

A cobordism hypothesis for flagged TQFTs

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What was a TQFT again?

Specific axiomatization of Schrödinger picture QFT
in the topological case:

a d -dim TQFT Z assigns:

$Z(M^d) \in \mathbb{C}$ partition function

closed

$Z(X^{d-1}) \in \text{Vect}_{\mathbb{C}}$ state space

$Z(X^{d-2}) \in \text{Cat}_{\mathbb{C}}$

$\vdots$

$$Z\left(\begin{array}{c} x_2 \\ \text{---} \\ \text{---} \\ x_1 \end{array}\right) : Z(x_1) \rightarrow Z(x_2)$$

$(d-1)\text{-dim} \quad d\text{-dim}$

stacking law: $Z(x_1 \sqcup x_2) \simeq Z(x_1) \otimes Z(x_2)$

gluing law: $Z\left(\begin{array}{c} \text{---} \\ \text{---} \end{array}\right) \circ Z\left(\begin{array}{c} \text{---} \\ \text{---} \end{array}\right) = Z\left(\begin{array}{c} \text{---} \\ \text{---} \end{array}\right)$

Symmetric monoidal functor

$Z: \left(\text{Bord}_n^X, \sqcup \right)_P \rightarrow \mathcal{C}$
 $(\infty, n) \text{-cat}$

Geometry (orientation, spin), tangential structure
 $X = \text{BSpin}(n) \quad X = \text{BSpin}(n) \quad X \rightarrow \text{BO}(n)$
 $\uparrow \quad \nearrow$
 $M \quad TM$

$\uparrow$
 Sym mon $(\infty, n) \text{-cat}$

Example $n=1, \overbrace{X=\text{BO}(1)}^{\text{unoriented}} Z(\text{pt}) \in \mathcal{C} = \text{Vect}_{\mathbb{C}}$

$Z(\quad): \text{pt} \sqcup \text{pt} \rightarrow \emptyset : Z(\text{pt}) \otimes Z(\text{pt}) \rightarrow Z(\emptyset) = 1$

nondeg sym bil form on $Z(\text{pt}) \in \text{F.d.}$
 $\nearrow$

$\mathbb{Z} = \mathbb{Z}/2 \Rightarrow \mathbb{O}(1) - \text{Fixed point datum for}$

$(\cdot)^V \hookrightarrow \text{core}_0(\text{Vect}_{\mathbb{C}})$
 $\text{id TQFTs } (X = \text{BO}(1)) \simeq \text{core}_0(\text{Vect}_{\mathbb{C}})^{\text{hO}(1)}$

Notation: $\text{core}_k \mathcal{C}$ maximal (∞, k) subcategory

Cobordism hypothesis:

1. $\text{TQFTs } (X = *; \mathcal{C}) \simeq \text{core}_0 \mathcal{C}$

Changing the Framing

2. $\text{TQFTs } (X; \mathcal{C}) \simeq (\text{core}_0 \mathcal{C}^{\text{dual}})^X$

from now on all is rigid
 dual

homotopy fixed points for the $\mathcal{O}(n)$ s

Bocn

-action

More generally, $\exists \mathcal{O}(K)$ -action on $\text{core}_{n-k} \mathcal{C}$

if $\mathcal{C}$ is rigid sym mon (∞, n) from $\mathbb{Z}$

$$\text{core}_{n-k} \text{Fun}_{\text{Fro}}^{\text{box}}(\text{Bord}_k^{X=*}, \mathcal{C})$$

Flagged categories

Def a flagged (∞, n) category is

$$\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots \rightarrow \mathcal{C}_n \quad \text{where}$$

• $\mathcal{C}_i$ is (∞, i) -cat

• $\mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ are ess surj on $\leq i$ -mors

Ex 1 $\text{Bord}_0^X \rightarrow \text{Bord}_1^X \rightarrow \text{Bord}_2^X \rightarrow \dots \rightarrow \text{Bord}_n^X$

In Bord_n^X has flagged duals

Ex 2 $\text{core}_0 \mathcal{C} \rightarrow \text{core}_1 \mathcal{C} \rightarrow \dots \rightarrow \text{core}_{n-1} \mathcal{C} \rightarrow \mathcal{C}$

