AN INTRODUCTION TO THE GEOMETRY OF ALEXANDROV SPACES

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Dedicated to Katsumi Shiratani on his sixtieth birthday

CONTENTS

§0. Introduction

This note was based on the lectures given at the Daewoo Workshop on Differential Geometry held at Kwang Won University, Chunchon, Korea from 13th till 17th July, 1992.

The purpose of this note is to introduce along with the works by A. D. Alexandrov, Y. Burago, M. Gromov and Perelman [A], [BGP] the Geometry of Alexandrov spaces to non-specialists of Riemannian geometry including graduate students with minimum back ground on Riemannian geometry. An Alexandrov space is a complete and locally compact length space with curvature bounded below or above and introduced by A.D.Alexandrov. The Busemann G-spaces are special Alexandrov spaces admitting geodesic completeness, where the notion of curvature bounded below or above are defined

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by a similar (but different) manner. The most important problem discussed by these pioneers was if the differentiability assumption in Riemannian results is really essential. This problem was one of the motivation of the fantastic book [B] by H.Busemann, and influenced to many geometers in 40-50th. decade who tried to prove Riemannian results under weaker differentiability assumptions. For instance, P.Hartman discussed geodesic parallel circles on C^2-Riemannian manifolds of dimension two, S.B.Myers proved his famous compactness theorem for complete C^3-Riemannian manifolds whose curvature is bounded below by a positive constant, and V.A.Toponogov extended the Myers compactness theorem to C^2-Riemannian manifolds, using the most powerfull and important tool:the Toponogov comparison theorem. Busemann extended the Cohn-Vossen theorem on the total curvature of complete open Riemannian 2-manifilds to the Busemann G-surfaces admitting total excess. Since 60th decade people discussed only C^∞-Riemannian manifolds and forgot this important problem, except perhaps A.D.Alexandrov and H.Busemann. This sleeping period lasted almost twenty years until a sudden break brought by M.Gromov. Inspired by a series of striking results by Gromov, Alexandrov spaces got footlight because they are obtained as the Hausdorff limits of complete Riemannian manifolds belonging to a certain class determined by geometry. An exciting recent work by Burago, Gromov and Perelman [BGP] is the most important one and contains many fruitful ideas. However it is not easy to read. The motivation of this note is to smooth their discussion in [BGP] and to make it understandable even for students. The discussion developed in sections 4, 5, 6, 7 and 9 of [BGP] is introduced here in sections 2, 3, 6, 7 and 8 with detailed proofs. The work of A. D. Alexandrov [A] is introduced in §§2,4.

The organization of this note is stated as follows. In §1 we introduce length spaces and Hausdorff topology on a class of compact metric spaces. The Gromov precompactness theorem and convergence theorem are explained as the background of giving the motivation of this topic (compare [GLP], [F]). In §2 Alexandrov spaces with curvature bounded below or above are introduced by using the same principle determined only by distance function. The notion of curvature bounded above was first introduced by A. D. Alexandrov [A] by using R_K-domain, which is different from ours. It turns out that they are equivalent (see Theroem 4.7 in §4). In §3 angles are naturally introduced on Alexandrov spaces with curvature bounded below. This is based on the fact that geodesics on such an Alexandrov space do not have branch. In §4 the notion of upper angles is defined for Alexandrov spaces with curvature bounded above. Examples of Alexandrov spaces with curvature bounded below or above are provided in §5. They all are obtained as the Hausdorff limits of Riemannian manifolds. The Toponogov comparison theorem is proved in §6. The idea of the proof of it is basically the same as the original one given by Toponogov. Some modifications are needed to adjust proof technique to Alexandrov space with curvature bounded below. A simpler than that in [BGP] will be exhibited here. The notion of strainers and strained points are introduced in §7, where we discuss the dimensions of Alexandrov spaces with curvature bounded below. In §8 we introduce the basic tools, such as tangent cones, the space of directions, cut locus and exponential map on Alexandrov spaces with curvature bounded below.

I would like to express my thanks to Hyeong In Choi for his constant encouragement and to T.Shioya, M.Tanaka and T.Yamaguchi for reading the first draft of this note and for their criticisms, and also to Miss Chae Won Park for her nice typing skill.

§1. Length Spaces

A length space X is by definition a locally compact and complete metric space with the Menger convexity. The Menger convexity of X means that for distinct points $x, y \in X$ there exists a point $z \neq x, y$ on X such that $d(x,z) + d(z,y) = d(x,y)$, where d is the distance function. By iterating this procedure we finally obtain by completeness of X a curve $\gamma : [0, d(x,y)] \to X$ joining x to y such that the length $L(\gamma)$ of γ is $d(x,y)$. Thus the Menger convexity is equivalent to state that there exists for every points $x, y \in X$ a curve γ joining x to y whose length realizes $d(x,y)$. We call such a γ a *geodesic*. We also denote by xy a geodesic joining x to y. The Hausdorff limit of complete Riemannian manifolds is a length space, (see Lemma 1.2).

We next define Hausdorff distance on the space of compact metric spaces (see [GLP]). For subsets A, B in a metric space Z we define

$$d_H^Z(A,B) := \inf\{\varepsilon > 0; B(A,\varepsilon) \supset B,\ B(B,\varepsilon) \supset A\},$$

where $B(A,\varepsilon) = \{x \in Z : d(x,A) < \varepsilon\}$ is an ε-ball around A. For metric spaces X, Y and Z we define

$$d_H(X,Y) := \inf\{d_H^Z(f(X), g(Y)); f : X \to Z \text{ and } g := Y \to Z \text{ are isometric embeddings}\}.$$

Here the infimum is taken over all metric spaces Z and all isometric embeddings of X, Y into Z. It is easy to check that $d_H(X,Y) = d_H(Y,X) \geq 0$ and $d_H(X,Y) = 0$ if and only if X is isometric to Y, and that the triangle inequality holds for d_H. Let $\mathcal{X}$ be the set of all isometry classes of compact metric spaces. Then $(\mathcal{X}, d_H)$ is a metric space.

A subset $Y \subset X$ is by definition an *ε-net* iff $B(Y,\varepsilon) = \bigcup_{y\in Y} B(y,\varepsilon) = X$. A subset $Y \subset X$ is said to be *ε-discrete* iff $d(y_1, y_2) \geq \varepsilon$ for every $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Every bounded set U in a length space admits a maximal ε-net for every $\varepsilon > 0$ which is ε-discrete.

We now define the Lipshitz distance between metric spaces X and Y. Let $d_L(X,Y)$ be

$$d_L(X,Y) := \begin{cases} \inf_{f:X\to Y}\left\{|\log\sup_{x_1\neq x_2}\frac{d_Y(f(x_1),f(x_2))}{d_X(x_1,x_2)}| \right. \\ \left. \quad +|\log\sup_{y_1\neq y_2}\frac{d_X(f^{-1}(y_1)f^{-1}(y_2))}{d_Y(y_1,y_2)}|\right\} \\ \infty \text{ (if there is no homeomorphism} \\ \quad f : X \to Y \end{cases}$$

Here the infimum is taken over all homeomorphisms between X and Y. The d_L defines the metric of the set of all isometry classes of compact metric spaces. We see from definition that $d_L(X,Y) = 0$ if and only if X is isometric to Y.

Proposition 1.1. *Let $\{X_i\}$ be a sequence of metric spaces converging to a metric space X with respect to the Hausdorff distance. For every $\varepsilon' > \varepsilon > 0$ and for every ε-net $\mathcal{N}(\varepsilon)$ in X there are ε'-nets $\mathcal{N}_i(\varepsilon')$ in X_i such that $\lim_{i\to\infty} {}_{d_L}\mathcal{N}_i(\varepsilon') = \mathcal{N}(\varepsilon)$. Conversely, if $\sup\{\mathrm{diam}(X_i),\ \mathrm{diam}(X)\} < \infty$ and if there exists for every $\varepsilon > 0$ and for every ε-net $\mathcal{N}(\varepsilon)$ in X, an ε-net $\mathcal{N}_i(\varepsilon)$ in X_i such that $\lim_{i\to\infty} {}_{d_L}\mathcal{N}_i(\varepsilon) = \mathcal{N}(\varepsilon)$, then $\lim_{i\to\infty} {}_{d_H}X_i = X$.*

The proof is omitted here (see Proposition 3.5 in [GLP]).

As a consequence of the above Proposition 1.1 we have the (see Proposition 3.8 in [GLP]).

Lemma 1.2. *If a sequence $\{X_i\}$ of length spaces converges to a complete metric space X with respect to d_H then X is a length space.*

Proof. We only need to prove the Menger convexity of X. Let $x, y \in X$ be distinct points. For an arbitrary small positive ε with $d(x,y) \gg \varepsilon$, there is a number $i(\varepsilon)$ such that $d_H(X_i, X) < \varepsilon$ for all $i > i(\varepsilon)$. There are isometric embeddings $f_i : X_i \to Z_i$ and $g_i : X \to Z_i$ for some metric space Z_i such that

$$d_H^{Z_i}(f_i(X_i), g_i(X)) < 2\varepsilon.$$

We then choose $x_i, y_i \in X_i$ for $i > i(\varepsilon)$ such that $d_H^{Z_i}(f_i(x_i), g_i(x)) < 2\varepsilon$ and $d_H^{Z_i}(f_i(y_i), g_i(y)) < 2\varepsilon$. If $z_i \in x_iy_i$ is the midpoint of x_iy_i in X_i, then there is a point $z_\varepsilon \in X$ such that $d^{Z_i}(f_i(z_i), g_i(z_\varepsilon)) < 2\varepsilon$. Therefore we have $d(x, z_\varepsilon) = d^{Z_i}(g_i(x), g_i(z_\varepsilon)) \le d^Z(g_i(x), f_i(x_i)) + d^Z(f_i(x_i), f_i(z_i)) + d^{Z_i}(f_i(z_i), g_i(z_\varepsilon)) < d(x_i, z_i) + 4\varepsilon = \frac{1}{2}d(x_i, y_i) + 4\varepsilon$.

Also we have

$$\begin{aligned} d(x_i, y_i) &\le d^{Z_i}(f_i(x_i), g_i(x)) + d^{Z_i}(g_i(x), g_i(y)) + d^{Z_i}(g_i(y), f_i(y_i)) \\ &< d(x,y) + 4\varepsilon. \end{aligned}$$

Therefore,

$$d(x, z_\varepsilon) < \frac{1}{2}d(x,y) + 6\varepsilon.$$

Similarly $d(y, z_\varepsilon) < \frac{1}{2}d(x,y) + 6\varepsilon$, and hence we find a point $z = \lim_{\varepsilon \to 0} z_\varepsilon$ with $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$. This proves Lemma 1.2. $\square$

The pointed Hausdorff convegence is discussed for noncompact length spaces. Let X_j and X be noncompact length spaces and $o_j \in X_j$, $o \in X$ be the base points. Then $\lim_{j\to\infty} {}_{d_H}(X_j, o_j) = (X; o)$ means that for all sufficiently large fixed $r > 0$ and for all $\varepsilon_j > 0$ with $\lim \varepsilon_j = 0$,

$$\lim_{j\to\infty} d_H(B(o_j, r+\varepsilon_j),\ B(o,r)) = 0.$$

A practical approach to the Hausdorff distance is stated as follows, (see [F]). A map $f : Y \to Z$ between compact metric spaces Y and Z is called an *ε-Hausdorff approximation map* for $\varepsilon > 0$ iff

$$|d_Z(f(y_1), f(y_2)) - d_Y(y_1, y_2)| < \varepsilon \quad \text{for } y_1, y_2 \in Y$$

and

$$B(f(Y), \varepsilon) = Z.$$

Here f is not reguired to be continuous. We then define

$$\widetilde{d}_H(Y, Z) := \inf \Big\{ \varepsilon > 0 : \text{there exsit } \varepsilon\text{-Hausdorff approximation maps } f : Y \to Z \text{ and } g : Z \to Y \Big\}$$

Then $\widetilde{d}_H$ satisfies that $\widetilde{d}_H(Y, Z) = \widetilde{d}_H(Z, Y) \geq 0$ and $\widetilde{d}_H(Y, Z) = 0$ if and only if Y is isometric to Z. Moreover $\widetilde{d}_H(Y, Z) \leq 2\{\widetilde{d}_H(Y, W) + \widetilde{d}_H(W, Z)\}$ holds for all compact metric spaces. Then $(\mathcal{X}, \widetilde{d}_H)$ gives a metrizable uniform structure. We may talk about the convergence with respect to $\widetilde{d}_H$ in $\mathcal{X}$.

Theorem 1.3. (The Gromov precompactness theorem). *For given $n \geq 2$, $\kappa \in \mathbf{R}$ and $D > 0$ we consider the $\mathcal{M}(n, \kappa, D)$ of all complete Riemannian n-manifolds where Ricci curvature is bounded below by $(n-1)\kappa$ and whose diameter is bounded above by D. Then the closure of $\mathcal{M}(n, \kappa, D)$ with respect to d_H in $\mathcal{X}$ is compact.*

We see that the Hausdorff limit of Riemannian manifolds belonging to $\mathcal{M}(n, \kappa, D)$ is a length space. If the class is restricted then the Hausdorff limit of Riemannian manifolds becomes a Riemannian manifold, as stated (see [GLP], [GW], [Pe]),

Theorem 1.4. (The Gromov convegence theorem). *For given integer $n \geq 2$ and $\kappa, D, V > 0$ let $\mathcal{M}(n, \kappa, D, V) \subset (\mathcal{X}, d_H)$ be the set of all complete Riemannian n-manifolds whose diameter is bounded above by D, volume bounded below by V and whose sectional curvature in absolute value is bounded a bove by κ. Then every convergent sequence $\{M_i\}$ in $\mathcal{M}(n, \kappa, D, V)$ with respect to d_H has a limit N which is a C^∞-compact n-manifold with $C^{1,\alpha}$-Riemannian metric for $0 < \alpha < 1$.*

In view of the above theorems it is important for the study of curvature and topology of Riemannian manifolds to investigate the topology of Alexandrov spaces. This also gives an important motivation for the study of Alexandrov spaces.

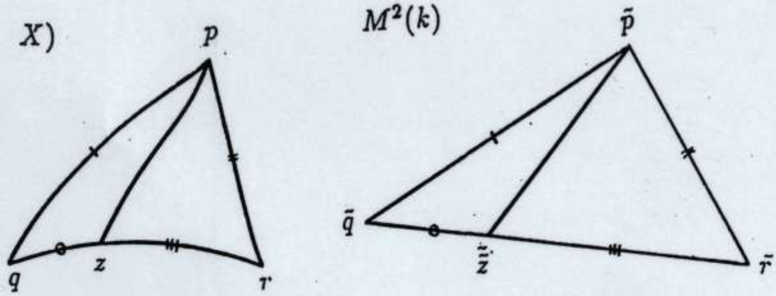


FIGURE 2-1.

§2. Alexandrov spaces

Let X be a complete locally compact length space. For a tripple of points $p, q, r \in X$ a geodesic triangle $\Delta(pqr)$ is by definition a tripple of geodesics joining these two points. We denote by $M^m(k)$ the m-dimensional complete simply connected space of constant sectional curvature k. For a geodesic triangle $\Delta(pqr)$ in X we denote by $\Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$ a geodesic triangle sketched in $M^2(k)$ whose corresponding edges have equal lengths as $\Delta(pqr)$. If $k > 0$ we always assume for a moment that the circumference of $\Delta(pqr)$ is less them $2\pi/\sqrt{k}$. This assumption in the case of positive lower curvature bound will be removed later in Theorem 6.2 by showing that every $\Delta(pqr)$ has its circumference not greater than $2\pi/\sqrt{k}$ if X has curvature bounded below by $k > 0$.

Definition 2.1. The definition of $\mathrm{Curv}(X) \geq k$ ($\mathrm{Curv}(X) \leq k$ respectively). X is said to have curvature bounded below (above, respectively) by k (and hencefore this will be denoted by $\mathrm{Curv}(X) \geq k$ ($\mathrm{Curv}(X) \leq k$, respectively)) iff for every point $x \in X$ there exists an open set U_x around x such that for every geodesic triangle $\Delta(pqr)$ whose edges are contained entirely in U_x the corresponding geodesic triangle $\Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$ sketched in $M^2(k)$ has the following property: For every point $z \in qr$ and for $\tilde{\tilde{z}} \in \tilde{q}\,\tilde{r}$ with $d(q, z) = d(\tilde{q}, \tilde{\tilde{z}})$ we have (see Figure 2-1)

$$d(p, z) \geq d(\tilde{p}, \tilde{\tilde{z}}),\ (d(p, z) \leq d(\tilde{p}, \tilde{\tilde{z}}),\ \text{respectively})$$

We denote by $\angle\tilde{p}\tilde{q}\tilde{r}$ the angle at $\tilde{q}$ of $\Delta(\tilde{p}\tilde{q}\tilde{r})$. For convenience we often write $\tilde{\Delta}(pqr)$ instead of $\Delta(\tilde{p}\tilde{q}\tilde{r})$ and also $\tilde{\angle}pqr$ instead of $\angle\tilde{p}\tilde{q}\tilde{r}$.

We now discuss X with curvature bounded below. Let $\mathrm{Curv}(X) \geq k$. Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be geodesics emanating from a point $p = \alpha(0) = \beta(0)$ and $\Delta_{st} := \Delta(\alpha(s)p\beta(t))$ for $0 \leq s \leq a$ and $0 \leq t \leq b$. Set $q = \alpha(a)$ and $r = \beta(b)$. Let $\theta_k(s,t)$ be the angle at $\tilde{p}$ of $\tilde{\Delta}_{st} := \Delta(\tilde{\alpha}(s)\tilde{p}\tilde{\beta}(t))$. Then the Alexandrov convexity property for angles at p is stated as follows.

2.2. The local version of the Alexandrov convexity (concavity) property.

For every $x \in X$ there exists an open set U_x around x such that for any geodesic triangle $\Delta(pqr)$ contained entirely in U_x having α and β the angle $\theta_k(s,t)$ is monotone non-increaing in the following sense (see Figure 2-2)

$$\theta_k(s_1,t_1) \geq \theta_k(s_2,t_2) \text{ for } 0 \leq s_1 \leq s_2 \leq a,\ 0 \leq t_1 \leq t_2 \leq b$$

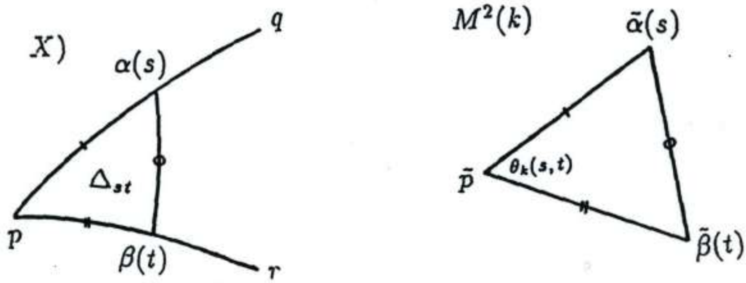


FIGURE 2-2.

Notice that if a complete Riemannian manifold M has its sectional curvature founded below by k, then (the local version of) Alexandrov convexity property holds.

Proposition 2.3. *$\mathrm{Curv}(X) \geq k$ holds if and only if the local version of Alexandrov convexity holds for every point $p \in X$ and for every geodesics α, β emanating from p.*

Proof. It is clear to show that $\mathrm{Curv}(X) \geq k$ implies the Alexandrov convexity property. Assume that the local version of Alexandrov convexity property holds. Let $p \in X$ and α, β be geodesics emanating from p. It suffices to prove that if $\theta_k(s_1, t) \geq \theta_k(s_2, t)$ for $0 \leq s_1 \leq s_2 \leq a$, then $d(\beta(t), \alpha(s_1)) \geq d(\tilde{\beta}(t), \tilde{\tilde{\alpha}}(s_1))$ for $\tilde{\tilde{\alpha}}(s_1) \in \tilde{p}\,\tilde{\alpha}(s_2)$ with $d(\tilde{\tilde{a}}(s_1), \tilde{p}) = d(p, \alpha(s_1)) = s_1$.

Let $S(\tilde{p}, s_1)$ be the geodesic (smooth) circle in $M^2(k)$ around $\tilde{p}$ with radius s_1, and parametrized by angle $\theta \in [0, 2\pi]$. By identifying θ with the point $w \in S(\tilde{p}, s_1)$ such that $\angle w\tilde{p}, \tilde{\beta}(t) = \theta$, we observe that $\theta \mapsto d(\tilde{\beta}(t), \theta)$ is strictly increasing in $\theta \in (0, \pi)$ (see Figure 2-3).

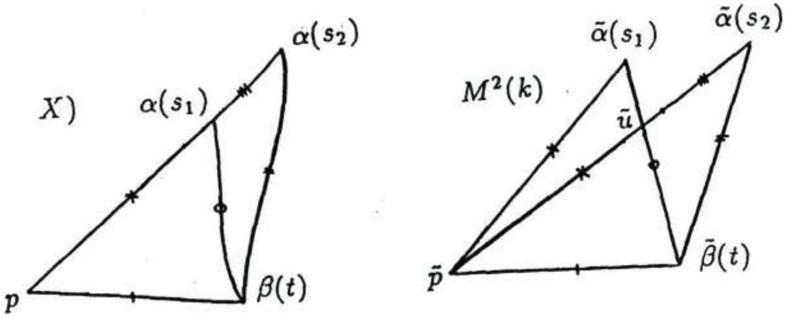


FIGURE 2-3.

If $\tilde{u} \in \tilde{p}\,\tilde{a}(s_2)$ is the point of intersection with $S(\tilde{p}, s_1)$, then $d(\tilde{\beta}(t), \tilde{u}) \leq d(\tilde{\beta}(t), \tilde{a}(s_1))$ follows from the assumption that $\theta_k(s_1, t) \geq \theta_2(s_2, t)$. Here we use the property that

$$\theta \mapsto d(\theta, \tilde{\beta}(t)), \quad \theta \in [0, \pi] \tag{2-1}$$

is strictly increasing, and

$$\theta \mapsto \angle \tilde{p}\theta\tilde{\beta}(t) \tag{2-2}$$

is strictly decreasing for $t \geq s_1$.

Thus the proof is complete. $\square$

Remark 2.1. It follows from definition that if $k_1 \geq k_2$, then $\mathrm{Curv}(X) \geq k_1$ implies $\mathrm{Curv}(X) \geq k_2$.

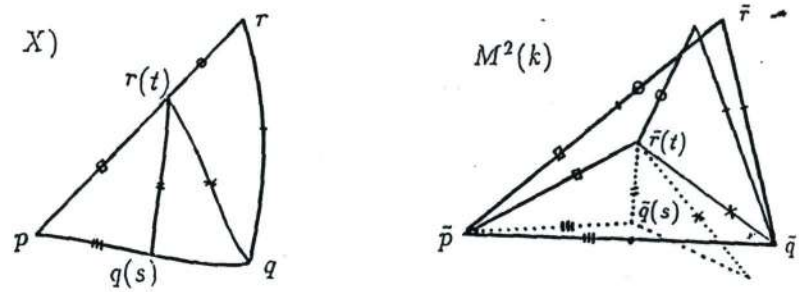


FIGURE 2-4.

Lemma 2.4. *If* $\mathrm{Curv}(X) \geq k$, *then any geodesic in* X *does not have a branch point.*

Proof. Suppose that there exists a branch point x of some geodesic, e.g., x belongs to an ν iterior point of geodesics pr and pq such that $pr \cap pq = px$ and such that $xr \subset pr$, $xq \subset pq$ and $xr \cap xq = \{x\}$. Choose points $r_1 \in xr$ and $q_1 \in xq$ such that $d(x, r_1) = d(x, q_1)$. Clearly, the corresponding triangle $\tilde{\Delta}(pq_1r_1)$ is a nondegenerate isoceles triangle. If $\hat{x}_1 \in \tilde{p}\tilde{r}_1$ and $\tilde{x}_2 \in \tilde{p}\tilde{\varepsilon}_1$ are chosen such that $d(\tilde{p}, \hat{x}_1) = d(p, x) = d(\tilde{p}, \hat{x}_2)$, then $d(\hat{x}_1, \hat{x}_2) > 0$ leads to a contradiction to the assumption $\mathrm{Curv}(X) \geq k$. This proves Lemma 2.4. $\square$ $\tilde{r}$

We now discuss the case where X has curvature bounded above by k. It follows from Definition 2.1 that if $\Delta(pqr) \subset U_x$ and if $q(s) \in pq$ and $r(t) \in pr$ are chosen such that

$$d(p, q(s)) = s \in [0, d(p,q)], \; d(p, r(t)) = t \in [0, d(p,r)]$$

and if $\tilde{\tilde{q}}(s) \in \tilde{p}\,\tilde{q}$ and $\tilde{\tilde{r}}(t) \in \tilde{r}\,\tilde{p}$ are chosen such that

$$d(\tilde{p}, \tilde{\tilde{q}}(s)) = s, \; d(\tilde{p}, \tilde{\tilde{r}}(t)) = t,$$

then $d(q(s), r(t)) \leq d(\tilde{\tilde{q}}(s), \tilde{r}(t)) \leq d(\tilde{\tilde{q}}(s), \tilde{\tilde{r}}(t))$,(see Figure 2-4).
$\tilde{\Delta}(pqr(t)) = \Delta(\tilde{p}\,\tilde{q}\,\tilde{r}(t))$, $\tilde{\Delta}(pqr) = \Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$, $\tilde{\Delta}(pq(s)r(t)) = \Delta(\tilde{p}\,\tilde{q}(s)\,\tilde{r}(t))$

Therefore if $\omega_k(s,t)$ is the angle at $\tilde{p}$ of $\Delta(\tilde{p}\,\tilde{q}(s)\,\tilde{r}(t))$, (e.g., $\omega_k(s,t) = \angle\tilde{r}(t)\,\tilde{p}\,\tilde{q}(s)$), then $\omega_k(s,t)$ is monotone non-decreasing in s and t, and hence the limit of $\omega_k(s,t)$ as $s,t \to 0$ exists. We define the upper angle $\bar{\angle}rpq$ as this limit in §4. However upper angles do not have nice properties because of the existence of branch points on geodesics. The behavior of geodesics on an Alexandrov space with $\mathrm{Curv}(X) \le k$ is quite different from the case of lower curvature bound.

