The differential topology and geometry of universal supergravity

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Context and background

Main goal

My main goal with this talk is to discuss the global differential geometry and topology of four-dimensional universal supergravity and its application to U-duality.

Our goal is to understand the mathematical structure of universal supergravity, in the sense of characterizing the underlying geometric topological object on which the theory is formulated in terms of globally defined differential operators acting on prescribed infinite dimensional spaces and study its mathematical applications.

Key ingredient

The implementation of the DSZ integrality condition of universal supergravity in terms of the appropriate sheaf cohomology and its geometric interpretation.

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- L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre'and T. Magri, N=2 Supergravity and N=2 Super Yang-Mills Theory on General Scalar Manifolds: Symplectic Covariance, Gaugings and the Momentum Map, J. Geom. Phys. 23, 1997.
- L. Andrianopoli, R. D'Auria, S. Ferrara, U-Duality and Central Charges in Various Dimensions Revisited, Int. J. Mod. Phys. A13 (1998).
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Ungauged supergravity in four dimensions

Four-dimensional supergavity is a supersymmetric theory of gravity that plays a prominent role in the description of the effective dynamics to Type-II string theory compactified on a Calabi-Yaus or certain conformally balanced complex manifolds.

- Kähler manifolds and moduli spaces in algebraic geometry.
- Projective special Kähler geometry and Quaternionic-Kähler manifolds.
- Maps of special type into complex and Quaternionic-Kähler manifolds.
- Homogeneous spaces and exceptional Lie groups.
- Spinorial Lipschitz structures and differential spinors.
- Gauge-theoretic moduli problems and dynamical systems.
- Evolution flows and initial data sets.

These ingredients appear in local supergravity and give rise to open mathematical problems and potential applications in differential geometry/topology.

Procedure

Constructing a global geometric model of supergravity requires:

1. Determining its configuration space in terms of connections and global sections of the appropriate fiber bundles (submersions, gerbes or algebroids, among others) equipped with the appropriate geometric structures.
2. Determining the equations of motion and Killing spinor equations of the theory in terms of global differential operators acting on the corresponding spaces of sections (sophisticated spinorial Lipschitz structures required!).
3. Determining the automorphism group of the system of partial differential equations defining the theory: U - duality!.

Developing the mathematical formulation of supergravity allows to:

1. Study geodesic extensions and completeness of supergravity solutions.
2. Study of the Cauchy problem and evolution flow of supersymmetric configurations, evaluating well-posedness and short-time existence.
3. Study the moduli space of its supersymmetric solutions, and in particular the moduli of supersymmetric initial data on a Cauchy hypersurface.
4. Potential applications of supersymmetric initial data sets to the construction of smooth invariants on low-dimensional manifolds.
5. Compute and characterize the continuous and arithmetic U-duality groups.
6. Understand the topology and geometry of supergravity U-folds.
7. Construct the universal c-map.

Toy model: Maxwell theory

Consider the vacuum Maxwell equations on (U, g) with $U \subset \mathbb{R}^4$ contractible:

$$\mathrm{d} *_g \mathrm{d}A = 0 \quad \Leftrightarrow \quad \mathrm{d} *_g F = 0\,, \quad \mathrm{d}F = 0$$

for $A \in \Omega^1(U)$ and $F \in \Omega^2(U)$ respectively. These two systems are equivalent: A and $F = \mathrm{d}A$ well-defined modulo gauge transformations.

Suppose we want to promote this formulation to an oriented Lorentzian four-manifold (M, g) of arbitrary topology. Which formulation do we choose?

1. Formulation $\mathrm{d} *_g \mathrm{d}A = 0$ with $A \in \Omega^1(M)$ is insufficient: Dirac's monopole!!
2. Formulation $\mathrm{d} *_g F = 0$ is more general since $F \in \Omega^2_{cl}(M) \nRightarrow F = \mathrm{d}A$.

Issue resolved: we go for formulation (2). This is however also problematic:

Aharonov-Bohm effect!!

There is a way to solve this conundrum: implement Dirac quantization. An element $F \in \Omega^2_{cl}(M)$ defines a real cohomology class in $H^2(M, \mathbb{R})$. Consider the full lattice:

$$L := j(H^2(M, \mathbb{Z})) \subset H^2(M, \mathbb{R}),$$

and restrict the space of field strengths to those satisfying $\frac{1}{2\pi}[F] \in L$. Then, Weil's Theorem states that for every such element ω there exists a principal $\mathrm{U}(1)$with connection A such that $\omega = F_A$: Maxwell theory is promoted to a theory of connections on a principal bundle, whose holonomy takes care of the Aharonov-Bohm effect and which may not exist as globally defined potentials on M!

We want to do the analog construction for the universal bosonic sector of four-dimensional ungauged supergravity, which also contains an abelian gauge theory.

Goal: Dirac-Schwinger-Zwanziger integrality condition in universal supergravity!