($\mathcal{C}$ (∞, n) -cat sym mon rigid)

flagged duals ✓

Ex 3 • $\text{Alg}(\text{Vect}_{\mathbb{C}})$ is 1-cat of algs & homomorphism

• $\text{Bim}(\text{Vect}_{\mathbb{C}})$ is 2-cat of algs, bimodules & bimodule maps

$$\text{core}_0 \text{Alg}(\text{Vect}_{\mathbb{C}}) \rightarrow \text{core}_1 \text{Bim}(\text{Vect}_{\mathbb{C}}) \rightarrow \text{Bim}(\text{Vect}_{\mathbb{C}})$$

$$\varphi: A \rightarrow B \mapsto B_{\varphi}$$

Flagged duals

Idea: If $C_0 \xrightarrow{\text{ess sur}} C$ is a sym mon Flagged $(0,1)$ -cat,
 $\text{core}_0 C$ has $O(1)$ -action
 provide a lift $x^\vee \in C_0$ of the dual of $x \in C$ to C_0

Informal def The data of Flagged duals on

$$C_0 \rightarrow \dots \rightarrow C_{n-1} \rightarrow C_n \text{ consists of}$$

compatible $O(n-k) \hookrightarrow C_k$ (lifting) $O(n-k) \hookrightarrow \text{core}_k C_n$

Q Does the Flagged n -cat

Ex $\text{core}_0 \text{Alg}(\text{Vect}_\mathbb{C}) \rightarrow \text{Bim}(\text{Vect}_\mathbb{C})$ have Flagged duals?
FAS $\xrightarrow{O(2)} \text{core}_0 \text{Bim}(\text{Vect}_\mathbb{C})$

$O(1)$ part: $A \mapsto A^{\text{op}}$ lifts to $\text{Alg}(\text{Vect}_\mathbb{C})$ ✓
 $M \mapsto M^{\text{op}}$

$SO(2)$ part is problematic: $SO(2) = \mathbb{B}\mathbb{Z} \rightarrow \text{Aut}(\text{core}_0 \text{Bim}(\text{Vect}_\mathbb{C}))$
 $\mathbb{Z} \ni 1 \mapsto (A \mapsto A^{\text{op}})$
 $\text{id}_A \mapsto \text{id}_A$
 Frobenius structure on A with Nakayama φ
 $\mathbb{Z}$ bimodule iso
 $\mathbb{Z}$ $\xrightarrow{\text{Serre aut}} \mathbb{Z}$
 $SO(2)$ -action

Frobenius str on A :

$\lambda: A \rightarrow \mathbb{C}$ s.t. $\langle a, b \rangle := \lambda(ab)$ is non-deg

$$\lambda(ab) = \lambda(e(b)a)$$

Ex $\text{Alg}^{\text{fr}}(\text{Cat}_\mathbb{C}) \rightarrow \text{Bim}^{\text{fr}}(\text{Cat}_\mathbb{C})$

2-cat

Morita 2-cat

Assuming conjectures, can give this flagged $\mathcal{O}(3)$ -duals

Def A Flagged TQFT with target $C_0 \rightarrow \dots \rightarrow C_n$

is a $\text{Bord}_0^{X_0} \rightarrow \text{Bord}_1^{X_1} \rightarrow \dots \rightarrow \text{Bord}_n^{X_n}$

$\downarrow \mathcal{O}(n) \quad \downarrow \mathcal{O}(n-1) \quad \downarrow$
 $C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n$

Explicitly, i -dim bordisms $\Rightarrow i$ -morphisms in C_i
 $i \leq n$

i -dim mapping cyl $\Rightarrow i$ -mers in C_{i-1}

$\mathcal{O}(K)$ -equiv: rotating extra K directions in a K -fold mapping cylinder becomes $\mathcal{O}(K)$ -action on C_{i-K}

Th (Flagged cob hyp)

$\{ \text{Flagged TQFTs } \text{Bord}_\bullet^{X_\bullet} \rightarrow C_\bullet \} \simeq C_\bullet$

$C_n^{X_n}$

Cor 2d ~~flagged~~ oriented TQFTs = Sym Froh algs up to Morita

3d _____ = Spherical fusion cat

$\mathcal{N}_{\leq 2} \mathcal{O}(3) = \mathbb{Z}_2 \times B\mathbb{Z}_2$