Strong quasipositivity, Thurston-Bennequin invariants, and arc index

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(partly) joint w/
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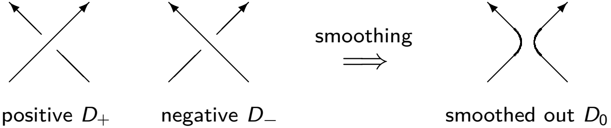
1. Introduction, Definitions, etc.

Link/Knot Diagrams

All link diagrams and links are assumed oriented.

Crossings in an oriented diagram D of a knot K are called



(1)

$c_\pm(D)$: number of positive, resp. negative crossings of D

$c(D) = c_+(D) + c_-(D)$: number of crossings in D.

$w(D) = c_+(D) - c_-(D)$: writhe of D.

$s(D)$: number of circles after smoothing all crossings of D.
(Such circles are called Seifert circles.)

$c(K)$: crossing number of K

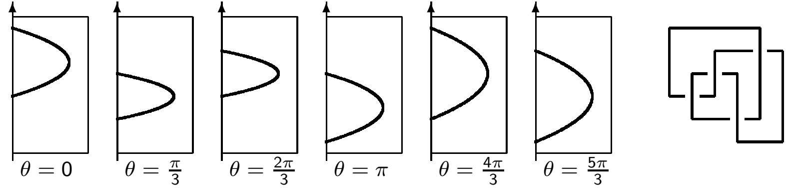
$$c(K) = \min\{c(D) \mid D \text{ is a diagram of } K\}$$

Definition 1.1

$!D, !K$ mirror image. *If* $K = !K$, *then* K *is* achiral *(or amphicheiral).*

Arc presentation and arc index

An **arc presentation** of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called **pages**, with a common boundary line in such a way that each half plane contains a properly embedded single arc.



The minimal number of pages among all arc presentations of a link L is called the **arc index** of L and is denoted by $\boldsymbol{a(L)}$.

Remark 1.2

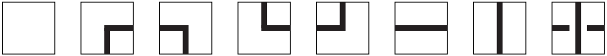
$$a(L_1 \# L_2) = a(L_1) + a(L_2) - 2$$

(Cromwell; at least for knots $L_{1,2}$ *also follows from results of Torisu/Etnyre-Honda and Dynnikov-Prasolov on the maximal Thurston-Bennequin invariant; below.)*

Grid diagram and arc presentation

A **grid diagram** is a knot diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges.

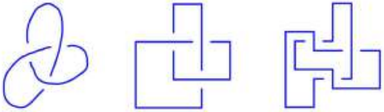
Up to adjustment, grid diagram $\iff$
mosaic that can be composed in the plane by the tiles



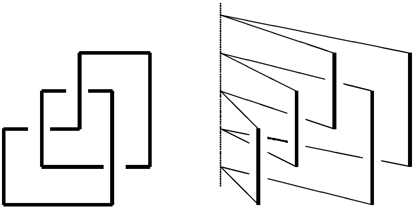
(2)

(à la Kauffman-Lomonaco, 和靜-. . .)

It is not hard to see that every knot admits a grid diagram.

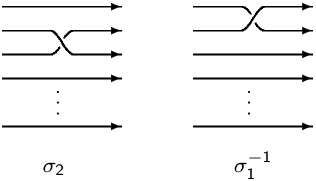


The figure below explains that arc presentations and grid diagrams correspond one-to-one.



Braids and the Artin Generators $\sigma_1, \ldots, \sigma_{n-1}$

For $n \geq 2$, the **braid group** B_n has **Artin generators** $\sigma_1, \ldots, \sigma_{n-1}$,



and relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, & |i-j| \geq 2. \end{aligned}$$

Closure $\hat{\beta} = L$ for $\beta \in B_n$ can be regarded as a diagram of L, for a *word* of β in σ_i.
Then $w(\hat{\beta}) =$ *exponent sum* of β, and $s(\hat{\beta}) = n$ number of strings.