The monotone non-decreasing property of $w_k(s,t)$ in s and t is called *the local version of the Alexandrov concavity property* for pq and pr. We do not have the global version of the Alexandrov concavity property. The following Proposition is clear and its proof is omitted.

Proposition 2.5. *$\mathrm{Curv}(X) \le k$ is equivalent to state that the local version of the Alexandrov concavity property holds for every point $x \in X$ and for every geodesic xy and xz contained in U_x.*

Lemma 2.6. (The existence of fundamental length). *If $\mathrm{Curv}(X) \le k$, then any points $p, q \in U_x$ for some $x \in X$ are joined by a unique geodesic. In particular X does not admit sufficiently small geodesic biangles.*

Proof. Suppose α and β are distinct geodesics joining p to q in U_x. Here p and q are chosen in δ-ball $B(x,\delta)$ around x, where $B(x,2\delta)$ is contained entirely in U_x. If $r \in \beta$, $r' \in \alpha$ are the midpoints of them, then a contradiction is derived. In fact the corresponding geodesic triangle $\tilde{\Delta}(pqr)$ is degenerate in $M^2(k)$, and hence $d(r,r') \le d(\tilde{r},\tilde{r}') = 0$. This proves Lemma 2.6. □

Lemma 2.7. (The existence of strongly convex balls). *Let $\mathrm{Curv}(X) \le k$. Let $R > 0$ be a positive constant ($R < \pi/2\sqrt{k}$, if $k > 0$) with the property that $B(x,2R) \subset U_x$. Then $B(x,R)$ is strongly convex in the sense that any points in $B(x,R)$ can be joined by a unique geodesic lying in $B(x,R)$. In particular, the distance function to x is convex in $B(x,R)$.*

The convexity property of $d(x,\cdot)$ on $B(x,R)$ is a direct consequence of the Definition 2.1 of $\mathrm{Curv}(X) \le k$, because a continuous midconvex function is convex. The proof is omitted.

§3. Angles

Throughout this section let X be an Alexandrov space with $\mathrm{Curv}(X) \geq k$ for some $k \in \mathbf{R}$. For two geodesics α and β emanating from a point p, the monotone property of the angle $\theta_k(s,t)$ at $\tilde{p}$ of the triangle $\tilde{\Delta}(p\alpha(s)\beta(t))$ corresponding to $\Delta(p\alpha(s)\beta(t))$, for sufficiently small s,t implies the existence of the limit of $\theta_k(s,t)$ as $s,t \to 0$. This makes it possible to define a natural angle at p between α and β.

Definition 3.1. The angle $\angle qpr$ for $q \in \alpha$, $r \in \beta$ is defined by

$$\angle qpr := \lim_{s,t\to 0} \theta_k(s,t).$$

We observe from Definition 3.1 that

$$\lim_{s,t\to 0} \theta_k(s,t) = \lim_{s\to 0} \theta_k(s,s) = 2\sin^{-1}\frac{1}{2}\left(\lim_{s\to 0}\frac{d(\alpha(s),\beta(s))}{s}\right).$$

Moreover,

$$2\sin^{-1}\frac{1}{2}\left(\lim_{s\to 0}\frac{d(\alpha(s),\beta(s))}{s}\right) = \lim_{s\to 0}\theta_0(s,s). \tag{3-1}$$

Remark. If r is an interior point of a geodesic pq, then $\angle prq = \pi$. We shall prove in Lemma 3.5 that if $x \in X\backslash pq$ then $\angle xrp + \angle xrq = \pi$.

Making use of the property of angles as stated in the above Remark, we have the local version of the Toponogov comparison theorem.

Theorem 3.1. (The local version of the Toponogov theorem). *If $\mathrm{Curv}(X) \geq k$, and if $\Delta(pqr)$ is sufficiently small, then*

$$\angle pqr \geq \tilde{\angle} pqr,\ \angle qrp \geq \tilde{\angle} qrp,\ \angle rpq \geq \tilde{\angle} rpq.$$

The proof of Theorem 3.1 is straightforward and omitted here. We also have an equivalent statement of Theorem 3.1 which is called hinge theorem.

Theorem 3.1'. (Hinge theorem). *Let $Curv(X) \geq k$ and $\alpha : [0,a] \to X$, $\beta : [0,b] \to X$ be geodesics with $\alpha(0) = \beta(0)$. If $\alpha^* : [0,a] \to M^2(k)$, $\beta^* : [0,b] \to M^2(k)$ are geodesics with $\alpha^*(0) = \beta^*(0)$ and $\angle(\dot\alpha^*(0), \dot\beta^*(0)) = \angle\alpha(a)p\beta(b)$, then*

$$d(\alpha(s), \beta(t)) \leq d(\alpha^*(s), \beta^*(t))$$

for all $s \in [0,a]$ and $t \in [0,b]$.

In Riemannian geometry the local version of the Toponogov theorem is equivalent to the Alexandrov convexity property if the sectional curvature of a complete Riemannian manifold is bounded below by k. However, an angle between two geodesics in an Alexandrov space is not defined without curvature assumptions. Therefore Theorem 3.1 is not equivalent to the (local version of) Alexandrov convexity property. By assuming the existence of angles with certain properties, we shall prove the

Proposition 3.2. *Assume that a length space X has the property that the angle $\angle pqr$ at q of qp and qr exists in such a way that if p is an interior point of qr, then $\angle xpq + \angle xpr = \pi$ holds for all $x \in X$. Assume further that for every $x \in X$ there exists an open set U_x around x such that if $\Delta(pqr)$ is contained entirely in U_x, then*

$$\angle pqr \geq \tilde\angle pqr, \; \angle qrp \geq \tilde\angle qrp, \; \angle rpq \geq \tilde\angle rpq.$$

Then for every point $m \in qr$ we have

$$d(p,m) \geq d(\tilde p, \hat m),$$

where $\hat m \in \tilde q \tilde r$ is taken such that $d(q,m) = d(\tilde q, \hat m)$.

Proof. Suppose that $d(p,m) < d(\tilde p, \hat m)$ holds for some point $m \in qr$ and for some $\Delta(pqr)$,(see Figure 3-1).

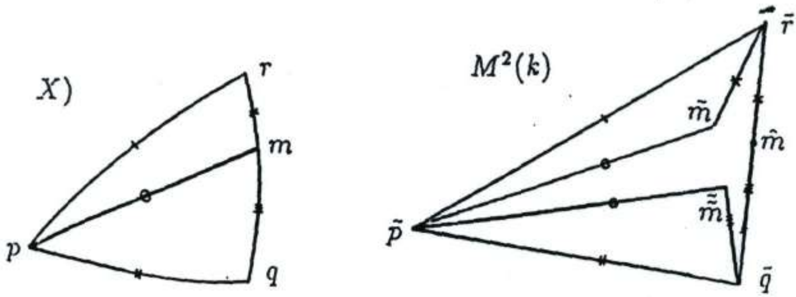


FIGURE 3-1.

$\tilde\Delta\, pqr = \Delta(\tilde p\,\tilde q\,\tilde r)$, $\tilde\Delta pmr = \Delta(\tilde p\,\tilde m\,\tilde r)$, $\tilde\Delta pqm = \Delta(\tilde p\,\tilde q\,\tilde{\tilde m})$

It follows from what we have supppsed that $\tilde m$ and $\tilde{\tilde m}$ are contained in $\Delta(\tilde p\,\tilde q\,\tilde r)$, and hence (2-1) implies that $\angle\tilde p\,\hat m\,\tilde q < \angle\tilde p\,\tilde{\tilde m}\,\tilde q$. Therefore the local version of the Toponogov theorem implies that $\angle\tilde p\,\tilde{\tilde m}\,\tilde q \le \angle pmq$, and thus we have $\angle pmq > \angle\tilde p\,\hat m\,\tilde q$. Similarly we have $\angle pmr > \angle\tilde p\,\hat m\,\tilde r$ from $\Delta\, pmr$ and $\Delta\,\tilde p\,\tilde r\,\hat m$. Summing up these two angles gives

$$\pi = \angle pmr + \angle pmq > \angle\tilde p\,\hat m\,\tilde r + \angle\tilde p\,\hat m\,\tilde q = \pi,$$

a contradiction. This proves Proposition 3.2. □

The angles of X with $\mathrm{Curv}(X) \ge k$ has the following properties.

Lemma 3.3. *Let $\{p_i\}, \{q_i\}, \{r_i\}$ be sequences of points in X such that $\lim p_i = p$, $\lim q_i = q$ and $\lim r_i = r$ and such that $\lim p_iq_i = pq$, $\lim p_ir_i = pr$, $\lim q_ir_i = qr$. Then we have*

$$\angle qpr \le \lim_{i\to\infty}\inf \angle q_ip_ir_i.$$

Proof. For an arbitrary fixed $\varepsilon > 0$ we choose a sufficiently small $\delta > 0$ such that if $y \in pq$ and $z \in pr$ satisfy $\delta > d(p, y) =: s$, $\delta > d(p, z) =: t$, then

$$\theta_k(s,t) \le \angle qpr \le \theta_k(s,t) + \varepsilon.$$

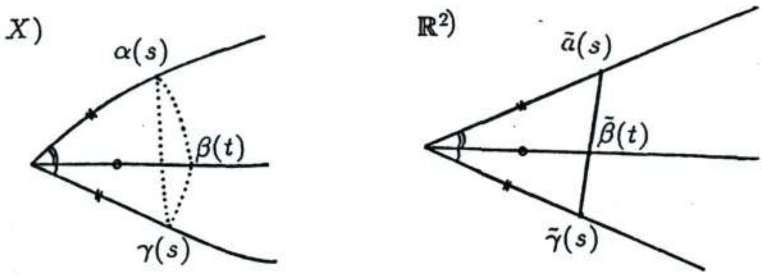


FIGURE 3-2.

Since p_iq_i and p_ir_i both converge to pq and pr there exists a number i_ε such that if $i > i_\varepsilon$, and if $y_i \in p_iq_i$ and $z_i \in p_ir_i$ are taken such that $s = d(p_i, y_i)$, $t = d(p_i, z_i)$, then

$$|\theta_k^i(s,t) - \theta_k(s,t)| < \varepsilon \quad \text{for all} \quad s,t \in (0,\delta).$$

Here $\theta_k^i(s,t)$ is the angle at $\tilde{p}_i$ of $\tilde{\Delta}(p_iq_iz_i)$. Therefore we get

$$\lim_{s,t\to 0} \theta_k^i(s,t) = \angle\, q_ip_ir_i \geq \theta_k^i(s,t) > \theta_k(s,t) - \varepsilon > \angle\, qpr - 2\varepsilon.$$

This concludes the proof since $\varepsilon > 0$ is arbitrary. $\square$

Lemma 3.4. *Let α, β, γ be geodesics emanating from $p \in X$, and take a point a on α, b on β and c on γ. Then*

$$\angle\, apc \leq \angle\, apb + \angle\, bpc.$$

Proof. From the definition of angles we see that they do not exceed π. Thus we only need to prove the case where $\angle\, apb + \angle\, bpc < \pi$. All geodesic triangles under consideration are sufficiently small and shrinking to a point p. In view of the equation (3-1) we may consider corresponding triangles sketcked on $\mathbf{R}^2$. For a sufficiently small $s > 0$ we take a triangle $\Delta(\tilde{p}\,\tilde{a}(s)\,\tilde{r}(s))$ on $\mathbf{R}^2$ such that $d(\tilde{p},\, \tilde{a}(s)) = d(\tilde{p},\, \tilde{r}(s)) = s$ and $\angle\, \tilde{\alpha}(s)\tilde{p}\,\tilde{\gamma}(s) = \angle\, apb + \angle\, bpc$ (see Figure 3-2).

Let $\tilde{\beta}(t) \in \tilde{\alpha}(s)\tilde{\gamma}(s)$ be taken such that

$$\angle \tilde{\alpha}(s)\tilde{p}\,\tilde{\beta}(t) = \angle apb,\ \angle \tilde{\gamma}(s)\tilde{p}\,\tilde{\beta}(t) = \angle bpc.$$

Notice that $\Delta(\tilde{p}\,\tilde{\alpha}(s)\tilde{\gamma}(s))$ forms a nondegenerate isosceles triangle because of $\angle apb + \angle bpc < \pi$.

Notice also that t/s is constant. Then

$$fd(\tilde{\alpha}(s), \tilde{\beta}(t))s = \lim_{s \to 0} \frac{d(\beta(t), \alpha(s))}{s},\ fd(\tilde{\beta}(t), \tilde{\gamma}(s))s = \lim_{s \to 0} \frac{d(\beta(t), \gamma(s))}{s}$$

and

$$d(\tilde{\alpha}(s), \tilde{\beta}(t)) + d(\tilde{\beta}(t), \tilde{\gamma}(s)) = d(\tilde{\alpha}(s), \tilde{\gamma}(s)).$$

We now denote by $\theta_k(s,t,u)$ the angle of a geodesic triangle in $M^2(k)$ with edge lengths s, t and u opposite to the edge of length u. Then

$$\lim_{\varepsilon \to 0} \theta_k(\varepsilon s, \varepsilon t, \varepsilon u) = \theta_0(s,t,u) \ \text{ and } \ \theta_0(\varepsilon s, \varepsilon t, \varepsilon u) = \theta_0(s,t,u).$$

With this notation we see

$$\angle apc = \lim_{s \to 0} \theta_k(s, s, d(\alpha(s), \gamma(s))) = \lim_{s \to 0} \theta_0(1, 1, \frac{d(\alpha(s), \gamma(s))}{s}),$$

and

$$\begin{aligned}\lim_{s \to 0} \frac{d(\alpha(s), \gamma(s))}{s} &\le \lim_{s \to 0} \frac{d(\alpha(s), \beta(t)) + d(\beta(t), \gamma(s))}{s} \\ &= \frac{d(\tilde{\alpha}(s), \tilde{\beta}(t))}{s} + \frac{d(\tilde{\beta}(t), \tilde{\gamma}(s))}{s} \\ &= \frac{d(\tilde{\alpha}(s), \tilde{\gamma}(s))}{s}.\end{aligned}$$

Therefore we have

$$\angle apc \le \theta_0(1, 1, \frac{d(\tilde{\alpha}(s), \tilde{\gamma}(s))}{s}) = \angle apb + \angle bpc.$$

This proves Lemma 3.4. □

Lemma 3.5. *If $r \in X$ is an interior point of a geodesic pq and if $x \in X$, $x \neq r$, then*

$$\angle prx + \angle qrx = \pi.$$

Proof. It follows from Lemma 3.4 that $\angle prx + \angle qrx \geq \angle prq = \pi$. Here $\angle prq = \pi$ is trivial from the definition of angles. Thus we only need to show

$$\angle prx + \angle qrx \leq \pi.$$

For sufficiently small $s > 0$ we choose points p_s, $q_s \in pq$ and $x_s \in rx$ such that $d(r, p_s) = d(r, q_s) = d(r, x_s) = s$. Let $\tilde{\Delta}(p_s q_s x_s) = \Delta(\tilde{p}_s \tilde{q}_s \tilde{x}_s)$. Let $\tilde{r} \in \tilde{p}_s \tilde{q}_s$ be the midpoint of $\tilde{p}_s \tilde{q}_s$ and set $\tilde{\Delta}(p_s r x_s) = \Delta(\tilde{p}_s \tilde{r} \tilde{x}'_s)$, $\tilde{\Delta}(q_s r x_s) = \Delta(\tilde{q}_s \tilde{r} \tilde{x}''_s)$ (see Figure 3-3). From assumption for curvature we see

$$d(\tilde{r}, \tilde{x}_s) \leq d(\tilde{r}, \tilde{x}'_s),\ d(\tilde{r}, \tilde{x}_s) \leq d(\tilde{r}, \tilde{x}''_s).$$

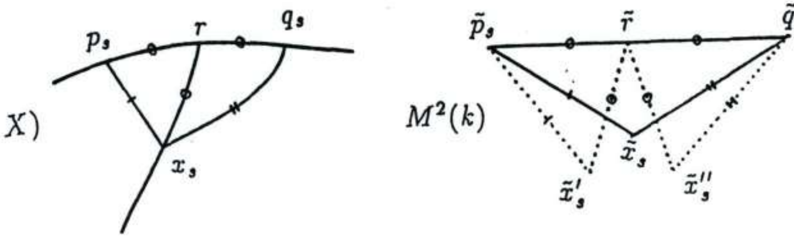


FIGURE 3-3.

By means of (2-1), $d(\tilde{q}_s, \tilde{x}_s) \leq d(\tilde{q}_s, \tilde{x}'_s)$ and $d(\tilde{p}_s, \tilde{x}_s) \leq d(\tilde{p}_s, \tilde{x}''_s)$. These inequalities mean that $\Delta(\tilde{p}_s \tilde{r} \tilde{x}'_s)$ and $\Delta(\tilde{q}_s \tilde{r} \tilde{x}''_s)$ do not intersect at their interior points, and hence $\angle \tilde{p}_s \tilde{r} \tilde{x}'_s + \angle \tilde{q}_s \tilde{r} \tilde{x}''_s \leq \pi$ holds for all sufficiently small $s > 0$. Therefore $\angle prx + \angle qrx = \lim_{s \to 0} (\angle \tilde{p}_s \tilde{r} \tilde{x}'_s + \angle \tilde{q}_s \tilde{r} \tilde{x}''_s) \leq \pi$. This proves Lemma 3.5. □

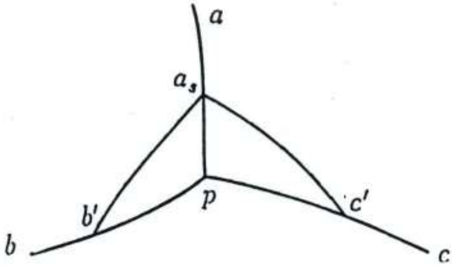


FIGURE 3-4.

Lemma 3.6. *For three distinct geodesics pa, pb and pc we have*

$$\angle apb + \angle bpc + \angle cpa \leq 2\pi.$$

Proof. Fix interior points b' of pb and c' of pc and a_s of pa with $d(p, a_s) = s$ for small $s > 0$, (see Figure 3-4). Then Lemma 3.4 implies that $\angle b'a_sc' \leq b'a_sp + \angle c'a_sp$ and hence $\angle aa_sb' + \angle aa_sc' + \angle b'a_sc' \leq (\angle aa_sb' + \angle b'a_sp) + (\angle aa_sc' + \angle c'a_sp) = 2\pi$. The last equality is due to Lemma 3.5.

Lemma 2.4 now implies that $\lim_{s\to 0} b'a_s = b'p$, $\lim_{s\to 0} c'a_s = c'p$. Apply Lemma 3.3 to $\{b'a_s\}$ and $\{c'a_s\}$ to obtain that

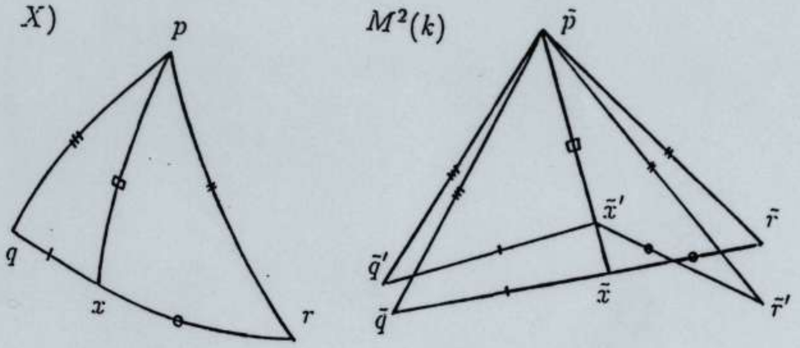
$$\liminf_{s\to 0} \angle aa_sb' \geq \angle apb' \ (= \angle apb)$$
$$\liminf_{s\to 0} \angle aa_sc' \geq \angle apc' \ (= \angle apc)$$
$$\liminf \angle b'a_sc' \geq \angle b'pc' \ (= \angle bpc).$$

This proves Lemma 3.6. $\square$

Consider the following property for a length space X.

The Four Points Property. For every $x \in X$ there is an open set U_x around x such that if $a, b, c, d \in X$ and $ab, bc, cd, da, ac, bd \subset U_x$ then

$$\tilde{\angle}\, bac + \tilde{\angle}\, cad + \tilde{\angle}\, dab \leq 2\pi.$$



$\tilde{\Delta}(pxr) = \Delta(\tilde{p}\,\tilde{x}'\,\tilde{r}')$, $\tilde{\Delta}(pxq) = \Delta(\tilde{p}\,\tilde{x}'\,\tilde{q}')$, $\tilde{\Delta}(pqr) = \Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$

FIGURE 3-5.

lemma 3.7. *Let X be a length space. Then $Curv(X) \geq \kappa$ is equivalent to the Four Points Property.*

Proof. Assume that X satisfies $\text{Curv}(X) \geq \kappa$. Then the Alexandrov convexity property holds, and the local version of the Toponogov comparison theorem is valid. Therefore we have

$$\tilde{\angle}\, bac + \tilde{\angle}\, cad + \tilde{\angle}\, dab \leq \angle\, bac + \angle\, cad + \angle\, dab.$$

Since the right hand side of the above inequality does not exceed 2π by Lemma 3.6, the Four Points Property is satisfied.

Assume that the Four Points Property holds on X. Take an interior point $x \in qr$ for a sufficiently small triangle $\Delta(pqr)$. From assumption it follows that

$$\tilde{\angle}\, pxq + \tilde{\angle}\, pxr + \tilde{\angle}\, qxr \leq 2\pi.$$

Since $\Delta(qxr)$ is degenerate, so is $\tilde{\Delta}(qxr)$ and thus we observe $\tilde{\angle}\, qxr = \pi$. The above inequality reduces to

$$\tilde{\angle}\, pxq + \tilde{\angle}\, pxr \leq \pi.$$

Suppose that $d(p,x) < d(\tilde{p},\tilde{x})$, for some $\tilde{x} \in \tilde{q}\tilde{r}$ and $x \in qr$ with $d(q,x) = d(\tilde{q},\tilde{x})$, where $\tilde{\Delta}(pqr) = \Delta(\tilde{p}\tilde{q}\tilde{r})$. Let $\tilde{\Delta}(pxr) = \Delta(\tilde{p}\tilde{x}'\tilde{r}')$, $\tilde{\Delta}(pxq) = \Delta(\tilde{p}\tilde{x}'\tilde{q}')$ and $\tilde{x}' \in \tilde{p}\tilde{x}$ be chosen such that $d(p,x) = d(\tilde{p},\tilde{x}')$, (see Figure 3-5). Cosine rule implies $0 = \cos\angle\tilde{p}\tilde{x}\tilde{q} + \cos\tilde{p}\tilde{x}\tilde{r}$. From $d(\tilde{p},\tilde{x}') < d(p,x)$ it follows that $\cos\angle\tilde{p}\tilde{x}'\tilde{q}' + \cos\angle\tilde{p}\tilde{x}'\tilde{r}' < 0$, and hence $\tilde{\angle}pxq + \tilde{\angle}pxr = \angle\tilde{p}\tilde{x}'\tilde{q}' + \angle\tilde{p}\tilde{x}'\tilde{r}' > \pi$, a contradiction. □

Remark. In 2.7;[BGP] it is stated that a length space X has the property $\mathrm{Curv}(X) \geq k$ if and only if every point $x \in X$ has a neighborhood U_x with the property that if $p, q, r, z \in U_x$ are any points then they are embedded isometrically into $M^3(k')$ for some $k' \geq k$. Here the k' depends on the choice of four points in U_x, (for detail, see [ABN]). This property is not discussed here since it is not used in this note.

§4. Upper Angles

The work of A. D. Alexandrov [A] is introduced to define upper angles between geodesics on a length space. Throughout this section let Y be a length space and $p, q, r \in Y$ be distinct points. Geodesics emanating from p and joining to q and r are expressed by $s \mapsto q(s)$, $t \mapsto r(t)$ where s and t are arc length parameters.

