THETA FUNCTIONS IN HIGHER DIMENSION

For the case of the elliptic curve in Weierstrass's notations theta functions appear as the sigma function σ which is holomorphic and has a simple zero. The quotients of two products of the translates of the sigma function is used to form an elliptic function with assigned multiplicative data of poles and zeroes. The sigma function σ is obtaind by integrating the Weierstrass p function twice and change the sign and take the exponential so that we end up with a simple zero. This process, of course, destroys the periodicity. Instead we get a periodicity factor which is the exponential of a polynomial of degree one. So in the case of theta functions on the higher dimensional Euclidean space $\mathbb{C}^n$ (or a complex vector space V of complex dimension n when we do not want to specify any coordinate system) with a lattice D we define the theta function as an entire function $F(x)$ which for $u \in D$ satisfies $F(x+u) = F(x)\exp(2\pi\sqrt{-1}\,g(x,u))$ for some polynomial $g(x,u)$ of degree one in x. The function $g(x,u)$ is defined on $V \times D$ and its value is defined only up to an additive integer. We separate the homogeneous part and constant part of $g(x,u)$ and write $g(x,u) = L(x,u) + J(u)$, where $L(x,u)$ is $\mathbb{C}$-linear in x. A theta function will be later interpreted as a holomorphic section of a holomorphic line bundle on the manifold V/D. In this interpretation $\exp(2\pi\sqrt{-1}(L(x,u)+J(u)))$ will appear as the transition function of the line bundle and satisfies the compatibility condition for transition functions. We now formulate the compatibility conditions in our setting here.

$$\begin{aligned} F(x+u+v) &= F((x+v)+u) = F(x+v)\exp(2\pi\sqrt{-1}(L(x+v,u)+J(u))) \\ &= F(x)\exp(2\pi\sqrt{-1}(L(x,v)+J(v)))\exp(2\pi\sqrt{-1}(L(x+v,u)+J(u))). \end{aligned}$$

On the other hand,

$$F(x+u+v) = F(x+(u+v)) = F(x)\exp(2\pi\sqrt{-1}(L(x,u+v)+J(u+v))).$$

Hence

$$L(x,u+v)+J(u+v) \equiv L(x,v)+J(v)+L(x+v,u)+J(u) \mod \mathbb{Z}.$$

We separate this condition into the homogeneous part and the constant part by setting $x=0$. By setting $x=0$ in the above equation we get

$$J(u+v) \equiv J(v) + L(v,u) + J(u) \mod \mathbb{Z} \tag{1}$$

and plucking this back into the equation we get

$$L(x, u+v) \equiv L(x, v) + L(x, u) \mod \mathbb{Z}.$$

Actually the last congruence relation is an identity because of the $\mathbb{C}$-linearity of $L(x, u)$. The reason is as follows. Since $L(x, u+v) - (L(x, v) + L(x, u))$ is always an integer, its partial derivatives

$$\frac{\partial}{\partial x_\nu}(L(x, u+v) - (L(x, v) + L(x, u)))$$

with respect to x must vanish. On the other hand, since $L(x, u)$ is homogeneous in x of degree one, we have by Euler's equation

$$L(x, u) = \sum_{\nu=1}^{n} x_\nu \frac{\partial}{\partial x_\nu} L(x, u).$$

It follows that

$$\begin{aligned} &L(x, u+v) - (L(x, v) + L(x, u)) \\ &= \sum_{\nu=1}^{n} x_\nu \frac{\partial}{\partial x_\nu}(L(x, u+v) - (L(x, v) + L(x, u))) = 0 \end{aligned}$$

and we have the $\mathbb{Z}$-linearity of $L(x, u)$ in u for $u \in D$. We can extend the domain of definition of $L(x, u)$ to $V \times V$ so that $L(x, u)$ is $\mathbb{C}$-linear in $x \in V$ and $\mathbb{R}$-linear in $u \in V$. All we have to do is to construct for fixed x the $\mathbb{R}$-linear function $L(x, u)$ in u whose values in a basis of V over $\mathbb{R}$ consisting of elements of D agree with the values of the original function $L(x, u)$. After we finish handling the homogeneous part $L(x, u)$, we now return to the constant part $J(u)$ which satisfies the compatibility condition (1). The function $J(u)$ is not linear in u in that congruence. The obstruction is $L(u, v)$. We can modify it to make it linear by using the usual polarization for the quadratic form $L(u, v)$. First we note that from (1) we have

$$L(u, v) \equiv L(v, u) \mod \mathbb{Z} \quad \text{for } u, v \in D. \tag{2}$$

When we are dealing with congruences, we actually have a quadratic form $L(u, v)$. Define $K(u) = J(u) - \frac{1}{2}L(u, u)$. Then we have $K(u+v) \equiv K(u) + K(v) \mod \mathbb{Z}$. Since the value of $K(u)$ is defined only up to an additive integer, we can now extend the definition of $K(u)$ so that it is an $\mathbb{R}$-linear function on V. All we have to do again is to choose a basis of V over $\mathbb{R}$ consisting of elements of D and use the values of $K(u)$ on those basis elements to define the $\mathbb{R}$-linear function on V. We now want to get more information about the homogeneous part $L(x,u)$. The congruence (2) says that $L(u,v)$ is symmetric on $D \times D$ modulo $\mathbb{Z}$. We introduce (2 times) the skew-symmetric part $E(x,y) = L(x,y) - L(y,x)$ of $L(x,y)$ to measure its failure to be symmetric. Since $L(x,y)$ is $\mathbb{C}$-linear in $x \in V$ and $\mathbb{R}$-linear in $y \in V$, we know that $E(x,y)$ is $\mathbb{R}$-linear in both x and y. Moreover, E assumes integral values on $D \times D$. Since we can choose basis elements of V from D, it follows that E is $\mathbb{R}$-valued. We would like to investigate the possibility of writing E as the imaginary part of a Hermitian form on V. Forget our notation E for the time being. In general a Hermitian form $H(x,y) = S(x,y) + \sqrt{-1}E(x,y)$ has symmetric real part $S(x,y)$ and skew-symmetric part $E(x,y)$. The $\mathbb{C}$-linearity of $H(x,y)$ in x can be expressed in terms of relations between $S(x,y)$ and $E(x,y)$ as follows.

$$\begin{aligned}
S(\sqrt{-1}x,y) + \sqrt{-1}E(\sqrt{-1}x,y) &= H(\sqrt{-1}x,y) \\
= \sqrt{-1}H(x,y) &= -E(x,y) + \sqrt{-1}S(x,y)
\end{aligned}$$

is equivalent to $S(x,y) = E(\sqrt{-1}x,y)$. So in general given a skew-symmetric form $E(x,y)$ a necessary and sufficient condition for it be the imaginary part of a Hermitian form is that the form $S(x,y)$ defined as $E(\sqrt{-1}x,y)$ is symmetric in x and y. Now let us go back to our notation $E(x,y)$. It is not just any skew-symmetric form, but is the skew-symmetric part of a form $\mathbb{C}$-linear in the first argument and $\mathbb{R}$-linear in the second argument. We claim that this property insures that it is the imaginary part of a Hermitian form. (Of course the converse is clearly true because a Hermitian form is $\mathbb{C}$-linear in its first argument and is $\mathbb{R}$-linear in its second argument.) We now use the $\mathbb{C}$-linearity of $L(x,y)$ in x to verify the symmetry of $S(x,y) = E(\sqrt{-1}x,y)$ in x and y.

$$\begin{gathered}
S(x,y) - S(y,x) = E(\sqrt{-1}x,y) - E(\sqrt{-1}y,x) \\
= L(\sqrt{-1}x,y) - L(y,\sqrt{-1}x) - (L(\sqrt{-1}y,x) + L(x,\sqrt{-1}y)) \\
= \sqrt{-1}L(x,y) + \sqrt{-1}L(\sqrt{-1}y,\sqrt{-1}x) - \sqrt{-1}L(y,x) - \sqrt{-1}L(\sqrt{-1}x,\sqrt{-1}y) \\
= \sqrt{-1}(E(x,y) - E(\sqrt{-1}x,\sqrt{-1}y)) \\
.
\end{gathered}$$

