The Atiyah-Jänich Theorem

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The goal of this paper is to prove the Atiyah-Jänich Theorem which links two fundamental areas of mathematics: functional analysis and algebraic topology. This paper will be divided into two parts: in the first part we will give the necessary analytic background culminating with the statement of The Atiyah-Jänich Theorem; in the second part we will prove this theorem and then give an application to homotopy theory.

1 Analytic Background

The central concepts in this section are the notions of Fredholm operators and the index of Fredholm operators. Before delving into rigorous mathematics, we begin with some motivating remarks inspired from [3].

A fundamental problem in mathematics is to solve equations of the form $Tf = g$, where T is a linear map, g is a known quantity and f is an unkown quantity. For example, suppose $r \in \mathbb{C}^n$ is a vector and $A \in M_n(\mathbb{C})$ is a matrix. In linear algebra, one is interested in solving the equation $Ax = r$. More generally, if $T : V \to W$ is a linear map between vector spaces and $w \in W$, the problem is then to solve the equation $Tv = w$. A slightly more interesting example comes from the theory of differential equations:

Example 1.1 (The Heat Equation)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and let $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ be the Laplacian. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : \mathbb{R}^n \times I \to \mathbb{R}$ is said to be a solution of the heat equation if:

$$\frac{\partial f}{\partial t} = \Delta f.$$

Hence, the heat equation reduces to solving the equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) f = 0.$$

In general, many differential equations can be written in the form $Tf = g$, where T is a suitable differential operator.

When we are given an equation of the form $Tf = g$, we may ask the following two questions:

- Does there exist a solution to this equation ? (**Existence**)
- How unique is the solution to this equation ? (**Uniqueness**)

It turns out that in general answering these questions in isolation is difficult. However, there is a quantity called the index, that gives us information about both the existence and uniqueness of the solution simaltaneously. We now give a very rough template for defining an index in practice.

As before, suppose we are interested in solving the equation $Tf = g$. In this situation, there will often be a number M measuring the existence of solutions to this equation and a number N measuring the uniqueness of solutions to this equation. We then define the index of T to be

$$\text{ind } T := N - M.$$

Ideally, we would like to be able to compute M and N separately. However, in practice this turns out to be quite difficult. On the other hand, while the index gives us less information than the numbers M and N, computing the index is often easier. We now give a simple example of the above.

Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear map. Given a fixed $w \in W$, suppose we want to solve $Tv = w$ for $v \in V$. Then asking about existence of solutions to $Tv = w$ is tantamount to asking about the surjectivity of T. A number that measures the surjectivity of T is $\dim(\text{coker } T)$. Similarly, asking about the uniqueness of solutions to $Tv = w$ is tantamount to asking about the injectivity of T and a number that measures the injectivity of T is $\dim(\ker T)$. Thus, following the template above, we are led to define

$$\text{ind } T := \dim(\ker T) - \dim(\text{coker } T).$$

However, note that we have the following proposition:

Proposition 1.2. *Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Then*

$$\textit{ind } T = \textit{dim } V - \textit{dim } W$$

Proof. Note that

$$\begin{aligned}
\text{ind } T &= \dim(\ker T) - \dim(\text{coker } T) \\
&= \dim(\ker T) + \dim(\text{im } T) - \dim(W) \\
&= \dim V - \dim W \qquad \text{(by the rank-nullity theorem).}
\end{aligned}$$

□

Thus, the index of T doesn't depend on T; it just depends on the dimensions of V and W. To get more interesting results, we are naturally led to instead considering infinite dimensional vector spaces and we thus enter the realm of functional analysis. We begin by recalling the most studied object in this branch of mathematics: the notion of a Hilbert space.

Definition 1.3 (Hilbert space)**.** A $\mathbb{C}$-vector space V is called an inner product space if it is equipped with a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $x, y \in V$ and $c \in \mathbb{C}$

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle cx, y \rangle = c\langle x, y \rangle$
- $\overline{\langle x, y \rangle} = \langle x, y \rangle$
- $\langle x, x \rangle > 0$ for all $x \neq 0$.