For an arbitrary fixed constant k let $\tilde{\Delta}(pq(s)r(t)) = \Delta(\tilde{p},\tilde{q}(s)\tilde{r}(t))$ be the triangle in $M^2(k)$ corresponding to $\Delta(pq(s)r(t))$. Then the *upper angle* $\bar{\angle}(pq,pr)$ at p of pq and pr is defined by

$$\bar{\angle}(pq,pr) := \lim_{s,t\to 0}\sup \tilde{\angle}q(s)pr(t).$$

It follows from definition that $\bar{\angle}(pq,pr) \in [0,\pi]$ and this angle is independent of k. With these notations we first prove the

Lemma 4.1. *If we set*

$$\cos \tilde{\angle} q(s)pr(t) =: \frac{s - d(q(s), r(t))}{t} + \varepsilon,$$

then, $\lim\limits_{\frac{t}{s}\to 0} \varepsilon = 0$.

Proof. By setting $\omega := \tilde{\angle} q(s)pr(t)$ and $u := d(r(t), q(s))$ we use the cosine rule for hyperbolic trigonometry to obtain

$$\cos \omega = \frac{\cosh kt - \cosh ku}{\sinh ks \cdot \sinh kt} + \frac{\cosh kt(\cosh ks - 1)}{\sinh ks \cdot \sinh kt}$$

Making use of $\cosh kt - \cosh ku = 2 \sinh \frac{k(t-u)}{2} \sinh \frac{k(t+u)}{2}$ and

$\cosh ks - 1 = 2 \sinh^2 \frac{ks}{2}$, $\sinh ks = 2 \sinh \frac{ks}{2} \cosh \frac{ks}{2}$ the above equations reduces to

$$\cos \omega = \frac{2 \sinh \frac{k(s-u)}{2} \cdot \sinh \frac{k(s+u)}{2}}{\sinh ks \cdot \sinh kt} + \frac{\sinh \frac{kt}{2} \cdot \cosh ks}{\sinh ks \cdot \cosh \frac{kt}{2}}.$$

Because of $t \to 0$ and $\frac{t}{s} \to 0$ the second term of the right hand side in the above equation tends to zero. Since $|s - u| \le t$ and $|1 - \frac{u}{s}| = \frac{t}{s} \to 0$, we get $\lim\limits_{\substack{s\to 0 \\ \frac{t}{s}\to 0}} \dfrac{\sinh \frac{k(s+u)}{2}}{\sinh ks} = 1$ and $\lim\limits_{s\to 0} \dfrac{2 \sinh \frac{k(s-u)}{2}}{\sinh kt} = \dfrac{s-u}{t}$. Therefore if $\varepsilon := \cos \omega - \frac{s-u}{t}$, then $\lim\limits_{\substack{t\to 0 \\ \frac{t}{s}\to 0}} \varepsilon = 0$. This proves Lemma 4.1. $\square$

We now want to prove the

Lemma 4.2.

$$\tilde{\angle}(pq, pr) = \overline{\lim_{t\to 0}} \sup_{s\in[0, d(p,q)]} \tilde{\angle} r(t)pq(s).$$

A basic inequality used in this section is:

$$s \mapsto s - d(r(t), q(s)), \quad s \in [0, d(pq)] \tag{4-1}$$

is monotone nondecreasing for each $t \in [0, d(p,r)]$.

Lemma 4.1 and (4-1) imply that

$$(4\text{-}1)^* \qquad \tilde{\angle} r(t)pq(s) \leq \tilde{\angle} r(t)pq(s'),\ 0 \leq s' \leq s \leq d(p,q).$$

Proof. In view of above discussion we observe that

$$\tilde{\angle}(pq, pr) \leq \varlimsup_{t\to 0} \left\{ \sup_{s\in[0,d(p,q)]} \tilde{\angle} r(t)pq(s) \right\}.$$

For the proof of $\tilde{\angle}(pq,pr) \geq \varlimsup_{t\to 0}\left\{\sup_{s\in[0,d(p,q)]} \tilde{\angle} r(t)pq(s)\right\}$, we choose for $t > 0$ and $\varepsilon(t) \downarrow 0$ an $s(t) > 0$ such that

$$|\tilde{\angle} r(t)pq(s(t)) - \sup_{s\in[s,d(pq)]} \tilde{\angle} r(t)pq(s)| < \varepsilon(t).$$

If there exists a decreasing sequence $\{t_i\} \downarrow 0$ such that $\lim\limits_{i\to\infty} s(t_i) = 0$, then we have

$$\begin{aligned} \tilde{\angle}(pq,pr) &= \lim_{s,t\to 0} \sup \tilde{\angle} r(t)pq(s) \\ &\geq \varlimsup_{i\to\infty} \left[\tilde{\angle} r(t)pq(s(t_i)) - \varepsilon(t_i)\right] \\ &= \varlimsup_{t\to 0} \left\{ \sup_{s\in[0,d(p,q)]} \tilde{\angle} r(t)pq(s) \right\}. \end{aligned}$$

Therefore, we only consider the case where there exists a positive number $a > 0$ such that $s(t) \geq a$ for all $t > 0$. It suffices to prove

$$\tilde{\angle}(pq,pr) \geq \varlimsup_{\substack{t\to 0 \\ s\geq a}} \tilde{\angle} r(t)pq(s).$$

If $t \mapsto \overline{s}(t)$ is chosen so as to satisfy $\overline{s}(t) > 0$, $\lim\limits_{t\to 0} \overline{s}(t) = 0$ and $\lim\limits_{t\to 0} \dfrac{t}{\overline{s}(t)} = 0$, then we may consider $s(t) \geq a > \overline{s}(t)$ for all small $t > 0$. Then (4-1) and the cosine rule of plane trigonometry imply the existence of an $\varepsilon'' = \varepsilon''(t) > 0$ with $\lim\limits_{t\to 0} \varepsilon''(t) = 0$ such that

$$\tilde{\angle} r(t)pq(s(t)) \le \tilde{\angle} r(t)pq(s(t)) + \varepsilon''.$$

By taking the limit of the above inequality, we have

$$\begin{aligned}\overline{\lim_{t\to 0}} \left\{ \sup_{s\ge a} \tilde{\angle} r(t)pq(s) \right\} &\le \overline{\lim_{t\to 0}} \left\{ \tilde{\angle} r(t)pq(\overline{s}(t)) + \varepsilon''(t) \right\} \\ &\le \bar{\angle}(pq, pr).\end{aligned}$$

This proves Lemma 4.3. $\square$

As a direct consequence of the above discussion we have the

Corollary 4.4. *With the same notations as in Lemma 4.2 we have*

(1) $\cos \bar{\angle}(pq, pr) \le \underline{\lim}_{\substack{t\to 0\\ \frac{t}{s}\to 0}} \dfrac{s - d(q(s), r(t))}{t}$

(2) *For every fixed* $s \in [0, d(p,q)]$ *we have*

$$\frac{d}{dt} d(q(s), r(t))\Big|_{t=0} \ge \cos \bar{\angle}(pq, pr).$$

(3) *If x is an interior point of pq and if $y \ne x$, then* $\bar{\angle}\, pxy + \bar{\angle}\, qxy \ge \pi$.

(4) *If pa, pb and pc are geodesics emanating from p, then*

$$\bar{\angle}\, apb + \bar{\angle}\, bpc \ge \bar{\angle}\, apc.$$

Proof. The proofs of (1), (2) are clear from the discussion in the proof of the previous Lemma, and omitted. To prove (3) we choose sufficiently small $s, t > 0$ such that $p_s =: p(s) \in px$, $q_s =: q(s) \in qx$, $y_t =: y(t) \in xy$ as in Figure 3-3. For a fixed $s > 0$ we have from (1)

$$\cos \bar{\angle} pxy + \cos \bar{\angle}\, qxy \le \lim_{\substack{t\to 0\\ \frac{t}{s}\to 0}} \frac{2s - \{d(p(s), y(t)) + d(q(s), y(t))\}}{t} \le 0.$$

For the proof of (4) we take $a(s) \in pa$, $b(t) \in pb$, $c(u) \in pc$ such that $d(p, a(s)) = s$, $d(p, b(t)) = t$, $d(p, c(u)) = u$, and corresponding points in $M^2(k)$ so as to satisfy $\tilde{\Delta}(pa(s)b(t)) = \Delta(\tilde{p}\,\tilde{a}(s)\,\tilde{b}(t))$, $\tilde{\Delta}(pb(t)c(u)) = \Delta(\tilde{p}\,\tilde{b}(t)\,\tilde{c}(u))$. Then

$$\tilde{\angle}\,apb + \tilde{\angle}\,bpc = \lim_{s,t\to 0}\sup \angle\,\tilde{a}(s)\,\tilde{p}\,\tilde{b}(t) + \lim_{t,u\to 0}\sup \angle\,\tilde{b}(t)\,\tilde{p}\,\tilde{c}(u).$$

If t is chosen so as to satisfy $\tilde{b}(t) \in \tilde{a}(s)\tilde{c}(u)$, then

$$\tilde{\angle}\,apb + \tilde{\angle}\,bpc \geq \lim_{s,u\to 0}\sup \angle\,\tilde{a}(s)\tilde{p}\,\tilde{c}(u).$$

If $\tilde{\Delta}(pa(s)c(u)) = \Delta(\tilde{p}\,\tilde{a}(s)\hat{c}(u))$, then $d(\tilde{a}(s), \tilde{c}(u)) \geq d(\tilde{a}(s), \hat{c}(u))$ implies that

$$\lim_{s,u\to 0}\sup \angle\,\tilde{a}(s)\tilde{p}\,\tilde{c}(u) \geq \lim_{s,u\to 0}\sup \angle\,\tilde{a}(s)\tilde{p}\,\hat{c}(u) = \tilde{\angle}apc.$$

This proves Corollary 4.4. $\square$

Lemma 4.5. *For a nondegenerate geodesic triangle $\Delta(pqr)$ in X we have*

$$\frac{d}{dt}\tilde{\angle}\,r(t)pq(s)$$

$$\geq \frac{\cos\tilde{\angle}pq(s)r(t) - \cos\tilde{\angle}pq(s)r(t)}{\sin\tilde{\angle}(pq(s)r(t))} \times \begin{cases} \frac{\sqrt{-k}}{\sinh\sqrt{-k}\,s} & \text{if } k < 0 \\ \frac{1}{s} & \text{if } k = 0 \\ \frac{\sqrt{k}}{\sin\sqrt{k}\,s} & \text{if } k > 0 \end{cases}$$

Proof. Let $\tilde{\Delta}(pq(s)r(t))$ be the corresponding triangle to $\Delta(pq(s)r(t))$ sketched in $M^2(K)$, where $K = -k^2 < 0$, $K = k^2 > 0$ or $K = 0$. Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be upper angles at p, $q(s)$, $r(t)$ between $pq(s)$ and $pr(t)$, $q(s)p$ and $q(s)r(t)$, $r(t)q(s)$ and $r(t)p$ respectively and $\alpha_K, \beta_K, \gamma_K$ the corresponding angles of $\tilde{\Delta}(pq(s)r(t))$ in $M^2(K)$. Set $z = z(s,t) = d(q(s), r(t))$.

Then these angles are functions on s and t and we assert

$$\begin{aligned} \frac{\partial \alpha_K}{\partial s} &\geq \frac{\cos\bar{\beta} - \cos\beta_K}{\sin\beta_K} \cdot \frac{k}{\sinh ks} \quad (\text{if } K < 0) \\ &\geq \frac{\cos\bar{\beta} - \cos\beta_K}{\sin\beta_K} \cdot \frac{1}{s} \quad (\text{if } K = 0) \\ &\geq \frac{\cos\bar{\beta} - \cos\beta_K}{\sin\beta_K} \cdot \frac{k}{\sin ks} \quad (\text{if } K > 0) \end{aligned}$$

For the proof of assertion we may set $x = ks$, $y = kt$, $z = kz$. The hyperbolic cosine rule implies

$$\cosh z = \cosh x \cdot \cosh y - \sinh x \cdot \sinh y \cdot \cos \alpha_K .$$

The monotone property of z implies that $z = z(s,t)$ is differentiable almost all s and t values. Thus

$$\begin{aligned}\sinh z \cdot \frac{\partial z}{\partial x} &= \sinh x \cdot \cosh y - \cosh x \cdot \sinh y \cdot \cos \alpha_K \\ &\quad + \sinh x \cdot \sinh y \cdot \sin \alpha_K \cdot \frac{\partial \alpha_K}{\partial x}\end{aligned}$$

makes sense for almost all s and t.

The second term on the right hand side of the above equation can be replaced by $\dfrac{\cosh x \cdot \cosh y - \cosh z}{\sinh x}$, and hence

$$\begin{aligned}\frac{\partial z}{\partial x} \cdot \sinh z &= \sinh x \cdot \cosh y - \frac{\cosh x(\cosh x \, \cosh y - \cosh z)}{\sinh x} \\ &\quad + \sinh x \cdot \sinh y \cdot \sin \alpha_K \cdot \frac{\partial \alpha_K}{\partial x}\end{aligned}$$

Using again the cosine rule,

$$\cosh y = \cosh x \cdot \cosh z - \sinh x \cdot \sinh z \cdot \cos \beta_K$$

and substituting this into the right hand side,

$$\frac{\partial z}{\partial x} \cdot \sinh z = \frac{\sinh x \cdot \sinh z \cdot \cos \beta_K}{\sinh x} + \sinh x \cdot \sinh y \cdot \sin \alpha_K \cdot \frac{\partial \alpha_K}{\partial x}.$$

The sine rule implies that

$$\sinh y \cdot \sin \alpha_K = \sinh z \cdot \sin \beta_K ,$$

and hence

$$\frac{\partial z}{\partial x} = \cos \beta_K + \sinh x \cdot \sin \beta_K \cdot \frac{\partial \alpha_K}{\partial x}$$

From Corollary 4.4, (2) we have

$$\frac{\partial z}{\partial x} \geq \cos \bar{\beta}.$$

Therefore by rewriting $x = ks$,

$$\frac{\partial \alpha_K}{\partial s} \geq \frac{\cos \bar{\beta} - \cos \beta_K}{\sin \beta_K} \cdot \frac{k}{\sin ks}.$$

This proves Lemma 4.5. □

Lemma 4.6. *For a non-degenerate geodesic triangle $\Delta(pqr)$ and for every $\varepsilon > 0$ there exists a constant $a = a(\varepsilon, K, \Delta)$ such that if*

$$\tilde{\angle} pq(s)r(t) - \bar{\angle} pq(s)r(t) \geq \varepsilon,$$

then there exists an $s' \in (0, s)$ such that

$$\tilde{\angle} q(s)pr(t) - \tilde{\angle} q(s')pr(t) > a \log \frac{s}{s'}.$$

Proof. Setting $\beta_K := \tilde{\angle} pq(s)r(t)$ and $\bar{\beta} := \bar{\angle} pq(s)r(t)$ we see from Lemma 4.5 that

$$\frac{\cos\bar{\beta} - \cos\beta_K}{\sin\beta_K} \geq \frac{\cos(\beta_K - \varepsilon) - \cos\beta_K}{\sin\beta_K}$$
$$= \sin\varepsilon - (1 - \cos\varepsilon)\cot\beta_K.$$

From $\beta_K \geq \varepsilon$ follows $-\cot\beta_K \geq -\cot\varepsilon$, and hence

$$\frac{\cos\bar{\beta} - \cos\beta_K}{\sin\beta_K} \geq \sin\varepsilon - (1 - \cos\varepsilon)\cot\varepsilon = \tan\frac{\varepsilon}{2}.$$

Therefore we get

$$\frac{\partial\alpha_K}{\partial s} \geq \tan\frac{\varepsilon}{2} \cdot \frac{k}{\sinh ks} > 0, \quad s \in [0, d(p, q)],$$

and in particular we find a constant $b > 0$ such that

$$\frac{ks}{\sinh ks} \geq b \quad \text{for all} \quad s \in [0, d(p, q)],$$

and hence

$$\frac{\partial\alpha_K}{\partial s} \geq (b\tan\frac{\varepsilon}{2}) \cdot \frac{1}{s} = (b\tan\frac{\varepsilon}{2}) \cdot \frac{d}{ds}(\log s).$$

We conclude the proof by

$$\alpha_K(s, t) - \alpha_K(s', t) \geq \int_{s'}^{s} a \cdot \frac{d}{ds}(\log s)ds = a(\log s - \log s').$$

□

Theorem 4.7. *For every geodesic triangle $\Delta(pqr)$ let $\tilde{\Delta}(pqr)$ be the corresponding triangle sketched in $M^2(K)$ (for $K = -k^2, 0$ or k^2). Let $\bar{\alpha} := \bar{\alpha}(s,t)$, $\bar{\beta}(s,t)$, $\bar{\gamma}(s,t)$ be upper angles of $\Delta(p\,q(s)r(t))$ at $p, q(s), r(t)$ and $\alpha_K(s,t)$, $\beta_K(s,t)$, $\gamma_K(s,t)$ the corresponding angles of $\tilde{\Delta}(p, q(s)r(t))$. If*

$$\nu := lub\{\bar{\alpha}(s,t) + \bar{\beta}(s,t) + \bar{\gamma}(s,t) - (\alpha_K(s,t) + \beta_K(s,t) + \gamma_K(s,t)) \\ : 0 \le s \le d(p,q),\ 0 \le t \le d(p,r)\},$$

then

$$\bar{\alpha}(s,t) - \alpha_K(s,t) \le \nu \quad \text{for all} \quad (s,t) \in [0, d(p,q)] \times [0, d(p,r)].$$

Proof. If $\Delta(pqr)$ is degenerate, then the conclusion is obvious. We may assume that $\Delta(pqr)$ is nondegenerate. Suppose the conclusion is false. Then there are $t > 0$ $s > 0$, and $\varepsilon > 0$ such that

$$\bar{\alpha}(s,t) - \alpha_K(s,t) \ge \nu + 2\varepsilon.$$

By assumption we have

$$(\beta_K(s,t) - \bar{\beta}(s,t)) + (\gamma_K(s,t) - \bar{\gamma}(s,t)) \ge \bar{\alpha}(s,t) - \alpha_K(s,t) - \nu \ge 2\varepsilon,$$

and hence either

$$\beta_K(s,t) - \bar{\beta}(s,t) \ge \varepsilon$$

or else

$$\gamma_K(s,t) - \bar{\gamma}(s,t) \ge \varepsilon.$$

Lemma 4.6 and $\gamma_K(s,t) - \bar{\gamma}(s,t) \ge \varepsilon$ imply the existence of a constant $a > 0$ and $t' \in (0,t)$ such that

$$\alpha_K(s,t) - \alpha_K(s,t') > a \log \frac{t}{t'} > 0,$$

and similarly from $\overset{K}{\beta_{\cancel{k}}}(s,t) - \bar{\beta}(s,t) \ge \varepsilon$ implies the existence of a constant $a > 0$ and $s' \in (0,s)$ such that

$$\alpha_K(s,t) - \alpha_K(s',t) > a \log \frac{s}{s'}.$$

In any case, ~~By taking the mean value~~, we find an $s' \in (0, s)$ and $t' \in (0, t)$ such that

$$\alpha_K(s,t) - \alpha_K(s',t') > a \log \frac{ts}{t's'} > 0. \tag{4-2}$$

Thus we see

$$\bar{\alpha}(s',t') - \alpha_K(s',t') > \bar{\alpha}(s,t) - \alpha_K(s,t) \geq \nu + 2\varepsilon.$$

The triangle $\Delta(pq(s')r(t'))$ plays the same role as $\Delta(pq(s)r(t))$, and the above argument shows the existence of $t'' \in (0, t')$ and $s'' \in (0, s')$ such that

$$\bar{\alpha}(s'',t'') - \alpha_K(s'',t'') > \nu + 2\varepsilon \tag{4-3}$$

and such that

$$\alpha_K(s,t) - \alpha_K(s'',t'') > a \log \frac{ts}{t''s''}. \tag{4-4}$$

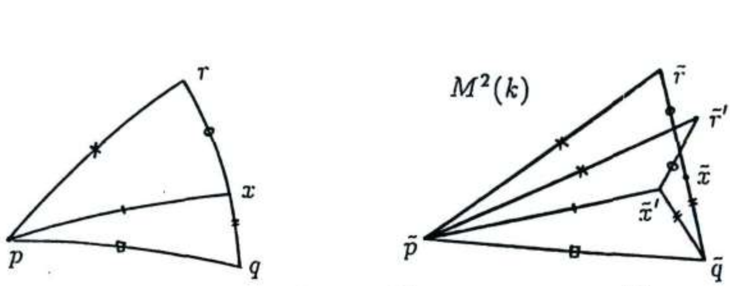
In view of $(4\text{-}1)^*$ there exists a positive constant $C > 0$ which is the lower bound of all the product $s''t''$ with the properties (4-3) and (4-4). Then there exists a sequence of pairs (s_n, t_n) of positive numbers such that $\lim_{n\to\infty} s_n t_n = C$, and $\{s_n\}$, $\{t_n\}$ are decreasing and satisfy (4-3) and (4-4). By setting $\bar{s} = \lim_{n\to\infty} s_n$ and $\bar{t} = \lim_{n\to\infty} t_n$ we see from continuity of logarismic function that $(\bar{s}, \bar{t})$ satisfies (4-3) and (4-4). The above argument shows the existence of $\bar{s}' < \bar{s}$ and $\bar{t}' < \bar{t}$ such that $(\bar{s}', \bar{t}')$ fulfills (4-3) and (4-4), a contradiction to the choice of $C > \bar{s}'\bar{t}'$. This proves Theorem 4.7. $\square$

Alexandrov defined in [A: §3] the notion of curvature of a length space Y bounded above by k as follows.

Definition 4.8. The curvature of Y is bounded above by k iff at each point $y \in Y$ there exists a convex neighborhood U_y in such a way that if $\Delta(pqr)$ is contained entirely in U_y, then

$$\tilde{\angle} qpr + \tilde{\angle} pqr + \tilde{\angle} qrp - (\angle qpr + \angle pqr + \angle qrp) \leq 0$$

(If $k > 0$ then the circumference of every geodesic triangle is assumed to be less than $2\pi/\sqrt{k}$).



$\tilde\Delta(pqr) = \Delta(\tilde p\,\tilde q\,\tilde r)$, $\tilde\Delta(prs) = \Delta(\tilde p\,\tilde r'\,\tilde s')$, $\tilde\Delta(pqs) = \Delta(\tilde p\,\tilde q\,\tilde s')$

FIGURE 4-1.

We now want to show that a length space Y satisfying Definition 4-8 is equivalent to Y satisfying the Alexandrov concavity property. The Alexandrov concavity property implies Definition 4.8 because the function $\alpha_k(s,t)$ is monotone nondecreasing and $\bar\angle qpr = \alpha_k(d(pq), d(pr)) \geq \tilde\angle qpr$. Assuming Definition 4.8 for Y, Theorem 4.7 yields $\bar\angle qpr \leq \tilde\angle qpr$, $\bar\angle pqr \leq \tilde\angle qpr$ and $\bar\angle qrp \leq \tilde\angle qrp$. If follows from Corollary 4.4-(3) that if $x \in qr$, then $\tilde\Delta(pqx) = \Delta(\tilde p\,\tilde q\,\tilde x')$ and $\tilde\Delta(pxr) = \Delta(\tilde p\,\tilde x'\,\tilde r')$ has the property that (see Figure 4-1)

$$\pi \leq \bar\angle pxr + \bar\angle pxq \leq \angle \tilde p\,\tilde x'\,\tilde r + \angle \tilde p\,\tilde x'\,\tilde q,$$

and hence we have

$$d(p,x) \leq d(\tilde p,\, \tilde x)$$

for $\tilde x \in \tilde q\,\tilde r$ with $d(\tilde x, \tilde q) = d(x,q)$.

Remark. It turns out that we only need to take the limit in the definition of upper angle for an Alexandrov space with curvature bunded above, as stated

$$\bar\angle\, qpr = \lim_{s,t\to 0} \tilde\angle q(s)pr(t).$$

In fact the monotone property of $\alpha_k(s,t)$ for $s \in (0, d(p,q)]$, $t \in (0, d(p,r)]$ has been established by showing the equivalence of Definition 4.8 and the Alexandrov concavity property.

For a sufficiently small geodesic triangle $\Delta = \Delta(pqr)$ in Y and for a constant k we denote by Δ^k a geodesic triangle in $M^2(k)$ corresponding to Δ, and by $\sigma(\Delta^k)$ the area of $\Delta^k \subset M^2(k)$. We denote the angles by

$$\begin{array}{lll} \bar{\alpha} := \bar{\angle}\, qpr, & \bar{\beta} := \bar{\angle}\, pqr, & \bar{\gamma} := \bar{\angle}\, qrp \\ \alpha_k := \tilde{\angle} qpr, & \beta_k := \tilde{\angle} pqr, & \gamma_k := \tilde{\angle} rqp. \end{array}$$

these

Let $\delta_k(\Delta) := \bar{\alpha} + \bar{\beta} + \bar{\gamma} - (\alpha_k + \beta_k + \gamma_k)$. With there notations an equivalent condition for Y to have curvature bounded above is stated as follows.