Since the left-hand side is real and the right-hand side is purely imaginary, it follows that both sides vanish and $S(x, y)$ is symmetric in x and y. Now we form the Hermitian form $H(x, y) = S(x, y) + \sqrt{-1}E(x, y)$. Two times the skew-symmetric part of $-\frac{\sqrt{-1}}{2}H$ is its imaginary part E. So L and $-\frac{\sqrt{-1}}{2}H$ both have the same skew-symmetric part. Or what is the same thing their difference is symmetric. Recall that we get to this part by considering the compatibility condition for the periodicity factor $\exp(2\pi\sqrt{-1}\, g(x, u))$ and we separate into the homogeneous part and the constant part to get L and J and then we decide that K is a better entity than J because of its $\mathbb{R}$-linearity. When we use L and K, we lose no information. However, when we consider the skew-symmetric E part of L and then construct from it the Hermitian form H, the entities L and H are not equivalent. We can get from L to H, but would have trouble reconstructing L from H. The Hermitian form H is better, because instead of $\mathbb{C}$-linearity of L in its first argument and $\mathbb{R}$-linearity in its second argument we have the better condition of $\mathbb{C}$-linearity of H in its first argument and anti$-\mathbb{C}$-linearity in its second argument. The price we pay is that some information is lost. We claim that for our purpose such a loss of information is immaterial.

We consider theta functions because we want to take the quotient of two products of translates of such theta functions to get a meromorphic function on V/D. When the theta function has no zero, the meromorphic function we get on V/D would be constant and is uninteresting. So we can ignore theta functions which have no zeroes. We consider those theta functions trivial. One way to construct such theta functions is to look at the exponential of polynomials of degree two. Then the periodicity factors would clearly be the exponential of a polynomial of degree one. So we define a theta function as *trivial* if it is of the form $\exp(2\pi\sqrt{-1}(b(x, x) + \lambda(x) + c))$, where $b(x, y)$ is $\mathbb{C}$-bilinear and $\lambda(x)$ is $\mathbb{C}$-linear and $c \in \mathbb{C}$. Two theta functions are called *equivalent* if their quotient is a trivial theta function.

Now let us come back to the question of information loss when we pass from L to H. This information loss disappears modulo trivial theta functions. Intuitively this is the case because the difference between L and $-\frac{\sqrt{-1}}{2}H$ is symmetric and since both L and $-\frac{\sqrt{-1}}{2}H$ are $\mathbb{C}$-linear in their first arguments their difference is actually a symmetric $\mathbb{C}$-bilinear form which can serve as (2 times) $b(x, y)$. By considering equivalence modulo trivial theta functions we have the bonus of the choice of $\lambda(x) + c$. Note that c is useless because it simply multiplies the trivial theta function by a constant. We will use this choice to make K real-valued. Consider an $\mathbb{R}$-linear function $\ell(x) = p(x) + \sqrt{-1}q(x)$ on V, where $p(x)$ and $q(x)$ are both real-valued. For $\ell(x)$ to be $\mathbb{C}$-linear it is necessary and sufficient that $p(x) = q(\sqrt{-1}x)$. Now write $K(x) = p(x) + \sqrt{-1}q(x)$ with $p(x)$ and $q(x)$ both real-valued. Then we can define $\lambda(x) = q(\sqrt{-1}x) + \sqrt{-1}q(x)$ and $K(x) - \lambda(x) = p(x) - q(\sqrt{-1}x)$ is real-valued and $\lambda(x)$ is $\mathbb{C}$-linear. Let

$$L(x,u) - \left(-\frac{\sqrt{-1}}{2}H(x,u)\right) = 2\,b(x,u).$$

Define the trivial theta function $F_0(x) = \exp(2\pi\sqrt{-1}(b(x,x) + \lambda(x))$. Then

$$F_0(x+u) = F_0(x)\exp\left(2\pi\sqrt{-1}\left(2b(x,u) + b(u,u) + \lambda(u)\right)\right)$$

and

$$\begin{aligned}
&\left(\frac{F}{F_0}\right)(x+u) \\
&= \left(\frac{F}{F_0}\right)(x)\exp\left(2\pi\sqrt{-1}\left(L(x,u) - 2b(x,u) + J(u) - b(u,u) - \lambda(u)\right)\right) \\
&= \left(\frac{F}{F_0}\right)(x)\exp\left(2\pi\sqrt{-1}\left(-\frac{\sqrt{-1}}{2}H(x,u) + J(u) - b(u,u) - \lambda(u)\right)\right) \\
&= \left(\frac{F}{F_0}\right)(x)\exp\left(2\pi\sqrt{-1}\left(H(x,u) + K(u) + \frac{1}{2}L(u,u) - b(u,u) - \lambda(u)\right)\right) \\
&= \left(\frac{F}{F_0}\right)(x)\exp\left(2\pi\sqrt{-1}\left(-\frac{\sqrt{-1}}{2}H(x,u) + (K(u) - \lambda(u)) - \frac{\sqrt{-1}}{4}H(u,u)\right)\right).
\end{aligned}$$

Here $K(u) - \lambda(u)$ is real-valued. We now define a theta function as *normalized* if $L(x,y) = -\frac{\sqrt{-1}}{2}H(x,y)$ and $K(x)$ is real-valued on V. We have just proved that any theta function is equivalent to a normalized theta function. Moreover, any two normalized theta function in the same equivalence class differs by a multiplicative constant. The reason is that the quotient of two normalized theta functions in the same class must be of the form $\exp(2\pi\sqrt{-1}(b(x,x) + \lambda(x) + c)$ and is normalized. Since the periodicity factor is $\exp(2\pi\sqrt{-1}(2b(x,u) + b(u,u) + \lambda(u))$. So $b(x,y)$ is at the same time $\mathbb{C}$-bilinear and Hermitian and must vanish. The function $\lambda(x)$ is at the same time $\mathbb{C}$-linear and real-valued and so must also vanish. The normalized theta function is now just the constant $\exp(2\pi\sqrt{-1}\,c)$. We have just seen that a normalized theta function transforms according to

$$F(x+u) = F(x)\exp\left(2\pi\sqrt{-1}\left(-\frac{\sqrt{-1}}{2}H(x,u) + K(u) - \frac{\sqrt{-1}}{4}H(u,u)\right)\right).$$

Since K is both real-valued and $\mathbb{R}$-linear, we can define

$$\psi(x) = \exp(2\pi\sqrt{-1}\,K(x))$$

and get a character on the additive group V (which means a homomorphism from the additive group V to the multiplicative group of all complex numbers of absolute value 1). We call the pair (H,ψ) the *Hermitian character* of the normalized theta function. It determines how the normalized theta function transforms under translation by an element of the lattice D.

The Hermitian form of any non identically zero normalized theta function is positive semidefinite. The reason is as follows. Let

$$h(x) = F(x)\exp\left(-\frac{\pi}{2}H(x,x)\right).$$

Then

$$\begin{aligned}
h(x+u,x+u) &= F(x+u)\exp\left(-\frac{\pi}{2}H(x+u,x+u)\right)\\
= F(x)\exp&\left(2\pi\sqrt{-1}\left(-\frac{\sqrt{-1}}{2}H(x,u)+K(u)-\frac{\sqrt{-1}}{4}H(u,u)\right)\right)\cdot\\
&\cdot\exp\left(-\frac{\pi}{2}H(x,x)-\frac{\pi}{2}\mathrm{Re}\,H(x,u)-\frac{\pi}{2}H(u,u)\right)\\
&= h(x)\exp\left(\pi\sqrt{-1}\mathrm{Im}\,H(x,u)+2\pi\sqrt{-1}\,K(u)\right).
\end{aligned}$$

So $|h(x)| = |h(x+u)|$ which implies that $|h(x)|$ is bounded on V. So

$$|F(x)| \le C\,\exp\left(\frac{\pi}{2}H(x,x)\right)$$

on V for some positive real number C. If H is not positive semidefinite, then $H(x,x) < 0$ for some $x \in V$. Since

$$|F(\lambda\,x)| \le C\,\exp\left(\frac{\pi}{2}|\lambda|^2 H(x,x)\right)$$

for all $\lambda \in \mathbb{C}$, it follows that the entire function $F(\lambda x)$ on $\mathbb{C}$ as a function of λ approaches zero as $|\lambda| \to \infty$, which implies by Liouville's theorem $F(\lambda x) \equiv 0$ for all $\lambda \in \mathbb{C}$. Since $H(x,x) < 0$, we can find an open neighborhood U of x in V so that $H(y,y) < 0$ for all $y \in U$. So $F(\lambda y) \equiv 0$ for $\lambda \in \mathbb{C}$ and $y \in U$. Being identically zero on a nonempty open subset of V the entire function F must be identically zero, which is a contradiction.

