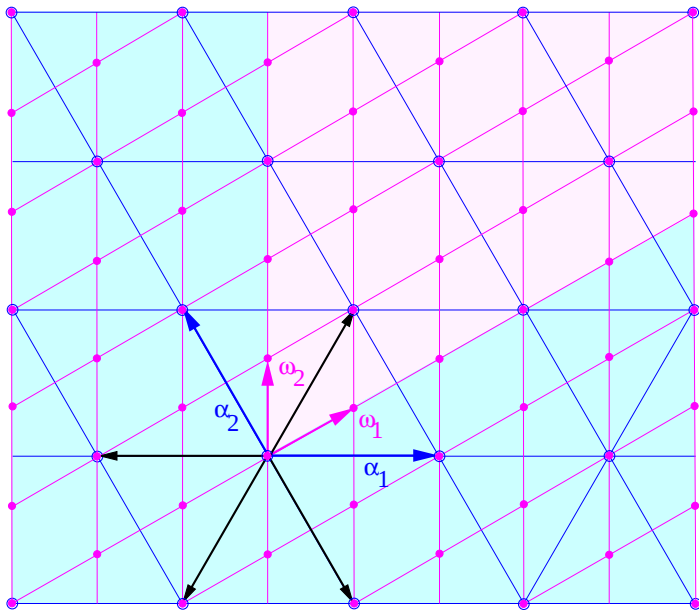




Berry Curvature, Spin, and Anomalous Velocity

Michael Stone

Institute for Condensed Matter Theory
University of Illinois



Talk based on:

Motivation

M.A.Stephanov, Y.Yin, *Chiral Kinetic Theory*, Phys. Rev. Lett. **109** 162001 (2012).

Our Work

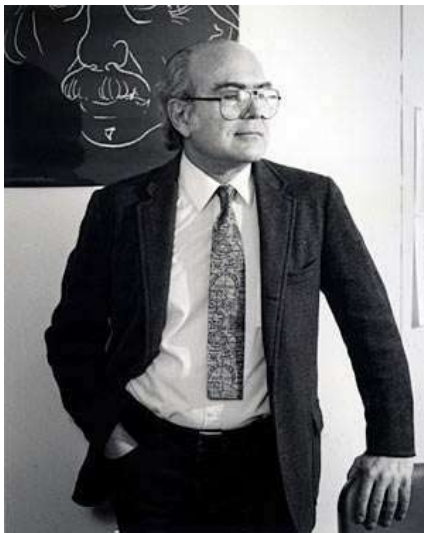
MS, V.Dwivedi, *A Classical Version of the Non-Abelian Gauge Anomaly* Phys. Rev. **D88** 045012 (2013).

V.Dwivedi, MS, *Classical chiral kinetic theory and anomalies in even space-time dimensions*, J. Phys. A **47** 025401 (2014).

MS, V.Dwivedi, T.Zhou, *Berry Phase, Lorentz Covariance, and Anomalous Velocity for Dirac and Weyl Particles*, arXiv:1406.0354

Also important

J.Y.Chen, D.T.Son, M.A.Stephanov, H.U.Yee, Y.Yin, *Lorentz Invariance in Chiral Kinetic Theory*, arXiv:1404.5963



Bruno Zumino 1923-2014

Outline

- 1 Covariant Berry Connection
 - Anomalous Velocity
 - WKB and Berry
 - Berry, Thomas, and Pauli-Lubanski
- 2 Relativistic Mechanics of Spinning Particles
 - Mathisson-Papatrou-Dixon equations
 - Anomalous velocity
 - Meaning of Conditions on Spin Tensor
- 3 Massless Case
 - A Gauge Invariance?
 - Wigner Translations
 - Physical Meaning of Wigner Translations
- 4 Conclusions

Outline

- 1 Covariant Berry Connection
 - Anomalous Velocity
 - WKB and Berry
 - Berry, Thomas, and Pauli-Lubanski
- 2 Relativistic Mechanics of Spinning Particles
 - Mathisson-Papatrou-Dixon equations
 - Anomalous velocity
 - Meaning of Conditions on Spin Tensor
- 3 Massless Case
 - A Gauge Invariance?
 - Wigner Translations
 - Physical Meaning of Wigner Translations
- 4 Conclusions

Anomalous Velocity

Luttinger, Blount, Niu, and others show that a Berry phase in the equations of motion of a Bloch quasiparticle $\Rightarrow$ **anomalous velocity**:

$$\begin{aligned}\dot{\mathbf{k}} &= -\frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{x}} + e(\dot{\mathbf{x}} \times \mathbf{B}), \\ \dot{\mathbf{x}} &= \frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{k}} - (\dot{\mathbf{k}} \times \boldsymbol{\Omega}).\end{aligned}$$

Anomalous Velocity

Luttinger, Blount, Niu, and others show that a Berry phase in the equations of motion of a Bloch quasiparticle $\Rightarrow$ **anomalous velocity**:

$$\begin{aligned}\dot{\mathbf{k}} &= -\frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{x}} + e(\dot{\mathbf{x}} \times \mathbf{B}), \\ \dot{\mathbf{x}} &= \frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{k}} - (\dot{\mathbf{k}} \times \boldsymbol{\Omega}).\end{aligned}$$

- Many applications!
- Want to use for Dirac and Weyl particles
- Can we make these equations covariant?