Theorem 4.9. *For a length space Y, the following (a) and (b) are equivalent.*

(a) *$Curv(Y) \leq K$.*

(b) *For every point $y \in Y$ there exists a strongly convex neighborhood U_y around y in such a way that for every sequence $\{\Delta_y\}$ of geodesic triangles in U_y shrinking to a point $p \in U_y$*

$$\lim_{j\to\infty} \sup \frac{\delta_0(\Delta_j)}{\sigma(\Delta_j^0)} \leq K.$$

Remark. From definition we observe $\delta_0(\Delta) - \delta_k(\Delta) = \alpha_k + \beta_k + \gamma_k - \pi$, and hence from the Gauss-Bonnet theorem we have

$$K \cdot \sigma(\Delta^k) = \delta_0(\Delta) - \delta_k(\Delta).$$

Proof of (a) $\Longrightarrow$ (b). From definition 3.1 it follows that $\delta_k(\Delta) \leq 0$ for all $\Delta \subset U_y$, and hence

$$K \cdot \sigma(\Delta^k) \geq \delta_0(\Delta).$$

Therefore, by setting $A_j := \sigma(\Delta_j^0)/\sigma(\Delta_j^k)$ we have

$$K \geq \frac{\delta_0(\Delta_j)}{\sigma(\Delta_j^k)} = A_j \cdot \frac{\delta_0(\Delta_j)}{\sigma(\Delta_j^0)}.$$

Clearly $\lim_{j\to\infty} A_j = 1$, and (b) is derived.

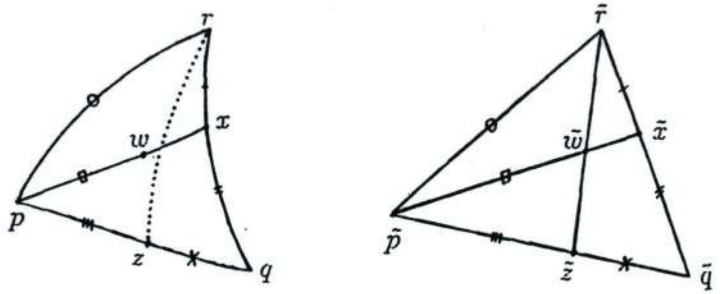


FIGURE 4-2.

Proof of (b) $\Longrightarrow$ (a). For an arbitrary fixed small positive number ε and for an arbitrary small $\Delta \subset U_y$, the condition (b) implies

$$\delta_0(\Delta) \leq (K+\varepsilon)\sigma(\Delta^0).$$

Thus we have

$$\begin{aligned}(K+\varepsilon)\sigma(\Delta^0) &= (K+\varepsilon)\cdot\sigma(\Delta^{k+\epsilon})\cdot\frac{\sigma(\Delta^0)}{\sigma(\Delta^{k+\epsilon})} \\ &= \delta_0(\Delta^{k+\epsilon})\cdot\frac{\sigma(\Delta^0)}{\sigma(\Delta^{k+\epsilon})}.\end{aligned}$$

Notice that if $K+\varepsilon \geq 0$, then $\sigma(\Delta^0) \leq \sigma(\Delta^{K+\epsilon})$ and if $K+\varepsilon \leq 0$, then $\sigma(\Delta^0) \geq \sigma(\Delta^{K+\epsilon})$, and hence

$$\begin{aligned}\bar{\alpha}+\bar{\beta}+\bar{\gamma}-\pi = \delta_0(\Delta) &\leq (K+\varepsilon)\sigma(\Delta^0) \\ &\leq (K+\varepsilon)\sigma(\Delta^{K+\epsilon}) = \alpha_k+\beta_k+\gamma_k-\pi.\end{aligned}$$

This proves Theorem 4.9. $\square$

The following result is due to Alexandrov [A].

Theorem 4.10. *Let $Curv(Y) \leq k$. If a geodesic triangle $\Delta(pqr)$ contained in U_y for some point $y \in Y$ has the property that one of the upper angles, say $\bar{\angle}qpr$ at p, is equal to $\tilde{\angle}qpr$, then there exists a unique smooth totally geodesic surface S in Y of constant curvature k which is bounded by $\Delta(pqr)$.*

Proof. It follows from assumption that if $x \in qr$ and $\tilde{x} \in \tilde{q}\tilde{r}$ are taken such that $d(q,x) = d(\tilde{q},\tilde{x})$, then $d(p,x) = d(\tilde{p},\tilde{x})$. In view of Corollary 4.4, (3) we have $\bar{\angle}pxq = \angle\tilde{p}\,\tilde{x}\,\tilde{q}$ and $\bar{\angle}pxr = \angle\tilde{p},\tilde{x}\,\tilde{r}$, and by letting $x \to q$, $x \to r$ we observe $\bar{\angle}pqr = \tilde{\angle}pqr$ and $\bar{\angle}prq = \tilde{\angle}prq$.

If $z \in pq$ and $\tilde{z} \in \tilde{p}\tilde{q}$ are taken such that $d(p,z) = d(\tilde{p}\tilde{z})$, then $d(r,z) = d(\tilde{r},\tilde{z})$ and $\tilde{r}\tilde{z}$ intersects $\tilde{p}\tilde{x}$ at a unique point $\tilde{w}$ (see Figure 4-2). If $w \in p\tilde{x}$ is a point with $d(p,w) = d(\tilde{p}\tilde{w})$, then $d(r,\tilde{w}) = d(\tilde{r}\,\tilde{w})$ and $d(z,w) = d(\tilde{z},\tilde{w})$ follows from the fact that $\tilde{\Delta}(prw) = \Delta(\tilde{p}\tilde{r}\,\tilde{w})$ and $\tilde{\Delta}(pzw) = \Delta(\tilde{p}\,\tilde{z}\,\tilde{w})$ having the same corresponding edge angles at p and $\tilde{p}$. Therefore $d(r,w)+d(w,z) = d(r,z)$ and hence $w \in rz$. This fact means that if $\alpha : [0, d(q,r)] \to X$ and $\beta : [0,pq] \to X$ are the edges with $\alpha(0) = \beta(0) = q$, $\alpha(d(q,r)) = r$, $\beta(d(p,q)) = p$, then a natural maps $f : [0,d(p,q)] \times [0,d(q,r)] \to X$ and $\tilde{f} : [0,d(p,q)] \times [0,d(q,r)] \to \Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$ is defined as follows. To each $(u,v) \in [0,d(p,q)] \times [0,d(q,r)]$ a point $f(u,v)$ (respectively, $\tilde{f}(u,v)$) is asigned as the intersection of geodesics $p\alpha(u) \cap r\beta(v)$ (respectively, $\tilde{p}\,\tilde{\alpha}(u) \cap \tilde{r}\,\tilde{\beta}(v)$), where $\tilde{a}$ and $\tilde{\beta}$ are the edges of $\Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$ corresponding to α and β. The two geodesics $p\alpha(u)$ and $q\beta(v)$ (respectively, $\tilde{p}\,\tilde{\alpha}(u)$ and $\tilde{q}\,\tilde{\beta}(v)$) divide Δ (respectiely, $\tilde{\Delta}$) into four small geodesic triangles, all these corresponding triangles have the properties that all the corresponding edge lengths and angles are the same. Therefore we see that $E := f \circ \tilde{f}^{-1} : \tilde{\Delta} \to X$ is an isometric embedding. This map is totally geodesic in the sense that any two points on $E(\tilde{\Delta})$ are joined by a unique geodesic which is contained entirely in $E(\tilde{\Delta})$. This proves Theorem 4.8. $\square$

Remark. The proof of Theorem 4.10 is valid for Alexandrov spaces with curvature bounded below. In this case the Alexandrov concarity property is replaced by the Alexandriv convexity and the property of complementary angles in Corollary 4.4 (3) is replaced by Lemma 3.5. A careful treatment is needed because the fundamental length does not exist. This property is proved in Lemma 6.4 and used for the proof of the lemma on narrow triangles.

§5. Examples

We shall exhibit examples of Alexandrov spaces with curvature bounded below and above.

5-1. First of all Alexandrov sapces with curvatue bounded below are givne as follows.

(1) Every complete Riemannian manifolds whose sectional curvature is bounded below are Alexandrov spaces of curvature bounded below.

(2) Every convex body in $\mathbf{R}^{n+1}$ has its boundary X with inner distance induced from $\mathbf{R}^{n+1}$ is an Alexandrov space whose curvature is bounded below by 0, but not bounded above at vertices. We denote it by

$$0 \leq \mathrm{Curv}(X) \leq +\infty.$$

(3) The double of unit balls $B^n(1) \cup_{\partial B^n(1)} B^n(1) = X$ joined along with their common boundary unit sphere $S^{n-1} \subset \mathbf{R}^n$ with inner distance induced from $\mathbf{R}^n$ is an Alexandrov space with

$$0 \leq \mathrm{Curv}(X) \leq +\infty.$$

(4) For a length space X with diameter $d(X) \leq \pi$ the cone $K(X)$ generated by X with vertex at $o \in K(X)$ is defined as follows.

$$K(X) := \{(x,t); x \in X, t \geq 0,\ (x,0) = o \text{ for all } x \in X\}.$$

The distance ρ of $K(X)$ is introduced by

$$\rho((x_1,t_1),(x_2,t_2)) := \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos d(x,y)}.$$

It is not difficult to check the triangle inequality for points on $K(X)$. There is a natural embedding $\varphi : X \to K(X)$ such that $\varphi(x) := (x,1)$ for $x \in X$. If $(\rho|\varphi(X))^*$ is the interior metric on X induced through ρ, then $d = (\rho|\varphi(x))^*$. In fact, if $x, y \in X$ then $(\rho|\varphi(X))^*(x,y) = \sqrt{2 - 2\cos d(x,y)} = 2\sin\frac{d(x,y)}{2}$. Therefore we have

$$\lim_{d(x,y)\to 0} \frac{(\rho|\varphi(X))^*(x,y)}{d(x,y)} = 1.$$

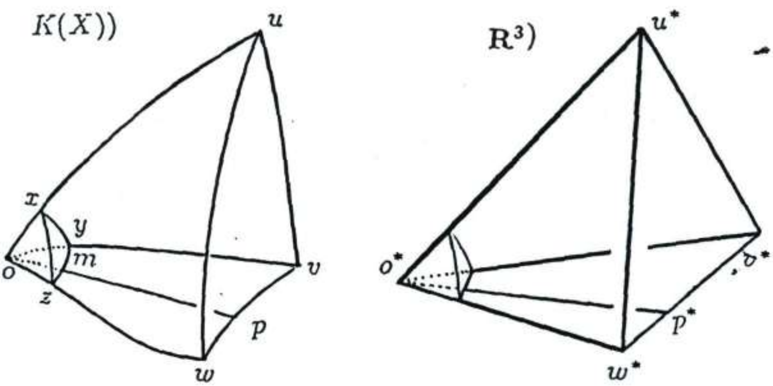


FIGURE 5-1.

Theorem 5.1. (Theorem 3.7 in [BGP]). *Let X be a length space with $d(X) \leq \pi$. Then $K(X)$ is a length space. Moreover the following (a) and (b) are equivalent.*

(a) *$K(X)$ has curvature bonnded below by 0,*

(b) *X has curvature bounded below by 1.*

Proof. For the proof of $K(X)$ being a length space it suffices to show that $K(X)$ satisfies the Menger convexity. Let $u = (x, s)$, $v = (y, t)$ be points on $K(X)$ such that $x, y \in X$ and $s, t \geq 0$. Let z be in interior point of a geodesic xy in X and $w \in K(X)$ with $W = (z, l)$ be taken in such a way that if $\Delta(o^*u^*v^*)$ is a plane triangle with $d(o^*, u^*) = s$, $d(o^*, v^*) = t$ and $\angle u^*o^*v^* = d(x, y)$ and if $w^* \in u^kv^*$ is taken such that $\angle u^*o^*w^* = d(x, z)$, then $l := d(o^*, w^*)$. From the definition of distance on $K(X)$ we see $\rho(u, w) = d(u^*, w^*)$, $\rho(v, w) = d(v^*, w^*)$ and $\rho(u, v) = d(u^*, v^*)$, and hence $w \in K(X)$ satisfies $\rho(u, w) + \rho(w, v) = \rho(u, v)$.

The above discussion means that for every $x, y \in X$ and for every geodesic xy the set $\{w = (z, t) : z \in xy,\ t \geq 0\}$ is isometric to the plane sector whose vertex angle is $d(x, y)$.

Finally we take $u = (x, s), v = (y, t), w = (z, l) \in K(X)$ and corresponding points $u^*, v^*, w^* \in \mathbf{R}^3$ with the origin o^* such that $d(o^*, u^*) = s$, $d(o^*, v^*) = t$, $d(o^*, w^*) = l$ and $\angle v^*o^*v^* = d(x, y)$, $\angle u^*o^*w^* = d(x, z)$, $\angle v^*o^*w^* = d(y, z)$, (see, Figure 5-1).

Let $\tilde{x} \in S^2(1) \cap o^*u^*$, $\tilde{y} := S^2(1) \cap o^*v^*$, $\tilde{z} = S^2(1) \cap o^*w^*$. Then $\tilde{\Delta}(xyz) = \Delta(\tilde{x}\,\tilde{y}\,\tilde{z})$. If $p = (m, h) \in vw$ is an arbitrary point, then $m \in yz$ and we find corresponding point $p^* = (\tilde{m}, h)$ to p on v^*w^* with $d(v^*, p^*) = d(v, p)$. Therefore $\mathrm{Curv}(K(X)) \geq 0$ implies $\rho(u, p) \geq d(u^*, p^*)$, and in particular $d(x, m) \geq d(\tilde{x}, \tilde{m})$ means that $\mathrm{Curv}(X) \geq 1$. This proves (a) $\Longrightarrow$ (b). The converse is now clear. This proves Theorem 5.1.

(5) the cone $K(\mathbf{R}P^{n-1})$ generated by the real projective $(n-1)$-space of constant curvature 1 is obtained as the pointed Hausdorff limit of the following sequence of Riemannian $(n+1)$-manifolds. Consider $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n \ni (t, x)$ and let $g_\varepsilon = \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ for every $\varepsilon > 0$ be a fixed point free isometry such that $g_\varepsilon(t, x) = (t + \varepsilon, -x)$. Let $\langle g_\varepsilon \rangle$ be the group of isometries over $\mathbf{R}^{n+1}$ generated by g_ε, and set $M_\varepsilon := \mathbf{R}^{n+1}/\langle g_\varepsilon \rangle$. The pointed Hausdorff limit of $\{M_\varepsilon\}_{\varepsilon \downarrow 0}$ is the $K(\mathbf{R}P^{n-1})$ with base point at the origin o^* of $\mathbf{R}^{n+1}$ and $o \in K(\mathbf{R}P^{n-1})$ its vertex.

$$\lim_{\varepsilon \to 0\ d_H} (M_\varepsilon, o^*) = (K(\mathbf{R}P^{n-1}), o).$$

This cone has an essential singularity at o, appeared as the result of a collapsing phenomenon.

(6) The spherical suspension. The spherical suspension $\Sigma(X)$ of an Alexandrov space X with $\mathrm{Curv}(X) \geq 1$ is obtained as follows. Let $\Sigma(X) := \{(x, t); x \in X,\ 0 \leq t \leq \pi, (x, 0) = (y, 0), (x, \pi) = (y, = \pi) \text{ for all } x, y \in X\}$. Then ρ the distence function on $\Sigma(X)$ is defined as the cosine rule of spherical trigonometry:

$$\cos \rho((x_1, t_1), (x_2, t_2)) = \cos t_1 \cdot \cos t_2 + \sin t_1 \cdot \sin t_2 \cdot \cos d(x, y).$$

By using the same isometry group $\langle g_\varepsilon \rangle$ acting freely on $\mathbf{R} \times S^{n-1}$, we observe that if $N_\varepsilon := \mathbf{R} \times S^{n-1}/\langle g_\varepsilon \rangle$ then $\lim_{\varepsilon \to 0\ d_H} N_\varepsilon = \sum(\mathbf{R}P^{n-1})$. $\square$

Remark 5.1. $X = B^2(1) \bigcup_{2B(1)} B^2(1)$ as in example (3) does not have curvature bounded above. In fact suppose that therse exists a $K > 0$ such that $\mathrm{Curv}(X) \leq K$. Let $p, q \in \partial B^2(1)$ be taken such that $d(p, q) < \pi/\sqrt{K}$. There exist two distinct geodesics $\alpha_1, \alpha_2 : [0, d(p, q)] \to X$ joining p to q with the angle $\angle(\alpha_1, \alpha_2)_p$ at p being equal to $2 \sin^{-1} \frac{d(p,q)}{2}$. If $m_i = \alpha_i(d(p, q)/2)$ for

$i = 1,2$, then $d(m_1, m_2) > 0$. If $\tilde{\Delta}(pm_1q) = \Delta(\tilde{p}\tilde{m}_1\tilde{q})$ and if $\tilde{m}_2$ is the midpoint of $\tilde{p}\tilde{q}$, then $\mathrm{Curv}(X) \leq K$ implies

$$0 < d(m_1, m_2) \leq d(\tilde{m}_1, \tilde{m}_2) = 0,$$

a contradiction.

5-2. We next show examples of Alexandrov spaces with curvature bounded above.

(7) Every complete Riemannian manifold with sectional curvature bounded above is an Alexandrov space with curvature bounded above.

(8) The complement X of a closed convex body in $\mathbf{R}^n$ is an Alexandrov space with curvature bounded above by 0 but not bounded below.

$$0 \geq \mathrm{Curv}(X) \geq -\infty.$$

(9) The double of the complement of $B^n(1) \subset \mathbf{R}^n$. Let X be such that

$$X := \{\mathbf{R}^n \backslash B^n(1)\} \bigcup_{\partial B^n(1)} \{\mathbf{R}^n \backslash B^n(1)\}.$$

Then $0 \geq \mathrm{Curv}(X) \geq -\infty$.

Remark 5.2. Let $X = \{\mathbf{R}^2 \backslash B^2(1)\} \bigcup_{\partial B^2(1)} \{\mathbf{R}^2 \backslash B^2(1)\}$. We then see from Figure 5-2 that $0 \geq \mathrm{Curv}(X)$.

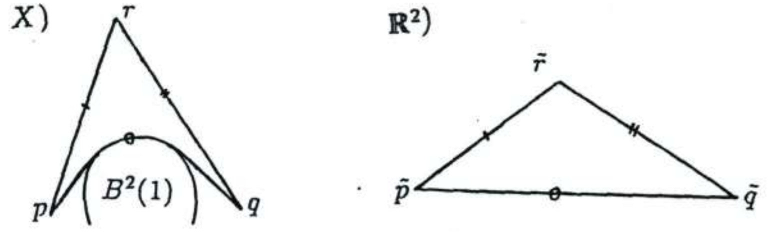


FIGURE 5-2.

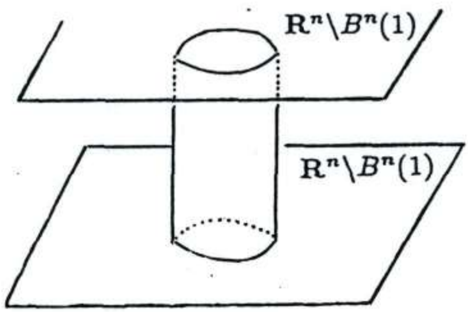


FIGURE 5-3.

Suppose that $\mathrm{Curv}(X) \geq -k^2$ for some $k > 0$. Let $\Delta(rpq)$ be an isosceles triangle in X with $p, q \in \partial B^2(1)$ and $\tilde{\angle} prq = \theta$ being sufficiently close to $\pi/2$ such that pr and qr are tangent to the unit circle at p and q respectively. Then we get $d(p,r) = d(q,r) = \cot\theta$ and $d(p,q) = \pi - 2\theta$. If $m \in pq$ is the midpoint of the geodesic pq, then $d(r,m) = \mathrm{cosec}\theta - 1$. Setting $h := d(\tilde{r}, \hat{m})$, where $\tilde{\Delta}(pqr) = \Delta(\tilde{p}\,\tilde{q}\,\tilde{r})$ and $\hat{m} \in \tilde{p}\,\tilde{q}$ is the midpoint of $\tilde{p}\,\tilde{q}$ in $M^2(k^2)$, we have

$$\cosh kh = \frac{\cosh(k\cot\theta)}{\cosh k(\frac{\pi}{2} - \theta)}.$$

Setting $t := \frac{\pi}{2} - \theta > 0$, the above equation is rewritten as

$$\cosh kh = \frac{\cosh(k\tan t)}{\cosh kt},$$

and $\mathrm{Curv}(X) \geq -k^2$ means that $ed(r,m) \geq d(\tilde{r}, \hat{m}) = h$. Thus we have

$$\frac{1-\cos t}{\cos t} \geq h.$$

On the other hand,

$$0 = \cosh kh \cdot \cosh kt - \cosh(k\tan t) \leq \cosh k(\frac{1-\cos t}{\cos t}) \cdot \cosh kt - \cosh(k\tan t).$$

However the last term is negative for sufficiently small positive t, a contradiction.

(10) $X = \{\mathbf{R}^n \backslash B^n(1)\} \cup S^{n-1} \times [0,1] \cup \{\mathbf{R}^n \backslash B^n(1)\}$ as in Figure 5-3 has curvature

$$0 \geq \mathrm{Curv}(X) \geq -\infty.$$

Remark 5.3. As is seen is the above examples X with curvature bounded above may allow its geodesics having branch points. Also X with curvature bounded below may admit suppiciently small geodesic biangles.

The following example (11) shows that the curvature property does not preserve on the Hausdorff limit of Riemannian manifolds.

(11) For given $\varepsilon > 0$ let M_ε be the surface obtained by gluing a flat cone with a flat forus $S^1(\varepsilon) \times S^1(\varepsilon)$ in such a way that $\varepsilon/4$-ball is removed from $S^1(\varepsilon) \times S^1(\varepsilon)$ and also from a flat cone whose center is at the vertex 0 and that M_ε has negative Gaussian curvature in a neighborhood around the $\varepsilon/4$-circle.

Taking a base point $o \in M_\varepsilon$ on the flat torus, we see that

$$\lim_{\varepsilon \to 0} {}_{d_H} (M_\varepsilon; o) = (\mathcal{C}; o^*)$$

The Gaussian curvature of M_ε is nonpositive but $\mathcal{C}$ has curvature bounded below by O.

In the case of lower curvature bound we have the

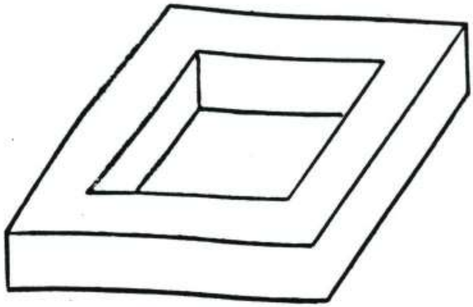
Theorem 5.2. (Grove-Petersen-Wu [GPW]). *If $\{M_j\}$ is a sequence of complete Riemannian n-manifolds of sectional curvature uniformly bounded below by k and* $\operatorname{diam}(M_j) \leq D$ *for a constant $D > 0$, and if X is its Hausdorff limit, then we have*

$$Curv(X) \geq k,$$

and

$$\dim X \leq n.$$

For the proof of $\dim X \leq n$ in Theorem 5.2 we need the notion of strainer by which the dimension of an Alexandrov space with curvature bounded below is well defined as a positive integer. This will be discussed in §7. More generally we can prove that if $\{X_i\}_i$ is a sequence of Alexandrov spaces such that $\operatorname{Curv}(X_i) \geq k$, and if X is the Hausdorff limit of $\{X_i\}$, then X is an Alexandrov space with $\operatorname{Curv}(X) \geq k$.



$\mathrm{Curv}(X) = +\infty$ at corners outside, $\mathrm{Curv}(X) = -\infty$ at corners inside

FIGURE 5-4.

A prismblock in $\mathbf{R}^3$ as in Figure 5-4 is an example of a length space whose curvature is neither bounded above nor below.

All examples as above are obtained as the Hausdorff limits of certain sequence of complete Riemannian manifolds. A natural question arising from these examples is if every Alexandrov space with curvature bounded below (or above) can be obtained as the Hausdorff limit of a certain sequence of complete Riemannian manifolds.

We next provide an example of a length space obtained as the Hausdorff limit of complete surfaces of nonpositive curvature in $\mathbf{R}^3$, at some points on which curvature is not defined.

(12) Let $o_1 = {}^t(-1,0,0)$ and $o_2 = {}^t(1,0,0)$ and C_1, C_2 be cones with vertices at o_1, o_2 generated by half-lines passing throug o_1 and o_2 (see Figure 5-5)

Let X be the union of $C_1 \cup o_1o_2 \cup C_2$. Clearly X is a length space obtained as the Hausdorff limit of complete surface of revolution aroung x-axis of non-positive curvature. Geodesics passing through o_1 and o_2 and containing these

Proof. We only need to prove that $\lim_{\lambda\to 0}(\lambda M; o) = K(M(\infty); \rho); o^*)$. Let $u, v \in K(M(\infty), \rho)$. Then $u = (x, a)$ and $v = (y, b)$ for some $x, y \in \mathcal{R}_o$ and for $a, b > 0$. Let $\gamma \in x$, $\sigma \in y$. Then $\tilde{\Delta}(o\gamma(\lambda^{-1}a)\sigma(\lambda^{-1}b))$ for sufficiently small posiitve λ sketched in $\mathbf{R}^2$ has an angle at $\tilde{o}$ $\theta(\lambda^{-1}a, \lambda^{-1}b)$, and $\lim_{\lambda\to 0}\theta(\lambda^{-1}a, \lambda^{-1}b) = d_\infty(x, y)$ follows from the definition of the Tits metric. Thus we conclude the proof by

$$\lim_{\lambda\to 0} d_{\lambda M}(\gamma(\lambda^{-1}a), \sigma(\lambda^{-1}b)) = \rho(u, v).$$

□

A similar result can also be proved for a finitely connected complete open surfaces admiting total curvature (finite or infinite). The ideal boundary of such a surface is investigated in [Sy-2] and [Sy-2], by which we can prove the following: Let $\mathcal{S}$ be a finitely connected complete open surface with total curvature (finite or infinite). Then for an arbitrary fixed base point $o \in \mathcal{S}$, we have

$$\lim_{\lambda\to 0} d_H(\lambda\mathcal{S}; o) = K((\mathcal{S}(\infty), \rho), o^*)$$

Here o^* is the vertex of the cone.