We are going to interprete the Hermitian form as the curvature of the line bundle of which the theta function is a holomorphic section. The line bundle which we call L is associated with the divisor of the normalized theta function F. We cover the complex manifold V/D by small open balls U_ν so that each open ball U_ν can be lifted homeomorphically to an open ball $U_{\nu 0}$ in V under the universal covering map $\pi : V \to V/D$. For $\ell \in D$ we let $U_{\nu\ell}$ be $U_\nu + \ell$. We consider the covering $\{\pi(U_{\nu\ell})\}$. The divisor on $\pi(U_{\nu\ell})$ is defined by the holomorphic function $F|U_{\nu\ell}$ after the identification of $U_{\nu\ell}$ with $\pi(U_{\nu\ell})$ under π. The transition function for the line bundle between $U_{\nu\ell}$ and $U_{\mu p}$ is given by

$$g_{\nu\ell,\mu p}(\pi(x)) = \frac{F(x+u)}{F(x)} = \exp\left(\pi\, H(x,u) + \frac{\pi}{2} H(u,u) + 2\pi\sqrt{-1}\, K(u)\right)$$

for $x \in U_{\mu p} \cap (U_{\nu\ell} - u)$, where $u = u_{\nu\ell,\mu p}$ is the unique element in D such that $U_{\mu p} + u$ intersects $U_{\nu\ell}$. We have to define a Hermitian metric for the line bundle. Let $h_{\nu\ell}(\pi(x)) = \exp(-\pi\, H(x,x))$ for $x \in U_{\nu\ell}$. We have to verify that $h_{\nu\ell}|g_{\nu\ell,\mu p}|^2 = h_{\mu p}$ on $\pi(U_{\nu\ell}) \cap \pi(U_{\mu p})$. This means that

$$\begin{aligned}\exp(-\pi H(x+u,x+u)) \left|\exp(\pi H(x,u) + \frac{\pi}{2} H(u,u) + 2\pi\sqrt{-1}\, K(u))\right|^2 \\ = \exp(-\pi\, H(x,x))\end{aligned}$$

for $x \in U_{\mu p}$ and $u = u_{\nu\ell,\mu p}$. The curvature of the line bundle is given by

$$\frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log |g_{\nu\ell,\mu p}|^2 = H.$$

Here we have made the identification of the (1,1)-form

$$\sqrt{-1} \sum_{j,k=1}^n a_{jk} dz_j \wedge d\overline{z_k}$$

with the Hermitian form

$$(\xi_j) \to \sum_{j,k=1}^{n} a_{jk}\xi_j\overline{\xi_k}.$$

In general, the curvature of the line bundle associated to a divisor can be written as positive (1,1)-current. In this case the curvature form is smooth and constant because of the nature of the theta function. So it must be positive semidefinite. This is the geometric explanation of the second argument for the positive semidefiniteness of the Hermitian form associated to an entire normalized theta function which is not identically zero.

When we have a theta function, we have an effective divisor on the compact manifold V/D. The converse is also true. We going to construct from a complex hypersurface of V/D a theta function. Let $\bar{X}$ be the complex hypersurface in V/D and X be its inverse image in V. Let $\pi : V \to V/D$ be the projection map. The idea of the construction is as follows. We consider the line bundle $\mathcal{L}$ over V/D assoicated to the divisor $\bar{X}$ of V/D and consider the canonical section $s_{\mathcal{L}}$ of the line bundle $\mathcal{L}$ so that the divisor of $s_{\mathcal{L}}$ is $\bar{X}$. Consider the pullback $\pi^*\mathcal{L}$ of $\mathcal{L}$ to V and we will produce an isomorphism between $\pi^*\mathcal{L}$ and the trivial line bundle of V so that the pullback $\pi^*s_{\mathcal{L}}$ of $s_{\mathcal{L}}$ becomes an entire function on V whose divisor is X and this entire function is the theta function we seek. We introduce Hermitian metric for $\mathcal{L}$ which gives a Chern form $\gamma_{\mathcal{L}}$. The metric of $\mathcal{L}$ pulls back to a metric of $\pi^*\mathcal{L}$ whose Chern form is $\pi^*\gamma_{\mathcal{L}}$. The isomorphism between $\pi^*\mathcal{L}$ and the trivial line bundle is constructed by first using a change of local fiber coordinates for $\pi^*\mathcal{L}$ so that the Chern form $\pi^*\gamma_{\mathcal{L}}$ becomes identically zero. Then the transition functions become locally constant and by using the simply connectedness we can get finally the isomorphism between $\pi^*\mathcal{L}$ and the trivial line bundle. To make the Chern form $\pi^*\gamma_{\mathcal{L}}$ zero by using new local fiber coordinates we first make $\gamma_{\mathcal{L}}$ harmonic, which is the same as the coefficients of $\gamma_{\mathcal{L}}$ being constant.

We cover V/D by small open balls U_ν so that on each U_ν the complex hypersurface $\bar{X}$ is defined by a holomorphic function φ_ν on U_ν. Let $U_{\nu 0}$ be an open ball in V such that U_ν is its image under the projection map $\pi : V \to V/D$. For $\ell \in D$ let $U_{\nu\ell} = U_{\nu 0} + \ell$. We lift $\varphi_{\nu\ell}$ to $U_{\nu\ell}$ via the projection map π. Let $g_{\mu\nu} = \frac{\varphi_\mu}{\varphi_\nu}$ on $U_\mu \cap U_\nu$. Then $(g_{\mu\nu})$ are the transition functions for the line bundle associated with the divisor $\bar{X}$. We want to get a (normalized) theta function whose divisor is X. We have seen that the

Hermitian matrix associated to the theta function is the curvature of the line bundle and so must be in the Chern class of the line bundle. The Chern class of a holomorphic line bundle is of type (1,1). We locate the Chern class in the following way. Let h_ν be a Hermitian metric of the line bundle. Then $h_\mu |g_{\mu\nu}|^2 = h_\nu$ and

$$d\,\log g_{\mu\nu} = \partial\,\log h_\nu - \partial\,\log h_\mu. \tag{3}$$

This step writes the $(1,0)$-form $d\,\log g_{\mu\nu}$ as the coboundary of $(1,0)$-forms. Geometrically, $\partial \log h_\nu$ is the connection. The curvature (*i.e.* the Chern form $\gamma_{\mathcal{L}}$) is given by (up to a normalizing constant) $\bar\partial\partial\,\log h_\nu$ and we know that this represents the class where the Hermitian matrix associated to the normalized theta function must be in. The Hermitian matrix associated to the normalized theta function has constant coefficients. So to identify it we simply produce a (1,1)-form in the same class as the curvature that has constant coefficients. (Actually this means taking the harmonic representative.) We write

$$\bar\partial\partial\,\log h_\nu = \sum_{j,k=1}^{n} f_{jk}(z) dz_j \wedge d\overline{z_k}.$$