The local Lagrangian of ungauged supergravity

Fix contractible open set $U \subset \mathbb{R}^4$. The bosonic sector of unaguged four-dimensional supergravity on U is defined by the following Lagrangian:

$$\mathcal{L} = -s^g + \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j + \mathcal{R}_{\Lambda\Sigma}(\phi)F^\Lambda_{A\,ab} *_g F^{\Sigma\,ab}_A + \mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{A\,ab}F^{\Sigma\,ab}_A$$

where $\phi^i \colon U \to \mathbb{R}$, $i = 1, \ldots, n_s$, are (scalar fields) and:

$$F^\Lambda = \mathrm{d}A^\Lambda \in \Omega^2(U)\,, \quad \Lambda = 1, \ldots n_v\,,$$

are the field strengths of the local gauge fields $A^\Lambda \in \Omega^1(U)$.

- $\mathcal{R} \colon V \to \mathrm{Mat}(n_v, \mathbb{R})\,,\ \mathcal{I} \colon V \to \mathrm{Mat}(n_v, \mathbb{R})\,,\ \mathcal{R}(\phi) := \mathcal{R} \circ \phi\,,\ \mathcal{I}(\phi) := \mathcal{I} \circ \phi$.
- Variables of the theory: (g, ϕ^i, A^Λ); $\mathcal{G}_{ij}$ and $\mathcal{I}_{\Lambda\Sigma}$ positive definite.
- The scalar couplings $\mathcal{G}_{ij}$ are considered as a Riemannian metric on the scalar manifold.

First big question: what are the A^Λ mathematically?

Local universal supergravity is uniquely determined by a choice of Riemannian metric $\mathcal{G}$ on V, and matrix valued functions $\mathcal{R}$ and $\mathcal{I}$. The matrix $\mathcal{I}$ generalizes the inverse of the squared coupling constant appearing in ordinary four-dimensional gauge theories, whereas $\mathcal{R}$ generalizes the *theta angle* of quantum chromodynamics. The functional S_I can be naturally written as a sum of three pieces:

$$\mathrm{S}_I = \mathrm{S}_I^e + \mathrm{S}_I^s + \mathrm{S}_I^v\,,$$

where:

- $\mathrm{S}_I^e \leftrightarrow$ Einstein-Hilbert term $\leftrightarrow$ Theory of Einstein metrics.
- $\mathrm{S}_I^s \leftrightarrow$ scalar sector $\leftrightarrow$ Theory of wave (harmonic) maps.
- $\mathrm{S}_I^v \leftrightarrow$ gauge sector $\leftrightarrow$ Yang-Mills-like theory.

Hence, bosonic supergravity simultaneously involves three classical theories in physics, coupling the Einstein-Hilbert term to a non-linear sigma model with Riemannian target space $(V, \mathcal{G}_{ij})$ and to a given number of *abelian gauge fields*.

Interludio: scalar manifolds four-dimensional supergravity.

The scalar manifolds that can appear in ungauged four-dimensional supergravity are highly constrained by supersymmetry, as the following table indicates.

Number of supersymmetries	Isometry type of $(V, \mathcal{G}_{ij})$
$\mathcal{N} = 1$	$\mathcal{M}_{KH}$
$\mathcal{N} = 2$	$\mathcal{M}_{PSK} \times \mathcal{M}_{QK}$
$\mathcal{N} = 3$	$\mathrm{SU}(3, n)/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(n))$
$\mathcal{N} = 4$	$\mathrm{SU}(1,1)/\mathrm{U}(1) \times \mathrm{SO}(6, n)/\mathrm{S}(\mathrm{O}(6) \times \mathrm{O}(n))$
$\mathcal{N} = 5$	$\mathrm{SU}(1,5)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5))$
$\mathcal{N} = 6$	$\mathrm{SO}^*(12)/\mathrm{U}(1) \times \mathrm{SU}(6)$
$\mathcal{N} = 8$	$\mathrm{E}_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$

Table: Local isometry type of the scalar manifolds of four-dimensional supergravity

The period matrices are also given in terms of very specific formulas dictated by supersymmetry, and are yet to be mathematically understood!

Geometric interpretation of the gauge sector

The Maxwell equations contain the key information that we shall need in the following:

$$\nabla^g_a(\mathcal{R}_{\Lambda\Sigma}(\phi)(*F^\Sigma_A)^{ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)(F^\Sigma_A)^{ab}) = 0$$

Define the *period matrix map* $\mathcal{N} := \mathcal{R} + i\mathcal{I} \colon V \to \mathbb{SH}_{n_v}$: local supergravity completely determined by a pair $(\mathcal{G}, \mathcal{N})$ on (U, V).

Since U is contractible we consider closed two-forms $F^\Lambda \in \Omega^2(U, \mathbb{R}^{n_v})$ as variables of the Maxwell equations. Condition $\mathrm{d}F = 0$ ensures $F = \mathrm{d}A$ for A unique modulo local gauge transformations. Denoting by Conf_c the set of all triples (g, ϕ, F) where F is a closed two-form valued in $\mathbb{R}^{n_v}$ we have a canonical bijection:

$$\frac{\mathrm{Conf}}{\sim} \xrightarrow{\simeq} \mathrm{Conf}_c\,, \qquad (g, \phi, [A]) \mapsto (g, \phi, F_A)\,,$$

where $(g_1, \phi_1, A_1) \sim (g_2, \phi_2, A_2)$ if $g_1 = g_2$, $\phi_1 = \phi_2$ and $A_1 = A_2 + \mathrm{d}f$ for a vector valued function $f \in C^\infty(U, \mathbb{R}^{n_v})$. Reformulation of the Maxwell equations:

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}(\mathcal{R}(\phi)F) = \mathrm{d}(\mathcal{I}(\phi) * F)\,,$$

with variables in Conf_c.