Anomalous Velocity

Luttinger, Blount, Niu, and others show that a Berry phase in the equations of motion of a Bloch quasiparticle $\Rightarrow$ **anomalous velocity**:

$$\begin{aligned}\dot{\mathbf{k}} &= -\frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{x}} + e(\dot{\mathbf{x}} \times \mathbf{B}), \\ \dot{\mathbf{x}} &= \frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{k}} - (\dot{\mathbf{k}} \times \boldsymbol{\Omega}).\end{aligned}$$

- Many applications!
- Want to use for Dirac and Weyl particles
- Can we make these equations covariant?

$$\dot{\mathbf{k}} = e(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \rightarrow \dot{k}_\mu = eF_{\mu\nu}\dot{x}^\nu, \quad \mu = 0, 1, 2, 3. \quad \checkmark$$

Anomalous Velocity

Luttinger, Blount, Niu, and others show that a Berry phase in the equations of motion of a Bloch quasiparticle $\Rightarrow$ **anomalous velocity**:

$$\begin{aligned}\dot{\mathbf{k}} &= -\frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{x}} + e(\dot{\mathbf{x}} \times \mathbf{B}), \\ \dot{\mathbf{x}} &= \frac{\partial \varepsilon(\mathbf{k}, \mathbf{x})}{\partial \mathbf{k}} - (\dot{\mathbf{k}} \times \boldsymbol{\Omega}).\end{aligned}$$

- Many applications!
- Want to use for Dirac and Weyl particles
- Can we make these equations covariant?

$$\dot{\mathbf{k}} = e(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \rightarrow \dot{k}_\mu = eF_{\mu\nu}\dot{x}^\nu, \quad \mu = 0, 1, 2, 3. \quad \checkmark$$

$$\dot{\mathbf{x}} = \mathbf{v}_\varepsilon - (\dot{\mathbf{k}} \times \boldsymbol{\Omega}) \rightarrow \dot{x}_i = v_{i,\varepsilon} + \Omega_{ij}\dot{k}^j, \quad i = 1, 2, 3. \quad ?$$

Covariant WKB for Dirac

Look for WKB solution of Dirac equation

$$(i\hbar\gamma^\mu(\partial_\mu + ieA_\mu/\hbar) - m)\psi = 0.$$

as

$$\psi(x) = a(x)e^{-i\varphi(x)/\hbar}, \quad a = a_0 + \hbar a_1 + \hbar^2 a_2 + \dots,$$

where

$$a_0(x) = u_\alpha(k(x))C^\alpha(x)$$

and $u_\alpha(k)$ (and later $v_\alpha(k)$) are solutions to

$$(\gamma^\mu k_\mu - m)u_\alpha(k) = 0$$

$$(\gamma_\mu k^\mu + m)v_\alpha(k) = 0$$

covariantly normalized so that

$$\bar{u}_\alpha u_\beta = \delta_{\alpha\beta} = -\bar{v}_\alpha v_\beta$$

Spin Transport Equation

Plug WKB solution into Dirac. Find that

$$\left[\delta_{\alpha\beta} \left(V^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{2} \frac{\partial V^\mu}{\partial x^\mu} \right) + \frac{ie}{2m} S_{\alpha\beta}^{\mu\nu} F_{\mu\nu} - i \mathbf{a}_{\alpha\beta, \nu} \dot{k}^\nu \right] C^\beta(x) = 0.$$

where

$$\frac{ie}{2m} S_{\alpha\beta}^{\mu\nu} F_{\mu\nu}$$

gives Larmor precession, and

$$\mathbf{a}_{\alpha\beta, \nu} = i \bar{u}_\alpha \frac{\partial u_\beta}{\partial k^\nu}, \quad \nu = 0, 1, 2, 3$$

is an unconventional, but **covariant** Berry connection.

Covariant Berry Curvature

Matrix-valued connection form

$$\mathbf{a}_{\alpha\beta,\nu} dk^\nu = i \bar{u}_\alpha \frac{\partial u_\beta}{\partial k^\nu} dk^\nu.$$

Curvature form

$$\mathfrak{F} = d\mathbf{a} - i\mathbf{a}^2.$$

Use Dirac equation to find

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2} (S_{\mu\nu})_{\alpha\beta} dk^\mu \wedge dk^\nu,$$

where

$$(S_{\mu\nu})_{\alpha\beta} = \bar{u}_\alpha \left(\frac{i}{4} [\gamma_\mu, \gamma_\nu] \right) u_\beta = i \bar{u}_\alpha \sigma_{\mu\nu} u_\beta.$$

Note that Dirac $\Rightarrow k^\mu S_{\mu\nu} = 0$.

