LABELED CONFIGURATION SPACES AND GROUP COMPLETIONS

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Abstract. Given a pair of an partial abelian monoid M and a pointed space X, let $C^M(\mathbf{R}^\infty, X)$ denote the configuration space of finite distinct points in $\mathbf{R}^\infty$ parametrized by the partial monoid $X \wedge M$. In this note we will show that if M is embedded in a topological abelian group and if we put $\pm M = \{a - b \mid a,\, b \in M\}$ then the natural map $C^M(\mathbf{R}^\infty, X) \to C^{\pm M}(\mathbf{R}^\infty, X)$ induced by the inclusion $M \subset \pm M$ is a group completion. This generalizes the result of Caruso [1] that the space of "positive and negative particles" in $\mathbf{R}^\infty$ parametrized by X is weakly equivalent to $\Omega^\infty \Sigma^\infty X$.

1. Introduction

In [5] we assigned to any space Y and any partial abelian monoid M the configuration space $C^M(Y)$ of finite subsets of Y with labels in M. As a set $C^M(Y)$ consists of those pairs (S, σ), where S is a finite subset of Y and σ is a map $S \to M$. But (S, σ) is identified with (S', σ') when $S \subset S'$, $\sigma' | S = \sigma$, and $\sigma'(x) = 0$ if $x \notin S$. It should be noted that the topology of $C^M(Y)$ depends not only on the topologies of Y and M, but also on the partial monoid structure of M.

For any pointed space X let

$$C^M(\mathbf{R}^\infty, X) = C^{X \wedge M}(\mathbf{R}^\infty).$$

Here $X \wedge M$ is regarded as a abelian partial monoid such that $x_1 \wedge a_1, \cdots, x_k \wedge a_k$ are summable if and only if $x_1 = \cdots = x_k$ and $a_1, \cdots, a_k$ are summable in M, and in such a case we have $x \wedge a_1 + \cdots + x \wedge a_k = x \wedge (a_1 + \cdots + a_k)$.

Let $E^M(X) = \Omega C^M(\mathbf{R}^\infty, \Sigma X)$, where ΣX is the reduced suspension of X. As $C^M(\mathbf{R}^\infty, X)$ is a continuous functor of X, there exists a natural map

$$C^M(\mathbf{R}^\infty, X) \to \Omega C^M(\mathbf{R}^\infty, \Sigma X) = E^M(X).$$

The results of [5] imply the following.

(1) The map $C^M(\mathbf{R}^\infty, X) \to E^M(X)$ is a group completion, that is, induces an isomorphism of Pontrjagin ring

$$H_\bullet(C^M(\mathbf{R}^\infty, X))[\pi^{-1}] \cong H_\bullet(E^M(X))$$

for any commutative coefficient ring, where $\pi = \pi_0 C^M(\mathbf{R}^\infty, X)$.

(2) The correspondence $X \mapsto \pi_\bullet E^M(X)$ defines a generalized homology theory on the category of pointed spaces.

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Various homology theories arise in this way. For example, the stable homotopy and the ordinary homology theories correspond, respectively, to the subsets $\{0,1\}$ and $\mathbf{N} = \{0,1,2,\dots\}$ of the additive group of integers $\mathbf{Z}$. (The former is a consequence of the Barratt-Priddy-Quillen theorem and the latter is the Dold-Thom theorem.) On the other hand, the connective K-homology theory arises from $\mathrm{Gr}(\mathbf{R}^\infty)$, the Grassmannian of finite-dimensional subspaces of $\mathbf{R}^\infty$. Here $\mathrm{Gr}(\mathbf{R}^\infty)_J$ consists of those tuples (V_j) such that V_i and V_j are perpendicular if $i \neq j$, and $\sum_{j\in J} V_j$ is defined to be the direct sum $\bigoplus_{j\in J} V_j$. (Compare [4].)