Given a period matrix map $\mathcal{N}\colon V \to \mathbb{SH}(n_v)$ we define a map:

$$G_{\mathcal{N}}\colon \mathrm{Conf}_c \to \Omega^2(U, \mathbb{R}^{n_v})\,, \quad (g, \phi, F) \to G_{\mathcal{N}}(g, \phi, F) := \mathcal{R}(\phi)F - \mathcal{I}(\phi) * F\,,$$

The Bianchi identities and Maxwell equations reduce to:

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}G_{\mathcal{N}}(g, \phi, F) = 0\,,$$

which in turn can be equivalently written simply as:

$$\mathrm{d}\mathcal{V}_{\mathcal{N}}(g, \phi, F) = 0\,,$$

where $\mathcal{V}_{\mathcal{N}}\colon \mathrm{Conf}_c \to \Omega^2(U, \mathbb{R}^{2n_v})$ denotes the map defined by:

$$\mathrm{Conf}_c \ni (g, \varphi, F) \mapsto \mathcal{V}_{\mathcal{N}}(g, \phi, F) = \begin{pmatrix} F \\ G_{\mathcal{N}}(g, \phi, F) \end{pmatrix} \in \Omega^2(U, \mathbb{R}^{2n_v})\,.$$

Note that $\mathcal{V}_{\mathcal{N}}(g, \phi, F)$ is valued in $\mathbb{R}^{2n_v}$ whereas F is valued in $\mathbb{R}^{n_v}$.

Geometric interpretation of the gauge sector

Not every $\mathbb{R}^{2n_v}$ valued two-form is in the image of $\mathcal{V}_{\mathcal{N}} \colon \mathrm{Conf}_c \to \Omega^2(U, \mathbb{R}^{2n_v})$.

Lemma

A vector valued two-form $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})$ can be written as:

$$\mathcal{V} = \mathcal{V}_{\mathcal{N}}(g, \phi, F) = \begin{pmatrix} F \\ G_{\mathcal{N}}(g, \phi, F) \end{pmatrix},$$

for a certain $(\mathcal{G}, \mathcal{N})$ and (g, ϕ, F) iff $*_g \mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V})$, where $\mathcal{J} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R})$ is the matrix-valued map defined as follows in terms of $\mathcal{N} := \mathcal{R} + i\mathcal{I} \colon V \to \mathbb{SH}_{n_v}$:

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R}),$$

and $\mathcal{J}(\phi) := \mathcal{J} \circ \phi \colon U \to \mathrm{Gl}(2n_v, \mathbb{R})$. In particular, we have $\mathcal{J}^2 = -1$.

The matrix-valued map $\mathcal{J} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R})$ can be understood as a fiber-wise complex structure on the trivial vector bundle of rank $2n_v$ over V.

Geometric interpretation of the gauge sector

Equation $*_g \mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V})$ is known as the *twisted self-duality condition*. The following proposition gives its geometric interpretation, which in turn hints at the global geometric interpretation of the twisted self-duality condition.

Lemma

Let ω be the standard symplectic form on $\mathbb{R}^{2n}$. A matrix-valued map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n})$ can be written as:

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} : V \to \mathrm{Aut}(\mathbb{R}^{2n})$$

for a local $(\mathcal{N})$ if and only if $\mathcal{J}|_p$ is a compatible taming of ω for every $p \in V$.

A complex structure J on $\mathbb{R}^{2n_v}$ is said to be a *compatible taming* of ω if:

$$\omega(J\xi_1, J\xi_2) = \omega(\xi_1, \xi_2)\,,\ \forall\ \xi_1, \xi_2 \in \mathbb{R}^{2n_v}\,,\ \omega(\xi, J\xi) > 0\,,\ \forall\ \xi \in \mathbb{R}^{2n_v} \setminus \{0\}\ .$$

Definition

A taming map $\mathcal{J}$ on V is a smooth map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$ such that $\mathcal{J}|_p$ is a taming on $\mathbb{R}^{2n_v}$ with respect to the standard symplectic structure on $\mathbb{R}^{2n_v}$.

One-to-one correspondence between taming maps and period matrix maps!

Definition

The configuration space $\mathrm{Conf}(\mathcal{G},\mathcal{J})$ of the local bosonic supergravity determined by $(\mathcal{G},\mathcal{J})$ on (U,V) is:

$$\mathrm{Conf}(\mathcal{G},\mathcal{J}) := \{(g,\phi,\mathcal{V}) \mid *_g \mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V})\}$$

Solution space $\mathrm{Sol}(\mathcal{G},\mathcal{J}) \subset \mathrm{Conf}_U(\mathcal{G},\mathcal{J})$. Maxwell equations $\mathrm{d}\mathcal{V} = 0$, $\mathcal{V} \in \Omega^2(M,\mathbb{R}^{2n_v})$.