§6. The Toponogov Comparison Theorem

Throughout this section let X be an Alexandrov space with curvature bounded below by k. We want to prove the

Theorem 6.1. (The Toponogov Comparison Theorem) *For every geodesic triangle $\Delta = \Delta(pqr)$ there exists a corresponding geodesic triangle $\tilde{\Delta} = \tilde{\Delta}(pqr)$ in $M^2(k)$ such that*

$$\angle qpr \geq \tilde{\angle} qpr,\ \angle pqr \geq \tilde{\angle} pqr,\ \angle qrp \geq \tilde{\angle} qrp.$$

Theorem 6.2. *If $k > 0$, then every geodesic triangle in X has circumference not greater than $2\pi/\sqrt{k}$. In particular, if there exists a geodesic triangle with circumference $2\pi/\sqrt{k}$, then the diameter $d(X)$ of X is equal to $\pi/\sqrt{k}$. Moreover X is isometric to either the spherical suspension $\Sigma(X_1)$ of some Alexandrov space X_1 with $\mathrm{Curv}(X_1) \geq k$ if $d(X) = \pi/\sqrt{k}$, or else the double suspension $\Sigma^2(X_2)$ of some Alexandrov space X_2 with $\mathrm{Curv}(X_2) \geq k$ if the geodesic triangle with length $2\pi/\sqrt{k}$ has all of angles equal to π.*

The following lemma on limit angles plays an essential role for the proof of Theorem 6.1. We employ the original idea of Toponogov (see [T-1]) for the case of Alexandrov spaces. Therefore some modification is needed. We employ the property of geodesic biangles which states that if $\{\alpha_i, \beta_i\}_{i=1,2,\ldots}$ is a sequence of geodesic biangles with common starting point $p = \alpha_i(0) = \beta_i(0)$ such that the limit of lengths of α_i converges to 0, then the limit of angles between α_i and β_i at p tends to 0 as $i \to \infty$ (see Lemma 8.5). ← unif w.r.t. $\ell(\alpha_i)$?

Lemma 6.3. (Lemma on limit angles) *Assume that $\mathrm{Curv}(X) \geq k$. Let γ be a geodesic with $\gamma(0) = p$ and $\gamma(a) = q$. For a point $r \in X$ if σ_j is a geodesic from $\gamma(s_j)$ to r for $s_j \in (0, a)$ with $s_j \downarrow 0$, then*

$$\lim_{s_i \to 0} \sup \angle r\gamma(s_j)q \leq \angle rpq,$$

where the right hand side in the above inequality means the angle at p between γ and any geodesic joining p to r.

Proof. Suppose that the conclusion is false. Then there exists a sequence $\{\sigma_j\}$ of geodesics joining $\gamma(s_j)$ to r and a geodesic τ joining p to r such that $\lim_{j\to\infty} s_j = 0$ and such that

$$\bar{\alpha} := \lim_{j\to\infty} \angle r\gamma(s_j)q > \angle rpq = \angle(\gamma, \tau) =: \alpha.$$

Let σ be the limit geodesic of $\{\sigma_j\}$ and fix a small positive $\varepsilon \ll \bar{\alpha} - \alpha$. Then there exists a large number $j(\varepsilon)$ such that if $j > j(\varepsilon)$, then there is a point $x_j \in \sigma_j$ with the property that

$$\angle x_j p\gamma(s_j) \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

In fact, we define $I := \{t \in [0, d(r, \gamma(s_j))]$; every geodesic $p\sigma_j(t)$ has the property $\angle\sigma_j(t)p\gamma(s_j) < \alpha - \varepsilon\}$. Clearly $0 \in I$ and there is an interval $I_0 = [0, t_j) \subset I$ such that $t_j < d(r, \sigma(s_j))$ and t_j belongs to the boundary of I. Then there exists either a geodesic $p\sigma_j(t_j)$ with $\angle\sigma(t_j)p\gamma(s_j) \in (\alpha - \varepsilon, \alpha + \varepsilon)$, or a geodesic biangle (α_j, β_j) with corners at p and $\sigma_j(t_j)$ whose angle at p is not less than 2ε. Suppose that there is a sequence of geodesic biangles $(\alpha_j, \beta_j)\}_j$ such that $\alpha_j(0) = \beta_j(0) = p$ and $\alpha_j(l_j) = \beta_j(l_j) = \sigma_j(t_j)$ and such that the angle at p is not less than 2ε, where $l_j = d(p, \sigma_j(t_j))$. Because $\{\sigma_j(t_j)\}_j$ converges to p we see $\lim_{j\to\infty} l_j = 0$. This contradicts to Lemma 8.5 in §8. Choose a point $y_j \in \tau$ and $z_j \in \sigma_j$ such that (see Figure 6-1) $d(p, x_j) = d(p, y_j) = d(x_j, z_j)$.

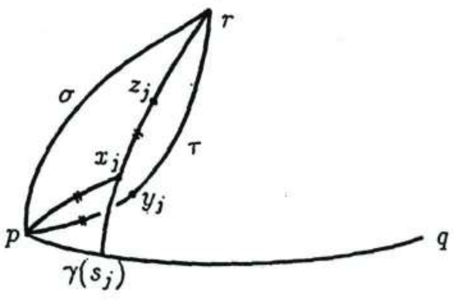


FIGURE 6-1.

From triangle inequality we see

$$d(p, x_j) + d(x_j, r) \geq d(p, r) = d(p, y_j) + d(y_j, r),$$

and hence

$$d(x_j, r) \geq d(y_j, r)$$

Similarly we get

$$d(r, y_j) + d(y_j, \gamma(s_j)) \geq d(r, \gamma(s_j)) = d(r, x_j) + d(x_j, \gamma(s_j)).$$

Since U_p contains the points $x_j, y_j, \gamma(s_j)$ for all $j \geq j(\varepsilon)$,

$$d(x_j, \gamma(s_j)) = d(y_j, \gamma(s_j)) + O(\varepsilon s_j)$$

and

$$d(p, z_j) = 2d(p, x_j) \cdot \sin\frac{\psi_j}{2} + O(\varepsilon s_j),$$ $\gamma_j = \angle p x_j z_j$

where $\psi_j := \angle p x_j \gamma(s_j)$ tends to $\bar{\alpha} - \alpha$ since $\lim \sigma_j = \sigma$ and the geodesic triangles $\{\Delta(p x_j \gamma(s_j))\}$, converge to a plane triangle with angles α, $\pi - \bar{\alpha}$ and $\bar{\alpha} - \alpha$. Summing up above computations we see

$\gamma_j \to \pi - (\bar{\alpha} - \alpha)$

$$d(r, y_j) = d(r, x_j) + O(\varepsilon s_j).$$

Finally we see

$$\begin{aligned}
d(p, r) &\leq d(p, z_j) + d(z_j, r) \\
&= d(p, z_j) + d(x_j, r) - d(x_j, z_j) \\
&= d(p, z_j) + d(r, y_j) - d(x_j, z_j) + O(\varepsilon s_j) \\
&= d(p, z_j) + d(pr) - d(y_j, p) - d(x_j, z_j) + O(\varepsilon s_j) \\
&= d(p, r) - 2d(p, x_j)(1 - \sin\frac{\psi_j}{2}) + O(\varepsilon s_j).
\end{aligned}$$

This is a contradiction for large j. $\square$

A geodesic triangle $\Delta(pqr)$ is called a *narrow triangle* iff

$$pq \cup pr \subset \bigcup_{x \in pq} U_x \cap \bigcup_{y \in pr} U_y$$

is satisfied. The lemma on narrow triangles (see Lemma 6.5) plays an essential role for the proof of Theorem 6.1 and the proof of it is different from the Riemannian case. For the moment we assume Lemma 6.5 and see how to proceed the proof of Theorem 6.1.

The proof of Theorem 6.1 by assuming Lemma 6.5..

Let $\Delta(pqr)$ be a geodesic triangle with edges α, β and γ in X. If $k > 0$, then we assume that the circumference of Δ is less than $2\pi/\sqrt{k}$. We prove

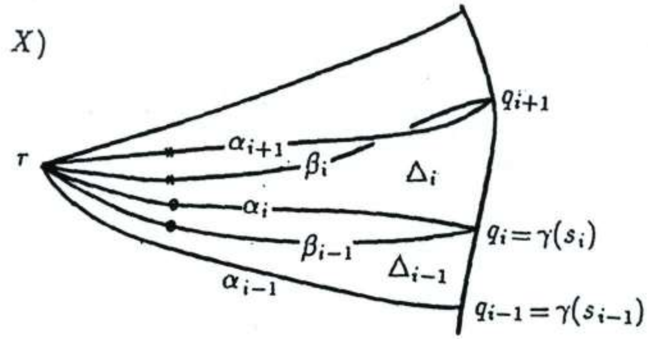


FIGURE 6-2.

$\angle rpq \geq \tilde{\angle} rpq$. Let $\gamma : [0, d(pq)] \to X$ be the edge joining $p = \gamma(0)$ to $q = \gamma(d(p,q))$. Choose a sufficiently fine partition $0 = s_0 < s_1 < \ldots, < s_N = d(p,q)$ of $[0, d(p,q)]$ in such a way that for every $i = 0, \cdots, N-1$ there is a narrow triangle $\Delta_i = \Delta(\gamma(s_i)\gamma(s_{i+1})r)$ with edges $\gamma \,|\, [s_i, s_{i+1}]$, $\alpha_i := r\gamma(s_i)$ and $\beta_i := r\gamma(s_{i+1})$ (see Figure 6-2). Here α_i for every $i = 0, \ldots, N-1$ is chosen as the limit of $r\gamma(s)$ as $s \downarrow s_i$, and therefore lemma 6.3 implies that $\angle(\beta_{i-1}, \gamma)|_{\gamma(s_i)} \geq \angle(\alpha_i, \gamma)|_{\gamma(s_i)}$. In particular α_0 may be different from α. Because Lemma 6.3 implies $\angle rpq = \angle(\alpha, \gamma)|_p \geq \angle(\alpha_0, \gamma)|_p$, we need to prove

$$\angle(\alpha_0, \gamma)|_p \geq \tilde{\angle} rpq.$$

By setting $\gamma(s_i) = q_i$ $(p = q_0,\ q = q_N)$ we see from the lemma on narrow triangle,

$$\angle rq_iq_{i+1} \geq \tilde{\angle} rq_iq_{i+1},\ \angle rq_{i+1}q_i \geq \tilde{\angle} rq_{i+1}q_i.$$

The sum of angles at q_i of Δ_{i-1} and Δ_i does not exceed π. The points $\tilde{r}, \tilde{q}_{i-1}, \tilde{q}_i$ and $\tilde{q}_{i+1}$ form a convex quadrangle, if they are placed in such a way that $\tilde{\Delta}_{i-1}$ and $\tilde{\Delta}_i$ do not overlap and they have the common edge $\tilde{r}\,\tilde{q}_i$, (see Figure 6-3).

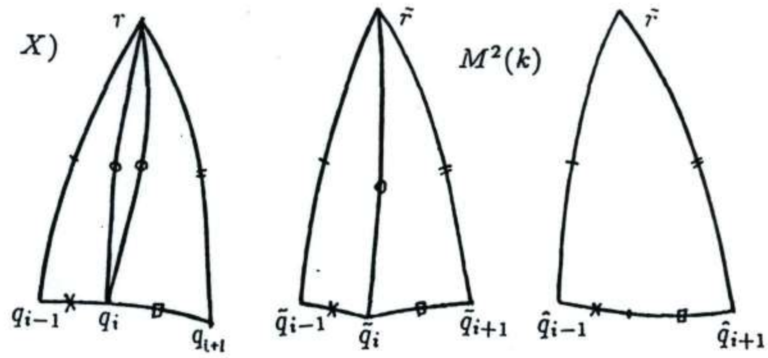


FIGURE 6-3.

Therefore if $\tilde{\Delta}(r\, q_{i-1}q_{i+1}) = \Delta(\tilde{r}\, \hat{q}_{i-1}\, \hat{q}_{i+1})$, then $\angle r q_{i-1} q_{i+1} \geq \angle \tilde{r} \tilde{q}_{i-1} \tilde{q}_i \geq \angle \tilde{r}\, \hat{q}_{i-1}\, \hat{q}_{i+1}$ and similarly, $\angle r q_{i+1} q_{i-1} \geq \angle \tilde{r}\, \hat{q}_{i+1} \hat{q}_{i-1}$. By iterating this procedure we have

$$\angle rpq \geq \angle(\alpha_0, \gamma)|_p \geq \tilde{\angle} r q_0 q_1 \geq \angle \tilde{r}\, \hat{q}_0 \hat{q}_2 \geq \cdots \geq \tilde{\angle} rpq.$$

This completes the proof of Theorem 6.1. □

We see from the above discussion that if $\tilde{\Delta}(rp\gamma(s)) = \Delta(\tilde{r}\, \tilde{p}\, \tilde{\gamma}(s))$, then $s \mapsto \tilde{\angle} rp\gamma(s)$ is monotone non-increasing in $s \in [0, d(p,q)]$. Therefore the Alexandrov convexity property holds for any two geodesics in X emanating from a common point.

For the proof of the lemma on narrow triangles we need Lemma 6.4 which deals with the critical case where the angles of a narrow triangle Δ are equal to those of the corresponding triangle $\tilde{\Delta}$.

Lemma 6.4. *Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be geodesics with $\alpha(0) = \beta(0) = p$ such that*

$$\alpha([0,a]) \cup \beta([0,b]) \subset \bigcup_{s\in[0,a]} U_{\alpha(s)} \cap \bigcup_{t\in[0,b]} U_{\beta(t)}.$$

Assume that there are $s_0 \in (0, a)$ and $t_0 \in (0, b)$. and a geodesic $\alpha(s_0)\beta(t_0)$ such that

(1) *the Alexandrov convexity property holds for all angles of $\Delta(p\alpha(s)\beta(t))$ for $s \in [0, s_0]$, $t \in [0, t_0]$.*

(2) $\angle p\beta(t_0)\alpha(s_0) = \tilde{\angle} p\beta(t_0)\alpha(s_0)$.

Then there exists a totally geodesic smooth surface of constant curvature k bounded by the geodesic triangle $\Delta(p\alpha(s_0)\beta(t_0))$.

Proof.

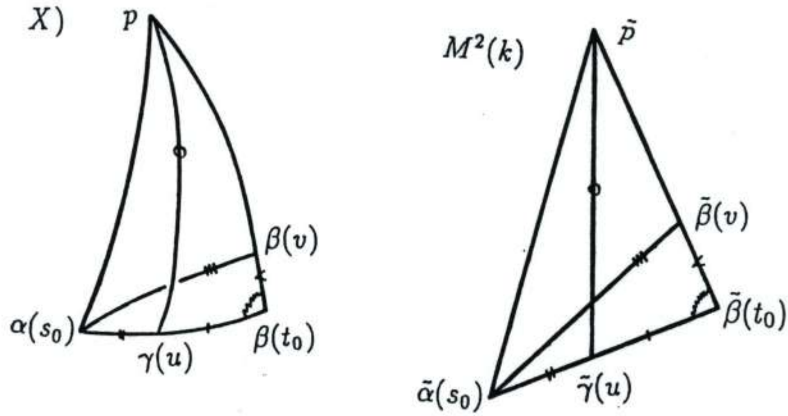


FIGURE 6-4.

Let $\gamma : [0, c] \to X$ be the edge of $\Delta(p\alpha(s_0)\beta(t_0))$ such that the angle beween γ and β at $\beta(t_0) = \gamma(0)$ is equal to $\tilde{\angle} p\beta(t_0)\alpha(s_0)$. For each $u \in [0, c]$ we see from (1) that

$$d(p, \gamma(u)) = d(\tilde{p}, \tilde{\gamma}(u))$$

and also for each $v \in [0, t_0]$,

$$d(\alpha(s_0), \beta(v)) = d(\tilde{\alpha}(s_0), \tilde{\beta}(v)),$$

and therefore we have

$$d(\gamma(u), \beta(v)) = d(\tilde{\gamma}(u), \tilde{\beta}(v)), \quad \text{for all} \quad (u, v) \in [0, c] \times [0, t_0].$$

First of all we assert that $\angle p\alpha(s_0)\beta(t_0) = \tilde{\angle} p\alpha(s_0)\beta(t_0)$ and $\angle \alpha(s_0)p\beta(t_0) = \tilde{\angle}\alpha(s_0)p\beta(t_0)$. Because $\Delta(p\beta(t_0)\gamma(u))$ and $\Delta(p\gamma(u)\alpha(s_0))$ have all angles not less than the corresponding angles, $\pi = \angle p\gamma(u)\beta(t_0) + \angle p\gamma(u)\alpha(s_0) \geq \tilde{\angle} p\gamma(u)\beta(t_0) + \tilde{\angle} p\gamma(u)\alpha(s_0) = \pi$ implies that for all $u \in (0, c)$ and for all $p\gamma(u) = \tau_u$

$$\angle p\gamma(u)\beta(t_0) = \tilde{\angle} p\gamma(u)\beta(t_0),\ \angle p\gamma(u)\alpha(s_0) = \tilde{\angle} p\gamma(u)\alpha(s_0).$$

Since $\lim_{u\to c} \tau_u = \alpha \,|\, [0, s_0]$, this proves

$$\angle p\alpha(s_0)\beta(t_0) = \tilde{\angle} p\alpha(s_0)\beta(t_0).$$

The same discussion leads to $\angle \alpha(s_0)p\beta(t_0) = \tilde{\angle}\alpha(s_0)p\beta(t_0)$.

The above fact implies that for every $u \in (0, c)$ and for every $v \in (0, t_0)$ the angles of $\Delta(\beta(t_0)\gamma(u)\beta(v))$ are the same as the corresponding angles.

Secondly, we prove that for an arbitrary fixed $u \in (0, c)$ and for a fixed geodesic $\tau_u : [0, l_u] \to X$ with $\tau_u(0) = p$, $\tau_u(l_u) = \gamma(u)$, $l_u = d(p, \gamma(u))$, there exists for each $v \in (0, t_0)$ a unique geodesic $\sigma_v = [0, m_v] \to X$ with $\sigma_v(0) = \beta(v)$, $\sigma_v(m_v) = \alpha(s_0)$, $m_v = d(\alpha(s_0), \beta(v))$ such that σ_v meets τ_u at a unique point. In fact, if $\tilde{\sigma}_v = [0, m_v] \to M^2(k)$ is a geodesic corresponding to σ_v, then $\tilde{\sigma}_v$ meets $\tilde{\tau}_u$ at a unique point, say, $\tilde{\tau}_n(z) = \tilde{\sigma}_v(w)$.

It follows from what is discussed in the last paragraph and from the Alexandrov convexity property for the angle at p of $\Delta(p\gamma(u)\beta(t_0))$ we have

$$d(\beta(v), \tau_u(z)) = d(\tilde{\beta}(v), \tilde{\tau}_u(z)),$$

and similarly from $\Delta(p\gamma(u)\alpha(s_0))$,

$$d(\alpha(s_0), \tau_u(z)) = d(\tilde{a}(s_0), \tilde{\tau}_u(z)),$$

and also

$$d(\beta(v), \alpha(s_0)) = d(\tilde{\beta}(v), \tilde{a}(s_0)).$$

This proves the second assertion.

Finally let σ_v for each $v \in [0, t_0]$ be the geodesic joining $\beta(v)$ to $\alpha(s_0)$ such that σ_v intersects τ_u at a point $\tau_u(z) = \sigma_v(w)$. Set

$$\mathcal{F}_u := \{x \in \sigma_v([0, d(\alpha(s_0), \beta(v))]); v \in [0, t_0]\}$$

where $\sigma_{t_0} = \gamma$. By the same manner as in the proof of Theorem 4.8 we obtain an isometric embedding of $\tilde{\Delta}$ into X which is smooth and totally geodesic. This proves Lemma 6.4. □

Remark. Under the assumptions in Lemma 6.4 we know that for every $s \in [0, s_0]$ and $t \in [0, t_0]$ the number of geodesics joining $\beta(t)$ to $\alpha(s)$ is equal to that of geodesics joining $\beta(t_0)$ to $\alpha(s_0)$, and is equal to that of totally geodesic smooth surfaces of constant curvature k bounded by $\alpha([0, s_0])$ and $\beta([0, t_0])$. Every geodesic $\alpha(s)\beta(t)$ lies on some $\mathcal{F}u$.

Lemma 6.5. (Lemma on narrow triangles) *Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be geodesic s with $\alpha(0) = \beta(0) = p$, $\alpha(a) = q$, $\beta(b) = r$ such that $\Delta(pqr)$ forms a nondegenerate geodesic triangle and such that*

$$\alpha([0,a]) \cup \beta([0,b]) \subset \bigcup_{s\in[0,a]} U_{\alpha(s)} \cap \bigcup_{t\in[0,b]} U_{\beta(t)}.$$

Then the Toponogov conparison theorem holds for every geodesic triangle $\Delta(p\alpha(s)\beta(t))$ and for all $s \in [0,a]$, $t \in [0,b]$.

Proof. For sufficiently small positive s and t we have $\Delta(p\alpha(s)\beta(t)) \subset U_p$. Let $s^* \in [0,a]$ and $t^* \in [0,b]$ be defined as follows: For every $s \in [0,s^*)$ and for every $t \in [0,t^*)$, the Toponogov comparison theorem holds for $\Delta(p\alpha(s)\beta(t))$ and $\alpha(s^*)\beta(t^*) \subset U_{\alpha(s^*)} \cap U_{\beta(t^*)}$. Definition 2.1 ensures that s^* and t^* exist with $s^* > 0$, $t^* > 0$. Let L be the least upper bound for the sum $s^* + t^*$ of such pairs (s^*, t^*). We only need to prove

$$L = a + b.$$

Suppose that $L < a+b$. Then there is a pair (s^*, t^*) such that $L = s^* + t^*$. Without loss of generality we may assume that

$$s^* < a.$$

There exists for a sufficiently small $\varepsilon > 0$ some numbers s' and t' near s^* and t^* such that for every $s \in [s^*, s^* + \varepsilon]$,

$$\alpha(s)\beta(t^*) \subset U_{\alpha(s')} \cap U_{\beta(t')}.$$

By means of the choice of s^* and t^* we see that $\Delta(p\alpha(s)\beta(t^*))$ for every $s \in (s^*, s^* + \varepsilon]$ does not have angles greater or equal to the corresponding ones, and hence we have from $\Delta(\beta(t^*)\alpha(s^*)\alpha(s)) \subset U_{\alpha(s')} \cap U_{\beta(t')}$,

$$\angle p\beta(t^*)\alpha(s) < \tilde{\angle} p\beta(t^*)\alpha(s). \tag{6-1}$$

Taking the limit of $\tilde{\angle}\, p\beta(t^*)\alpha(s)$ as $s \downarrow s^*$, we see $\angle\, p\beta(t^*)\alpha(s^*) \le \tilde{\angle}\, p\beta(t^*)\alpha(s^*)$. The opposite inequality is guaranteed by the choice of s^*, t^*, and hence for every geodesic $\alpha(s^*)\beta(t^*)$ we have

$$\angle\, p\beta(t^*)\alpha(s^*) = \tilde{\angle}\, p\beta(t^*)\alpha(s^*).$$

Thus Lemma 6.4 implies that $\Delta(p\alpha(s^*)\beta(t^*))$ for every geodesic $\alpha(s^*)\beta(t^*)$ has all angles equal to the corresponding angles of $\tilde{\Delta}(p\alpha(s^*)\beta(t^*))$ and that there exists a smooth totally geodesic surface of constant curvature k bounded by $\alpha([0, s^*])$, $\beta([0, t^*])$ and $\alpha(s^*)\beta(t^*)$. It should be noted that there is no such a geodesic $\alpha(s^*)\beta(t^*)$ that satisfies $\angle\, p\beta(t^*)\alpha(s^*) > \tilde{\angle}\, p\beta(t^*)\alpha(s^*)$. The existence of such a geodesic violates the choice of s^* and t^*.