The aim of this note is to show that when M is embedded in a topological abelian group then we can take $C^{\pm M}(\mathbf{R}^\infty, X)$ as a group completion of $C^M(\mathbf{R}^\infty, X)$, where $\pm M = \{a - b \mid a,\, b \in M\}$. More precisely, we will show

Theorem 1. *Let M be an arbitrary subset of a topological abelian group such that $0 \in M$. Then for any pointed space X the natural map $C^M(\mathbf{R}^\infty, X) \to C^{\pm M}(\mathbf{R}^\infty, X)$ induced by the inclusion $M \subset \pm M$ is a group completion.*

This implies that there is a natural isomorphism of homology theories

$$\pi_\bullet E^M(X) \cong \pi_\bullet C^{\pm M}(\mathbf{R}^\infty, X).$$

In particular, let $M = \{0,1\} \subset \mathbf{Z}$. Then $C^M(\mathbf{R}^\infty, X) = C(\mathbf{R}^\infty, X)$ is the standard configuration space of finite subsets of $\mathbf{R}^\infty$ parametrized by X. On the other hand, $C^{\pm M}(\mathbf{R}^\infty, X) = C^{\pm}(\mathbf{R}^\infty, X)$ is the space of positive and negative particles introduced by Mcduff [2]. Caruso has shown in [1] that $C^{\pm}(\mathbf{R}^\infty, X)$ is weakly homotopy equivalent to $\Omega^\infty\Sigma^\infty X$ if X is locally equi-connected. By using Theorem 1 this can be generalized, both in M and in X, as follows.

Theorem 2. *Let M be a finite set of integers such that $0 \in M$. Suppose that M contains at least one non-zero element and is stable under the involution $n \mapsto -n$. Then $C^M(\mathbf{R}^\infty, X)$ is weakly homotopy equivalent to $\Omega^\infty\Sigma^\infty X$ for any pointed space X.*

To see this let d be the greatest common divisor of the positive members of M. Then there are positive integers k and l such that

$$\pm\{0,d\} \subset (\pm)^k M \subset (\pm)^l\{0,d\} \subset (\pm)^{k+l-1} M$$

holds, for we have $(\pm)^l\{0,d\} = \{0, \pm d, \dots, \pm ld\}$. By Theorem 1 we have

$$\pi_\bullet C^{\pm\{0,d\}}(\mathbf{R}^\infty, X) \cong \pi_\bullet C^{(\pm)^k M}(\mathbf{R}^\infty, X).$$

We also have

$$\pi_\bullet C^M(\mathbf{R}^\infty, X) \cong \pi_\bullet C^{(\pm)^k M}(\mathbf{R}^\infty, X)$$

because $C^M(\mathbf{R}^\infty, X)$ is a grouplike Hopf-space.

Thus $C^M(\mathbf{R}^\infty, X)$ is weakly equivalent to $C^{\pm\{0,d\}}(\mathbf{R}^\infty, X)$. But $C^{\{0,d\}}(\mathbf{R}^\infty, X)$ is homeomorphic to the standard configuration space $C(\mathbf{R}^\infty, X)$, hence its group completion $C^{\pm\{0,d\}}(\mathbf{R}^\infty, X)$ is weakly equivalent to $\Omega^\infty\Sigma^\infty X$ by the Barratt-Priddy-Quillen theorem.

Corollary 3. *If M is a finite set of integers containing at lease one non-zero element then $\pi_\bullet E^M(X)$ is the stable homotopy of a pointed space X.*

2. Proof of Theorem 1

In [5] we showed that there exist a CW monoid $D(X \wedge M)$ and a weak equivalence $\Phi\colon D(X \wedge M) \to C^M(\mathbf{R}^\infty, X)$ natural in X. Let us briefly recall the definitions.

Let $\mathcal{Q}(M)$ be the topological category whose space of objects is the disjoint union $\coprod_{p\geq 0} M^p$, and whose morphisms from $(a_i) \in M^p$ to $(b_j) \in M^q$ are maps of finite sets $\theta\colon \{1,\dots,p\} \to \{1,\dots,q\}$ such that $b_j = \sum_{i\in\theta^{-1}(j)} a_i$ holds for $1 \leq j \leq q$. Let $Q(M)$ denote the classifying space of $\mathcal{Q}(M)$, that is, the realization of the nerve $[k] \mapsto N_k\mathcal{Q}(M)$. Then $Q(M)$ is a homotopy commutative monoid, because $\mathcal{Q}(M)$ is a permutative category with respect to the operation

$$(a_1,\dots,a_p)\cdot(b_1,\dots,b_q) = (a_1,\dots,a_p,b_1,\dots,b_q).$$

Given a pointed space X let $D(X \wedge M) = |S_\bullet Q(X \wedge M)|$ be the realization of the total singular complex of $Q(X\wedge M)$. Then $D(X\wedge M)$ inherits from $Q(X\wedge M)$ a monoid structure with respect to which the weak equivalence

$$D(X \wedge M) = |S_\bullet Q(X \wedge M)| \to Q(X \wedge M)$$

is a map of topological monoids. Note that $D(X \wedge M)$ is homeomorphic to the realization of the diagonal simplicial set

$$[k] \mapsto S_k N_k \mathcal{Q}(X \wedge M) = N_k\mathcal{Q}(S_k(X \wedge M)).$$