Note that an inner product gives rise to a norm by defining $||x|| = \sqrt{\langle x, x \rangle}$ and this norm gives rise to a metric by defining $d(x, y) = ||x - y||$. A inner product space H which is complete with respect to the metric induced by the inner product is called a Hilbert space.

For instance, the Euclidean space $\mathbb{C}^n$ equipped with the inner product

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

is a Hilbert space.

Definition 1.4. A Hilbert space H is called separable if it has a countable orthonormal basis.

Example 1.5 (l^2-space)**.** The space $l^2 := \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{C}, \sum_{i=1}^{\infty} |x_n|^2 < \infty\}$ equipped with the inner product

$$\langle (x_1, x_2, \cdots), (y_1, y_2, \cdots) \rangle := \sum_{i=1}^{\infty} x_i\overline{y_i}$$

is a separable Hilbert space. The unit vectors $e_i := (0, 0, \cdots, 1, 0, 0, \cdots)$ form a countable orthonormal basis.

A standard but surprising result from functional analysis is the following:

Theorem 1.6. *Any two infinite dimensional separable Hilbert spaces are isometrically isomorphic.*

For the rest of this paper, we fix an infinite dimensional separable Hilbert space H and by the above theorem, we may assume that $H = l^2$. An operator $T : H \to H$ is called bounded if there exists a $C \in \mathbb{R}_{>0}$ such that $\sup_{||x||=1} ||T(x)|| \leq C$. It is a fact from functional analysis that an operator is bounded if and only if it is continuous. The space of all bounded operators on H is denoted by $\mathcal{B}$.

Definition 1.7 (Fredholm operators)**.** A operator $T \in \mathcal{B}$ is called a Fredholm operator if $\ker T$ and $\operatorname{coker} T$ are finite dimensional vector spaces. Let $\mathcal{F}$ denote the set of all Fredholm operators on H.

We are now equipped to define the notion of an index in the infinite-dimensional situation.

Definition 1.8 (Fredholm index)**.** If $T \in \mathcal{F}$ is a Fredholm operator, the integer

$$\text{ind } T := \dim (\ker T) - \dim (\text{coker } T) \in \mathbb{Z}$$

is called the index of T.

Example 1.9 (Backward and Forward shifting operators)**.** Consider the backward shifting operator

$$T : H \to H, \quad (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

Then $\dim (\ker T) = 1$ and $\dim (\text{coker } T) = 0$ and so $\text{ind } (T) = 1 - 0 = 1$. Instead now consider the foward shifting operator

$$T' : H \to H, \quad (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

Then $\dim (\ker T') = 0$ and $\dim (\text{coker } T') = 1$ and so $\text{ind } (T') = 0 - 1 = -1$.

Proposition 1.10. *The index map $\mathcal{F} \to \mathbb{Z}$ given by $T \mapsto$ ind T is surjective.*

Proof. For any $k \in \mathbb{N}$, we can generalize the previous example to consider Fredholm operators which shift backward k times and Fredholm operators which shift forward k times; these operators have index k and $-k$ respectively. □

Proposition 1.11 (Algebraic Properties of Fredholm operators)**.** *Let $T : H \to H$ and $T' : H \to H$ be Fredholm operators.*

- *We have that $T' \circ T$ is Fredholm and*

$$\textit{ind } (T' \circ T) = \textit{ind } T + \textit{ind } T'$$

- *Recall that the adjoint $T^* : H \to H$ is the unique map that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$.*

  Then T^* is Fredholm and

$$\textit{ind } T^* = -\textit{ind } T.$$

Proof. The idea of the proof is to use the Snake Lemma; for more details see [4, Lemma 2.1.6] . □

We now proceed to explore the topological properties of Fredholm operators. Note that $\mathcal{B}$ becomes a metric space under the operator norm: $||T|| = \sup_{||x||=1} ||T(x)||$ and thus $\mathcal{F}$ also inherits the same metric space space structure.