Setting $q^* := \alpha(s^* + \varepsilon)$, we find a number $s_1 \in (0, s^*)$ with the property that the angle comparison holds for every geodesic triangle $\Delta(\beta(t^*)\alpha(s)q^*)$ for all $s \in (s_1, s^* + \varepsilon)$. In fact, $\Delta(\alpha(s^*)q^*\beta(t^*)) \subset U_{\alpha(s')}$ implies that $\Delta(\alpha(s)q^*\beta(t^*))$ is also contained in $U_{\alpha(s')}$ for all $s < s^*$ sufficiently close to s^*. Thus the angle comparison holds for such $\Delta(\alpha(s)q^*\beta(t^*))$. The inequality (6-1) implies that if s_1^* is the infimun of such $s \in (0, s^*)$ that satisfies the angle comparison for $\Delta(\alpha(s)q^*\beta(t^*))$, then $s_1^* > 0$.

Because $s_1^* > 0$ every geodesic triangle $\Delta(\alpha(s_2)q^*\beta(t^*))$ for $s_2 < s_1^*$ being taken sufficiently close to s_1^* has the property that the angle comparison does not hold for this triangle. The previous discussiòn then implies that every geodesic $\beta(t^*)\alpha(s_1^*)$ has the property that

$$\angle\, \beta(t^*)\alpha(s_1^*)q^* = \tilde{\angle}\, \beta(t^*)\alpha(s_1^*)q^*$$
$$\angle\, \alpha(s_1^*)\beta(t^*)q^* = \tilde{\angle}\, \alpha(s_1^*)\beta(t^*)q^*$$
$$\angle\, \alpha(s_1^*)q^*\beta(t^*) = \tilde{\angle}\, \alpha(s_1^*)q^*\beta(t^*)$$

and that there exists a smooth totally geodesic surface $\mathcal{F}^*$ of constant curvature k bounded by $\Delta(\beta(t^*)\alpha(s_1^*)q^*)$. Because $\angle\, p\alpha(s_1^*)\beta(t^*) = \tilde{\angle}\, p\alpha(s_1^*)\beta(t^*)$ the $\mathcal{F}^*$ can be extended to a smooth totally geodesic surface of constant curvature k bounded by $\alpha([0, s^* + \varepsilon])$, $\beta([0, t^*])$ and $g^*\beta(t^*)$. This fact means that $\angle\, p\beta(t^*)q^* = \tilde{\angle}\, p\beta(t^*)q^*$, a contradiction to (6-1) for $s = s^* + \varepsilon$. This proves Lemma 6.5. $\square$

The Alexandrov convexity and hinge theorem holds for all Δ, and stated as follows. This is a direct consequence of Theorem 6.1 and the proof is omitted.

Theorem 6.6. *Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be geodesics with $\alpha(0) = \beta(0) = p$. If $\theta_k(s,t)$ for $s \in [0,a]$ and $t \in [0,b]$ is the angle at $\tilde{p}$ of the triangle $\tilde{\Delta}(p\alpha(s)\beta(t))$, then $\theta_k(s,t)$ is monotone non-increasing. In particular if $\hat{\alpha}, \hat{\beta}$ are geodesics on $M^2(k)$ with the same starting point and the same lengths as α, β and have the same angle $\angle(\hat{\alpha}, \hat{\beta}) = \angle(\alpha, \beta)|p$, then*

$$d(\alpha(s), \beta(t)) \leq d(\hat{\alpha}(s), \hat{\beta}(t))$$

for all $s \in [0,a]$ and $t \in [0,b]$.

Proof of Theorem 6.2. First of all we prove that if $k > 0$, then $d(X) \leq \pi/\sqrt{k}$. Suppose that $d(X) > \pi/\sqrt{k}$. Then there are points $p, q \in X$ with $d(p,q) > \pi/\sqrt{k}$. Let $m \in pq$ be the midpoint of a geodesic pq and take a point x near m such that $x \notin pq$ and $\Delta(pmx)$ and $\Delta(qmx)$ have circumfence less than $2\pi/\sqrt{k}$. Then, the hinge theorem 6.6 implies that if $\tilde{p}, \tilde{m}, \tilde{q}$ are on a great circle and if $\hat{x} \in M^2(k)$ is taken such that $d(\tilde{m}, \hat{x}) = d(m,x)$, $\angle \tilde{p} \tilde{m} \hat{x} = \angle pmx$ and $\angle \tilde{q} \tilde{m} \hat{x} = \angle qmx$, then $d(\tilde{p}, \hat{x}) \geq d(p,x)$ and $d(\tilde{q}, \hat{x}) \geq d(q,x)$. It is clear that on $M^2(k)$, we have $d(\tilde{p}, \hat{x}) + d(\tilde{q}, \hat{x}) < d(\tilde{p}, \tilde{m}) + d(\tilde{q}, \tilde{m})$ (see Figure 6-5). Thus $d(p,q) > d(\tilde{p}, \tilde{x}) + d(\tilde{q}, \tilde{x}) \geq d(p,x) + d(q,x)$, a contradiction.

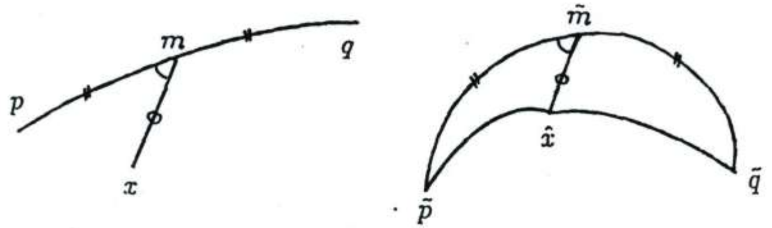


FIGURE 6-5.

We next prove that if $p, q \in X$ are chosen such that

$$d(p,q) = d(X),$$

then q is uniquely determined for p. In fact, suppose $q_1 \in X$ satisfies $q_1 \neq q$ and $d(p,q_1) = d(p,q) = d(X)$. By taking the midpoint m of qq_1 and choosing a constant $k_1 \in (0,k)$ such that the circumference of $\Delta(pqq_1)$ is less than (but sufficiently close to) $2\pi/\sqrt{k_1}$ we see that the corresponding geodesic triangle $\tilde{\Delta}(pqq_1)$ sketched on $M^2(k_1)$ has the property that if $\hat{m}$ is the midpoint of $\tilde{q}\,\tilde{q}_1$, then $d(\tilde{p},\tilde{m}) > d(\tilde{p},\tilde{q})$. Theorem 6.6 then implies that $d(p,m) \geq d(\tilde{p},\tilde{m})$. This is ridiculons.

Now, we prove that $\Delta(pqr)$ for every p, q and r in X has circumference not greater than $2\pi/\sqrt{k}$. We have already established that each edge has length not greater than $\pi/\sqrt{k}$. Suppose that the circumference L of $\Delta(pqr)$ is greater than $2\pi/\sqrt{k}$. There is an interior point m of qr such that $d(p,q) + d(q,m) = d(p,r) + d(r,m) = L/2$. Choosing a constant $k_1 \in (0,k)$ as in the last paragraph, we observe that

$$d(p,m) > L/2 > \pi/\sqrt{k},$$

which is a contradiction.

Finally, if the circumference of a triangle is $2\pi/\sqrt{k}$, then the above argument implies that $d(X) = \pi/\sqrt{k}$. Let $p, q \in X$ be such that

$$d(p,q) = \pi/\sqrt{k}.$$

For every point $r \in X\backslash\{p,q\}$ we have $d(p,r) + d(r,q) = \pi/\sqrt{k}$ and hence there exists a unique geodesic joining p to q and passing through r. Let

$$E := \{x \in X : d(p,x) = d(q,x)\}.$$

If $x, y \in E$ satisfy $d(x,y) \leq \pi/2\sqrt{k}$, then xy is contained entirely in E and moreover $\angle pxy = \angle qxy = \angle pyx = \angle qyx = \pi/2$. Therefore Lemma 6.4 implies the existence of a totally geodesic smooth surface of constant curvature k bounded by two geodesics joining p to q and passing through x and y. This means that X is isometric to the spherical suspension $\sum(X)$ of E, where E is a totally geodesic Alexandrov subspace with $\mathrm{Curv}(E) \geq k$. If $d(E) = \pi/\sqrt{k}$, then E is isometric to a spherical suspension. This proves Theorem 6.2. $\square$

The Toponogov splitting theorem holds for Alexandrov spaces with curvature bounded below by 0. A similar discussion is seen in Theorem 5.8 in [GP].

Theorem 6.7. (The Toponogov splitting theorem.) *If $Curv(X) \geq 0$ and if X admits a straight line $\gamma : \mathbf{R} \to X$, then X is isometric to the metric product $X_1 \times \mathbf{R}$, where X_1 is an Alexandrov space with $Curv(X_1) \geq 0$.*

A Busemain function $F_\alpha : X \to \mathbf{R}$ for a ray $\sigma : [0, \infty) \to X$ is used for the proof of Theorem 6.7. Let

$$F_\sigma(x) := \lim_{t \to \infty} [t - d(\sigma(t), x)].$$

Since $t - d(\sigma(t), x)$ is monotone non-increasing in $t \geq 0$ and bounded above by $d(\sigma(0), x)$, $F_\sigma(x)$ is well defined. The Alexandrov convexity property then implies that F_σ is midconvex in the following sense. If $\alpha : [0, l] \to X$ is any geodesic, then $F_\sigma \circ \sigma(s) + F_\sigma \circ \alpha(t) \geq 2F_\sigma \circ \alpha(\frac{1}{2}(s + t))$ for all $s, t \in [0, l]$. Since a continuous midcenvex function is convex, so is F_σ, e.g., F_σ is convex along every geodesic in X.

Proof of Theorem 6.7. Fix an arbitrary point $p \in X \setminus \gamma(\mathbf{R})$ and choose the arclength parameter of γ so as to satisfy that $d(p, \gamma(0)) = d(p, \gamma(\mathbf{R}))$. If $F_+ := F_{\gamma[0,\infty)}$ and $F_- := F_{\gamma|(-\infty,0]}$, then triangle inequality implies that $F_+(p) + F_-(p) \leq 0$. On the other hand Theorem 6.1 for $\Delta(p\gamma(0)\gamma(t))$ and $\Delta(p\gamma(0)\gamma(-t))$ implies that (by letting $t \to \infty$), $F_+(p) = F_-(p) = 0$. Namely we have $F_+ + F_- \equiv 0$, and hence there exists a unique straight line $\gamma_p : \mathbf{R} \to X$ passing through p along which both $F_+ \circ \gamma_p$ and $F_- \circ \gamma_p$ are linear. In particular γ_p and γ make the same angle $\frac{\pi}{2}$ with $p\gamma(0)$ at p and $\gamma(0)$ respectively. A slight modification of Lemma 6.4, then implies the existence of a flat totally geodesic strip bounded γ_p and γ. This fact means that a lenel set $F_+^{-1}(\{0\})$ is isometric to all the other level sets of F_+. Since $F_+^{-1}(\{t\})$ for every $t \in \mathbf{R}$ is totally geodesic, we see that $X_1 := F_+^{-1}(\{0\})$ is an Alexandrov space with $\text{Curv}(X_1) \geq 0$. This proves Theorem 6.7. $\square$

§7. Strainers and Dimension

Throughout this section let X be an Alexandrov space with $\text{Curv}(X) \geq k$. The purpose of this section is to prove that there exists an open dense set in X each point of which has an open set homeomorphic to an open set in $\mathbf{R}^n$ for some positive integer $n \leq \infty$. Thus the dimension of X is defined here.

Definition 7-1. A point $p \in X$ by definition an (n, δ)-strained point for a sufficiently small δ iff there exist n pairs of points $(a_i, b_i)_{i=1}^n \in M \setminus \{p\}$ such that for all $i, j = 1, \ldots, n$ with $i \neq j$,

$$\tilde{\angle}\, a_i p b_i > \pi - \delta,\ \tilde{\angle}\, a_i p a_j > \frac{\pi}{2} - \delta$$
$$\tilde{\angle}\, a_i p b_j > \frac{\pi}{2} - \delta,\ \tilde{\angle}\, b_i p b_j > \frac{\pi}{2} - \delta.$$

We also say that such an $(a_i, b_i)_{i=1}^n$ is an (n, δ)-strainer at p.

The original idea of strainers will go back to the proof of a differentiable sphere theorem (see [OSY]) which states that if a complete Riemannian n-manifold M has its sectional curvature $K_M \geq 1$ and if $d_H(M, S^n(1))$ is sufficiently close to 0, then M is diffeomorphic to S^n. In fact $S^n(1)$ has a global $(n+1, 0)$-strainer each pair a_i^*, b_i^* of which is obtained as the intersection of $S^n(1)$ with the i-th coordinate axis. Hausdorff closeness between M and $S^n(1)$ then implies that M has a global $(n+1, \delta)$-strainer, each pair a_i, b_i of which is the image of a_i^*, b_i^* under a Hausdorff approximation map. Then the map $\Phi : M \to \mathbf{R}^{n+1}$ defined by $\Phi(x) := (\cos d(a_1, x), \ldots, \cos d(a_{n+1} x))$ can be approximated by a smooth regular map whose image is C^1-close to $S^n(1)$ in $\mathbf{R}^{n+1}$. Applying this idea in a small neighborhood around a strained point $p \in X$, a bilipschitz homeomorphism between such a neighborhood and an open set in $\mathbf{R}^n$ will be established.

It should be remarked that for an (n, δ)-strainer (a_i, b_i) at p we observe from Theorem 6.1 nd Lemma 3.6 that

$$\text{(7-1)} \qquad \frac{\pi}{2} - \delta < \tilde{\angle}\, a_i p a_j \leq \angle\, a_i p a_j \leq 2\pi - \angle\, a_i p b_j - \angle\, a_j p b_j < \frac{\pi}{2} + 2\delta$$

Also the set of all (n, δ)-strained points is open in X. By means of the Alexandrov convexity property we see that if $(a_i, b_i)_{i=1}^n$ is an (n, δ)-strainer at p and if $a_i' \in pa_i$, $b_i' \in pb_i$, then so is $(a_i', b_i')_{i=1}^n$. We can therefore choose an (n, δ)-strainer at p as close to p as desired.

Also notice that n-pairs of points $(a_i, b_i)_{i=1}^n$ is an (n, δ)-strainer at p, if and only if for all $i, j = 1, \ldots, n$ with $i \neq j$

$$\text{(7-2)} \qquad \begin{cases} \angle\, a_i p b_i > \pi - \delta, & \angle\, a_i p b_j > \frac{\pi}{2} - \delta \\ \angle\, b_i p b_j > \frac{\pi}{2} - \delta, & \angle\, a_i p a_j > \frac{\pi}{2} - \delta. \end{cases}$$

In fact, if $(a_i, b_i)_{i=1}^n$ satisfies the above inequalities then by choosing $a_i' \in pa_i$, $b_i' \in pb_i$ sufficiently close to p, we observe that $|\angle\, a_i p b_i - \tilde{\angle}\, a_i' p b_i'|$ and $|\angle\, a_i p b_j - \tilde{\angle}\, a_i' p b_j'|$, e.t.c., are sufficiently close to zero, and $(a_i', b_i')_{i=1}^n$ satisfies the definition of an (n, δ)-strainer. The converse is clear from Theorem 6.1.

Definition 7.2. Let Y and Z be metric spaces and $U \subset Y$ an open set. A map $\varphi : U \to Z$ is by definition an ε-open for an $\varepsilon > 0$ iff for any $x \in U$ and for any $z \in Z$ such that if $\{w \in Y; d_Y(x,w) \le \frac{1}{\varepsilon} d_Z(\varphi(x),z)\} \subset U$ then there exists a point $y \in U$ such that $\varphi(y) = z$ and such that $d_Y(x,y) \le \frac{1}{\varepsilon} d_Z(\varphi(x),z)$.

An ε-open map $\varphi : U \to Z$ is open. In fact for any open set $V \subset U$ and for any fixed point $x \in V$ there exists an $r > 0$ such that $B(x,r) \subset U$. We only need to show that $B(\varphi(x), \varepsilon r) \subset \varphi(V)$. If $z \in B(\varphi(x), \varepsilon r)$, then there is a point $y \in U$ such that $\varphi(y) = z$ and such that $d_Y(x,y) \le \frac{1}{\varepsilon} d_Z(\varphi(x),z) < r$. This proves $B(\varphi(x), \varepsilon r) \subset \varphi(V)$.

If $\varphi : U \to Z$ is a continuous and $1-1$ ε-open map, then $\varphi : U \to \varphi(V)$ is a homeomorphism. Moreover φ^{-1} is locally Lipschitz with Lipschitz constant ε^{-1}. For the proof of φ^{-1} being locally Lipschitz homeomorphism with Lipschitz constant ε^{-1}, we set $r_p := d(p, Y \setminus U)$ for every $p \in U$ (if $U = Y$ then $r_p := \infty$). For a fixed $r \in (0, r_p)$ we set $V_1 := \varphi(B(p,r)) \cap B(\varphi(p), \frac{\varepsilon}{2}(r_p - r))$ and $V := \varphi^{-1}(V_1) = B(p,r) \cap \varphi^{-1}(B(\varphi(p), \frac{\varepsilon}{2}(r_p - r))$. We prove that $\varphi^{-1}|_{V_1}$ is Lipschitz continuous with Lipschitz constant ε^{-1}. Let $x, y \in V$. Then $\{z \in Y; d_Y(x,z) \le \frac{1}{\varepsilon} d_Z(\varphi(x), \varphi(y))\} \subset \{z \in Y; d_Y(x,z) \le r_p - r\}$ follows from $d_z(\varphi(x), \varphi(y)) \le d(V_1) \le \varepsilon(r_p - r)$, and the right hand side of the above implication is contained in $B(x, r_p - r) \subset B(B(p,r), r_p - r) = B(p, r_p) \subset U$. Thus

$$\frac{d_Y(\varphi^{-1}(w), \varphi^{-1}(z))}{d_Z(w,z)} \le \frac{1}{\varepsilon} \quad \text{for all} \quad w, z \in V_1.$$

Theorem 7.3. *If $p \in X$ is an (n,δ)-strained point with a strainer $(a_i, b_i)_{i=1}^n$ for $\delta < 1/2n$, then there exists an open set U around p. Further a map $\varphi : U \to \mathbf{R}^n$ defined by $\varphi(q) := (d(a_1,q), \dots, d(a_n,q)) \in \mathbf{R}^n$ is an $\frac{1-2n\delta}{\sqrt{n}}$-open map. Moreover, if U does not admit any $(n+1, \tau(\delta))$-strained point for $\tau(\delta)$ with $\lim_{\delta \to 0} \tau(\delta) = 0$, then there is an open set $V \subset U$ around p on which φ is a bilipschitz homeomorphism.*

The following lemma 7.4 is used for the proof of the second statement in Theorem 7.3.

Lemma 7.4. *Let $Curv(X) \geq 0$ and $a, b, c, d \in X$ be points such that (see Figure 7-2) for sufficiently small positive numbers ε and ε_1,*

$$d(b,d) < \varepsilon \cdot \min\{d(a,b), d(c,b)\}, \ \tilde{\angle} abc \geq \pi - \varepsilon_1$$
$$\frac{d(a,b)\cdot d(b,c)}{d(a,c)^2} > \varepsilon^{\frac{1}{3}}.$$

Then there is a constant $\tau(\varepsilon) := \sqrt{\varepsilon}+4\varepsilon$ such that $\angle abd - \tilde{\angle} abd$, $\angle adb - \tilde{\angle} adb$, $\angle cbd - \tilde{\angle} cbd$ and $\angle cdb - \tilde{\angle} cdb$ are all bounded above by $\tau(\varepsilon) + 2\varepsilon_1$.

Proof of Lemma 7.4. From assumption we may assume that $\tilde{\angle} bad \leq 2 \sin \tilde{\angle} bad \leq 2\frac{d(b,d)}{d(a,b)} < 2\varepsilon$, and similarly, $\tilde{\angle} bcd < 2\varepsilon$.

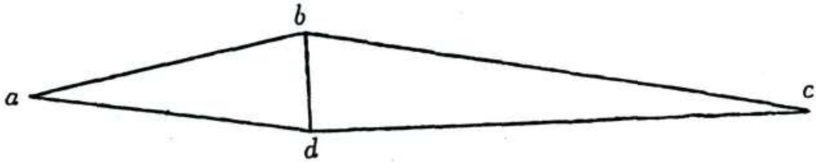


FIGURE 7-1.

From $\tilde{\Delta}(abd)$ and $\tilde{\Delta}(cbd)$ we see

$$\tilde{\angle} abd + \tilde{\angle} adb + \tilde{\angle} cbd + \tilde{\angle} cdb > 2\pi - 4\varepsilon. \tag{7-3}$$

We now assert that $\tilde{\angle} adc \geq \pi - \sqrt{\varepsilon} - \varepsilon_1$. In fact

$$\cos(\pi - \varepsilon_1) > \cos \tilde{\angle} abc = \frac{d(a,b)^2 + d(b,c)^2 - d(a,c)^2}{2d(a,b)\cdot d(b,c)}.$$

From assumption $|d(a,b) - d(a,d)| < d(b,d) < \varepsilon \cdot d(a,b)$ follows

$$(1-\varepsilon)d(a,b) \leq d(a,d) \leq (1+\varepsilon)d(a,b)$$

and similarly

$$(1-\varepsilon)d(c,b) \leq d(c,d) \leq (1+\varepsilon)d(c,b).$$

Therefore above three inequalities reduce to

$$-\cos\varepsilon_1 > \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \left\{\frac{d(a,d)^2 + d(c,d)^2 - d(a,c)^2}{2d(a,d)\cdot d(c,d)} - \frac{d(a,c)^2}{d(a,d)\cdot d(c,d)}\cdot \varepsilon(1+\frac{\varepsilon}{2})\right\}.$$

From assumption we have $\dfrac{d(a,c)^2}{d(a,d)d(c,d)} < \varepsilon^{-\frac{1}{2}}$ and hence

$$\tilde{\angle}\, adc \geq \pi - \sqrt{\varepsilon} - \varepsilon_1.$$

Theorem 6.1 then implies together with Lemma 3.6 that $\angle\, abd - \tilde{\angle}\, abd$ is bounded above by $\angle\, abd + \angle\, adb + \angle\, cbd + \angle\, cdb - (\tilde{\angle}\, abd + \tilde{\angle}\, adb + \tilde{\angle}\, cbd + \tilde{\angle}\, cdb) < (2\pi - \angle\, abc) + (2\pi - \angle\, adc) - (2\pi - 4\varepsilon) < (\pi + \varepsilon_1) + (\pi + \sqrt{\varepsilon} + \varepsilon_1) - (2\pi - 4\varepsilon) = \sqrt{\varepsilon} + 4\varepsilon + 2\varepsilon_1 = \tau(\varepsilon) + 2\varepsilon$. Thus the proof of Lemma 7.4 is complete. □

Corollary to Lemma 7.4. *In addition to the assumption as in Lemma 7.4 if $|d(a,b) - d(a,d)| < \varepsilon \cdot d(b,d)$ and if $x \in bd$, then*

$$\min\{\tilde{\angle}\, axb, \tilde{\angle}\, axd, \tilde{\angle}\, cxb, \tilde{\angle}\, cxd\} \geq \frac{\pi}{2} - \tau(\varepsilon) - 3\varepsilon_1.$$

Proof. The additional assumption means that $\tilde{\Delta}(abd)$ is sufficiently close to an isosceles triangle, and hence there is a $\tau_1(\varepsilon)$ with $\lim_{\varepsilon\to 0}\tau_1(\varepsilon) = 0$ such that $\tilde{\angle}\, abd < \frac{\pi}{2} + \tau_1(\varepsilon)$. Hence we get from Lemma 7.4 $\tilde{\angle}\, abx \leq \angle\, abx = \angle\, abd \leq \tilde{\angle}\, abd + (\tau(\varepsilon) + 2\varepsilon_1) \leq \frac{\pi}{2} + \tau_2(\varepsilon) + 2\varepsilon_1$. All the other angles are estimated by a similar manner. This proves Corollary to Lemma 7.4. □

Proof of Theorem 7.3. The norm of $\mathbf{R}^n$ is defined as $\|x\| := \sum_{i=1}^n |x_i|$ for every $x = (x_1, \ldots, x_n)$. We first assert that φ is $(1-2n\delta)$-open with respect to the above norm. Take a point $\bar{q} \in X$ sufficiently close to p and set $\bar{z} := \varphi(\bar{q})$. For any point $z \in \mathbf{R}^n$ taken sufficiently close to $\bar{z}$, we prove that there is a point $q \in X$ such that $\varphi(q) = z$ and $d(q,\bar{q}) \leq \frac{1}{1-2n\delta}\|z - \bar{z}\|$. The q is obtained as the limit of a sequence $\{q_j\}$, which we now want to construct. Since all

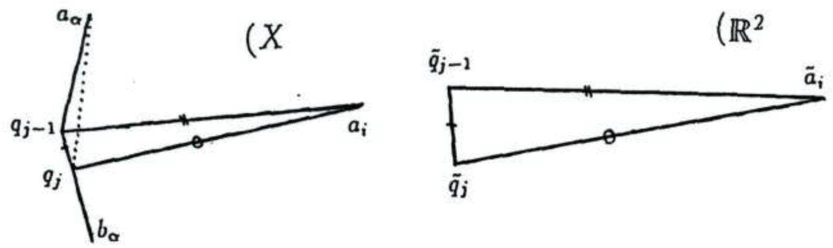


FIGURE 7-2.

points under consideration are taken sufficiently close to p, we may sketch all the corresponding geodesic triangles in $\mathbf{R}^2$. Set $q_1 := \bar q$ and assume that q_{j-1} is well defined. Then q_j is defined as follows: First, choose an $\alpha = 1, \ldots, n$ such that $|d(a_\alpha, q_{j-1}) - z^\alpha| = \max\{|d(a_\beta, q_{j-1}) - z^\beta| : \beta = 1, \ldots, n\}$. Let $q_j \in a_\alpha q_{j-1} \cup q_{j-1} b_\alpha$ be chosen such that $d(q_j, a_\alpha) = z^\alpha$. We may assume $q_j \in q_{j-1} b_\alpha$, since the other case is easier. The Alexandrov convexity implies that $\tilde\angle\, a_\alpha q_{j-1} q_j \geq \tilde\angle\, a_\alpha q_{j-1} b_\alpha \geq \pi - \delta$, and hence $d(a_\alpha, q_{j-1}) < d(a_\alpha, q_j)$. Thus $|d(a_\alpha, q_{j-1}) - z^\alpha| = d(a_\alpha, q_j) - d(a_\alpha, q_{j-1}) \gtrsim d(q_{j-1}, q_j) \cdot \cos\delta$ (see Figure 7.2)

why exists the point

+ Alexandrov cond.