Translation in the manifold V/D topologically preserves this class (because translation produces clearly a homotopy). So we can average this form over all translations and still end up with a form in the same class. We are doing this in topology and can forget the complex structure. When we average over all translations, it is the same as picking up the constant term in the Fouier series expansion in V/D and the result is harmonic. Let us go back to the equation (3). We are free to choose the metric h_ν. We are interested in getting a harmonic $\bar\partial\partial\,\log h_\nu$(*i.e.* one with constant coefficients) so that it is (up to a normalizing constant) the same as the Hermitian matrix associated to the normalized theta function we are trying to construct. To simply notations we let $\zeta_\nu = \partial\,\log h_\nu$. We are going to modify ζ_ν by replacing it by $\zeta_\nu - \zeta$ for some glocally ∂-closed (1,0)-form ζ so that $d\zeta_\nu$ is harmonic. Cohomologically, we expresses $d \log g_{\mu\nu}$ as the coboundary of $\zeta_\mu - \zeta$ instead of as the coundary of ζ_μ. Geometrically, we are replacing the connection ζ_μ by the connection $\zeta_\mu - \zeta$ so that the Laplacian of the new connection $\zeta_\mu - \zeta$ is harmonic. Clearly what we should do is to solve the equation $\Delta\zeta_\nu = \Delta\zeta$ for some global ζ. Here the Laplacian Δ is applied coefficientwise. We have

$$\Delta\zeta_\nu = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial\overline{z_j}} \zeta_\nu = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \eta_j,$$

where

$$\eta_j = \frac{\partial}{\partial \overline{z_j}} \zeta_\nu = \frac{\partial}{\partial \overline{z_j}} \zeta_\mu$$

is independent of ν because $d \log g_{\mu\nu} = \zeta_\nu - \zeta_\mu$ is holomorphic on $U_\mu \cap U_\nu$. (Actually η_j is simply defined by $\bar{\partial}\partial \log h_\nu = \sum_{j=1}^n d\overline{z_j} \wedge \eta_j$.) The entity $\Delta\zeta_\nu$ is globally defined and is independent of ν. We can solve the equation $\Delta\zeta_\nu = \Delta\zeta$ on V/D if and only if $\Delta\zeta_\nu$ is perpendicular to all (1,0)-forms with constant coefficients (as we can see by using Fourier series for each coefficient). This condition is met because $\Delta\zeta_\nu = \sum_{j=1}^n \frac{\partial}{\partial z_j} \eta_j$ is the sum of partial derivatives and we can easily use integration by parts to verify the condition. Note that we do not know at this point that ζ is ∂-closed. However, it turns out that this does not matter much in the process we use to construct the theta function. Let $\zeta'_\nu = \zeta_\nu - \zeta$. We form now $\omega = d\zeta'_\nu$. Then ω is harmonic and can be written as

$$\omega = \sum_{j,k=1}^n a_{jk} dz_j \wedge dz_k + \sum_{j,k=1}^n b_{jk} d\overline{z_j} \wedge dz_k,$$

because ζ'_ν is of type $(1,0)$ and the differential of any $(1,0)$-form is a sum of a $(2,0)$-form and a $(1,1)$-form. The purpose of having ω is to be able to write it, in the explicit manner we want, as the differential of some 1-form on $\mathbb{C}^n$.

We want to go backward to recover the theta function. The way to do it is to sacrifice periodicity to get a function out of a form. The way to go about it is to integrate. Let

$$\psi = \sum_{j,k=1}^n a_{jk} z_j dz_k + \sum_{j,k=1}^n b_{jk} \overline{z_j} dz_k.$$

Then $d\psi = \omega$ on $\mathbb{C}^n$. This is the first step of the integration process. Then $d(\zeta'_\nu - \psi) = 0$ on $U_{\nu\ell}$. The setup to modify ζ'_ν by a global $(1,0)$-form is to make the modified ζ'_ν closed. By Poincare's lemma, we can find a smooth function $f_{\nu\ell}$ on $U_{\nu\ell}$ so that $df_{\nu\ell} = \zeta'_\nu - \psi$. Since the right-hand side is of type (1,0), we know that $f_{\nu\ell}$ is holomorphic on $U_{\nu\ell}$. All the time we keep the 1-form under consideration to be of type $(1,0)$ so that when it is closed it is locally the differential of a local holomorphic function. On $U_{\nu\ell} \cap U_{\mu p}$ we have

$$df_{\nu\ell} - df_{\mu p} = \zeta'_\nu - \zeta'_\mu = d \log g_{\mu\nu} = d \log(\varphi_\mu / \varphi_\nu).$$

Hence $\varphi_\nu \exp(f_{\nu\ell})$ and $\varphi_\mu \exp(f_{\mu p})$ differ by a constant factor on $U_{\nu\ell} \cap U_{\mu p}$. So we can start with some fixed $U_{\nu\ell}$ and continue analytically to a holomorphic function F on all of V which differs from $\varphi_\mu \exp(f_{\mu p})$ only by a constant factor. We now want to look at the transformation property of F under translation by an element of D. Take $x \in U_{\nu\ell}$ and let $u = p - \ell$. Then

$$F(x+u) = c_1 \varphi_\nu(x) \exp\left(f_{\nu p}(x+u)\right)$$

on $U_{\nu p}$ and

$$F(x) = c_2 \varphi_\nu(x) \exp\left(f_{\nu\ell}(x)\right)$$

on $U_{\nu\ell}$ for some constants c_1 and c_2. Thus

$$\begin{aligned} d \log \frac{F(x+u)}{F(x)} &= d\, f_{\nu p}(x+u) - d\, f_{\nu\ell}(x) \\ &= \left(\zeta'_\nu(x) - \psi(x+u)\right) - \left(\zeta'_\nu(x) - \psi(x)\right) \\ &= \psi(x) - \psi(x+u) \\ &= -\sum_{j,k=1}^{n} a_{jk} u_j dz_k - \sum_{j,k=1}^{n} b_{jk} \overline{u_j} dz_k. \end{aligned}$$

Integrating once again yields

$$F(x+u) = F(x) \exp(L(x,u) + J(u)),$$

where

$$L(x,u) = -\sum_{j,k=1}^{n} a_{jk} x_k u_j - \sum_{j,k=1}^{n} b_{jk} x_k \overline{u_j}$$

is $\mathbb{C}$-linear in x and $J(u)$ is the constant of integration. Since F differs from $\varphi_\nu \exp(f_{\nu\ell})$ on $U_{\nu\ell}$ only by a constant factor, we know that the divisor of F agrees with the divisor X.

We now want to handle the question of the null space of the positive semidefinite Hermitian form H. Let F be a normalized theta function which is not identically zero and let H be the Hermitian form associated to F. Let V_H be the null space of H, *i.e.* V_H is the set of all $x \in V$ such that $H(x,x) = 0$. From

$$0 \leq H(\lambda\, x + y, \lambda\, x + y) = 2 \text{ Re } \left(\lambda\, H(x,y)\right) + H(y,y)$$

for all $\lambda \in \mathbb{C}$, we know that $H(x,y) = 0$ for $x \in V_H$ and $y \in V$. Since H is positive semidefinite, we know that V_H is a $\mathbb{C}$-vector space. First we observe that the value of $F(x)$ depends only on the coset of V_H where x is in. In other words, $F(x+y) = F(x)$ for $y \in V_H$. This follows from the consideration of $F(x)\exp(-\frac{\pi}{2}H(x,x))$. We have seen that

$$\left|F(x+u)\exp\left(-\frac{\pi}{2}H(x+u,x+u)\right)\right| = \left|F(x)\exp\left(-\frac{\pi}{2}H(x,x)\right)\right|$$

for all $u \in D$ and so $|F(x)| \leq C\,\exp(-\frac{\pi}{2}H(x,x))$ for some positive number C. For $y \in V_H$ we have $H(x+\lambda\,y,\,x+\lambda\,y) = H(x,x)$ for $x \in V$ and hence

$$|F(x+\lambda y)| \leq C\,\exp\left(-\frac{\pi}{2}H(x,x)\right)$$

for all $\lambda \in V$. By Liouville's theorem we know that $F(x+\lambda y)$ is independent of λ and hence $F(x+y) = F(x)$. Secondly we claim that the image of D in V/V_H under the natural projection $V \to V/V_H$ is discrete. This is a result of the fact that the imaginary part E of H assumes integral values on $D \times D$. The characterization of $x \in V_H$ is that $H(y,x) = 0$ for all $y \in V$. Since