Lemma 1.12. *If $T \in \mathcal{B}^\times$ is invertible, then $\mathcal{B}^\times$ contains an open ball around around T of radius $\frac{1}{||T||}$. In particular, the set of invertible elements $\mathcal{B}^\times$ is open in $\mathcal{B}$.*

Proof. See [2, Proposition 2.11]. □

Theorem 1.13. *The index map $\mathcal{F} \to \mathbb{Z}$ given by $T \mapsto ind\, T$ is continuous, where $\mathbb{Z}$ is equipped with the discrete topology.*

Proof. Let $T \in \mathcal{F}$. Let $J : (\ker T)^{\perp} \to H$ denote the inclusion map. Since coker $J \cong \ker T$, we have that

$$\text{ind } J = \dim \ker J - \dim \text{coker } J = 0 - \dim \ker T = -\dim \ker T.$$

Let $Q : H \to \text{im } T$ be the projection map. Since $\ker Q \cong \text{coker } T$, we have that

$$\text{ind } Q = \dim \ker J - \dim \text{coker } J = \dim \text{coker } T - 0 = \dim \text{coker } T.$$

Thus,

$$\text{ind } QTJ = \text{ind } Q + \text{ind } T + \text{ind } J = 0. \tag{1}$$

Now, by basic functional analysis we have that $QTJ : (\ker T)^{\perp} \to \text{im } T$ is invertible. Let $\epsilon := 1/||(QTJ)^{-1}||$ and pick $T' \in \mathcal{F}$ such that $||T - T'|| < \frac{\epsilon}{||Q|| \cdot ||J||}$. It thus follows that

$$||QTJ - QT'J|| = ||Q(T - T')J|| \leq ||Q|| \cdot ||T - T'|| \cdot ||J|| < \epsilon.$$

Hence, Lemma 1.12 give us that $QT'J$ is invertible and so

$$\text{ind } QT'J = \text{ind } Q + \text{ind } T' + \text{ind } J = 0. \tag{2}$$

Comparing equations (1) and (2) yields that ind T = ind T'. Hence, the index map is locally constant and therefore continuous. □

An immediate consequence of this theorem is that the index is constant on the path-connected components of $\mathcal{F}$.
In fact, the converse is also true: any two Fredholm operators having equal index are in the same path-connected component (see [2, Theorem 3.18] for more details). Thus,

Theorem 1.14. *The Fredholm index induces a bijection:*

$$\{\textit{path-connected components of } \mathcal{F}\} \to \mathbb{Z}.$$

Proof. The remarks before the theorem imply that the map is well-defined and injective and we have already seen in Proposition 1.10 that this map is surjective. □

We now try to interpret Theorem 1.14 from a topological point of view. Firstly, we note that $K(\text{pt}) = \mathbb{Z}$. Next, if X is a topological space, let $[X, \mathcal{F}]$ denote the homotopy classes of maps $X \to \mathcal{F}$. Then

$$[\text{pt}, \mathcal{F}] \cong \{\text{path-connected components of } \mathcal{F}\}$$

. Hence, from this point of view, we can write Theorem 1.14 as

$$[\text{pt}, \mathcal{F}] \cong K(\text{pt}).$$

The Atiyah-Jänich Theorem is far reaching generalization of this fact:

Theorem 1.15. *If X is any compact Hausdorff topological space, then then there is an index map*

$$\textit{ind: } [X, \mathcal{F}] \to K(X)$$

and this map is an isomorphism of monoids (hence groups).

