The topology and algebra of $SO(n-1) \to SO(n) \to S^{n-1}$

Herman's seminar 7-31-13

jim stasheff

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Outline

The bundle $SO(n-1) \to SO(n) \to S^{n-1}$

For detailed explanations/proofs see Hatcher section 3D p.292+

Special cases

$SO(1)$ is a point.
$SO(2)$, the rotations of R^2, is both homeomorphic and isomorphic as a group to S^1, thought of as the unit complex numbers.
$SO(3)$ is homeomorphic to RP^3
$SO(4)$ is homeomorphic to $S^3 \times SO(3) \simeq S^3 \times RP^3$

Remarkably, in general, $SO(n)$ inherits a cell structure from a product of RP^k's.

The bundle structure

Regard S^n as $e^n \cup e^0$.

Use the geometry to describe a contraction of e^n, e.g. along lines of longitude.

Choose a metric connection for the bundle, then choose the corresponding lift of e^n to get $SO(n) \cup e^n$

Extend to $SO(n) \times e^n$ using the multiplication in $SO(n+1)$.

Alternatively, regard the sphere as the union of two hemispheres e^n_+ and e^n_- and play the same game one each, then notice that over S^{n-1} the two lifts $SO(n) \times S^{n-1} \times \to SO(n+1)$ differ by a transition function given by $S^{n-1} \to SO(n)$.

A cell structure for $SO(n)$ from a product of RP^k's

In particular, there is a map

$$RP^4 \times RP^3 \times RP^2 \times RP^1 \to SO(5)$$

which is the product using the $SO(5)$ group multiplication of individual maps $RP^k \to SO(5)$

We already have $RP^3 = SO(3)$

Recall that the minimal cell structure for RP^k consists of a single cell e^i for each $i \leq k$. Let D^i denote the closed i ball. Then the attaching map $\phi^i : \partial D^i \to RP^{i-1}$ is the 2-sheeted covering.

To simplify notation, we will write P^i for RP^i.

To each nonzero vector $v \in \mathbb{R}^n$, we can associate the reflection $r(v) \in O(n)$ across the hyperplane consisting of all vectors orthogonal to v. Consider the composition $\rho(v) = r(v)r(e_1)$.Since $\rho(v)$ depends only on the line spanned by v, ρ defines a map $P^{n-1} \to SO(n)$. This map is injective since $r(v)$ determined an injection of P^{n-1} into $O(n) - SO(n)$. We may think of ρ as embedding P^{n-1} as a subspace of $SO(n)$. Notice that restriction to the top cell e^{n-1} of P^{n-1} gives an alternate description to be compared to the argument using a metric connection. I leave it to the geometers to tell me if they agree on the nose. What is the relation between this P^{n-1} and a Pontryagin cycle?

More generally, for a sequence $I = (i_1, \cdots, i_m)$ with each $i_j < n$, we define a map $\rho : P^I = P^{i_1} \times \cdots \times P^{i_m} \to SO(n)$ to be the product in $SO(n)$ of the $\rho(v_j)$. The product of the appropriate $\rho(v_j)$ applied to characteristic maps for the top- dimensional cells of the P^{i_j} will give cells $D^I = D^{i_1} \times \cdots \times D^{i_m}$ for $SO(n)$. Of special interest are the sequences $I = (i_1, \cdots, i_m)$ satisfying $n > i_1 > \ > i_m > 0$. These sequences will be called *admissible*, as will the sequence consisting of a single 0.

Theorem

The maps $D^I \to SO(n)$, for I ranging over all admissible sequences, are the characteristic maps of a CW structure on $SO(n)$ for which the map $P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is cellular.

The homology $H_\bullet(SO(n))$ and the cohomology $H^\bullet(SO(n))$

Though I don't know if it is of any use to you, the homology $H_\bullet(SO(n);\mathbb{Z}_2)$ follows easily from the way the cell structure was described and the fact that the attaching map of the top cell in P^i is the 2-sheeted covering.

theorem The Pontryagin ring $H_\bullet(SO(n);\mathbb{Z}_2)$ is the exterior algebra $\Lambda_{\mathbb{Z}_2}[e^1,,e^{n-1}]$. We know that there is at least an additive isomorphism $H_\bullet(SO(n);\mathbb{Z}_2) \equiv \Lambda_{\mathbb{Z}_2}[e^1,\ldots,e^{n-1}]$ since their admissible products form a basis. The inclusion $P^iP^i \subset P^iP^{i-1}$ then implies that the Pontryagin product $(e_i)^2$ is 0. The graded commutativity relation follows from the inclusion $P^iP^j \subset P^jP^i$ for $i < j$.