$$H(y,x) = E(\sqrt{-1}y,x) + \sqrt{-1}E(y,x),$$

the characterization of $x \in V_H$ can be reformulated as $E(y,x) = 0$ for all $y \in V$. Since E assumes integral values on $D \times D$, we conclude that V_H is spanned by $V_H \cap D$ over $\mathbb{R}$. So we can find a basis $e_1, \cdots, e_n$ for V such that each e_j is in D and $e_1, \cdots, e_k$ form a basis for V_H by choosing $e_1, \cdots, e_k$ first. There is a positive integer N such that $N\,D \subset \sum_{j=1}^n \mathbb{Z}\,e_j$. Then the projection of D in V/V_H is contained in the discrete set $\frac{1}{N}\sum_{j=k+1}^n \mathbb{Z}\overline{e_j}$, where $\overline{e_j}$ is the image of e_j in V/V_H. This means that the proper setting for the normalized theta function is actually in the quotient space V/V_H with the lattice $(D+V_H)/V_H$. In this proper setting the Hermitian matrix associated to the normalized theta function is positive definite.

Now we know that whenever we have a theta function we have a positive definite Hermitian matrix H whose imaginary part assumes integral values at the lattice points. Now we want to look at the converse. We start out with the Hermitian matrix and try to construct a theta function. The Hermitian matrix is determined by its imaginary part. A Hermitian matrix $H(x,y) = S(x,y) + \sqrt{-1}\,E(x,y)$ is positive definite if and only its real part $S(x,y)$ is positive definite, simply because $H(x,x) = S(x,x)$. We start out with the following definition. A *Riemann form* E on V with respect to D is a real-valued $\mathbb{R}$-bilinear form $E : V \times V \to \mathbb{R}$ such that E is skew-symmetric, assumes integral values on $D \times D$, and $(x, y) \to E(\sqrt{-1}x, y)$ is positive symmetric. We say that the Riemann form E is *nondegenerate* if there exists no nonzero $x \in V$ such that $E(x, y) = 0$ for all y in V. (It is the same as saying that the matrix representing E is nonsingular.) We want to construct theta functions from a nondegenerate Riemann form. First we look at the so-called Frobenius decomposition, which chooses a good basis of D adapted to the nondegenerate skew-symmetric integral valued form E. We want to find elements $e_1, v_1, \cdots, e_n, v_n$ of D such that

(1) $D = \sum_{j=1}^{n} (\mathbb{Z}e_j + \mathbb{Z}v_j)$,

(2) $E(e_j, v_j) = d_j > 0$ with $d_1|d_2|\cdots|d_n$ (meaning d_{j-1} dividing d_j),

(3) $\mathbb{Z}e_j + \mathbb{Z}v_j$ is perpendicular to $\mathbb{Z}e_k + \mathbb{Z}v_k$ with respect to E for $j \neq k$.

The way to get the Frobenius decomposition is to first choose e_1 and v_1 in D so that $E(e_1, v_1)$ assumes the smallest positive value d_1. Let $V_1' = \mathbb{R}e_1 + \mathbb{R}v_1$ and V_1 be the set of all $x \in V$ such that $E(x, y) = 0$ for all $y \in V_1'$. We claim that $V_1 \cap V_1' = 0$, otherwise some nonzero element x in V_1 is in V_1' which means that some $\lambda e_1 + \mu v_1$ with λ and μ not both zero is perpendicular to e_1 and v_1 with respect to E. From $E(e_1, v_1) = d_1 \neq 0$ and $E(e_1, e_1) = E(v_1, v_1) = 0$ we know that this is not possible. Let $D_1 = D \cap V_1$ and $D_1' = D \cap V_1'$. We claim that $D = D_1 + D_1'$. Take $u \in D$. We want to find $a, b \in \mathbb{Z}$ such that $u - ae_1 - bv_1$ is perpendicular to D_1 with respect to E. We know that d_1 divides $E(u, e_1)$, otherwise $E(u, e_1) = qd_1 + r = q\,E(e_1, v_1) + r$ for some positive integer $r < d_1$ and another integer q, leading to the contradiction that $E(u + qv_1, e_1) = r < d_1$. Likewise we conclude that d_1 divides $E(u, v_1)$. We write $E(u, e_1) = -bd_1$ and $E(u, v_1) = ad_1$. Then $u - ae_1 - be_1$ is perpendicular to V_1' with respect to E. Since E is nondegenerate, we know that $E|V_1$ is also nondegenerate. Now we apply the previous argument to $E|V_1$ and use induction on $\dim_{\mathbb{R}} V$ to get $e_1, v_1, \cdots, e_n, v_n$. The last thing we have to verify is that d_{j-1} divides d_j. It suffices to verify that d_1 divides d_2. Suppose the contrary. Then we have $d_2 = q\,d_1 + r$ for some positive integer $r < d_1$ and some other integer q, leading to the contradiction that $E(e_2 - q\,e_1,\, v_2 + v_1) = d_2 - qd_1 = r < d_1$.

We would like to remark that $e_1, \cdots, e_n$ form a $\mathbb{C}$-basis of V. Suppose the contrary. Let $W = \mathbb{R}e_1 + \cdots + \mathbb{R}e_n$. Then $W \cap \sqrt{-1}W \neq 0$. Take a nonzero $y \in W$ with $\sqrt{-1}y \in W$. Since any two elements of W is perpendicular with respect to E, we conclude that $E(\sqrt{-1}y, y) = 0$, contradicting the fact that $(x, y) \to E(\sqrt{-1}x, y)$ is symmetric positive. We now work with coordinates with respect to the $\mathbb{C}$-basis $e_1, \cdots, e_n$ of V. From E we have the Hermitian form $H(x, y) = E(\sqrt{-1}x, y) + \sqrt{-1}E(x, y)$ from the assumption that $(x, y) \to E(\sqrt{-1}x, y)$ is symmetric in x and y. Since the imaginary part E of H vanishes on $\mathbb{R}e_1 + \cdots + \mathbb{R}e_n$, we know that H assumes only real values on $\mathbb{R}e_1 + \cdots + \mathbb{R}e_n$, which means that when we express H in terms of the basis $e_1, \cdots, e_n$ the entries of the matrix is real symmetric. In other words,

$$H\left(\sum_{j=1}^n z_j e_j, \sum_{j=1}^n w_j e_j\right) = \sum_{j,k=1}^n h_{jk} z_j \overline{w_k}$$

with $h_{jk} = H(e_j, e_k)$ real and symmetric. Likewise, H is real symmetric on $\mathbb{R}v_1 + \cdots + \mathbb{R}v_n$.

We are going to construct a theta function by infinite series, following the way the Jacobian theta function is constructed for the case of complex dimension one. Recall the Jacobian theta function $\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kiw}$, where $q = e^{\pi\tau i}$ and τ is the quotient $\frac{\omega_2}{\omega_1}$ of the two periods ω_1 and ω_2. We rewrite the infinite series

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2k\sqrt{-1}\,w}$$

in the form

$$\sum_{k=-\infty}^{\infty} \exp\left(k^2\pi\tau\sqrt{-1} + 2k\sqrt{-1}\,w + k\pi\sqrt{-1}\right)$$

and combine the two terms $2k\sqrt{-1}\,w + k\pi\sqrt{-1}$ together by using $s = \frac{w}{\pi} + \frac{1}{2}$ and we end up with

$$\sum_{k=-\infty}^{\infty} \exp(k^2\pi\tau\sqrt{-1} + 2ki\pi s).$$

Now in the higher dimensional case $\mathbb{C}^n/D$ we do something completely analogous. Instead of the integer k we use the n-tuple of integers t with components $t_1, \cdots, t_n \in \mathbb{Z}$ written as a column n-vector. Instead of $w \in \mathbb{C}$ we use the n-tuple of complex numbers s with components $s_1, \cdots, s_n \in \mathbb{C}$ written as a column n-vector. Instead of τ in the upper half plane we use the matrix