Quasipositive and Strongly Quasipositive Braids/Links

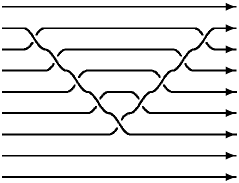
Definition 1.3

A a braid $\beta \in B_n$ and its closure $\hat{\beta} = L$ are called ***quasipositive*** *(Rudolph, Boileau-Orevkov, etc.) if β is of the form*

$$\beta = \prod_{k=1}^{l} w_k \sigma_{i_k} w_k^{-1} \, , \tag{3}$$

which is product of conjugates of σ_i's (but not *their inverses).*
A braid β and its closure $\hat{\beta} = L$ are called ***strongly quasipositive*** *if β is of the form as in (3) with*

$$w_k \sigma_{i_k} w_k^{-1} = \sigma_i \ldots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \ldots \sigma_i^{-1} = \sigma_{i,j} \tag{4}$$



$$= \sigma_{2,7} \in B_9$$

Positive Links

Definition 1.4

A link diagram D is **positive** *if $c_-(D) = 0$ (i.e., $w(D) = c(D)$). A link L is positive if it has a positive diagram.*

It is known that

$$\{\text{quasipositive}\} \supset \{\text{strongly quasipositive}\} \overset{\substack{\text{Rudolph,}\\ \text{Nakamura}}}{\supset} \{\text{positive}\}\,,$$

and the inclusions are proper.

Example 1.5

Torus links $T_{p,n} = T(p,n) =$ *closure of* $(\sigma_1 \cdots \sigma_{p-1})^n \in B_p$ *(see later §5) are positive.*

We also say D, L are *negative*, if $!D, !L$ are positive.

Band Representation of Braids

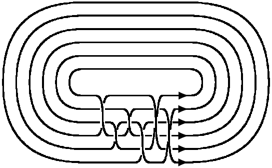
Notice that $\sigma_{i,i+1} = \sigma_i$. B_n is generated by $\sigma_{i,j}, 1 \le i < j \le n$, which are called the **band generators**.

A braid representative

$$\beta = \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1} \tag{5}$$

is called a **band representation** (Birman-Ko-Lee).The closure $L = \hat{\beta}$ of a band representation β naturally spans a Seifert surface.

Example of the 6-braid $\beta = \sigma_{1,4}\sigma_{3,5}\sigma_{2,4}\sigma_{3,6}\sigma_{1,5}\sigma_{2,6}$.



(6)

A word (5) is a **band representation** and its Seifert surface is a **braided surface** (Rudolph '83) of $L = \hat{\beta}$. A minimal genus braided surface is called **Bennequin surface** (Birman-Menasco '91).

More specifically, a braid of the form

$$\beta = \prod_{k=1}^{l} \sigma_{i_k, j_k} \tag{7}$$

is a **strongly quasipositive braid (word)** and a **positive band representation** of β. It gives a **strongly quasipositive surface** of $L = \hat{\beta}$ (like in (6)).

Bennequin's inequality

Theorem 1.6 (Bennequin '84)

For $\hat{\beta} = L$*, we have* $-\chi(L) \geq w(\hat{\beta}) - s(\hat{\beta})$

Corollary 1.7

Every strongly quasipositive surface is a Bennequin surface.

Definition 1.8

L *is* Bennequin-sharp *if*

$$-\chi(L) \;=\; \max_{\hat{\beta}=L} w(\hat{\beta}) - s(\hat{\beta})$$

Corollary 1.9

L *is strongly quasipositive* $\Longrightarrow$ L *is Bennequin-sharp.*

Problem 1.10 (Bennequin sharpness problem)

L *is strongly quasipositive* $\iff$ L *is Bennequin-sharp.*

Braid Indices

Definition 1.11

- *Let* $\boldsymbol{b(L)}$ *be the* ***braid index*** *of* L*, the minimal number of strings of a braid representative of* L*.*
- *Let* $\boldsymbol{b_b(L)}$ *be the* ***Bennequin braid index*** *of* L*, the minimal number of strings to span a Bennequin surface of* L*.*
- *When* L *is strongly quasipositive, let* $\boldsymbol{b_{sqp}(L)}$ *be the minimal number of strings to span a strongly quasipositive surface of* L*.*

$$b_{sqp}(L) \geq b_b(L) \geq b(L)$$

Example 1.12 (Hirasawa-S. '03)

$b_b(K) > b(K)$ *can occur.*

Question 1.13 (Rudolph)

Is $b_{sqp}(L) = b(L)$ *for every strongly quasipositive link* L*?*

It is true, e.g., for all prime s.q.p. knots up to 16 crossings.