2 The connection with K-theory

In this section, our goal will be to prove the Atiyah-Jänich Theorem. As a first step, we will need to construct an index map $[X, \mathcal{F}] \to K(X)$. For each $T : X \to \mathcal{F}$, write $T_x := T(x)$ i.e. for each $x \in X$, T_x is linear map $H \to H$. A natural idea is to form

$$\bigsqcup_{x\in X} \text{ker } T_x \ \ \& \ \bigsqcup_{x\in X} \text{coker } T_x$$

and define the index map

$$\left[\bigsqcup_{x\in X} \text{ker } T_x\right] - \left[\bigsqcup_{x\in X} \text{coker } T_x\right] \in K(X).$$

However, this construction will not give us a vector bundle as the following example shows.

Example 2.1 (Dimension Jumping)**.** For instance, let $X = [-1, 1]$ and $H = \mathbb{C}$ and suppose for each $x \in X$ we have a linear operator

$$T_x : H \to H, \ \ h \mapsto x \cdot h.$$

Then

$$\begin{cases} \text{dim ker } T_x = 1 & \text{if } x = 0 \\ \text{dim ker } T_x = 0 & \text{if } x \neq 0 \end{cases}$$

and

$$\begin{cases} \text{dim coker } T_x = 1 & \text{if } x = 0 \\ \text{dim coker } T_x = 0 & \text{if } x \neq 0 \end{cases}$$

The solution to avoid this problem is to add dimensions avoid the jump. Before giving the construction of this solution, we explain the basic idea. Suppose $T : H \to H$ is a Fredholm operator. Then ker $T \cong \mathbb{C}^d$. Define a new Fredholm operator

$$\tilde{T} : H = \text{ker } T \oplus (\text{ker } T)^\perp \to \mathbb{C}^d \oplus H.$$

Note that ker $\tilde{T}$ is trivial and

$$\text{coker}\tilde{T} \cong \frac{\mathbb{C}^d \oplus H}{\mathbb{C}^d \oplus \text{im } T} \cong \text{coker } T.$$

Thus,

$$\text{ind } T = d - \text{dim coker } T = d - \text{coker } \tilde{T}$$

and also note that ind $T = d+$ind $\tilde{T}$. In other words, we took a Fredholm operator and added some extra dimensions to the codomain to get a new injective Fredholm operator. Moreover, we can recover the index of the Fredholm operator from the new Fredholm operator.
Now suppose $T : X \to \mathcal{F}$ is family of Fredholm operators. We will create a new family of

injective operators $\tilde{T} : X \to \mathcal{F}$ by adding extra dimensions to the codomain. By compactness of X, we will do this by adding only a finite number of dimensions. It will turn out that by forcing the kernel to be trivial, the family of cokernels of $\tilde{T}$ becomes a vector bundle. As before, we can recover the index of T from $\tilde{T}$ and our generalized index map will be of the form

$$[X \times \mathbb{C}^d] - [\text{coker } \tilde{T}].$$

We now proceed to make the above ideas precise by proving the following two lemmas. Firstly, as a notational remark, we mention that $\mathcal{B}(H_1, H_2)$ will denote the set of bounded linear operators between two Hilbert spaces H_1 and H_2.

Lemma 2.2. *Let $T \in \mathcal{F}$ and let V be a closed subspace of finite codimension such that $V \cap \ker T = \{0\}$. Then $H/T(V)$ is finite dimensionsal, $T(V)$ is closed in H and $H/T(V)$ is isomorphic to a subspace W of H.*

There exists an open neighbourhood U of $\mathcal{B}$ such that for all $S \in U$,

1. *$V \cap \ker S = 0$.*
2. *$S(V)$ is closed in H.*
3. *The subspace $W \subseteq H$ projects isomorphically onto $H/S(V)$.*
4. *$\bigcup_{S \in U} H/S(V)$ topologized as a quotient space of $U \times H$ is a trivial vector bundle over U.*

Proof. Note that H/V and coker T are finite dimensional by assumption and that T induces a surjective map $H/V \to T(H)/T(V)$. Thus, $T(H)/T(V)$ is finite dimensional as well. Thus, from the exact sequence

$$0 \to T(H)/T(V) \to H/T(V) \to \text{coker } T \to 0$$

we conclude that $H/T(V)$ is finite dimensional as well. It is a fact from functional analysis that Fredholm operators map closed subspaces to closed subspaces (see [1, page 154]) and hence $T(V)$ is closed. Thus, $H = T(V) \oplus T(V)^\perp$ and setting $W := T(V)^\perp$ shows that $H/T(V) \cong W$ is isomorphic to a subspace of H.