Z in the Siegel upper half space. So $Z = X + \sqrt{-1}Y$ is a symmetric $g \times g$ matrix with complex entries such that $Y = \operatorname{Im} Z > 0$. Instead of

$$\sum_{k=-\infty}^{\infty} \exp(k^2\pi\tau\sqrt{-1} + 2k\sqrt{-1}\,\pi s),$$

we define

$$\Theta(s, Z) = \sum_{t\in\mathbb{Z}^g} \exp\left(\pi\sqrt{-1}\,(t'Zt + 2t's)\right).$$

The condition $Y = \operatorname{Im} Z > 0$ guarantees convergence of the infinite series. Now let us go back to our construction of the theta function from the nondegenerate Riemann form. To use the method of infinite series we need to produce this element Z of the Siegel upper half space. The lattice for the function $\Theta(s, Z)$ is generated by $e_1, \cdots, e_n, Ze_1, \cdots, Ze_n$, where $e_1, \cdots, e_n$ is the standard $\mathbb{C}$-basis for $\mathbb{C}^n$. An obvious thing to do is to equate $e_1, \cdots, e_n, Ze_1, \cdots, Ze_n$ with $e_1, \cdots, e_n, v_1, \cdots, v_n$ and to define Z by $Z = (z_{jk})$ with $v_j = \sum_{k=1}^n z_{jk}e_k$. It is almost right, but not yet completely right. The elements $e_1, \cdots, e_n, v_1, \cdots, v_n$ are obtained from the Frobenius decomposition of E. To keep track of what is going on, we want to see what the skew-symmetric form E for Z is. So we have to consider the transformation law for $\Theta(s, Z)$ under translation by an element of

$$\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n + \mathbb{Z}(Ze_1) + \cdots + \mathbb{Z}(Ze_n).$$

The transformation law is $\Theta(s + g) = \Theta(s)$ and

$$\Theta(s + Zg) = \exp\left(\pi\sqrt{-1}\,(-g'Zg - 2g's)\right)\Theta(s)$$

for the column vectors $s \in \mathbb{C}^n$ and $g \in \mathbb{Z}^n$ from $L(s, g) = \frac{1}{2}(-g'Zg - 2g's)$. We see that the function $L(x, u)$ is given by $L(x, u) = 0$ and $L(x, Zu) = -u'x$ for $u \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$. We now skew-symmetrize L and get E. In the first place, from $L(u, u) = 0$ and $L(Zu, Zu) = -u'Zu$ for $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ and the symmetry of Z that the skew-symmetric part E of Z vanishes on $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ and also vanishes on $\mathbb{Z}(Ze_1) + \cdots + \mathbb{Z}(Ze_n)$. To get E we need only consider

$$E(e_j, Ze_k) = L(e_j, Ze_k) - L(Ze_k, e_j) = -e_k' \cdot e_j = -\delta_{jk},$$

where δ_{jk} is the Kronecker delta. So we get the Frobenius basis

$$e_1, \cdots, e_n, Ze_1, \cdots, Ze_n$$

and the factors $d_1, \cdots, d_n$ are all 1. So when all the $d_j' s$ are equal to 1, our guess of defining Z by $v_j = -\sum_{k=1}^n z_{jk} e_k$ is correct. However, when not all the $d_j' s$ are equal to 1, we need some modification. In the first place our infinite series

$$\Theta(s, Z) = \sum_{t \in \mathbb{Z}^g} \exp\left(\pi\sqrt{-1}\,(t' Z t + 2t' s)\right)$$

can only produce the case of all the $d_j' s$ equal to 1. We need a slightly different series. Recall that the original Jacobian theta series was produced by the method of undetermined coefficients. To get the case we want, we simply apply again the method of undetermined coefficients in our case, which we are going to do later. It turns out that the correct way to define z_{jk} is to use $v_j = -d_j \sum_{k=1}^n z_{jk} e_k$. The intuitive reason is that we replace v_j by $\frac{v_j}{d_j}$ so that the new d_j becomes 1. Now we forget our motivation and just use $v_j = -d_j \sum_{k=1}^n z_{jk} e_k$ as the definition for the matrix $Z = (z_{jk})$ and start to verify that it is symmetric and its imaginary part is positive definite. The trick for the verification is the use of

$$F(x, y) = H(x, y) - H(x, \bar{y}) = -2\sqrt{-1}\, H(x, \mathrm{Im}\ y).$$

Here the complex conjugate $\bar{y}$ of y and the imaginary part $\mathrm{Im}\, y$ of y are both respect to the basis $e_1, \cdots, e_n$. So if $y = \sum_{j=1}^n y_j e_j$, then $\bar{y} = \sum_{j=1}^n \bar{y}_j e_j$ and $\mathrm{Im}\, y = \sum_{j=1}^n (\mathrm{Im}\, y_j)\, e_j$. This form $F(x, y)$ satisfies two properties. The first one is that $F(x, y) = F(y, x)$ for $x, y \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. This is because (i) $H(x, \bar{y})$ is $\mathbb{C}$-bilinear and symmetric in both x and y for x, y in V and (ii) the imaginary part of $H(x, y)$ vanishes for $x, y \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ (from $E(v_j, v_k) = 0$ for all $1 \le j, k \le n$) and so $H(x, y) = \mathrm{Re}\, H(x, y)$ is symmetric in x and y for $x, y \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. The second property is that the form

$$\mathrm{Re}\, F(x, y) = 2\, \mathrm{Im}\, H(x,\, \mathrm{Im}\, y) = 2\, H(\mathrm{Im}\, x,\, \mathrm{Im}\, y)$$

is positive semidefinite (the second equation coming from the fact that the coefficients of H are real with respect to $e_1, \cdots, e_n$). Now since $H(x, \bar{y})$ is symmetric in x and y, it follows that

$$\begin{aligned} F(e_\ell, v_k) &= H(e_\ell, v_k) - H(e_\ell, \bar{v}_k) = H(e_\ell, v_k) - H(v_k, \bar{e}_\ell) \\ &= H(e_\ell, v_k) - H(v_k, e_\ell) = 2\sqrt{-1}\, E(e_\ell, v_k) \end{aligned}$$

(because by definition e_ℓ is real and $\overline{e_\ell} = e_\ell$) and

$$F(v_j, v_k) = -d_j \sum_{\ell=1}^n z_{j\ell} F(e_\ell, v_k)$$

$$= -d_j \sum_{\ell=1}^{n} z_{j\ell} 2\sqrt{-1}\, E(e_\ell, v_k) = -d_j z_{jk} 2\sqrt{-1}\, d_k.$$

The symmetry of $F(x, y)$ for $x, y \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ implies that symmetry of $Z = (z_{jk})$. To get the positivity of the imaginary part of Z, we use

$$(\ddagger) \qquad F\left(\sum_{j=1}^{n} \lambda_j v_j, \sum_{j=1}^{n} \lambda_j v_j\right) = -2\sqrt{-1} \sum_{j,k=1}^{n} \lambda_j d_j z_{jk} \lambda_k d_k.$$

Since by taking the real part of (‡) we have

$$\operatorname{Re} F(\sum_{j=1}^{n} \lambda_j v_j, \sum_{j=1}^{n} \lambda_j v_j) = 2 \sum_{j,k=1}^{n} \lambda_j d_j \left(\operatorname{Im} z_{jk}\right) \lambda_k d_k,$$

it follows from

$$\operatorname{Re} F(x, y) = 2\, H\left(\operatorname{Im} x,\, \operatorname{Im} y\right)$$

that $\operatorname{Re} F\left(\sum_{j=1}^{n} \lambda_j v_j, \sum_{j=1}^{n} \lambda_j v_j\right) > 0$ if the imaginary part of $\sum_{j=1}^{n} \lambda_j v_j$ is nonzero. We know from the $\mathbb{R}$-linearly independence of $e_1, \cdots, e_n, v_1, \cdots, v_n$ that $\sum_{j=1}^{n} \lambda_j v_j$ is not in $\mathbb{R}e_1 + \cdots + \mathbb{R}e_n$ whenever $\lambda_1, \cdots, \lambda_n$ are not all zero, which means that the imaginary part of $\sum_{j=1}^{n} \lambda_j v_j$ is nonzero.