HOMFLY-PT polynomial

$$P\ :\ \{\text{oriented links}\} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

Convention (Morton), in (1) (and with $\bigcirc$ = unknot):

$$v^{-1}P(D_+) - vP(D_-) = zP(D_0)\,, \qquad P(\bigcirc) = 1\,. \qquad (8)$$

Powers of v, z are always odd/even for even/odd components of L.

Question 1.14 (Jones '85)

$P(K) = 1 \implies K = \bigcirc$?

Example 1.15

If a knot K is quasipositive and achiral ($K = !K$) then $P(K) = 1 \overset{Qu.\ 1.14}{\Longrightarrow} K = \bigcirc$.

Theorem 1.16 (Morton-Franks-Williams)

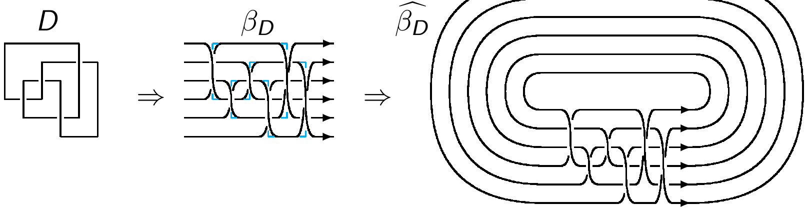
$b(L) \geq 1 + \frac{1}{2}\,\mathrm{span}_{\,v} P(L)\,.$

2. grid diagram $\Longleftrightarrow$ band representation

Definition 2.1

For a knot K and an integer t, let $\boldsymbol{A(K,t)}$ be the boundary of the t-framed K-knotted annulus. (Convention: $t =$ linking number.)

Nutt's construction: Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band representation, denoted by β_D. Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D} = A(K, \lambda(D))$ for some integer $\lambda(D)$.



Definition 2.2

The ***Thurston-Bennequin invariant*** *of a grid diagram D is defined by the formula*

$$TB(D) = w(D) - \#(\text{NW-corners}(D))$$

Lemma 2.3

For any grid diagram D of K,

$$\lambda(D) = -TB(D)\,. \tag{9}$$

Let $TB(K)$ be the **maximal Thurston-Bennequin invariant** of K and

$$\lambda(K) := \min\{\ t \mid A(K,t) \text{ is strongly quasipositive }\}$$

.

Corollary 2.4 (Rudolph)

(For $K \neq \bigcirc$*)* $\lambda(K) = -TB(K)$ *and*
$A(K, t)$ *is strongly quasipositive* $\iff \quad t \geq -TB(K)$.

Remark 2.5

$\lambda(\bigcirc) = 0$ *but* $-TB(\bigcirc) = 1$. *(But the unknot can be easily dealt with everywhere below on an* ad hoc *basis.)*

Theorem 2.6 (Dynnikov-Prasolov)

If D *is a minimal (i.e., size* $a(K)$*) grid diagram of* $K(\neq \bigcirc)$*, then* $\lambda(D) = \lambda(K)$.

Corollary 2.7

$\lambda(K) + \lambda(!K) = a(K)$. *(* $! =$ *mirror image)*

Proposition 2.8

$$\min\{b(A(K,t)) \mid t \geq \lambda(K)\} \leq b(A(K, \lambda(K))) \leq a(K)$$

Conjecture 2.9

$$\min\{b(A(K,t)) \mid t \in \mathbb{Z}\} = \min\{b(A(K,t)) \mid t \geq \lambda(K)\} = a(K)$$

The braid index of a link is obviously not less than the sum of the braid indices of constituent components.

Corollary 2.10 (Cromwell '95)

$$2b(K) \leq a(K).$$

It is known that $a(K) \leq c(K) + 2$ (Bae-C Park '00) and $a(K) \leq c(K)$ for K non-alternating (Jin- W Park '10).

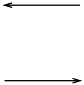
Corollary 2.11

We have $b(K) \leq c(K)/2 + 1$ *(Ohyama '93).*
If K *is non-alternating, then* $b(K) \leq c(K)/2$.