For each $S \in \mathcal{B}$, define a continuous linear operator

$$\phi_S : V \oplus W \to H$$

by $\phi_S(v, w) := S(v) + w$. We thus get a map

$$\phi : \mathcal{B} \to \mathcal{B}(V \oplus W, H), \;\; S \mapsto \phi_S.$$

Since ϕ_T is an isomorphism and since isomorphisms in $\mathcal{B}(V \oplus W, H)$ form an open set by Lemma 1.12, there is a neighbourhood U of T in $\mathcal{B}$ such that ϕ_S is an isomorphism for all $S \in U$. It is now straighforward to check that conditions 1-4 in the lemma are satisfied. □

Lemma 2.3. *Let X be a compact topological space and $T : X \to \mathcal{F}$ be a continuous map. Then there exists a closed subspace $V \subseteq H$ of finite codimension so that:*

(i) $V \cap ker\ T_x = \{0\}$ for any $x \in X$

(ii) The family of vector spaces $\bigcup_{x\in X} H/T_x(V)$, toplogized as a quotient space of $X \times H$, is a vector bundle over X.

Proof. For each $x \in X$, take $V_x = (\ker T_x)^\perp$. Then T_x maps V_x isomorphically onto $T_x(H)$. By Lemma 2.2, there exists a neighbourhood $\mathcal{U}_x$ of T_x in $\mathcal{B}$ such that for each $S \in \mathcal{U}_x$, $V_x \cap \ker S = \{0\}$. Let $U_x = T^{-1}(\mathcal{U}_x \cap \mathcal{F})$. Thus, if $y \in U_x$, then $V_x \cap \ker T_y = \{0\}$. Since X is compact, we can choose a finite covering $U_{x_1}, U_{x_2}, \ldots, U_{x_k}$ of X. Hence, by construction, $V := \bigcap_{j=1}^k V_{x_j}$ satisfies (i). By Lemma 2.2, we have that $\bigcup_y H/T_y(V)$ is locally trivial when y varies in a neighbourhood of x and so (ii) holds. □

The vector bundle $\bigcup_{x\in X} H/T_x(V)$ appearing in Lemma 2.3 will be denoted by $H/T(V)$.

Definition 2.4 (Index)**.** The index of a continuous family $T : X \to \mathcal{F}$ is defined by

$$\text{ind}\ T = [H/V] - [H/T(V)] \in K(X),$$

where V is chosen as in the previous lemma and H/V denotes the trivial bundle of $X \times H/V$.

Example 2.5. Let $X = \text{pt}$. Then T is a single Fredholm operator $H \to H$. We can choose $V = (\ker T)^\perp$. Then H/V is the trivial bundle with fiber $\ker T$ and $H/T(V)$ is the trivial bundle with fiber $\text{coker}\ T$. Thus, $\text{ind}\ T = [\ker T] - [\text{coker}\ T]$ and if we identify $K(\text{pt}) \cong \mathbb{Z}$, then we recover our previous concept of an index.

Remark 2.6. The definition of the index is independent of the choice of V satisfying condition (i) in Lemma 2.3. If W is another choice, note that $V \cap W$ is also a choice and so we may assume that $W \subseteq V$. Then we have the following short exact sequence of vector bundles

$$0 \to V/W \to H/W \to H/V \to 0$$

and

$$0 \to V/W \to H/T(W) \to H/T(V).$$

Thus, $[H/T(V)] - [H/T(W)] = -[V/W] = [H/V] - [H/W]$ which translates to $[H/V] - [H/T(V)] = [H/W] - [H/T(W)]$ showing that the definition of the index does not depend on the choice of V.