After all the above preparatory statements in linear algebra, we are ready to use the method of undetermined coefficients to construct theta functions. The result is given in the following theorem of Frobenius.

Theorem. Let V be a vector space over $\mathbb{C}$ and let D be a lattice in V and let $L : V \times V \to \mathbb{C}$ be $\mathbb{C}$-linear in the first variable and $\mathbb{R}$-linear in the second variable such that $E(x, y) = L(x, y) - L(y, x)$ is a nondegenerate Riemann form for the lattice D. Let $K : V \to \mathbb{R}$ be $\mathbb{R}$-linear. Then the set of all entire theta functions on V with respect to D having type (L, K) form a vector space over $\mathbb{C}$ with dimension equal to the Pfaffian of E with respect to D (which is $d_1 \cdots d_n$).

We now prove the theorem. Since L is symmetric in $\mathbb{R}\, e_1 + \cdots + \mathbb{R}\, e_n$ (due to the vanishing of $E(e_j, e_k)$), we can extend the restriction of L on $\mathbb{R}\, e_1 + \cdots + \mathbb{R}\, e_n$ to a $\mathbb{C}$-bilinear map L_1 on $V \times V$. We extend also the restriction of K on $\mathbb{R}\, e_1 + \cdots + \mathbb{R}\, e_n$ to a $\mathbb{C}$-linear map K_1 on V. We replace L by $L - L_1$ and K by $K - K_1$. This replacement is the same as multiplying the theta function by a trivial theta function (defined by L_1 and K_1). So without loss of generality we can assume that both L and K vanish on $\mathbb{R}\, e_1+\cdots+\mathbb{R}\, e_n$. The space of all entire theta functions of type (L, K) is the same as the space of all entire functions satisfying $\theta(z+e_j)=\theta(x)$ and

$$\theta(z+v_j)=\theta(z)\exp\left(2\pi\sqrt{-1}(z_j d_j+c_j)\right)$$

for some $c_j \in \mathbb{C}$. The condition $\theta(z+e_j)=\theta(z)$ implies that θ is periodic with periods e_j. So we can write

$$\theta(z)=\sum_{m\in\mathbb{Z}^n} a(m)\exp\left(2\pi\sqrt{-1}\,\langle m, z\rangle\right),$$

where $\langle m, z\rangle$ is the inner product of m and z. We have

$$\theta(z+v_j)=\sum_{m\in\mathbb{Z}^n} a(m)\exp(2\pi\sqrt{-1}\,\langle m, z+v_j\rangle)$$

$$\theta(z)\exp(2\pi\sqrt{-1}(z_j d_j+c_j)=\sum_{m\in\mathbb{Z}^n} a(m)\exp(2\pi\sqrt{-1}(\langle m+d_j e_j, z\rangle+c_j).$$

Hence

$$a(m-d_j e_j)=a(m)\exp(2\pi\sqrt{-1}(\langle m, v_j\rangle-c_j)).$$

To solve for $a(m)$, we let $a(m)=\exp(2\pi\sqrt{-1}\, b(m))$. The value of $b(m)$ is defined modulo $\mathbb{Z}$. In the following equations we use identity instead of congruence modulo $\mathbb{Z}$ by assuming that the value of $b(m)$ is chosen to give us the identity instead of congruence. Then

$$b(m-d_j e_j)-b(m)+c_j=\langle m, v_j\rangle=-\langle m, d_j Z e_j\rangle.$$

We claim that

$$b(m)=\left\langle m, Z\left(\sum_{k=1}^n m_k e_k\right)\right\rangle+\sum_{k=1}^n \frac{m_k c_k}{d_k}+h(s),$$

where $(s_1,\cdots,s_n)\in\mathbb{Z}^n$ with $0\le s_j<d_j$ satisfies $m_j\equiv s_j \bmod d_j$ and $h(s)$ is an indeterminate to be chosen arbitrarily. Let us now verify this claim.

$$b(m-d_j e_j)=\left\langle m-d_j e_j, Z\left(\sum_{k=1}^n m_k e_k\right)\right\rangle+\sum_{k=1}^n \frac{m_k c_k}{d_k}-c_j+h(s).$$

$$b(m - d_j e_j) - b(m) + c_j = -d_j \left\langle e_j, Z\left(\sum_{k=1}^{n} m_k e_k\right)\right\rangle$$

$$= -d_j \left\langle Ze_j, \sum_{k=1}^{n} m_k e_k \right\rangle = -d_j \left\langle Ze_j, m\right\rangle = -\left\langle m, d_j Ze_j\right\rangle .$$

Note that

$$b(m) = \sum_{j,k=1}^{n} m_j z_{jk} m_k + \sum_{k=1}^{n} \frac{m_k c_k}{d_k} + h(s).$$

The positive definiteness of the imaginary part of $Z = (z_{jk})$ guarantees the convergence of the series

$$\theta(z) = \sum_{m\in\mathbb{Z}^n} a(m) \exp\left(2\pi\sqrt{-1}\left\langle m, z\right\rangle\right).$$

The choice of $h(s_1, \cdots, s_n)$ with $(s_1, \cdots, s_n) \in \mathbb{Z}^n$ and $0 \leq s_j < d_j$ means that the complex dimension of the space of such theta functions is $d_1 \cdots d_n$. This finishes the proof of the theorem.

The theta function θ that we have constructed may be the theta function for a larger lattice. Let D' be the set of all $v \in V$ and $\frac{\theta(x+v)}{\theta(x)}$ is nowhere zero holomorphic. By taking $\log \frac{\theta(x+v)}{\theta(x)}$ and using the transformation law for translations by elements of D' we conclude that $\log \frac{\theta(x+v)}{\theta(x)}$ is of quadratic growth and hence must be a polynomial of degree at most 2. The set D' is discrete, because the imaginary part of H (which is nonsingular) have to assume integral values at $D' \times D'$ and in particular assume integral values at $D \times D'$.

If all the theta functions we have constructed are theta functions for a larger lattice, then the number of linearly independent functions is equal to the volume of the fundamental domain of D' measured by the volume form which is the n-fold exterior power of the imaginary part of H. This number is smaller than the number of linearly independent theta functions we have constructed, because the number we have obtained is equal to the volume of the fundamental domain of D measured by the volume form which is the n-fold exterior power of the imaginary part of H. We actually claim that we can find a theta function with the property that $D' = D$. Fix a point $x \in V$. We consider the union A of lattices D^* such that $D \subset D^*$ and

$\operatorname{Im} H : D^* \times D^* \to \mathbb{Z}$. The set D^*/D is finite. Pick $u_1, \cdots, u_k \in D^*$ forming a complete set of representatives for D^*/D. For each u_j we can find some theta function θ_j such that the zero-set of $\theta_j(x+u_j)$ is not equal to the zero-set of $\theta_j(x)$. The set Z_j of all theta functions is a closed proper subvariety of the finite dimensional vector space of all theta functions. Thus we can find a theta function θ such that the zero-set of $\theta(x+u_j)$ is not equal to the zero-set of $\theta(x)$ for $1 \leq j \leq k$.

Projective Embedding by Theta Functions. Let $D \subset V = \mathbb{C}^n$ be a lattice and H be a nondegenerate Riemann form and ψ be a character. From (H, ψ) we have a line bundle L over V/D.

Theoerm. $\Gamma(V/D,\, L^3)$ embeds V/D into $\mathbf{P}_N$, where $N = \dim_{\mathbb{C}} \Gamma(V/D,\, L^3) - 1$.