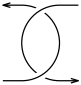
3. (Strong) quasipositivity of Whitehead doubles

Definition 3.1

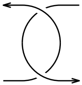
Whitehead double $\boldsymbol{W_\pm(K,t)}$ *of a knot* K *with framing* t *and positive/negative clasp.*



$A(K,t)$

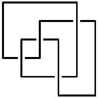


$W_-(K,t)$

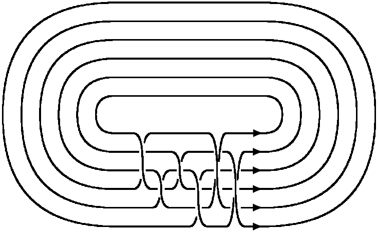


$W_+(K,t)$

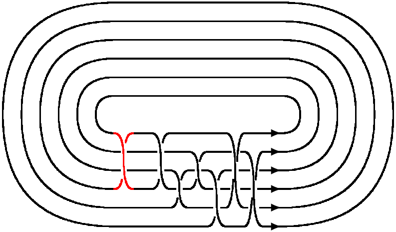
Example 3.2



4_1



$A(4_1,3)$



$W_+(4_1,3)$

Question 3.3

When is $W_{\pm}(K,t)$ strongly quasipositive?

We use Corollary 2.4 and Rudolph's result on Murasugi sum to prove

Theorem 3.4

$W_{-}(K,t)$ is never strongly quasipositive.
$W_{+}(K,t)$ is strongly quasipositive $\iff t \geq -TB(K)$.

Quasipositivity

We have no complete decision, but general results.

Theorem 3.5 (S.)

For K alternating, each $W_+(K,t)$ is quasipositive $\iff$ it is strongly quasipositive.

Proof uses, *i.a.*, ℓ-sharpness (below), Casson-Gordon-Gilmer (with thanks to Chuck Livingston), and Hedden's evaluation of the jump number for the τ invariant.

Example 3.6 (S.-Orevkov)

$W_+(K,0)$ *can be quasipositive but not strongly quasipositive for some (non-alternating) knots K:* pretzel knots $K = \mathcal{P}(3,3,-3) = 9_{46}$ *and* $K = \mathcal{P}(4,3,-3) = 10_{140}$.

We do not know if any $W_-(K,t)$, except

$$K = \bigcirc\,, \quad t = 0 \tag{10}$$

can be quasipositive. But:

Theorem 3.7 (S.)

(a) For every knot K, there is at most one t so that $W_-(K,t)$ is quasipositive.

(b) If $K \neq \bigcirc$ is alternating, negative, or quasipositive and not slice ($\Longleftarrow$ positive), then $\nexists\, t$.

(c) For any composite K with an alternating or positive knot factor, $\nexists\, t$.

(d) If K is achiral and $W_-(K,t)$ is quasipositive, then $t = 0$ and $P(W_-(K,t)) = 1$ ($P=$ HOMFLY-PT polynomial) $\overset{Qu.\ 1.14}{\Longrightarrow}$ (10).

Computation 3.8

For which "low" (≤ 16) crossing knots K can $W_{-}(K,t)$ be q.p.?

- *"fast-track" Kauffman* mod 2 *test: excludes composite knots K and all prime knots except ≈ 2200*
- *Computing $P(W_{-}(K,t))$ (for some t) reduces this list to 69 knots K, and $t = 0$ for all of them.*
- *They are all slice, so "slice-torus" obstructions fail.*
- *(z-)truncated 2-cable HOMFLY of $W_{-}(K,t)$ (it is a degree-4 satellite of K!) via the "needle" skein algorithm excluded 51 and reduces the list to 18.*

We are trying now:

(a) *More terms via the needle skein algorithm (unpromising).*

(b) *Mironov-Morozov-Singh universal formula: 5 are pretzel $K = \mathcal{P}(k, 3, -3)$ ($k = 6, \ldots, 10$), and 2 more are length-3-Montesinos.*

(c) *a new "braid-stack-delooping-move" algorithm that allows v-degree truncations*

Jump numbers

We can obtain the Livingston-Naik ('06) result for jump numbers of slice-torus invariants.

Application to Bennequin-sharpness Problem 1.10:

Corollary 3.9

When K is a positive knot, then every $W_\pm(K,t)$ is strongly quasipositive $\iff$ it is Bennequin-sharp

Remark 3.10

This is also true for alternating K, but for (somewhat) different reasons (Proposition 4.9). These later arguments apply for $A(K,t)$ as well, which does not work here because the "slice-torus" business restricts to knots.

4. HOMFLY-PT polynomial and ℓ-invariant

Assume convention (8).

Definition 4.1 (width of “the pan”; trim the panhandle)

K knot.

$$\ell(K) := 1 + \frac{1}{2} \operatorname{span}_v P(A(K,t))$$

when t is chosen so that

$$\min \deg_v P(A(K,t)) < 0 < \max \deg_v P(A(K,t))$$

(Keep in mind: powers of v, z are always odd/even for even/odd components.)