The local U-duality group

Duality transformations of the local theory are crucial to construct the global geometric formulation of the universal sector of bosonic supergravity.

• A duality transformation is a symmetry of the local supergravity equations which do not involve diffeomorphisms of U and that may not preserve the action functional.

• Extends to supergravity the well-known electromagnetic duality transformations occurring in standard electromagnetism and are key for its connection to string theory.

First recall that $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ has a natural action:

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v}) \to \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v})$$
$$(g, \phi, \mathcal{V}) \mapsto (g, f \circ \phi, \mathfrak{A}\,\mathcal{V}) \,,$$

This action does not preserve in general $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ of the local supergravity associated to a given local electromagnetic structure $(\mathcal{G}, \mathcal{J})$.

The local U-duality group

The failure of $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ to preserve the configuration space of a given local supergravity determined by $(\mathcal{G}, \mathcal{J})$ can be precisely characterized.

Theorem

For every $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$*, the map* $\mathbb{A}_{f,\mathfrak{A}}$ *induces by restriction a bijection:*

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Conf}(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}(f_* \mathcal{G}, \mathcal{J}^f_{\mathfrak{A}})\,,$$

such that it further restricts to a bijection of the corresponding spaces of solutions:

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Sol}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sol}(f_* \mathcal{G}, \mathcal{J}^f_{\mathfrak{A}})\,,$$

where $f_* \mathcal{G}$ *is the push-forward of* $\mathcal{G}$ *by* $f \colon V \to V$ *and* $\mathcal{J}^f_{\mathfrak{A}} := \mathfrak{A}\,(\mathcal{J} \circ f^{-1})\mathfrak{A}^{-1}$

$\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ defines a correspondence between *different* theories!

Corollary

Let $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ *such that* $f \in \mathrm{Iso}(V, \mathcal{G})$ *and:*

$$\mathcal{J}^f_{\mathfrak{A}} = \mathfrak{A}\,(\mathcal{J} \circ f^{-1})\,\mathfrak{A}^{-1} = \mathcal{J}\,.$$

Then $\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Conf}(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}(\mathcal{G}, \mathcal{J})$ *is a bijection that preserves* $\mathrm{Sol}(\mathcal{G}, \mathcal{J})$.

We define the local *U-duality group* of the supergravity theory defined by $(\mathcal{G}, \mathcal{J})$ precisely as the subgroup of $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ preserving $\mathrm{Conf}(\mathcal{G}, \mathcal{J})$.

Definition

The local electromagnetic U-duality group, or U-duality group for short, is:

$$\mathrm{U}(\mathcal{G}, \mathcal{J}) := \left\{(f, \mathfrak{A}) \in \mathrm{Iso}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A}\,\mathcal{J}\,\mathfrak{A}^{-1} = \mathcal{J} \circ f\right\}\,.$$

The U-duality group should be trivial for all interesting supergravities! Not the case!!
The opposite is true: for $\mathcal{N} > 2$ the U-duality group is as large as it can be!

The local U-duality group

Define $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) := \{\mathfrak{A} \in \mathrm{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A}\,\mathcal{J}\,\mathfrak{A}^{-1} = \mathcal{J}\}$. Then:

$$1 \to \mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) \to \mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Iso}_{pr}(V, \mathcal{G}) \to 1\,,$$

where $\mathrm{Iso}_{pr}(V, \mathcal{G}) \subset \mathrm{Iso}(V, \mathcal{G})$ is the subgroup of the isometry group of $(V, \mathcal{G})$ that is obtained by projecting $\mathrm{U}(\mathcal{G}, \mathcal{J})$ onto its first component. On the other hand:

$$\mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \subset \mathrm{Sp}(2n_v, \mathbb{R})\,, \qquad (f, \mathfrak{A}) \mapsto \mathfrak{A}\,,$$

fits into $1 \to \mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) \to \mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \to 1$, where
$\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) := \{f \in \mathrm{Iso}(V, \mathcal{G}) \mid \mathcal{J} \circ f = \mathcal{J}\}$.

Corollary

If $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) = \mathrm{Id}$ *then* $\mathrm{U}(\mathcal{G}, \mathcal{J}) = \mathrm{Iso}_{pr}(V, \mathcal{G}) \subset \mathrm{Iso}(V, \mathcal{G})$. *If* $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) = \mathrm{Id}$ *is trivial then* $\mathrm{U}(\mathcal{G}, \mathcal{J}) \hookrightarrow \mathrm{Sp}(2n_v, \mathbb{R})$ *canonically. If both* $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) = \mathrm{Id}$ *and* $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) = \mathrm{Id}$ *then the U-duality group* $\mathrm{U}(\mathcal{G}, \mathcal{J})$ *is canonically isomorphic to a subgroup of* $\mathrm{Iso}(V, \mathcal{G})$ *embedded in* $\mathrm{Sp}(2n_v, \mathbb{R})$.

