layer of the base to the next are not equivalences. The monoid $C(A_*)$ acts on the sequence fibre wise, so one can form

$$C(A_*)//C(A_*) \to C(D_*, D_\mu)//C(A_*) \to C(D_1, S_1 \cup D_\mu) .$$

Now all the fibres are contractible, so the attaching maps are necessarily equivalences, and we have $C(D_*, D_\mu)//C(A_*) \simeq C(D_1, S_1 \cup D_\mu)$. That complete the proof.

§4 Γ-spaces with G-action

In [9] I defined a category Γ whose objects are the natural numbers $\underline{0}, \underline{1}, \underline{2}, \ldots$. It is equivalent to the dual of the category of finite pointed sets. A Γ-space is a contravariant functor $A : \Gamma \to$ (spaces) such that the natural map $A(\underline{n}) \to A(\underline{1}) \times \ldots \times A(\underline{1})$ (induced by the n obvious maps $\underline{1} \to \underline{n}$ in Γ) is a homotopy equivalence for each n.

Clearly there is an equivariant generalization of this concept, obtained by replacing spaces by G-spaces, and requiring $A(\underline{n}) \to A(\underline{1})^n$ to be a G-homotopy-equivalence.(*) There are many obvious examples, analogous to those of [9].

If A is a G-Γ-space, and X is a G-space with base-point one can construct a G-space $X \otimes A$. It is not hard to show that if each space $A(\underline{n})$ is a G-ANR and each degeneracy map $A(\underline{n} - \underline{1}) \to A(\underline{n})$ is a cofibration then the functor $X \mapsto X \otimes A$ has the property P1 of §2, i.e. it takes G-triangulable spaces to G-ANR's. But in default of interesting applications I shall not prove that here. The property P2 is more interesting. We have

<u>Proposition (4.1)</u> If $Y \to X$ is a G-fibration then the sequence

$$Y \otimes A \to X \otimes A \to (X/Y) \otimes A$$

is a G-fibration sequence providing Y is G-connected.

If the Γ-space A is the natural numbers $\mathbb{N}$ then $X \otimes A = SP^\infty(X)$, so this provides an alternative proof of (2.1).

<u>Proof of (4.1)</u> (For more details of this argument of [12].)

Bearing in mind the Puppe sequence $Y \to X \to X \cup CY \to SY$

(*) G must be allowed to act on the set $\underline{n}$ here, i.e. we require $A(S) \to Map(S; A(\underline{1}))$ to be a G-homotopy-equivalence for each finite G-set S.

more

we shall prove first that there is a G-fibration sequence

$$X \otimes A \to (X \cup CY) \otimes A \to (SY) \otimes A .$$

We observe that $X \cup CY$ and SY are the realizations of simplicial spaces

$$X \leftleftarrows X \vee Y \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} X \vee Y \vee Y \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} \ldots\ldots$$

and

$$(\text{point}) \leftleftarrows Y \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} Y \vee Y \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} \ldots\ldots$$

respectively. (Both simplicial spaces are completely degenerate above degree 1.) The map $X \cup CY \to SY$ is induced by the projections $X \vee$

From (4.1) we can deduce the following result asserting that a G-Γ-space gives use to an equivariant spectrum by exactly the argument by which Theorem A was obtained from (2.1).

Proposition (4.3) (a) There is a G-homotopy-equivalence

$$S^W \otimes A \to \Omega^V(S^{V \oplus W} \otimes A)$$

whenever V and W are real G-modules such that $W^G \neq 0$.

(b) If $A(\underline{1})$ is group-like in the sense of §1 then

$$A(\underline{1}) = S^0 \otimes A \simeq \Omega^V(S^V \otimes A) \quad \text{for all } V.$$

(c) If $A(\underline{1})$ satisfies the conditions (b) and (c) preceding (1.5) there is a G-homology-equivalence

$$A(\underline{1})_\infty \to \Omega^V(S^V \otimes A).$$

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St. Catherine's College,
Oxford.