Theorem 4.2

We have a lower bound

$$\ell(K) \leq a(K)\,. \tag{11}$$

Question 4.3

Is $\ell(K_1\#K_2)=\ell(K_1)+\ell(K_2)-2$*? (Compare with Remark 1.2.)*

Considering the Morton-Beltrami lower bound from the Kauffman polynomial $F(a,z)$:

$$\mathrm{MB}\,(K) := \mathrm{span}_a\, F(K) + 2 \leq a(K)\,, \tag{12}$$

we *conjecture* that ℓ is better.

Conjecture 4.4

$$\mathrm{MB}\,(K) \leq \ell(K)\,. \tag{13}$$

Conjecture 4.4 is true for

- prime knots K up to 16 crossings,
- torus knots (Section 5),
- alternating knots (as an equality; in a moment).

Definition 4.5

K is ℓ-sharp *if $\ell(K) = a(K)$.*

For prime knots K up to 12 crossings, K is ℓ-sharp except for 10_{132}, four 11 crossing knots 11_{379}, 11_{424}, 11_{455}, 11_{459}, and 21 further examples of 12 crossings.

For an *alternating* knot K, the inequality (12) is exact, so the following "fits" with Conjecture 4.4.

Proposition 4.6 (Diao-Morton(-...) '24)

Alternating knots are ℓ-sharp.

This motivates us to study ℓ-sharpness.
Among applications, we have our generalization of Diao-Morton:

Proposition 4.7

Assume K is a non-trivial ℓ-sharp knot. Then

$$b(A(K,t)) = \begin{cases} \lambda(K) - t & \text{if } t \leq \lambda(K) - a(K) \\ a(K) & \text{if } \lambda(K) - a(K) \leq t \leq \lambda(K) \\ t - \lambda(K) + a(K) & \text{if } t \geq \lambda(K) \end{cases}$$

Corollary 4.8

If K is ℓ-sharp, then Conjecture 2.9 holds for K.

Proposition 4.9

If K is ℓ-sharp, then Bennequin sharpness Problem 1.10 and Question 1.13 are resolved affirmatively for all $L = A(K,t)$ and $W_\pm(K,t)$.

Also $b_b(L) = b(L)$, except possibly *one* t for $L = W_-(K,t)$.
Difficult examples: Computation 3.8, incl. $W_-(\mathcal{P}(k,3,-3),0) \ldots$
But: can be avoided when K is alternating.

TB invariant estimate

For $\lambda(K)$, we have (Fuchs-Tabachnikov, Ferrand, T. Tanaka, ...)

$$\lambda(K) \geq \mathrm{FT}\,(K) := 1 - \min \deg_a F(K) \quad \begin{pmatrix} \text{also fixes our mirroring} \\ \text{convention for } F \end{pmatrix} \tag{14}$$

Note: compatibly with Corollary 2.7,

$$\mathrm{FT}\,(K) + \mathrm{FT}\,(!K) = \mathrm{MB}\,(K)\,.$$

Proposition 4.10 ("Frying eggs in the pan")

We can determine a number $\theta(K)$ *from* $P(A(K,t))$ *so that*

$$\theta(K) \leq \lambda(K) \leq \theta(K) + a(K) - \ell(K)\,, \qquad \theta(K) + \theta(!K) = \ell(K)$$

ℓ-invariant version of (14);
in particular $\theta(K) = \lambda(K)$ when K is ℓ-sharp.

The following refines Conjecture 4.4:

Conjecture 4.11

$$\mathrm{FT}\,(K) \leq \theta(K)\,.$$

So that we have (with purple being expected):

$$\begin{array}{ccccc}
\lambda(K) & + & \lambda(!K) & = & a(K) \\
\geq & & \geq & & \geq \\
\theta(K) & + & \theta(!K) & = & \ell(K) \\
\geq & & \geq & & \geq \\
\mathrm{FT}\,(K) & + & \mathrm{FT}\,(!K) & = & \mathrm{MB}\,(K)
\end{array}$$

Conjecture 4.11 is true for

- prime knots K up to 16 crossings,
- mirrored (i.e., negative) torus knots,
- positive (incl. torus) knots (as an equality),
- alternating knots (as an equality).

Further methods and computations

We can further use (truncated!) degree-3-satellite HOMFLY and 2-cable Kauffman polynomials to get practically always sharp lower bounds on $a(K)$, $\lambda(K)$, ...