Lemma 2.7. *If $f : X' \to X$ and $T : X \to \mathcal{F}$ are continuous, then*

$$f^*(ind\ T) = ind\ (T \circ f).$$

Proof. Note that if $V \cap \ker T_x = \{0\}$ for all $x \in X$, then $V \cap \ker T_{f(x')} = \{0\}$ for all $x' \in X'$. Hence, a choice of the subspace $V \subseteq H$ is also a choice for $T \circ f$. Thus,

$$\begin{aligned} f^*(\text{ind}\ T) &= f^*([H/V] - [H/T(V)]) = [f^*(H/V)] - [f^*(H/T(V))] \\ &= [H/V] - [H/(T \circ f)(V)] = \text{ind}\ (T \circ f). \end{aligned}$$

□

Proposition 2.8. *If $S, T : X \to \mathcal{F}(H)$ are homotopic, then ind $S =$ ind T. Hence, the map*

$$\text{index: } [X, \mathcal{F}] \to K(X)$$

is well-defined.

Proof. Since S and T are homotopic, there exists a homotopy $F : X \times [0,1] \to \mathcal{F}$ such that $F \circ i_0 = S$ and $F \circ i_1 = T$, where for $j \in \{0,1\}$ we define $i_j : X \to X \times I$ by $i_j(x) = (x, j)$. Since $i_0^* = i_1^*$ in K-theory, we conclude that

$$\text{ind } S = \text{ind } (F \circ i_0) = i_0^*(\text{ind } (F)) = i_1^*(\text{ind } (F)) = \text{ind } (F \circ i_1) = \text{ind } T.$$

□

Note that we can equip $[X, \mathcal{F}]$ with the structure of a monoid. Namely, given $T, S \in [X, \mathcal{F}]$, we can define the composition $S \circ T : X \to \mathcal{F}$, via $ST_x = S_x \circ T_x$, where the latter $\circ$ denotes composition of Fredholm operators on H. The identity of the monoid is given by the constant map $X \to \mathcal{F}$ defined via $x \mapsto \{\text{id}\}$.

Proposition 2.9. *The map*

$$\text{ind: } [X, \mathcal{F}] \to K(X)$$

is a homomorphism of monoids.

Proof. Firstly, we note that the constant map $X \to \mathcal{F}$ defined via $x \mapsto \{\text{id}\}$ gets mapped to the trivial bundle. Next, we take $T, S \in [X, \mathcal{F}]$ and let $V, W \subseteq H$ be choices of closed subspaces for T and S respectively. Note that $H = W \oplus W^\perp$ and let $\pi : H \to W$ and $\pi^\perp : H \to W^\perp$ be orthogonal projections. Then for each $t \in [0,1]$, $\text{Id} - t\pi^\perp : H \to H$ is a Fredholm operator. Then the homotopy $h : X \times [0,1] \to \mathcal{F}$ given by $h(x,t) = (\text{Id} - t\pi^\perp) \circ T_x$ shows that T and $\pi \circ T$ are homotopic. We may therefore assume that $T_x(H) \subseteq W$. In particular, $T_x(V) \subseteq W$ for all $x \in X$ which implies that $V \cap \ker\, (S_x \circ T_x) \subseteq V \cap \ker\, (T_x) = \{0\}$. Hence, V is a choice of a closed subspace for ST. Now note that we have the following short exact sequences of vector bundles:

$$0 \to W/T(V) \to H/T(V) \to H/W \to 0$$

and

$$0 \to W/T(V) \to H/ST(V) \to H/S(W) \to 0.$$

Thus,

$$\begin{aligned}
\text{ind } ST &= [H/V] - [H/ST(V)] \\
&= [H/V] - [W/T(V)] - [H/S(W)] \\
&= [H/V] - [H/T(V)] + [H/W] - [H/S(W)] \\
&= \text{ind } T + \text{ind } S.
\end{aligned}$$

□

Theorem 2.10 (The Atiyah Jänich Theorem)**.** *If X is a compact Hausdorff topological space, then the homomorphism:*

$$\textit{ind: } [X, \mathcal{F}] \to K(X)$$

is an isomorphism.