The previous corollary puts on firm grounds the validity of a folklore statement made in the literature which states that the *U-duality group consists of a copy of the isometry group of the scalar manifold into the appropriate symplectic group.*

We want to construct a global *geometric model* that reduces locally to local supergravity.

Guding principle

Supergravity needs to implement the local electromagnetic U-duality groups in the sense that it must be possible to understand the theory as being the result of *gluing* the local supergravity theories *á la Čech*. This point is especially important for the resulting theory to describe *supergravity U-folds* in a geometric context and to make contact with the string theory and its compactification backgrounds.

The underlying topological structure.

Instead of going through the process of constructing geometric bosonic supergravity I will present it in its final form. Supergravity is uniquely determined by:

- A scalar bundle $(\pi, \mathcal{H}, \mathcal{G})$, a submersion $\pi\colon \mathcal{X} \to M$ equipped with a flat Ehresmann connection and a a vertically Riemannian metric $\mathcal{G}$.
- A duality bundle $\Delta := (\mathcal{S}, \omega, \mathcal{D})$ consisting of a symplectic vector bundle $(\mathcal{S}, \omega)$ over $\mathcal{X}$ equipped with a flat symplectic connection $\mathcal{D}$.
- A compatible taming $\mathcal{J}$ on $(\mathcal{S}, \omega, \mathcal{D})$ preserved by the extended parallel transport of $\mathcal{H}$ and $\mathcal{D}$ on the ddouble fibration structure $\mathcal{S} \to X \to M$.

Definition

Electromagnetic structure $\Theta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$.

Scalar-electromagentic structure $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Theta)$.

An uncountable infinity of inequivalent supergravities.

Isomorphism classes of duality structures on a fixed scalar manifold $\mathcal{M}$ are in general not unique and depend on the fundamental group of $\mathcal{M}$. The classical theory of flat vector bundles shows that isomorphism classes of duality structure are in one to one correspondence with a character variety:

$$\mathrm{Hom}(\pi_1(\mathcal{X}), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R})\,.$$

The fact that character varieties yield in general *continuous* moduli spaces suggests the possibility of constructing an uncountable infinity of inequivalent geometric bosonic supergravities which are however all locally equivalent!

Self-duality in four Lorentzian dimensions

For every Lorentzian metric g on M and scalar section φ we define:

$$\star_{g,J^{\varphi}} : \Lambda T^* M \otimes \mathcal{S}^{\varphi} \to \Lambda T^* M \otimes \mathcal{S}^{\varphi}$$

by $\star_{g,J^{\varphi}}(\alpha \otimes s) = *_g \alpha \otimes J^{\varphi}(s)$ on homogeneous elements. Restricted to two-forms:

$$\star^2_{g,J^{\varphi}} = 1$$

Novel notion of (anti) self-duality in four Lorentzian dimensions!

We can split the bundle of two-forms taking values in $\mathcal{S}^+$ in eigenbundles of $\star_{g,J^{\varphi}}$:

$$\Lambda^2 T^* M \otimes \mathcal{S}^{\varphi} = (\Lambda^2 T^* M \otimes \mathcal{S}^{\varphi})_+ \oplus (\Lambda^2 T^* M \otimes \mathcal{S}^{\varphi})_-$$

Definition

Elements of $\Omega^2_+(M, \mathcal{S}^{\varphi})$ are positive polarized self-dual two-forms and elements.

The configuration space.

The configuration space of the unique classical universal supergravity determined by $\Phi = (\pi, \mathcal{H}, \mathcal{G}, \Theta)$ is given by:

- A Lorentzian metric g on M.
- A *scalar section* $\varphi \colon M \to \mathcal{X}$.
- A positively-polarized two-form $\mathcal{V} \in \Omega^2_+(M, \mathcal{S}^\varphi)$ with values in $\mathcal{S}^\varphi$.

We obtain the global definition of the configuration space of the theory:

$$\mathrm{Conf}(M, \Phi) := \left\{(g, \varphi, \mathcal{V}) \mid g \in \mathrm{Lor}(M)\,,\ \varphi \in \Gamma(\pi)\,,\ \mathcal{V} \in \Omega^2_+(M, \mathcal{S}^\varphi)\right\}\,.$$

Classical configuration space, it is given in terms of field strengths!

Note the *coupled nature* of the configuration space!

The fundamental form

In Φ we do not require $\mathcal{D}$ to be compatible with $\mathcal{J}$, which is crucial for applications.

Definition

The fundamental form Ψ_Θ of Θ is $\Psi_\Theta :=: \mathcal{D}\mathcal{J} \in \Omega^1(\mathcal{X}, \mathrm{End}(\mathcal{S}))$.

Ψ_Θ measures the local deviation of the gauge sector of the supergravity theory defined by Θ from a Maxwell theory with n_v gauge fields with constant gauge couplings and theta angles.