Arc indices and maximal TB invariants for 13 crossing knots are now available on KnotInfo (*new site:* `https://knotinfo.org`) and for 14 crossings on my website.
`http://www.stoimenov.net/stoimeno/homepage/ptab/index.html`

(12) seems to perform well when $a(K)$ is large compared to $c(K)$. It is always exact in the case $a(K) = c(K) + 2$ ($\iff$ K is alternating). So far there is no example answering negatively this

Question 4.12

$$c(K) = a(K) \implies (12) \textit{ is exact } (\overset{\text{Conj. 4.4}}{\Longrightarrow} K \textit{ is } \ell\textit{-sharp})?$$

5. Torus knots

[This section is joint with H. Sati & V.Singh, NYU Abu Dhabi, and A. Mironov, Lebedev Institute and Moscow, ITEP.]

Recall: $T_{p,n} = T(p,n) =$ closure of $(\sigma_1 \cdots \sigma_{p-1})^n$ (positive)

Assumption: $(p,n) = 1$ (*knot* case; torus *links* are *slightly* more complicated. . .), $n > p(> 1)$.

Theorem 5.1 (Etnyre-Honda)

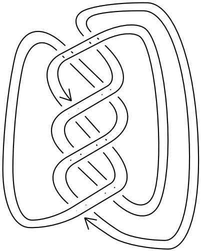
$a(T(p,n)) = p + n$

Theirs was the only proof, geometric (and '$\leq$' is "easy").
Goal: is there an algebraic proof using our approach?

For $K = T_{p,n}$, fix framing: $t = -(p-1)n$

Example 5.2

$p = 3,\ n = 4$



Let $X = [m; M]_v$ mean that $\min \deg_v X = m$ and $\max \deg_v X = M$.

Theorem 5.3 ("Panhandle Theorem" of Mironov-Sati-Singh-S.)

$$P(A(T_{p,n}, (1-p)n)) = [1-2p; 2p-1]_v + \underbrace{(1-p)zv\frac{v^{2n} - v^{2p}}{v^2 - 1}}_{\textit{panhandle}}$$

Example 5.4

$p = 2$, $n = 5$ (v-powers L-R, z-powers U-D)

20	1	-1	9						
	-3	3	9	-21	16	-4			
	-3	9	24	-71	50	-5	-1	-1	-1
	-3	3	22	-84	63	-1			
	-3	1	8	-45	37				
	-3	1	1	-11	10				
	-1	1		-1	1				

Remark 5.5

This is a different panhandle from the one in the definition of $\ell(K)$ (which occurs for arbitrary K when t is very large or very small).

Corollary 5.6

$$\ell(T(p,n)) = a(T(p,n)) = p + n\,, \tag{15}$$

i.e., $T_{p,n}$ is ℓ-sharp. (Compare with (16).)

Remark 5.7

Formula for $F(T(p,n))$ (Yokota '93, Labastida-Perez '95):

$$\Longrightarrow \quad \mathrm{MB}\,(T(p,n)) = \begin{cases} 2p & p \text{ odd} \\ p+n & p \text{ even} \end{cases} \tag{16}$$

Corollary 5.8

$\min \deg_a F(T_{p,n}) = (p-1)(n-1)$ + *Tanaka ((14) is exact)* $\Longrightarrow$
$\lambda(T_{p,n}) = -pn + p + n \overset{(15),\ Corr.\ 2.7}{\Longrightarrow} \lambda(!T_{p,n}) = pn$ *(Etnyre-Honda)*

Then we get all geometric applications we wanted, for example (but not only):

Corollary 5.9

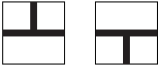
Let $K = T_{p,n}$ or $K =!T_{p,n}$. And let $L = A(K,t)$ or $L = W_{\pm}(K,t)$ (for some $t \in \mathbb{Z}$). Then

$$L \text{ is quasipositive} \iff L \text{ is strongly quasipositive} \iff$$
$$\iff L \text{ is Bennequin-sharp}$$

6. Open projects

Problem 6.1

- *Define ℓ-invariant for links (work in progress, based on torus links. . .)*
- *More general ("grid-graph") theory for braided surfaces of lower χ.*
  Grid-graph is PL spatial embedding of a trivalent graph whose diagram can be built up with the tiles in (2), *and the two extra tiles*



but not