Proof. We first begin with surjectivity. Pick an arbitrary element $[E] - [\underline{\mathbb{C}^k}]$, where E is a vector bundle over X and $\underline{\mathbb{C}^k}$ denotes the trivial vector bundle $X \times \mathbb{C}^k$. We have seen in the seminar that we can find a vector bundle E' over X such that $E \oplus E'$ is the trivial vector bundle $X \times \mathbb{C}^N$ for some N. For each $x \in X$, let π_x be the projection $\mathbb{C}^N = E_x \oplus E'_x \to E_x$. Note that $\mathbb{C}^N \otimes H$ is a Hilbert space whose elements are finite sums $\sum_{j=1}^m c_j \otimes v_j$ for $c_j \in \mathbb{C}^N$ and $v_j \in H$. If $f_1, \ldots, f_N$ is an orthonormal basis for $\mathbb{C}^N$ and $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H, then $\{f_i \otimes e_j\}$ is an orthonormal basis for $\mathbb{C}^N \otimes H$, where the inner product is given by $\langle f_i \otimes e_j, f_r \otimes e_s\rangle = \langle f_i, f_r\rangle \cdot \langle e_j, e_s\rangle$. Note that $\mathbb{C}^N \otimes H$ and H are isomorphic, since both are separable infinite dimensional Hilbert spaces.

Let $n < N$ be the rank of E and let $f_1, \ldots, f_n$ be the basis of E_x. Recall that we have defined the denote the backward shift operator S_k with index k. Note that S_0 is the identity, S_1 is surjective and ker S_1 is generated by e_1. Define a map $P_E : X \to \mathcal{F}(\mathbb{C}^N \otimes H) \cong \mathcal{F}$ by

$$P_E(x) = \pi_x \otimes S_1 + (\mathrm{id} - \pi_x) \otimes S_0.$$

If $c = \sum_{i=1}^N \lambda_i f_i$ with $\lambda_i \in \mathbb{C}$, and $v \in H$, then

$$(\pi_x \otimes S_1)(c \otimes v) = f_1 \otimes \lambda_1 S_1(v) + \cdots + f_n \otimes \lambda_n S_1(v)$$

and

$$((\mathrm{id} - \pi_x) \otimes S_0)(c \otimes v) = f_{n+1} \otimes \lambda_{n+1} v + \cdots + f_N \otimes \lambda_N v.$$

Since S_1 is surjective and $\{f_i \otimes e_j\}$ is a basis for $\mathbb{C}^N \otimes H$, this shows that $P_E(x)$ is surjective. Note also by construction (and since ker S_1 is generated by e_1), we have that ker $P_E(x)$ is generated by $\{f_1 \otimes e_1, \ldots, f_n \otimes e_1\}$ and hence is isomorphic to E_x. Thus, ind $P_E = [E]$.
Let $Q_k : X \to \mathcal{F}$ be the constant map $Q_k(x) = F_k$, where F_k is the k-shift forward operator with index $0 - k = -k$. Thus,

$$\text{ind } Q_k = -[\mathbb{C}^k].$$

. Thus,

$$\text{ind } (P_E \circ Q_k) = \text{ind } P_E + \text{ind } Q_k = [E] - [\underline{\mathbb{C}^k}].$$

Hence, ind is surjective as desired.