Definition

Let Φ be a scalar-electromagnetic bundle. The classical supergravity determined by Φ is defined by the following set of differential equations:

- The Einstein equations:

$$\mathrm{Ric}^g - \frac{g}{2} R^g = \frac{g}{2}\,\mathrm{Tr}_g(\varphi^*_{\mathcal{H}}\mathcal{G}) - \varphi^*_{\mathcal{H}}\mathcal{G} + 2\mathcal{V} \oslash_{\Theta^\varphi} \mathcal{V}\,,$$

- The scalar equations:

$$\nabla \mathrm{d}^v \varphi = \frac{1}{2}(\mathcal{V}, (\Psi^\varphi)^{\sharp_g}\mathcal{V})\,.$$

- The electromagnetic (or Maxwell) equations:

$$\mathrm{d}_{D^\varphi}\mathcal{V} = 0\,,$$

with variables $(g, \varphi, \mathcal{V}) \in \mathrm{Conf}(M, \Phi)$.

Recall that:

$$\mathrm{d}_{D^\varphi} : \Omega^2(M, \mathcal{S}^\varphi) \to \Omega^3(M, \mathcal{S}^\varphi)\,, \quad \mathrm{d}^v\varphi \in \Omega^1(M, \mathbb{V}^\varphi)\,.$$

We identify the classical U-duality group of supergravity associated to Φ with:

$$\mathrm{U}(\Phi) := \{u \in \mathrm{Aut}_\pi(\mathcal{S}) \mid \omega_u = \omega\,,\ \mathcal{D}_u = \mathcal{D}\,,\ \mathcal{H}_u\,,\ \mathcal{G}_u = \mathcal{G}\,,\ \mathcal{J}_u = \mathcal{J}\}\ ,$$

which fits in the short exact sequence:

$$1 \to \mathrm{Aut}_b(\Theta) \to \mathrm{U}(\Phi) \to \mathrm{Aut}_b^o(\pi, \mathcal{H}, \mathcal{G}) \to 1\,,$$

whence $\mathrm{U}(\Phi)$ is finite-dimensional and it can markedly differ from the U-duality group of the local theory usually considered in the literature!

Theorem

$\mathrm{U}(\Phi)$ *preserves* $\mathrm{Sol}(M, \Phi)$.

This theorem established the classical U-duality group as being a *solution-generating technique* in classical universal supergravity, as expected.

From field strengths to gauge potentials

We introduced classical supergravity as a theory of triples $(g, \varphi, \mathcal{V}) \in \mathrm{Conf}(\Phi)$, hence the gauge sector is a theory of field strengths $\to$ problematic *experimentally*.

We refine the construction as to obtain a theory of *potentials*, not field strengths.

- Question: How do we identify the *right* notion of gauge potential?
- Answer: the DSZ integrality condition in classical supergravity.

Key idea: the DSZ integrality condition in classical supergravity defines a locally constant sheaf and a sheaf cohomology class that, when interpreted geometrically, determines a class of principal bundles whose connections are our gauge fields.

Gauge potentials $\leftrightarrow$ connections on Siegel bundles

The DSZ integrality condition in classical supergravity

Given the classical supergravity theory determined by Φ we proceed as follows.

- Fix an *integral duality structure*;a bundle Λ of full lattices in $\mathcal{S}$, preserved by the parallel transport of D and such that ω is integer-valued on Λ.
- Associated to Φ and Λ we construct a smooth bundle $\mathcal{X} := \mathcal{S}/\Lambda$ of polarized Abelian varieties endowed with a flat Ehresmann connection whose parallel transport preserves the symplectic structure of the torus fibers.

The sheaf $\mathfrak{S}_{\mathcal{X}}$ of smooth flat sections of $\mathcal{X}$ fits into a short exact sequence of sheaves of Abelian groups defined on M for every scalar section $\varphi \colon \Gamma(\pi)$:

$$0 \to \mathfrak{S}^{\varphi}_{\Lambda} \xrightarrow{j^{\varphi}} \mathfrak{S}^{\varphi}_{\Delta} \to \mathfrak{S}^{\varphi}_{\mathcal{X}} \to 0\,.$$

This induces a long exact sequence in sheaf cohomology:

$$\ldots \to H^1(M, \mathfrak{S}^{\varphi}_{\mathcal{X}}) \to H^2(M, \mathfrak{S}^{\varphi}_{\Lambda}) \xrightarrow{j^{\varphi}_{*}} H^2(M, \mathfrak{S}^{\varphi}_{\Delta}) \to H^2(M, \mathfrak{S}^{\varphi}_{\mathcal{X}}) \to \ldots\,.$$

In particular: $j^{\varphi}_{*} \colon H^2(M, \mathfrak{S}^{\varphi}_{\Lambda}) \to H^2(M, \mathfrak{S}^{\varphi}_{\Delta})$.

The charge lattice of the integral scalar-electromagnetic bundle (Ξ, Λ) relative to φ is:

$$L^{\varphi}_{\Lambda} := j^{\varphi}_{*}(H^2(M, \mathfrak{S}^{\varphi}_{\Lambda})) \subset H^2(M, \mathfrak{S}^{\varphi}_{\Delta}),$$

Elements of this lattice are *integral cohomology classes*.

Compare to $j(H^2(M, \mathbb{Z})) \subset H^2(M, \mathbb{R})$ in Maxwell theory!