Let us define $\Phi\colon D(X \wedge M) \to C^M(\mathbf{R}^\infty, X)$ to be the composite

$$D(X \wedge M) = |N_\bullet \mathcal{Q}(S_\bullet(X \wedge M))| \xrightarrow{\Phi'} |S_\bullet C^{X\wedge M}(\mathbf{R}^\infty)| \to C^{X\wedge M}(\mathbf{R}^\infty)$$

where Φ' is a weak equivalence constructed in [5, §4]. Since Φ is a weak equivalence of Hopf-spaces, Theorem 1 follows from

Proposition 4. *The natural map $D(X \wedge M) \to D(X \wedge \pm M)$ induced by the inclusion $M \subset \pm M$ is a group completion.*

The rest of the note is devoted to the proof of this proposition.

Given a map of topological monoids $f\colon D \to D'$ let $B(D, D')$ denote the realization of the category $\mathcal{B}(D, D')$ whose space of objects is D' and whose space of morphisms is the product $D \times D'$, where $(d, d') \in D \times D'$ is regarded as a morphism from d' to $f(d)\cdot d'$. Then there is a sequence of maps

$$D' = B(0, D') \to B(D, D') \to B(D, 0) = BD$$

induced by the maps $0 \to D$ and $D' \to 0$ respectively. Observe that BD is the standard classifying space of the monoid D and $B(D, D)$ is contractible when f is the identity.

Let us consider the commutative diagram

$$\begin{array}{ccccc}
D(X\wedge M) & \longrightarrow & B(D(X\wedge M), D(X\wedge M)) & \longrightarrow & BD(X\wedge M) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle B(1,i)} & & \| \\
D(X\wedge \pm M) & \longrightarrow & B(D(X\wedge M), D(X\wedge \pm M)) & \longrightarrow & BD(X\wedge M)
\end{array} \tag{2.1}$$

in which the upper and the lower sequences are associated with the identity and the inclusion $i\colon D(X \wedge M) \to D(X \wedge \pm M)$, respectively.

Lemma 5. *The natural map $D(X \wedge M) \to \Omega BD(X \wedge M)$ is a group completion.*

This follows from the fact that $D(X \wedge M)$ is a homotopy commutative, hence admissible, monoid.

Lemma 6. *The lower sequence in the diagram (2.1) is a homotopy fibration sequence with contractible total space.*

Proposition 4 can be deduced from this, because $D(X \wedge M) \to D(X \wedge \pm M)$ is equivalent to the group completion map $D(X \wedge M) \to \Omega BD(X \wedge M)$ under the equivalence $D(X \wedge \pm M) \simeq \Omega BD(X \wedge M)$.

Proof of Lemma 6. Let us write $D = D(X \wedge M)$ and $D' = D(X \wedge \pm M)$. Since D acts on D' through homotopy equivalences, the diagram

$$\begin{array}{ccc} D' & \longrightarrow & B(D, D') \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B(D, 0) \end{array}$$

is homotopy cartesian by Proposition 1.6 of [3]. This implies that the lower sequence in the diagram (2.1) is a homotopy fibration sequence.

It remains to prove that $B(D, D')$ is contractible.

Notice that $B(D, D')$ is homeomorphic to the realization of the diagonal simplicial set

$$[k] \mapsto E_k = N_k\mathcal{B}(D_k, D'_k)\,.$$

where $D_k = N_k\mathcal{Q}(S_k(X \wedge M))$ and $D'_k = N_k\mathcal{Q}(S_k(X \wedge \pm M))$. Hence $B(D, D')$ is a CW-complex whose 0-cells correspond to elements of D'_0, 1-cells to pairs from $D_1 \times D'_1$, and so on. In particular, a pair consisting of $(S \to \theta_* S) \in D_1$ and $(T \to \psi_* T) \in D'_1$ determines a path in $B(D, D')$ joining $d_0 T$ to $d_1(\theta_* S \cdot \psi_* T)$.