To prove injectivity, we will make use of the following fact: If $A \to B$ is a surjective homomorphism of monoids with trivial kernel, and if further B is a group, then it is injective. Hence, it suffices to show that the kernel of ind is trivial. Thus, we want to show that if $T \in [X, \mathcal{F}]$ satisfies ind $T = 0$, then T is homotopic to the constant map $X \to \mathcal{F}$ defined via $x \mapsto \{\mathrm{id}\}$. We now use Kuiper's theorem which states that if $\mathcal{F}^\times$ denotes the set of invertible Fredholm operators, then $[X, \mathcal{F}^\times]$ is a singleton. Hence, it suffices to show that if $T \in [X, \mathcal{F}]$ satisfies ind $T = 0$, then T is homotopic to a map $T' : X \to \mathcal{F}^\times$.

Now ind $T = 0$ implies that $[H/V] - [H/T(V)] = 0$, where $V \subseteq H$ is a closed subspace of finite codimension as required in the definition of the index. We can choose a trivial bundle $\underline{\mathbb{C}^k}$ such that $H/V \oplus \underline{\mathbb{C}^k} \cong H/T(V) \oplus \underline{\mathbb{C}^k}$. Choose a closed subspace W of V such that $V/W \cong \mathbb{C}^k$; then $H/V \oplus \underline{\mathbb{C}^k} \cong H/W$ and $H/T(V) \oplus \underline{\mathbb{C}^k} \cong H/T(W)$ as vector bundles. We thus get an isomorphism $\alpha : H/W \to H/T(W)$. Again, using the fact that Fredholm operators map closed subspaces to closed subspaces (see [1, page 154]), we conclude that $T_x(W)$ is closed and so $H = T_x(W) \oplus T_x(W)^{\perp}$. We thus have a continuous map $\beta : H/T(W) \to H$ that maps each fiber $H/T_x(W)$ isomorphically onto $T_x(W)^{\perp}$. Thus, $\beta \circ \alpha : H/W \to H$ gives a continuous map $T' : X \to \mathcal{B}(H/W, H)$ that maps each fiber H/W at x onto $T_x(W)^{\perp}$. Since $V \cap \ker T_x = 0$ and $W \subseteq V$, T_x maps W isomorphically onto $T_x(W)$. Thus, $T'_x \oplus T_x : H = H/W \oplus W \to T_x(W)^{\perp} \oplus T_X(W) = H$ is an isomorphism.
We thus get a continuous map $T' \oplus T : X \to \mathcal{F}^{\times}$ defined by $x \mapsto T'_x \oplus T_x$. The homotopy $G : X \times I \to \mathcal{F}$ defined by $G(x,t) := (tT'_x) \oplus T_x$ shows that T and $T' \oplus T$ are homotopic as desired (It is a fact from functional analysis that if f is Fredholm operator and g is finite rank operator, then $f \oplus g$ is also Fredholm; hence the above homotopy is indeed well-defined). Hence, the index map is injective. This completes the proof of the Atiyah-Jänich Theorem. □

As an immediate consequence of the the Atiyah-Jänich Theorem is the following:

Corollary 2.11. *The monoid $[X, \mathcal{F}]$ is in fact a group and the map ind is a group isomorphism.*

To demonstrate the power of the Atiyah-Jänich theorem, we end this paper with the following application.

Theorem 2.12 (Homotopy groups of Fredholm operators)**.** *We have that*

$$\pi_n(\mathcal{F}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Proof. Take $X = S^n$ in Atiyah-Jänich theorem to get $[S^n, \mathcal{F}] \cong K(S^n)$. Hence, by Bott-Periodicity

$$[S^n, \mathcal{F}] = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \text{ even} \\ \mathbb{Z} & \text{if } n \text{ odd.} \end{cases}$$

Since elements in the n-th homotopy group consist of based maps from S^n to $\mathcal{F}$ and since $\{\text{connected components of } \mathcal{F}\} \cong \mathbb{Z}$,

$$\pi_n(\mathcal{F}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

□

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