From (7-1) we have $\frac{\pi}{2} - \delta < \tilde\angle\, q_j q_{j-1} a_i \leq \angle\, q_j q_{j-1} a_i < \frac{\pi}{2} + 2\delta$ for every $i \neq \alpha$. The cosine rule for plane triangles then implies that for $i \neq \alpha$,

$$
\begin{aligned}
&|d(a_i, q_j) - d(a_i, q_{j-1})| \\
&\leq \frac{d(q_j, q_{j-1})}{d(a_i, q_j) + d(a_i, q_{j-1})} \left\{ d(q_j, q_{j-1}) + 2d(a_i, q_{j-1}) \cdot |\cos \tilde\angle a_i q_{j-1} q_j| \right\}.
\end{aligned}
$$

Since $-\sin 2\delta < \cos \tilde\angle(a_i q_{j-1} q_j) < \sin\delta$, the right hand side of the above inequality is bounded above by

$$
\frac{d(q_j, q_{j-1})^2 + 2d(a_i, q_{j-1}) \cdot d(q_j, q_{j-1}) \cdot \sin 2\delta}{d(a_i, q_{j-1}) + d(a_i, q_j)} < 2\delta \cdot d(q_{j-1}, q_j).
$$

Here we notice $d(q_j, q_{j-1}) \ll \delta$ by the choice of z, and hence $\dfrac{2d(a_i, q_{j-1})}{d(a_i, q_{j-1}) + d(a_i, q_j)} \doteqdot 1$. Therefore we have for every $i \neq \alpha$,

$$
\begin{aligned}
\text{(7-4)} \qquad & |d(a_i, q_j) - z^i| \\
&\leq |d(a_i, q_j) - d(a_i, q_{j-1})| + |d(a_i, q_{j-1}) - z^i| \\
&\leq 2\delta d(q_j, q_{j-1}) + |d(a_i, q_{j-1}) - z^i| \\
&\leq \frac{2\delta}{\cos\delta} |d(a_\alpha, q_{j-1}) - z^\alpha| + |d(a_i, q_{j-1}) - z^i|.
\end{aligned}
$$

By setting $\lambda_j := \|\varphi(q_j) - z\|$, we have from the choice of α that $\lambda_j = \sum_{i\neq\alpha} |d(a_i, q_j) - z^i|$, and also

$$\lambda_j \leq \frac{2\delta}{\cos\delta}(n-1)|d(a_\alpha, q_{j-1}) - z^\alpha| + \lambda_{j-1} - |d(a_\alpha, q_{j-1}) - z^\alpha|.$$

Therefore, we have from $\delta < \frac{1}{2n}$ that

$$
\begin{aligned}
\lambda_{j-1} - \lambda_j &\geq \{1 - \frac{2\delta(n-1)}{\cos\delta}\} |d(a_\alpha, q_{j-1}) - z^\alpha| \\
&\geq \{\cos\delta - 2\delta(n-1)\} d(q_j, q_{j-1}) \\
&> (1 - 2n\delta) d(q_j, q_{j-1}).
\end{aligned}
$$

$1.\ \delta \leq \delta_0 \wedge \frac{1}{2n}$

On the other hand, from the choice of α it follows that

$$\lambda_{j-1} = \sum_{i=1}^{n} |d(a_i, q_{j-1}) - z^i| \leq n|d(a_\alpha, q_{j-1}) - z^\alpha|.$$

Using above two inequalities we get

$$
\begin{aligned}
\lambda_j &\leq \lambda_{j-1} - (1 - \frac{2\delta(n-1)}{\cos\delta}) \cdot |d(a_\alpha, q_{j-1}) - z^\alpha| \\
&\leq \lambda_{j-1} - (1 - \frac{2\delta(n-1)}{\cos\delta}) \frac{\lambda_{j-1}}{n}
\end{aligned}
$$

Here we used the assumption $\cos\delta > \delta > \cos\frac{1}{2n} > 1 - \frac{1}{2n}$, and hence $\frac{n-1}{\cos\delta} < \frac{n-1}{1-\frac{1}{2n}} < n$. This means that

$$\lambda_j \leq \lambda_{j-1}\left[1 - (1 - \frac{2\delta(n-1)}{\cos\delta})\frac{1}{n}\right] < \lambda_{j-1}(1 - \frac{1-2\delta}{n}).$$

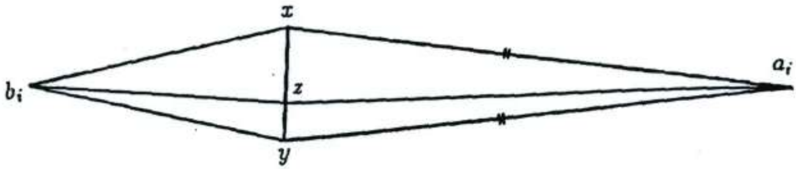


FIGURE 7-3.

Therefore $\{\lambda_j\}$ is a strictly decreasing sequence and has a limit. To show that $\{q_j\}$ is a Cauchy sequence we compute $d(q_k, q_\ell) \le \sum_{j=k+1}^{\ell} d(q_{j-1}, q_j) <$ $\sum_{\alpha=k+1}^{\ell} \frac{\lambda_{j-1} - \lambda_j}{1 - 2n\delta} = \frac{\lambda_{k+1} - \lambda_\ell}{1 - 2n\delta}$. Therefore $\{q_j\}$ has a limit, say, $q \in X$ such that $\varphi(q) = z$. This proves the open property of φ.

The local Lipschitz constant of φ^{-1} is oftained by

$$\frac{d(\varphi^{-1}(\bar{z}), \varphi^{-1}(z))}{\|\bar{z} - z\|} = \frac{d(\bar{q}, q)}{\|\bar{z} - z\|} \le \frac{\sum^{\infty} d(q_{j-1}, q_j)}{\|\bar{z} - z\|}$$
$$< \frac{\lambda_1}{1 - 2n\delta} \cdot \frac{1}{\|\bar{z} - z\|} = \frac{1}{1 - 2n\delta},$$

and hence

$$\frac{d(\varphi^{-1}(\bar{z}), \varphi^{-1}(z))}{d_{\mathbf{R}^n}(\bar{z}, z)} \le \frac{1}{\sqrt{n}(1 - 2n\delta)}.$$

Clearly φ is continuous. For the proof of the final statement we only need to show that φ is 1-1 if there is no $(n+1, 4\delta)$-strained point in some neighborhood around p. Let $V \subset X$ be a neighborhood around p such that every point in V is a strained point with a strainer $(a_i, b_i)_{i=1}^n$ and such that

$$d(V) \ll \min\{d(p, a_i), d(p, b_i); i = 1, \dots, n\}.$$

Suppose that $\varphi|V$ is not 1-1. Then there are points $x, y \in V$, $x \ne y$ such that (see Figure 7-3)

$$d(a_i, x) = d(a_i, y), \quad i = 1, \dots, n.$$

Then, Corollary to Lemma 7.4 implies that there is a $\tau(\delta)$ with $\lim_{\delta\to 0}\tau(\delta) = 0$ such that if $z \in xy$ is the midpoint of xy, then all the angles $\tilde{\angle}\, a_i z x$, $\tilde{\angle}\, a_i z y$, $\tilde{\angle}\, b_i z x$, $\tilde{\angle}\, b_i z y$ are bounded below by $\pi/2 - \tau(\delta)$ for all $i = 1, \dots, n$. Setting $a_{n+1} := x$ and $b_{n+1} = y$, we see that $(a_i, b_i)_{i=1}^{n+1}$ is an $(n+1, \tau(\delta))$-strainer at $z \in V$, a contradiction. This proves Theorem 7.3. $\square$

A point $x \in X$ is said to be a manifold point iff there exists an open set around x which is bilipschitz homeomorphic to an open set in $\mathbf{R}^n$. As is seen in the proof of Theorem 7.3 we can construct an (n, δ)-strainer at a point p' sufficiently close to any given point $p \in X$. Thus we have proved the

Corollary to Theorem 7.3. *The set of all manifold points on X forms an open and dense set in X.*

We now want to discuss the topological dimension of X. We consider a normal space Y. The covering dimension $\dim Y$ of a normal space Y is defined as follows. We say that $\dim Y \le n$ if and only if for every finite open cover $G_1, \cdots, G_s$ of Y there exists a refinement $H_1, \cdots, H_s$ with $\bigcup_{i=1}^{s} H_i = Y$ such that $H_i \subset G_i$ for $i = 1, \dots, s$ and such that $\bigcap_{j=1}^{n+2} H_{i_j} = \phi$ for every subclass $\{H_{i_j}\}$ of $n+2$ members of $H_1, \cdots, H_s$. We say that $\dim Y = n$ if and only if $\dim Y \le n$ and $\dim Y \le n-1$ does not hold.

The large (respectively, small) inductive dimension $\operatorname{Ind} Y$ (respectively, $\operatorname{ind} Y$) of Y is defined as follows. $\operatorname{Ind} Y = -1$ (respectively, $\operatorname{ind} Y = -1$) if and only if $y = \phi$. Suppose that $\operatorname{Ind} Y \le n-1$ (respectively, $\operatorname{ind} Y \le n-1$) has been defined for a normal space Y. Then $\operatorname{Ind} Y \le n$ (respectively, $\operatorname{ind} Y \le n$) if and only if for every closed set $F \subset Y$ and for every open set $G \subset Y$ with $F \subset G$ there exists an open set $H \subset Y$ such that $F \subset H \subset G$ and $\operatorname{Ind}(\bar{H}\setminus H) \le n-1$ (respectively, for every $y \in Y$ and for every open set $G \subset Y$ with $y \in G$ there exists an open set $H \subset Y$ such that $x \in H \subset G$ and $\operatorname{ind}(\bar{H}\setminus H) \le n-1$). It is well known that these dimensions are all topological invariants.

If Y is metric, then $\dim Y = \operatorname{Ind} Y$ is due to Katětov and Morita. If Y is separable metric, then $\operatorname{ind} Y = \operatorname{Ind} Y$ (see [HW]). Moreover we have the countable sum theorems for separable metric spaces as follows (see [E]). If a separable metric space Y is expressed as a countable sum of closed sets

$F_1, \cdots$, and if $\dim F_i \le n$ for all $i = 1, 2, \cdots$, then $\dim Y \le n$. Similarly, if $Y = \bigcup_{i=1}^{\infty} U_i$ for open sets $U_1, \cdots$, and if $\dim U_i \le n$ for all $i = 1, 2, \cdots$, then $\dim Y \le n$.

Now our length space X satisfies the second countability axiom since it is locally compact metric, and it is separable. Therefore we have

$$\dim X = \operatorname{Ind} X = \operatorname{ind} X.$$

It is not easy to deal with the large and small inductive dimensions of X, because the treatment of the boundary of an open set H is complicated. We only deal with the covering dimension. We want to prove that

$$\dim A = \dim_H A$$

for every bounded open or closed set $A \subset X$, where $\dim_H A$ is the Hausdorff dimension of A. Notice that the Hausdorff dimension is *not* a topological invariant. The existence of a strained point in A with an (n, δ)-strainer implies that $\dim_H A \ge n$. In order to establish $\dim A = \dim_H A = n$ by assuming the existence of an (n, δ)-strainer in A, the following notion of a strained number at a point is needed.

Definition 7.5. A nonnegative integer n is by definition the strained number at $p \in X$ iff for every $\delta > 0$ and for every neighborhood U of p there exists an (n, δ)-strained point in U and U does not contain any $(n+1, \delta)$-strained point. The strained number at p is by definition ∞ iff there is no such n.

We now introduce the rongh volume and rough dimension of a bounded set A in a metric space Y. Let $\Phi_\varepsilon(A)$ for $\varepsilon > 0$ be the set of all converings of A such that

$$\Phi_\varepsilon(A) := \{\{B_i\}; d(B_i) < \varepsilon,\ A \subset \cup B_i\}.$$

The a-dimensional Hausdorff volume of A is given as

$$\mathcal{H}_a(A) := \lim_{\varepsilon \to 0} \inf_{\{B_i\} \in \Phi_\varepsilon(A)} \sum_i d(B_i)^a,$$

and the Hausdorff dimension of A as

$$\dim_H A := \inf\{a > 0; \mathcal{H}_a(A) = 0\} = \sup\{a > 0; \mathcal{H}_a(A) = \infty\}$$

Clearly $\mathcal{H}_a(\cup A_j) \le \sum_j \mathcal{H}_a(A_j)$ and $\mathcal{H}_a$ defines an outer measure. We then see that $\mathcal{H}_a(A) > 0$ implies $\dim_H A \ge 0$, and $\dim_H A > a$ implies that $\mathcal{H}_a(A) = +\infty$, $\mathcal{H}_a(A) < \infty$ implies that $\dim_H A \le a$ and finally $\dim_H(A) < a$ implies $\mathcal{H}_a(A) = 0$. Similarly the rough a-dimensional volume $V_{r_a}(A)$ of A is defined by

$$V_{r_a}(A) := \lim_{\varepsilon \to 0} \sup \varepsilon^a \beta_A(\varepsilon),$$

where $\beta_A(\varepsilon)$ is the maximal (finite) number of ε-discrete points contained in A. We then see from the definition that if $0 < V_{r_a}(A) < +\infty$, then $V_{r_b}(A) = 0$ for all $b > a$ and $V_{r_c}(A) = \infty$ for all $c < a$. Therefore there exists for A a change-over point a_0 such that $V_{r_a}(A) = \infty$ for all $a < a_0$ and $V_{r_b}(A) = 0$ for all $b > a_0$. The rough dimension $\dim_r A$ of A is defined by $\dim_r A := \inf\{a > 0; V_{r_a}(A) = 0\} = \sup\{a > 0; V_{r_a}(A) = \infty\}$. It is easy to check that:(1) $V_{r_a}(A) > 0$ implies $\dim_r A \ge a$, (2) $\dim_r A > a$ implies $V_{r_a}(A) = \infty$, (3) $V_{r_a}(A) < \infty$ implies $\dim_r A \le a$, $\dim_r A < a$ implies $V_{r_a}(A) = 0$, and $A \subset B$ implies $\dim_r A \le \dim_r B$.

Lemma 7.6. *Let X be an Alexandrov space with $Curv(X) \ge k$. Then $V_{r_a}(A) \ge \frac{1}{2^a}\mathcal{H}_a(A)$ for every $a > 0$ and for every bounded set $A \subset X$. In particular, we have $\dim_r A \ge \dim_H A$.*

Proof. Let $\{x_i\}_{i=1}^{\beta_A(\varepsilon)}$ be the maximal system of ε-discrete points in A. Then $\{B(x_i, \varepsilon)\}_{i=1}^{\beta_\varepsilon(\varepsilon)}$ is a covering of A and belongs to $\Phi_{2\varepsilon}(A)$. Thus $\beta_X(\varepsilon) \ge \inf_{\{B_i\} \in \Phi_{2\varepsilon}(A)} \#\{B_i\}$, and hence

$$\begin{aligned}
V_{r_a}(A) &= \lim_{\varepsilon \to 0} \sup \varepsilon^a \beta_A(\varepsilon) \\
&\ge \lim_{\varepsilon \to 0} \sup \inf_{\{B_i\} \in \Phi_{2\varepsilon}(A)} \varepsilon^a \cdot \#\{B_i\} \\
&\ge \lim_{\varepsilon \to 0} \inf_{\{B_i\} \in \Phi_{2\varepsilon}(A)} \sum_i \left(\frac{d(B_i)}{2}\right)^a \\
&= \frac{1}{2^a} \lim_{\varepsilon \to 0} \inf_{\{B_i\} \in \Phi_{2\varepsilon}(A)} \sum_i d(B_i)^a \\
&= \frac{1}{2^a} \mathcal{H}_a(A).
\end{aligned}$$

□

Remark. (1) We denote by λX the scaling of X by $\lambda > 0$. Then $V_{r_a}(A; \lambda d) = \lambda^a \cdot V_{r_a}(A; d)$ and also $\mathcal{H}_a(A, \lambda d) = \lambda^a \cdot \mathcal{H}_a(A)$. Both $\dim_H$ and $\dim_r$ are invariant under the scaling of metric.

Remark. (2) If a map $f : Y \to Z$ is Lipschitz on a bounded set $A \subset Y$, then $\dim_H f(A) \le \dim_H A$ and $\dim_r f(A) \le \dim_r A$. In particular if f is a bilipschitz homeomorphism then

$$\dim_H(f(A)) = \dim_H A, \quad \dim_r f(A) = \dim_r A.$$

In fact, we may consider, by taking a suitable scaling of Y, that f is a contracting map. Then, $\beta_{f(A)}(\varepsilon) \le \beta_A(\varepsilon)$ implies $V_{r_a}(f(A)) \le V_{r_a}(A)$, and hence $\dim_r f(A) \le \dim_r A$. If $\{B_i\} \in \Phi_\varepsilon(A)$, then $\{f(B_i)\} \in \Phi_\varepsilon(f(A))$ implies $\mathcal{H}_a(f(A)) \le \mathcal{H}_a(A)$. The rest of the proof is now clear.

Lemma 7.7. *Let X be an Alexandrov space with $\mathrm{Curv}(X) \ge k$. If $u, v \in X$ and $U, V \subset X$ are open sets around u and v whose diameters are bounded, then*

$$\dim_r U = \dim_r V.$$

Proof. From Corollary to Theorem 7.3 and Lemma 7.6 we see that $\dim_r V > 0$. For a fixed $a \in (0, \dim_r V)$ we have $V_{r_a}(V) = \lim_{\varepsilon \to 0} \sup \varepsilon^a \beta_V(\varepsilon) = +\infty$, and hence we find for any $c > 0$ a decreasing positiv sequence $\{\varepsilon_i\}$ tending to 0 such that for all i,

$$\varepsilon_i^a \cdot \beta_V(\varepsilon_i) \ge c.$$

Choose an ε_i-discrete points $\{c_1, \dots, c_{N_i}\}$ in V such that $N_i = \beta_V(\varepsilon_i)$. Choose an $R > 0$ such that $B := B(U, R) \subset U$ and set $D := \sup\{d(u, x) : x \in V\}$ <<

For each $j = 1, \dots, N_i$ we take a point $b_j \in uc_j \cap B$ such that $d(u, b_j) = \frac{R}{D} d(u, c_j)$, (see Figure 7-4). The Alexandrov convexity property then applies to $\Delta(uc_jc_k)$ to obtain a constant $K = K(k, D, R)$ such that

$$d(b_j, b_k) \ge \frac{R}{D} K \cdot d(c_j, c_k), \quad j, k = 1, \dots, N_i.$$

This means that (by setting $\varepsilon_i' := \frac{RK}{D} \cdot \varepsilon_i$)

$$(\varepsilon_i')^a \beta_U(\varepsilon_i') \ge \mathrm{Const}(R, D, k) \cdot \varepsilon_i^a \cdot N_i \ge \mathrm{Const} \cdot c.$$

Therefore $V_{r_a}(U) = \infty$ since c is arbitrary large and hence $\dim_r U \ge \dim_r V$ is proved. The opposite inequality is obtained by the symmetric property of the discussion. □

Problem. Let X be an Alexandrov space with $\mathrm{Curv}(X) \geq k$. Assume that the covering dimension of X is $n < \infty$. Then is it true that the strained-number is n on an open dense set of X?

§8. Fundamental Tools

In this section let X be an n-dimensional $(2 \leq n < \infty)$ Alexandrov space with curvature bounded below by k. We discuss the space of directions at a point on X, the tangent cone $K(\sum_p)$ at p, the cut locus $C(p)$ to p and the exponential map $\exp_p : D_p \to X$, where $D_p \subset K(\sum_p)$ is a set which is star-shaped with respect to the vertex o of $K(\sum_p)$. The boundary of X^n is also discussed.

8.1 The existence of angles of geodesics emanating from a point p makes it possible to define an equivalence relation among all geodesics emanating from p. Two geodesics px and py are said to be *equivalent* iff $\angle xpy = 0$. We denote by $px \sim py$ iff $\angle xpy = 0$. Namely, $px \sim py$ iff one is contained in the other (for, geodesics have no branch points). Let $\sum'_p := \{pq; q \in X \backslash \{p\}\}/\sim$. The angles $\angle$ define the metric of $\sum'_p$. Let $\sum_p$ be the completion of $(\sum'_p, \angle)$, and $K(\sum_p)$ the cone with vertex at o generated by $(\sum_p, \angle)$. The $K(\sum_p)$ equipped with the distance defined in Example (4) in §5 is a length space, where the diameter of $\sum_p$ is not greater than π, (see Corollary 8.6).

Notice that if M is a Riemannian manifold then the tangent space T_pM to M at a point p is obtained as the pointed Hausdorff limit of the scaling of metric. Namely we have

$$(T_pM; O) = \lim_{\lambda \to \infty} {}_{d_H} (\lambda M; p).$$

Therefore the $\sum_p$ corresponds to the unit hypersphere $S_p(1) \subset T_pM$ centered at the origin O of T_pM, and the $K(\sum_p)$ to the tangent space to M at p. The $\{(\lambda M; p)\}_\lambda$ converges uniformly on every metric ball around p. The uniform convergence with respect to the pointed Hausdorff distance is guaranteed by the compactness of $S_p(1)$.

We want to prove that $(\sum_p, \angle)$ is a compact Alexandrov space of dimension $n-1$ with curvature bounded below by 1, and that $K(\sum_p)$ has curvature bounded below by 0. The compactness of $\sum_p$ is crucial to prove that the pointed Hausdorff limit of the scaling of metric of X is isometric to $K(\sum_p)$,

$$\lim_{\lambda \to \infty} (\lambda X; p) = (K(\Sigma_p); o).$$

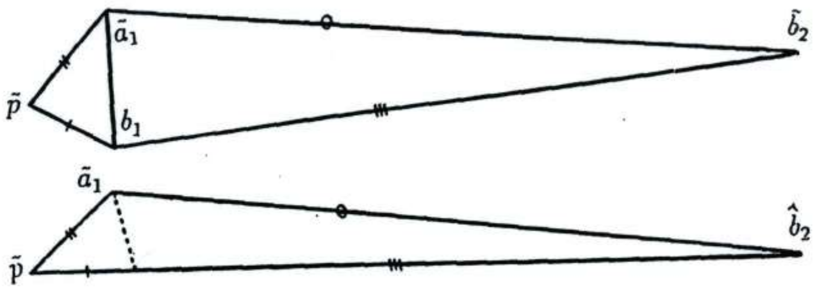


FIGURE 8-1.

Theorem 8.1. *If* $\dim X = n < \infty$ *and if* $\mathrm{Curv}(X) \geq k$, *then* $\sum_p$ *for every point* $p \in X$ *is compact.*

It suffices for the proof of Theorem 8.1 to show that $\sum'_p$ is precompact, e.g., that $\sum'_p$ admits finite ε-net and ε-discrete sets for all $\varepsilon > 0$. The following Lemmas 8.2 and 8.3 are needed for the proof of the precompactness of Σ'_p.