Let $\|E\|$ denote the thick realization of $[k] \mapsto E_k$. We shall show that the natural map $q\colon \|E\| \to |E| = B(D, D')$ is homotopic to the constant map. This implies that $B(D, D')$ is contractible, since q is a homotopy equivalence. (See [3, Appendix A].)

Let $r\colon \|E\| \to \|D'\|$ the map induced by the functor $\mathcal{B}(D, D') \to D'$, which takes a morphism (d, d') to the identity of $d \cdot d'$. Clearly the composite $jr\colon \|E\| \to \|E\|$ is homotopic to the identity. Hence q is homotopic to the constant map if so is $qj\colon \|D'\| \to B(D, D')$.

Let $\|D'\|_n$ denote the image of $\coprod_{k \le n} D'_k \times \Delta^k$ in $\|D'\|$, and let $qj_n\colon \|D'\|_n \to B(D, D')$ be the restriction of qj to $\|D'\|_n$. We construct a null homotopy h_n of qj_n by successively extending h_{n-1} for all $n \ge 0$.

For every element a of $\pm M$ choose $a^+ \in M$ and $a^- \in -M$ such that $a = a^+ + a^-$ holds. If $S = (x_j \wedge a_j) \in S_0(X \wedge \pm M)^p$ is an element of $D'_0 = \|D'\|_0$ then we put

$$S_+ = (x_j \wedge a_j^+) \in D_0, \quad S_- = (x_j \wedge a_j^-), \ \overline{S} = (x_j \wedge -a_j) \in D'_0.$$

Let $[S] \in B(D, D')$ denote the image of S under qj_0. If we regard S_+ and S_- as elements of D_1 and D'_1, respectively, via the degeneracy s_0 then the pair $(S_+, S_-) \in D_1 \times D'_1 = E_1$ determines a path in $B(D, D')$ joining $[S_-]$ to $[S_+ \cdot S_-]$.

On the other hand, the map $\nabla\colon \{1, \dots, 2p\} \to \{1, \dots, p\}$ such that $\nabla(j) = \nabla(p + j) = j$ $(1 \le j \le p)$ determines a path in $D' \subset B(D, D')$ joining $[S_+ \cdot S_-]$ to $[\nabla_*(S_+ \cdot S_-)] = [S]$. Hence we obtain a composite path in $B(D, D')$ joining $[S_-]$ to $[S]$, which we shall denote by the symbol

$$[S_-] \xrightarrow{\mu} [S]$$

Similarly, we have paths

$$[S_-] \xrightarrow{\gamma} [\mathbf{0}^p], \quad \emptyset \xrightarrow{\nu} [\mathbf{0}^p]$$

induced by the sequence $S_- \to \overline{S_-} \cdot S_- \to \nabla_*(\overline{S_-} \cdot S_-) = \mathbf{0}^p$ and the unique map $\nu\colon \emptyset \to \{1, \dots, p\}$, respectively.

Now we have a path $\alpha(S)\colon I \to B(D, D')$ joining S to $\emptyset$ induced by the chain

$$[S] \xleftarrow{\mu} [S_-] \xrightarrow{\gamma} [\mathbf{0}^p] \xleftarrow{\nu} \emptyset\,.$$

Thus the correspondence $(S, t) \mapsto \alpha(S)(t)$ defines a null homotopy of qj_0,

$$h_0\colon \|D'\|_0 \times I \to B(D, D')\,.$$

We shall extend h_0 to a null homotopy over $\|D'\|_1$. Let $\theta\colon S \to T$ be an element of D'_1, where $S \in S_1(X \wedge \pm M)^p$ and $T = \theta_* S \in S_1(X \wedge \pm M)^q$, and let $[\theta]$ be the composite of the 1-cell $I \to \|D'\|_1$ corresponding to θ with $qj_1\colon \|D'\|_1 \to B(D, D')$. Thus $[\theta]$ is a path in $B(D, D')$ joining $[d_0 S]$ to $[d_1 T]$.