It can be shown that L^{φ}_{Λ} is a full lattice in $H^2(M, \mathfrak{S}^{\varphi}_{\Delta})$. Given (Φ, Λ), we implement DSZ integrality condition by restricting the configuration space $\mathrm{Conf}(\Phi)$ to the subset:

$$\mathrm{Conf}(\Phi, \Lambda) \subset \mathrm{Conf}(\Phi)$$

obtained by imposing an *integrality condition* on the elements of $\mathrm{Conf}(\Phi)$. That is, we refine the configuration space and we select only those elements that admit the appropriate geometric interpretation in terms of gauge potentials.

The DSZ integrality condition in classical supergravity

Definition

The *integral configuration space* $\mathrm{Conf}(\Phi, \Lambda)$ defined by (Φ, Λ) is the set:

$$\mathrm{Conf}(\Phi, \Lambda) := \{(g, s, \mathcal{F}) \in \mathrm{Conf}(\Phi, \Lambda) \mid [\mathcal{F}] \in 2\pi L^{\varphi}_{\Lambda}\} \ .$$

Integral solution space $\mathrm{Sol}(\Phi, \Lambda) \stackrel{\mathrm{def.}}{=} \mathrm{Sol}(\Phi) \cap \mathrm{Conf}(\Phi, \Lambda)$.

Definition

The *framed integral configuration space* $\mathrm{Conf}(\mathfrak{V}, \Phi, \Lambda)$ with *framing* $\mathfrak{V}$ of the classical supergravity associated to (Φ, Λ),where $\mathfrak{V} \in H^2(\mathcal{X}, \mathfrak{S}_{\Lambda})$, is:

$$\mathrm{Conf}(\mathfrak{V}, \Phi, \Lambda) := \{(g, s, \mathcal{F}) \in \mathrm{Conf}(\Phi) \mid [\mathcal{F}] = 2\pi j^{\varphi}_{*}(\mathfrak{V}^{\varphi})\} \ ,$$

Framed integral solution space: $\mathrm{Sol}(\mathfrak{V}, \Phi, \Lambda) := \mathrm{Sol}(\Phi) \cap \mathrm{Conf}(\mathfrak{V}, \Phi, \Lambda)$.

Arithmetic U-duality group $\mathrm{U}(\Phi, \Lambda) := \{u \in \mathrm{U}(\Phi) \mid u(\mathcal{L}) = \mathcal{L}\}$.

The DSZ integrality condition in classical supergravity

Theorem

Let (Φ, Λ) be an integral scalar-electromagnetic bundle of type $\mathfrak{t}$. For every framed integral configuration space $\mathrm{Conf}(\mathfrak{V}, \Phi, \Lambda)$ there exists a polarized Siegel bundle $(P_{\mathfrak{t}}, \mathcal{J})$ on $\mathcal{M}$ such that $(\Delta, \Lambda) = \mathfrak{ad}(P_{\mathfrak{t}})$ and its twisted Chern class satisfies $c(P_{\mathfrak{t}}) = \mathfrak{V}$.

Given $\mathfrak{t} \in \mathrm{Div}^{n_v}$, we define $\mathrm{Aff}_{\mathfrak{t}} := \mathrm{U}(1)^{2n_v} \rtimes \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$ with multiplication rule:

$$(a_1, \gamma_1)\,(a_2, \gamma_2) = (a_1 + \gamma_1 a_2, \gamma_1\gamma_2)\,, \quad \forall\; a_1, a_2 \in \mathrm{U}(1)^{2n_v}\,, \quad \forall\; \gamma_1, \gamma_2 \in \mathrm{Sp}_{\mathfrak{t}}(2n_v, \mathbb{Z})\,.$$

The Siegel modular group $\mathrm{Sp}_{\mathfrak{t}}(2n_v, \mathbb{Z})$ is the automorphism group of the standard integral symplectic space $(\mathbb{R}^{2n_v}, \omega_{n_v}, \wedge_{\mathfrak{t}})$ of type $\mathfrak{t}$ and $\wedge_{\mathfrak{t}} := \mathbb{Z}^{n_v} \oplus \oplus_{i=1}^{n_v} t_i \mathbb{Z} \subset \mathbb{R}^{2n_v}$. $\mathrm{Aff}_{\mathfrak{t}}$ coincides with the group of affine symplectomorphisms of the $2n_v$-dimensional symplectic torus $(\mathbb{R}^{2n_v}/\Lambda_{\mathfrak{t}}, \Omega_{\mathfrak{t}})$, where $\Omega_{\mathfrak{t}}$ is induced by ω_{n_v}.

Definition

A *Siegel bundle* $P_{\mathfrak{t}}$ of type $\mathfrak{t}$ is a principal bundle over $\mathcal{M}$ with structure group $\mathrm{Aff}_{\mathfrak{t}}$.