Lemma 8.2. *Let* $\{pa_i\}_{i=1,\dots,n}$ *be a finite number of geodesics emanating from* p. *Then there exists for any fixed* $\delta > 0$ *a neighborhood* U *around* p *such that for every* $q \in pa_i \cap U$ *and for every* $r \in pa_j \cap U$ *we have*

$$\delta > \angle qpr - \tilde{\angle} qpr,\ \angle pqr - \tilde{\angle} pqr,\ \angle rpq - \tilde{\angle} rpq.$$

Proof. We only need to prove the case where $k = 2$. Setting $a := a_1$ and $b := a_2$ we choose an $R > 0$ such that $\angle a'pb' - \tilde{\angle} a'pb' < \delta/2$ hold for all $a' \in pa \cap B(p,R)$ and $b' \in pb \cap B(p,R)$. We then choose an R_1 with $0 < R_1 \ll \delta R$ such that for any points $a_1 \in pa \cap B(p,R_1)$, $b_1 \in pb \cap B(p,R_1)$ and for $b_2 \in pb$ with $d(p,b_2) = R$, we have

$$\tilde{\angle} a_1pb_1 - \tilde{\angle} a_1pb_2,\ \ \tilde{\angle} a_1b_2b_1 - \tilde{\angle} a_1b_2p \leq \delta/2.$$

We then apply the Gauss-Bonnet theorem to the triangles $\tilde{\Delta}(a_1pb_1)$, $\tilde{\Delta}(a_1b_2b_1)$ and $\tilde{\Delta}(a_1pb_2) := \Delta(\tilde{p}\tilde{a}_1\hat{b}_2)$ to get, (see Figure 8-1)

$$
\begin{aligned}
&(\tilde{\angle} a_1pb_1 - \tilde{\angle} a_1pb_2) + (\tilde{\angle} a_1b_2b_1 - \tilde{\angle} a_1b_2p) \\
&= (\tilde{\angle} pa_1b_2 - \tilde{\angle} pa_1b_1 - \tilde{\angle} b_1a_1b_2) + (\pi - \tilde{\angle} pb_1a_1 - \tilde{\angle} a_1b_1b_2).
\end{aligned}
$$

only if $k = c$

From the Alexandrov convexity property we see that all the bracket terms in the above equality are nonnegative. Replacing π by $\angle pb_1a_1 + \angle a_1b_1b_2$ we observe that $\angle pb_1a_1 - \tilde{\angle} pb_1a_1$ is bounded above by the left hand side of the above equality, which is less than δ. Becaus of the symmetry of the discussion we also have $\angle pa_1b_1 - \tilde{\angle} pa_1b_1 < \delta$. This proves Lemma 8.2. □

Lemma 8.3. *Fix an $n \geq 1$. For every integer m with $0 \leq m \leq n$ and for every $\varepsilon > 0$ there exist a positive number $\delta = \delta(m, \varepsilon)$ and an integer $N = N(m, \varepsilon)$ such that if X is an n-dimensional Alexandrov space with Curv$(X) \geq k$ then X does not admit points p, a_i, b_i and c_j for $1 \leq i \leq m$ and for $1 \leq j \leq N$ with the properties that*

$$
(8\text{-}1)_m \qquad
\begin{cases}
(a_i, b_i)_{i=1}^m \text{ is an } (m, \delta)\text{-strainer at } p \\
\tilde{\angle} c_jpa_i > \frac{\pi}{2} - \delta,\ \tilde{\angle} c_jpb_i > \frac{\pi}{2} - \delta \\
\tilde{\angle} c_jpc_{j'} > \varepsilon \text{ for all } i = 1, \ldots, m \text{ and all } j, j' = 1, \ldots, N.
\end{cases}
$$

Proof. We fix n and employ the reverse induction on $m = 0, \ldots, n$. First of all consider the case $m = n$. We then set $N(n, \varepsilon) = 1$ and $\delta := 1/1000n$. Suppose that points p, a_i, b_i and c on X satisfies $(8\text{-}1)_n$. Then a point $p' \in pc$ near p admits an $(n+1, 2\delta)$-strainer consisting of a_i, b_i and p, c. This is a contradiction to $\dim X = n$.

Suppose by inductive assumption that Lemma 8.2 is true for $m = n, \ldots, m+1$. Then there exist for any $\varepsilon > 0$ a $\delta = \delta(m+1, \varepsilon)$ and $M = N(m+1, \varepsilon)$ with the required properties. We now fix $\delta(m, \varepsilon)$ and $N(m, \varepsilon)$ as follows.

(a) $\delta(m, \varepsilon) < \varepsilon \cdot \delta(m+1, \varepsilon/2) \cdot \frac{1}{1000}$

(b) $\frac{1}{2}(N(m, \varepsilon) - 3N(m+1, \varepsilon/2)) > \frac{1000 \cdot N(m+1, \frac{\varepsilon}{2})}{\delta(m, \varepsilon)}$

Thus $\delta(m, \varepsilon) \ll 1$ and $N(m, \varepsilon) \gg 1$. Suppose there exist for $\varepsilon > 0$ points p, a_i, b_i and c_j in X with $i = 1, \ldots, m$ and $j = 1, \ldots, N = N(m, \varepsilon)$ such that they satisfy $(8\text{-}1)_m$ for $\delta = \delta(m, \varepsilon)$. We want to construct an $(m+1, 2\delta)$-strainer. Setting $a_{m+1} := c_N$ and $b_{m+1} := p$ we observe that if $p' \in$

$a_{m+1}b_{m+1}$ is taken near b_{m+1}, then the points p', a_i, b_i, for $i = 1, \dots, m+1$ has the properties that $p := p'$ is an $(m+1, 2\delta(m,\varepsilon))$-strained point with strainer $(a_i, b_i)_{i=1,\dots,m+1}$. By the assumption of induction on $m+1$, X does not admit points p, a_i, b_i, c_j satisfying $(8\text{-}1)_{m+1}$. Thus, if we set

$$\begin{aligned}
A &:= \#\{j : \tilde{\angle} a_{m+1}pc_j > \frac{\pi}{2} - 3\delta(m,\varepsilon), \tilde{\angle} b_{m+1}pc_j > \frac{\pi}{2} - 3\delta(m,\varepsilon)\},\\
B &:= \#\{j; \tilde{\angle} a_{m+1}pc_j \le \frac{\pi}{2} - 3\delta(m,\varepsilon)\},\\
C &:= \{j; \tilde{\angle} b_{m+1}pc_j \le \frac{\pi}{2} - 3\delta(m,\varepsilon)\},
\end{aligned}$$

then $A + B + C = N(m,\varepsilon) - 1$ and $A \le N(m+1,\varepsilon)$ follows from inductive assumption on $m+1$. It follows from the choice of $N(m,\varepsilon)$ that $B + C > N(m,\varepsilon) - 3N(m+1,\frac{\varepsilon}{2})$. Without loss of generality we may assume $B \ge C$. Then $B \ge \frac{1}{2}(N(m,\varepsilon) - 3N(m+1,\frac{\varepsilon}{2})) > 1000 \cdot N(m+1,\frac{\varepsilon}{2}) \cdot \delta(m,\varepsilon)$. By setting $\varphi_j := \angle a_{m+1}pc_j$ for $j = 1, \dots, N(m,\varepsilon) - 1$ we find a subset $J \subset \{1, 2, \dots, N(m,\varepsilon) - 1\}$ such that $\#J \ge N(m+1,\frac{\varepsilon}{2})$ and such that $|\varphi_j - \varphi_{j'}| < \frac{\delta(m,\varepsilon)}{100}$ for all $j, j' \in J$. Because $\varphi_j \ge \tilde{\angle} a_{m+1}pc_j$ and $d(p, a_{m+1})$ and $d(p, c_j)$ are sufficiently small, we see $|\varphi_j - \tilde{\varphi}_{j'}| < \delta(m,\varepsilon)$, and in particular $\#\{j; \varphi_j \le \frac{\pi}{2} - \delta(m,\varepsilon)\} \ge B > 1000 \cdot N(m+1,\frac{\varepsilon}{2}) \cdot \delta(m,\varepsilon)^{-1}$. By dividing $[0, \frac{\pi}{2} - \delta(m,\varepsilon)]$ into $1000 \cdot \delta(m,\varepsilon)^{-1}$ subintervals with equal lengths, we find a subinterval I which contains at least $N(m+1,\frac{\varepsilon}{2})$ members of $\varphi_{j's}$, and the length of I is less than $\delta/100$. We now fix a point $p_1 \in pa_{m+1}$ sufficiently close to p and choose a point $\bar{c}_j \in pc_j$ for each $j \in J$ in such a way that $d(p, \bar{c}_j)\cos\varphi_j = d(p, p_1)$ (see Figure 8-2).

By means of Lemma 8.3 we see that

$$\begin{aligned}
\tilde{\angle} a_{m+1}p_1\bar{c}_j &> \frac{\pi}{2} - \delta(m+1,\frac{\varepsilon}{2})\\
\tilde{\angle} b_{m+1}p\bar{c}_j &> \frac{\pi}{2} - \delta(m+1,\frac{\varepsilon}{2})
\end{aligned}$$

for all $j \in J$.

We finally assert that the points $p_1, a_i, b_i, \bar{c}_j$ for $i = 1, \dots, m+1$ and for $j \in J$ satisfy $(8\text{-}1)_{m+1}$ (and a contradiction will be derived).

Clearly $(a_i, b_i)_{i=1}^{m+1}$ is an $(m+1, \delta(m+1,\frac{\varepsilon}{2}))$-strairer at p_1.

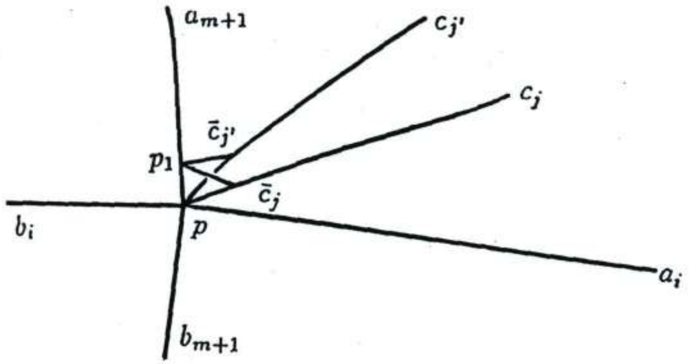


FIGURE 8-2.

We next show that $\tilde{\angle}\,\bar{c}_j p_1 \bar{c}_{j'} > \varepsilon/2$. Because $\bar{c}_j \in p_1 c_j$, $\bar{c}_{j'} \in p c_{j'}$ and $\angle c_j p c_{j'} > \varepsilon$ by what we have supposed and because $|\varphi_j - \varphi_{j'}| < \delta/100$ it follows from the continuity of angles that $\tilde{\angle}\,\bar{c}_j p_1 \bar{c}_{j'} > \frac{\varepsilon}{2}$.

We finally show that $\tilde{\angle}\,a_i p_1 \bar{c}_j > \frac{\pi}{2} - \delta(m+1, \frac{\varepsilon}{2})$ for all $i = 1, \dots, m$ and for all $j \in J$. In view of the Corollary to Lemma 7.4 we only need to prove that $|d(p_1, a_i) - d(\bar{c}_j, a_i)| \ll d(p_1, \bar{c}_j)$ for all $i = 1, \dots, m$ and for all $j \in J$. It follows from $(8\text{-}1)_m$ that $\frac{\pi}{2} - 2\delta < \tilde{\angle}\,a_i p p_1 < \frac{\pi}{2} + 4\delta$ and $\frac{\pi}{2} - 2\delta < \tilde{\angle}\,a_i p c_j < \frac{\pi}{2} + 4\delta$. Therefore we have $|d(a_i, p) - d(a_i, p_1)| < 4\delta \cdot d(p, p_1)$ and $|d(a_i, p) - d(\bar{c}_j, a_i)| < 4\delta \cdot d(p, \bar{c}_j)$, and hence $|d(p_1, a_i) - d(\bar{c}_j, a_i)| \leq |d(p_1, a_i) - d(p, a_i)| + |d(a_i, p) - d(\bar{c}_j, a_i)| < 4\delta(d(p, p_1) + d(p, \bar{c}_j))$.

On the other hand $|d(p, p_1) \cdot \tan\varphi_j - d(p_1, \bar{c}_j)| = O(\delta^2)$ and $|d(p, \bar{c}_j) \cdot \sin\varphi_j - d(p_1, \bar{c}_j)| = O(\delta^2)$ follows from the corresponding triangles on $M^2(k)$. Therefore $d(p, p_1) + d(p, \bar{c}_j) \geq d(p_1, \bar{c}_j)\{\cot\varphi_j + \operatorname{cosec}\varphi_j\} - O(\delta^2)$ and $\cot\varphi_j + \operatorname{cosec}\varphi_j < \frac{2}{\varepsilon}$. Since $\delta = \delta(m, \varepsilon)$ is chosen as in (a), we have

$$|d(p_1, a_i) - d(\bar{c}_j, a_i)| < d(p_1, \bar{c}_j) \cdot \frac{8\delta(m, \varepsilon)}{\varepsilon} < d(p_1, \bar{c}_j) \cdot \frac{8\delta(m+1, \frac{\varepsilon}{2})}{1000}.$$

This proves the final requirement for $(8\text{-}1)_{m+1}$, and a contradiction is derived. $\square$

Proof of Theorem 8.1. For every point $p \in X$ and for every $\varepsilon > 0$ let $\Phi(p, \varepsilon)$ be the maximal system of ε-nets which is ε-discrete in $\sum_p'$. By virtue of Lemma 8.3 for $m = 0$, we see that the number of elements in $\Phi(p, \varepsilon)$ does not exceed $N(n, \varepsilon) < +\infty$. This proves the precompactness of $\sum_p'$, and the proof of Theorem 8.1 is complete. □

Lemma 8.4. *If $\sum_p$ is isometric to $S^{n-1}(1)$, then $p \in X$ is a manifold point of X.*

Proof. Since $\sum_p$ is isometric to $S^{n-1}(1)$, there exists an $(n, 0)$-strainer $(a_i, b_i)_{i=1}^n$ of $\sum_p$ such that $d(a_i, b_i) = \pi$ and $d(a_i, b_j)$, $d(a_i, a_j)$, $d(b_i, b_j) = \pi/2$ for all $i, j = 1, \ldots, n$, $i \neq j$. The density property of $\sum_p'$ in $\sum_p$ then implies that there exists for any $\delta > 0$ an (n, δ)-strainer $(a_i', b_i')_{i=1}^n$ of $\sum_p'$ such that

$$d(a_i', b_i') > \pi - \delta, \; d(a_i', a_j'), d(a_i', b_j'), d(b_i', b_j') > \frac{\pi}{2} - \delta$$

for all $i, j = 1, \ldots, n$, $i \neq j$. Therefore p is a strained point with an (n, δ)-strainer generated by $(a_i', b_i')_{i=1}^n$. This proves Lemma 8.4. □

8-2. We now define the exponential map and cut locus at a point $p \in X$. Let $D_p \subset K(\sum_p) =: K_p$ be defined by

$D_p := \{(a, t) \in K_p$; there exists a geodesic px such that px belongs to the equivalence class of a and such that $t = d(p, x)\}$. Clearly, D_p is star-shaped with respect to the vertex o of K_p. The exponential map $\exp_p$ at p is defined as follows. $\exp_p : D_p \to X$, $\exp_p(a, t) := \exp_p ta = x$. Recall that $\sum_p$ is identified with $(\sum_p, 1) \subset K_p$.

Next we define the cut locus $C(p)$ to p by $C(p) := \{x \in X$; there exists a geodesic px which is not properly contained in any other geodesic emanating from $p\}$, and also $C_p \subset K_p$ by $C_p := \exp_p^{-1}(C(p))$. Clearly $C_p = \partial D_p$ and $\exp_p |(D_p \backslash C_p)$ is a homeomorphism between $D_p \backslash C_p$ and $X \backslash C(p)$, and $\exp_p^{-1} : X \to D_p$ is multivalued.

Theorem 8.5. *Let $p \in X$ be a fixed point. The pointed Hausdorff limit of the scaling $(\lambda X; p)$ of X is isometric to $(K_p; o)$;*

$$\lim_{\lambda \to \infty} {}_{d_H}(\lambda X; p) = (K_p; o).$$

Proof. For an arbitrary fixed $R > 0$ we denote by $B(R) \subset K_p$ the R-ball centered at o and $B_\lambda(R) \subset \lambda X$ the R-ball of λX centered at p. Thus $B_\lambda(R)$ coincides with $\lambda^{-1}R$-ball in X centered at p. We only need to prove that $\lim_{\lambda \to \infty} {}_{d_H} B_\lambda(R) = B(R)$. In view of Proposition 1.1 we construct for every ε-discrete net $\mathcal{N}(\varepsilon)$ in B an $(\varepsilon + c(\lambda))$-discrete net $\mathcal{N}_\lambda(\varepsilon + c(\lambda))$ in $B_\lambda(R)$ such that $\lim_{\lambda \to \infty} {}_{d_L} \mathcal{N}_\lambda(\varepsilon + c(\lambda)) = \mathcal{N}(\varepsilon)$. Here $c(\lambda) > 0$ converges to 0 as $\lambda \to \infty$. From compactness of $\sum_p$ we see that $\mathcal{N}(\varepsilon)$ is a finite set, say $\mathcal{N}(\varepsilon) = \{w_1, \dots, w_N\}$, where $w_i = (\xi_i, t_i)$ for $\xi_i \in \sum_p$ and $t_i \in [0, R]$. Since $\sum_p'$ is dense in $\sum_p$ there is a large number $\lambda(\varepsilon)$ such that there exists for all $\lambda > \lambda(\varepsilon)$ an $(\varepsilon - c(\lambda))$-discrete net $\{w_1^\lambda, \dots, w_N^\lambda\}$ in $\lambda D_p \cap B(R)$ such that $w_i^\lambda = (\xi_i^\lambda, t_i)$, $\xi_i^\lambda \in \sum_p'$ and $\rho(w_i, w_i^\lambda) < c(\lambda)$ for all $i = 1, \dots, N$. Because every geodesic triangle $\Delta(\exp_p \lambda^{-1} t_i \xi_i^\lambda, p, \exp_p \lambda^{-1} t_j \xi_j^\lambda)$ is small we have $(\lambda d(\exp_p \lambda^{-1} t_i \xi_i^\lambda, \exp_p \lambda^{-1} t_j \xi_j^\lambda) - \rho(w_i^\lambda, w_j^\lambda)| < c(\lambda)$ for all $i, j = 1, \dots, N$. Therefore by setting $z_i^\lambda := \exp_p \lambda^{-1} t_i \xi_i^\lambda \in B_\lambda(R)$ we see that $\mathcal{N}_\lambda(\varepsilon - c(\lambda)) := \{z_1^\lambda, \dots, z_N^\lambda\}$ is an $(\varepsilon - c(\lambda))$-discrete $(\varepsilon + c(\lambda))$-net of $B_\lambda(R)$ such that $\lim_{\lambda \to \infty} {}_{d_L} \mathcal{N}_\lambda(\varepsilon - ((\lambda)) = \mathcal{N}(\varepsilon)$. This proves Theorem 8.5. $\square$

Corollary 8.6. *The tangent cone K_p at each point $p \in X$ has curvature $\mathrm{Curv}(K_p) \geq 0$. If $\dim X \geq 2$, then $\mathrm{Curv}(\sum_p) \geq 1$.*

Proof. Because λX has curvature bounded below by $\lambda^{-2}k$, its pointed Hausdorff limit has curvature bounded below by 0. The rest follows from Theorem 5.1, where we agree that X with dimension 0 has $\mathrm{Curv}(X) \geq k$ and also 1-dimensional X has $\mathrm{Curv}(X) \geq k$ (in case of $k > 0$ the length of X is not longer than $2\pi/\sqrt{k}$).

Corollary 8.7. *Let $p \in X$. Then*

$$\dim K_p = \dim X \quad \text{and} \quad \dim \sum_p = \dim X - 1.$$

Proof. Fix an $R > 0$. The hinge theorem 3.1' implies that if p^* is a point on $M^n(k)$ and if $B^*(R) \subset M^n(k)$ is an R-ball around p^*, then

$$\exp_{p^*} \circ (\exp_p^\lambda | \lambda D_p \cap B(R))^{-1} : B_\lambda(R) \to B^*(R)$$

is an expanding map, and hence $\dim B_\lambda(R) \leq \dim B^*(R) = n$. If $(\xi_i, \eta_i)_{i=1}^{\dim B(R)}$ is a $(\dim B(R), \delta)$-strainer at a point $z \in B(R)$, then Lemma 1.2 and Theorem 8.5 imply that there exists a $\lambda(\delta) \gg 1$, such that $B_\lambda(R)$ admits a point $z_\lambda \in B_\lambda(R)$ and $(\dim B(R), 2\delta)$-strainer $(\xi_i^\lambda, \eta_i^\lambda)_{i=1}^{\dim B(R)}$ at z_λ such that $\lim_{\lambda \to \infty} {}_{d_L}(\xi_i^\lambda, \eta_i^\lambda) = (\xi_i, \eta_i)$ and $\lim_{\lambda \to \infty} z_\lambda = z$. Thus the strained number at z_λ for every $\lambda > \lambda(\delta)$ is not less than $\dim B(R)$, and hence $\dim B_\lambda(R) \geq \dim B(R)$. Since the dimension is invariant under homothety, this proves $\dim X = \dim \lambda X = \dim K_p$. The rest is now clear.

The following Lemma 8.8 is used for the proof of the Lemma on limit angles 6.3. Notice that discussion throughout this section is local in nature.

Lemma 8.8. *Let $(\alpha_i, \beta_i)_{i=1,2,\ldots}$ be a sequence of geodesic biangles such that $p = \alpha_i(0) = \beta_i(0)$, $q_i = \alpha_i(\ell_i) = \beta_i(\ell_i)$, $\ell_i = d(p, q_i)$. If $\lim_{i\to\infty} \ell_i = 0$ and if θ_i is the angle at p between α_i and β_i, then $\lim_{i\to\infty} \sup \theta_i = 0$.*

Proof. Let $X_i := \ell_i^{-1} X$, $i = 1, 2, \ldots$. Then we have $\lim_{i\to\infty} {}_{d_H}(X_i; p) = (K_p; o)$. If $\exp_p^i : \ell_i^{-1} D_p \to X_i$ is the exponential map at p of X_i, then $\alpha_i(t) = \exp_p^i t\xi_i$, $\beta_i(t) = \exp_p^i t\eta_i$ for $0 \leq t \leq 1$, where $\xi_i,\ \eta_i \in \sum_p$ is the tangential directions to α_i, β_i respectively. It follows from assumption that $\alpha_i(1) = \beta_i(1) = q_i$ for all $i = 1, 2, \ldots,$. If ξ and η are the limits of $\{\xi_i\}$ and $\{\eta_i\}$ (by taking a subsequence if necessary) then $\xi = \eta$. This proves Lemma 8.8. $\square$

In Riemannian case the injectivity radius of the exponential map of a complete Riemannian manifold is positive at each point on it. However, as is seen in the Example 5-B(3), this property does not hold at points on the boundary of a disk, because such a point is a vertex of geodesic biangles of arbitrary small lengths. Notice that the cut locus distance to $p \in X$ is not necessarity continuous in the tangential directions of geodesics emanating from p. The limit of cut points to p is not necessarily a cut point to p, because the regularity of the exponential map is not guaranteed. X may admit a sequence of geodesic biangles with a conver at $p \in X$ with arbitrary small angles at p, while their lengths do not converge to 0.

8-3. The boundary and interior points of X is now discussed. A point on a manifold M is an interior point iff there exists a neighborhood homeomorphic to an open set in a Euclidean space. However a point $p \in X$ may not have any neighborhood homeomorphic to an open set in $\mathbf{R}^n$. For instance, the vertex of a cone generated by $\mathbf{R}P^n$ is such a point. In view of $\sum_p$ being an Alexandrov space with $\mathrm{Curv}(\sum_p) \geq 1$ and with $\dim \sum_p = n-1$, we can define an interior (and a boundary) point of X by induction. Let $p \in X$ and p_1 an arbitrary point on $\sum_p$. Then the space $\sum_{p_1}$ of tangential directions at p_1 to $\sum_p$ is an $(n-2)$-dimensional Alexandrov space with $\mathrm{Curv}(\sum_{p_1}) \geq 1$. If p_{n-2} is an arbitrary point of $\sum_{p_{n-3}}$, then $\dim \sum_{p_{n-2}} = 1$. Because one-dimensional Alexandrov space is either a circle or a segment, we say that $p \in X$ is an *interior point* iff $\sum_{p_{n-2}}$ is a circle for any choice of points $p_1 \in \sum_p, p_2 \in \sum_{p_1}, \ldots, p_{n-2} \in \sum_{p_{n-3}}$. Also a point $p \in X$ is by definition a *boundary point* of X iff $\sum_{p_{n-2}}$ is a segment for some choice of points $p_1 \in \sum_p, \ldots, p_{n-2} \in \sum_{p_{n-3}}$.

We conclude this section by introducing a recent result due to Plaut [P-1].

Theorem 8.9. (Plaut;[P]). *X is isometric to the standard unit n-sphere if $\mathrm{Curv}(X) \geq 1$ and if X admits a global $(n+1,0)$-strainer $(a_i,b_i)_{i=1}^n$ such that $d(a_i,b_i) = \pi$ and $d(a_i,b_j)$, $d(a_i,a_j)$, $d(b_i,b_j) \geq \pi/2$.*

Proof. It follows from Theorem 6.2 that X is isometric to the spherical suspension $\sum(E_1)$ of the equidistance set $E_1 := \{x \in X; d(a_1,x) = d(b_1,x)\}$ and E_1 is a totally geodesic $(n-1)$-dimensional Alexandrov subspace with $\mathrm{Curv}(E_1) \geq 1$. Since E_1 admits a global $(n,0)$-strainver $(a_i,b_i)_{i\geq 2}$ E_1 is isometric to $\sum(E_2)$, where $E_2 := \{x \in E_1; d(a_2,x) = d(b_2,x)\} = \{x \in X; d(a_i,x) = d(b_i,x); i = 1,2,\}$. Thus we conclude the proof by induction. □

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