Then we have a diagram

$$\text{(2.2)}\qquad \begin{array}{ccccccccc}
[d_1T] & = & [d_1T] & \xleftarrow{\mu} & [d_1T_{-,0}] & \xrightarrow{\gamma} & [\mathbf{0}^q] & \xleftarrow{\nu} & \emptyset \\
{\scriptstyle [\theta]}\uparrow & & {\scriptstyle \xi}\uparrow & & {\scriptstyle \psi}\uparrow & & {\scriptstyle \xi^0}\uparrow & & \| \\
[d_0S] & \xrightarrow{1\cdot\nu} & [d_0S \cdot \mathbf{0}^q] & \xleftarrow{\mu\cdot\gamma} & [d_0S_{-,0} \cdot \overline{d_1T_+}] & \xrightarrow{\gamma\cdot\gamma} & [\mathbf{0}^p \cdot \mathbf{0}^q] & \xleftarrow{\nu} & \emptyset \\
\| & & \| & & {\scriptstyle 1\cdot\gamma}\downarrow & & \| & & \| \\
[d_0S] & \xrightarrow{1\cdot\nu} & [d_0S \cdot \mathbf{0}^q] & \xleftarrow{\mu\cdot 1} & [d_0S_{-,0} \cdot \mathbf{0}^q] & \xrightarrow{\gamma\cdot 1} & [\mathbf{0}^p \cdot \mathbf{0}^q] & \xleftarrow{\nu} & \emptyset \\
\| & & {\scriptstyle 1\cdot\nu}\uparrow & & {\scriptstyle 1\cdot\nu}\uparrow & & {\scriptstyle 1\cdot\nu}\uparrow & & \| \\
[d_0S] & = & [d_0S] & \xleftarrow{\mu} & [d_0S_{-,0}] & \xrightarrow{\gamma} & [\mathbf{0}^p] & \xleftarrow{\nu} & \emptyset
\end{array}$$

where ξ and ξ^0 are induced by the arrows

$$S \cdot \mathbf{0}^q \to \nabla(\theta \cdot 1)_*(S \cdot \mathbf{0}^q) = T, \quad \mathbf{0}^p \cdot \mathbf{0}^q \to \nabla(\theta \cdot 1)_*(\mathbf{0}^p \cdot \mathbf{0}^q) = \mathbf{0}^q$$

and ψ is the composite path

$$[d_0S_{-,0} \cdot \overline{d_1T_+}] \xrightarrow{\mu\cdot 1} [d_0S \cdot \overline{d_1T_+}] = [d_0(S \cdot s_0\overline{d_1T_+})] \to$$
$$[d_1(\nabla(\theta \cdot 1)_*(S \cdot s_0\overline{d_1T_+}))] = [\nabla_*(d_1T \cdot \overline{d_1T_+})] = [d_1T_{-,0}]\,.$$

One easily observes from (2.2) that there is a homotopy $[\theta] \simeq \emptyset$ which extends the one already defined on $\partial[\theta] = [d_0 S] \cup [d_1 T]$. Thus we can extend h_0 to a null homotopy h_1 over $\|D'\|_1$.

We need to extend the construction above to $\|D'\|_n$ for all $n \geq 0$. Suppose that for every $\mathcal{D} \in D'_k$ with $k < n$, there exists a null homotopy of the corresponding k-cell $[\mathcal{D}]\colon \Delta^k \to B(D, D')$ given by a chain of commutative diagrams

$$[\mathcal{D}] \to [\mathcal{D}_0] \leftarrow [\mathcal{D}_-] \to [\mathbf{0}^{\mathcal{D}}] \leftarrow \emptyset \tag{2.3}$$

which is compatible with face operators.

Let $\mathcal{D}' = (S(0) \xleftarrow{\theta_1} S(1) \xleftarrow{\theta_2} \cdots \xleftarrow{\theta_n} S(n))$ be an element of D'_n. Then the diagram similar to (2.2), but $d_0 S$ and $d_1 T$ are replaced by

$$d_0\mathcal{D}' = (d_0 S(1) \xleftarrow{\theta_2} \cdots \xleftarrow{\theta_n} d_0 S(n)) \in D'_{n-1}$$

and $d_1^n S(0)$ respectively, yields a null homotopy of the n-cell $[\mathcal{D}']$ which extends the ones already defined on its faces $[d_i\mathcal{D}']$. This implies that the null homotopy can be extended over $\|D'\|_n$, and hence completes the proof of the lemma. □

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