Definition

Let $\zeta := (\mathcal{M}, \mathcal{G}, P_{\mathfrak{t}}, \mathcal{J})$ be a polarized scalar-Siegel bundle over M. The *configuration space* of the supergravity defined by ζ is the set:

$$\mathrm{Conf}(\zeta) = \{(g, s, \mathcal{A}) \mid g \in \mathrm{Lor}(M)\,,\; s \in \Gamma(\pi)\,,\; \mathcal{A} \in \mathrm{Conn}(P^s_{\mathfrak{t}})\}\ .$$

The *universal bosonic sector* of four-dimensional supergravity determined on M by ζ is defined through the following differential system for triples $(g, s, \mathcal{A}) \in \mathrm{Conf}(\zeta)$:

- The Einstein equations: $\mathrm{Ric}^g - \frac{g}{2}\mathrm{R}^g = \frac{1}{2}\mathrm{Tr}_g(s^\ast_{\mathcal{C}}\mathcal{G})\, g - s^\ast_{\mathcal{C}}\mathcal{G} + 2\mathcal{F}_{\mathcal{A}} \oslash_{Q^s} \mathcal{F}_{\mathcal{A}}$.
- The scalar equations: $\nabla^{\Phi(g,s)}\mathrm{d}^{\mathcal{C}} s = \frac{1}{2}(\ast\mathcal{F}_{\mathcal{A}}, \Psi^s\mathcal{F}_{\mathcal{A}})_{g,Q^s}$.
- The Maxwell equations: $\star_{g,\mathcal{J}^s}\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$.

A is a connection on a principal bundle whose isomorphism type may depend on $s \in \Gamma(\pi)$!

The Maxwell equations of the bosonic gauge sector of local supergravity are given by a system of second-order partial differential equations for a number n_v of *electromagnetic* local gauge potentials whose curvatures satisfy a generalization of the Maxwell equations. This is locally equivalent to $\star_{g,\mathcal{J}^s}\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$, which reduces locally to a system of first-order partial differential equations for $2n_v$ local gauge fields, both *electric* and *magnetic*.

The Bianchi identity and polarized self-duality condition imply the following second order differential equation of Yang-Mills type:

$$\mathrm{d}_{\mathcal{D}^s} \star_{g,\mathcal{J}} \mathcal{F}_{\mathcal{A}} = 0\,.$$

These differ from the usual Yang-Mills equations since $\mathcal{F}_{\mathcal{A}}$ involves both electric and magnetic degrees of freedom while the equations themselves involve the pulled-back taming $\mathcal{J}^s$.

Definition

The gauge U-duality group $\mathrm{U}(\zeta)$ of the polarized scalar-Siegel bundle ζ is:

$$\mathrm{U}(\zeta) := \{u \in \mathrm{Aut}_\pi(P_\mathfrak{t}) \mid \mathcal{H}_u = \mathcal{H}\,,\ \mathcal{G}_u = \mathcal{G}\,,\ \mathcal{J}_u = \mathcal{J}\}$$

$$1 \to \mathrm{Aut}_b(P_\mathfrak{t}, \mathcal{J}) \to \mathrm{U}(\zeta) \to \mathrm{Aut}^o_b(\pi, \mathcal{H}, \mathcal{G}) \to 1$$

We have a canonical morphism of groups: $\mathfrak{ad}\colon \mathrm{U}(\zeta) \to \mathrm{U}(\Phi(\zeta))$ given by $u \mapsto \mathfrak{ad}_u$.

Definition

The continuous subgroup of the gauge U-duality group $\mathrm{U}(\zeta)$ is $C(\zeta) := \ker(\mathfrak{ad}) \subset \mathrm{U}(\zeta)$

$$C(\zeta) \hookrightarrow \mathrm{U}(\zeta) \xrightarrow{\mathfrak{ad}} \mathrm{U}(\Phi(\zeta))$$

Elements in $C(\zeta)$ behave as gauge transformations on a torus bundle and therefore act trivially on the curvature of any connection. In fact, the arithmetic U-duality group identifies with those gauge transformations of $P_\mathfrak{t}$ that act non-trivially on the adjoint bundle of $P_\mathfrak{t}$. This shows that U-dualities in supergravity are but gauge transformations of the underlying Siegel bundle, a fact that elucidates their geometric origin.

Further directions and open problems

1. Implement supersymmetry in universal supergravity: this implies in particular implementing the Kähler-Hodge, projective Special-Kähler, QK and Cartan geometries in the scalar bundle as well as the appropriate global constraints (not understood mathematically!) on the polarization of the underlying Siegel bundle.
2. Study the Cauchy problem for the globally hyperbolic solutions of universal supergravity and the associated flow equations.
3. Study the supersymmetric initial data sets and their potential application in low-dimensional differential topology.
4. Characterize the arithmetic U-duality groups occurring in supergravity.
5. Implement Dirac quantization on a general geometric supergravity, developing the appropriate model in differential cohomology.
6. Gauge universal supergravity and explore its global higher geometry by applying the theory developed by Kim, Saemann and others.
7. Explore a geometric model for Freudenthal duality in terms of the taming $\mathcal{J}$.
8. Investigate the supersymmetric gravitational waves of geometric supergravity and study their geodesic completeness, causality and boundaries.
9. Construct the universal supergravity c-map.

Thanks!