Proof. Let θ be a normalized theta function of type (H, ψ). Choose three points a, u, v of V and define

$$(*) \qquad \varphi(x; a, u, v) = \theta(x+u)\theta(x-a+v)\theta(x+a-u-v).$$

Though we have used three parameters a, u, v, actually we have only two, namely, u and $a - v$, because $u + (-a+v) + (a-u-v) = 0$. We need this equation to guarantee that $\varphi(x; a, u, v)$ is of type $(3H, \psi^3)$. The reason is that

$$\theta(x+u) = \theta(x) \exp\left(\pi\, H(x,u) + \frac{\pi}{2} H(u,u) + 2\pi\sqrt{-1}\, K(u)\right)$$

with $\psi(u) = \exp(2\pi\sqrt{-1}\, K(u))$. Thus

$$\prod_{j=1}^{3} \theta(x + a_j + u)$$

$$= \exp\left(\pi\, 3H\left(x + \frac{1}{3}\sum_{j=1}^{3} a_j, u\right) + \frac{\pi}{2} 3\, H(u,u) + 2\pi\sqrt{-1}\, 3\, K(u)\right) \prod_{j=1}^{3} \theta(x+a_j)$$

$$= \exp\left(\pi\, 3\, H(x,u) + \frac{\pi}{2} 3\, H(u,u) + 2\pi\sqrt{-1}\, 3\, K(u)\right) \prod_{j=1}^{3} \theta(x+a_j)$$

when $\sum_{j=1}^{3} a_j = 0$. We are going to use functions of the form (*) to do the embedding. One thing is that we have to make sure that the lattice is the proper one for the function θ. By this we mean the following. If $v \in V$ and $\frac{\theta(x+v)}{\theta(x)}$ is nowhere zero holomorphic, then $v \in D$. We can also assume that the zero-set of θ is irreducible. The reason is as follows. We decompose the zero-set of θ into irreducible hypersurface $Z_1, \cdots, Z_k$. We know that we can construct from each irreducible surface Z_k a theta function θ_j. We can get embeddings by constructing $\varphi_{j1}, \cdots, \varphi_{j\ell_j}$ from each θ_j and then we use $\prod_{j=1}^k \varphi_{j\nu_j}$ for the embeddings by elements of $\Gamma(V/D, L^3)$.

To get the projective embedding, the first thing we have to do is to make sure that given any $x \in V$ there exists some theta function of type $(3H, \psi^3)$ which is nonzero at x. We are going to get this theta function in the form (*) by choosing a, u, v suitably. Choose $a = x$. Choose v such that $\theta(v) \neq 0$ and then choose u choose that $\theta(a+u)\theta(2a-u-v) \neq 0$. We can do this, because both $\theta(a+u)$ and $\theta(2a-u-v)$ as functions of u are not identically zero and so their product as a function of u is not identically zero and we can find some u so that $\theta(a+u)\theta(2a-u-v) \neq 0$. This means that $\theta(x+u)\theta(x+a-u-v) \neq 0$ for $x = a$. Since $\theta(x-a+v) = \theta(v) \neq 0$, we know that $\varphi(x; a, u, v) = \theta(a+u)\theta(v)\theta(2a-u-v) \neq 0$. We now know that the map $V/D \to \mathbf{P}_N$ defined by $\Gamma(V/D, L^3)$ is a well-defined holomorphic map.

Next we want to show that the map $V/D \to \mathbf{P}_N$ defined by $\Gamma(V/D, L^3)$ distinguishes points. Fix $a \neq b$ in V not congruent modulo D. Since θ is not a trivial theta function we know that θ vanishes somewhere (otherwise $\log\theta$ is a well-defined holomorphic function on V and the transformation rule for translation by an element of D implies that the growth of $\log\theta$ is quadratic and $\log\theta$ must be a polynomial of degree ≤ 2). Moreover, since $b - a \notin D$, we know that either $\frac{\theta(x+b-a)}{\theta(x)}$ or its reciprocal is holomorphic and zero at x for some x. Since the zero-set of θ is irreducible, by exchanging the roles of a and b if necessary, we can assume that there exists $v \in V$ such that $\theta(v) = 0$ and $\theta(v-a+b) \neq 0$. Choose u such that $\theta(b+u)\theta(b+a-u-v) \neq 0$. Then at $x = a$ we have $\varphi(x; a, u, v) = \theta(a+u)\theta(v)\theta(2a-u-v) = 0$. At $x = b$ we have $\varphi(x; a, u, v) = \theta(b+u)\theta(b-a+v)\theta(b+a-u-v) \neq 0$. Thus the map is injective.

The last thing we have to prove is to show that the map has the rank n. We need the following lemma which will be proved later.

Lemma. Given θ. There exist $b_1, \cdots, b_n$ in V such that $\theta(b_j) = 0$ and $(d\theta)(b_1), \cdots, (d\theta)(b_n)$ (after being translated to the same point) are linearly independent.

From the lemma we have $b_1, \cdots, b_n$ in V. Fix $a \in V$. There exists $b_0 \in V$ with $\theta(b_0) \neq 0$. We can find u such that

$$\theta(a+u) \prod_{j=0}^{n} \theta(2a - u - b_j) \neq 0.$$

Let

$$\varphi_j(x) = \varphi(x; a, u, b_j) = \theta(x+u)\theta(x-a+b_j)\theta(x+a-u-b_j)$$

for $0 \leq j \leq n$. We claim that $\frac{\varphi_1}{\varphi_0}, \cdots, \frac{\varphi_n}{\varphi_0}$ form a local coordinate system at $x = a$. Since $\theta(x-a+b_j) = 0$ at $x = a$ for $1 \leq j \leq n$, when we take $d\varphi_j$ we must use $d\theta(x-a+b_j)$ which is equal to $d\theta(b_j)$ at $x = a$, otherwise we have only zero contribution. Thus $\frac{\varphi_1}{\varphi_0}, \cdots, \frac{\varphi_n}{\varphi_0}$ form a local coordinate system at $x = a$.

We now prove the lemma. Suppose the contrary. Then there exists $a \in V$ such that $d\theta(b; a) = 0$ for b with $\theta(b) = 0$. Look at the regular points of the zero-set of θ. Take locally a hyperplane $\mathbb{C}^{n-1}$ in $\mathbb{C}^n$ so that a is tangential to $\mathbb{C}^{n-1}$ and the zero-set of θ is the graph of a local holomorphic function on $\mathbb{C}^{n-1}$. Then the partial derivative of the local holomorphic function in the direction of a is zero and we conclude that the zero-set of θ is invariant under translation by a. This would contradict the assumption that, if $v \in V$ and $\frac{\theta(x+v)}{\theta(x)}$ is nowhere zero holomorphic, then $v \in D$, because we can choose $v \in \mathbb{C}a$ not in D. Actually even without this assumption we have a contradiction from the fact that H is nonsingular. The reason is as follows. For any $u \in D$, from

$$\theta(x+u) = \theta(x) \exp\left(\pi\, H(x,u) + \frac{\pi}{2} H(u,u) + 2\pi\sqrt{-1}\, K(u)\right)$$
$$\theta(x+a+u) = \theta(x+a) \exp\left(\pi\, H(x+a,u) + \frac{\pi}{2} H(u,u) + 2\pi\sqrt{-1}\, K(u)\right)$$

it follows that

$$(**) \qquad \frac{\theta(x+a+u)}{\theta(x+u)} = \frac{\theta(x+a)}{\theta(x)} \exp(\pi\, H(a,u)).$$

Since the function $\frac{\theta(x+a)}{\theta(x)}$ is nowhere zero holomorphic, we can take its logarithm. The function $\log \frac{\theta(x+a)}{\theta(x)}$ grows linearly according to (**) and hence is a polynomial $\lambda(x)$ of degree at most one. From (**) we obtain $\pi\, H(a,u) = \lambda(u)$. Thus $H(a,u)$ is $\mathbb{C}$-linear in u. This is possible only if we have $\sum_\alpha a_\alpha h_{\alpha\bar\beta} = 0$ for all β in the equation $H(a,u) = \sum_{\alpha,\beta} h_{\alpha\bar\beta} a_\alpha \overline{u_\beta}$, which contradicts the nonsingularity of H.