Pauli-Lubanski Tensor

Use mass-shell condition $E^2 \equiv k_0^2 = \mathbf{k}^2 + m^2$ to eliminate k_0 and find that

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2} \left(S_{ij} - \frac{k_i}{E} S_{0j} - S_{i0} \frac{k_j}{E} \right)_{\alpha\beta} dk^i \wedge dk^j,$$

Pauli-Lubanski Tensor

Use mass-shell condition $E^2 \equiv k_0^2 = \mathbf{k}^2 + m^2$ to eliminate k_0 and find that

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2} \left(S_{ij} - \frac{k_i}{E} S_{0j} - S_{i0} \frac{k_j}{E} \right)_{\alpha\beta} dk^i \wedge dk^j,$$

Expression in parentheses is a skew-symmetric tensor generalization of the
Pauli-Lubanski vector

Berry *versus* Llewellyn Thomas

Explicitly, in 3+1 dimensions we have

$$\mathfrak{F} = \frac{1}{2m^2\gamma} \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{(\mathbf{k} \cdot \boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)} \right) \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

Berry *versus* Llewellyn Thomas

Explicitly, in 3+1 dimensions we have

$$\mathfrak{F} = \frac{1}{2m^2\gamma} \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{(\mathbf{k} \cdot \boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)} \right) \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

What does this mean this physically?

Berry *versus* Llewellyn Thomas

Explicitly, in 3+1 dimensions we have

$$\mathfrak{F} = \frac{1}{2m^2\gamma} \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{(\mathbf{k} \cdot \boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)} \right) \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

What does this mean this physically? Look at connection

$$\begin{aligned} a_{\alpha\beta,i} \dot{k}^i &= \frac{1}{m^2(1+\gamma)} (\mathbf{k} \times \dot{\mathbf{k}}) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta} \\ &= \frac{\gamma^2}{1+\gamma} (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta}, \quad \boldsymbol{\beta} = \mathbf{k}/E = \mathbf{k}/m\gamma \\ &= -\boldsymbol{\omega}_{\text{Thomas}} \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta}. \end{aligned}$$

Berry *versus* Llewellyn Thomas

Explicitly, in 3+1 dimensions we have

$$\mathfrak{F} = \frac{1}{2m^2\gamma} \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{(\mathbf{k} \cdot \boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)} \right) \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

What does this mean this physically? Look at connection

$$\begin{aligned} a_{\alpha\beta,i} \dot{k}^i &= \frac{1}{m^2(1+\gamma)} (\mathbf{k} \times \dot{\mathbf{k}}) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta} \\ &= \frac{\gamma^2}{1+\gamma} (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta}, \quad \boldsymbol{\beta} = \mathbf{k}/E = \mathbf{k}/m\gamma \\ &= -\boldsymbol{\omega}_{\text{Thomas}} \cdot \left(\frac{\boldsymbol{\sigma}}{2} \right)_{\alpha\beta}. \end{aligned}$$

Covariant Berry-transport is Thomas precession

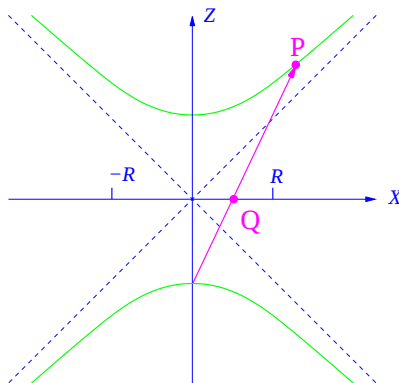
Nishina, Thomas, Hund



Yoshio Nishina, Llewellyn Thomas, Friedrich Hund

Thomas *versus* Lobachevsky

Thomas precession is parallel transport on the positive-energy mass-shell:



Embedding of three-dimensional Lobachevsky space into four-dimensional Minkowski space. The arrow shows the stereographic parametrization of the embedded space by the Poincaré ball $x_1^2 + x_2^2 + x_3^2 < R^2$.

Non-covariant WKB

With $u_\alpha^\dagger u_\beta = \delta_{\alpha\beta} = v_\alpha^\dagger v_\beta$, have

$$\left\{ \delta_{\alpha\beta} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{1}{2} \operatorname{div} \mathbf{v} \right) + N_{\alpha\beta} \right\} C^\beta(\mathbf{x}, t) = 0,$$

with

$$N_{\alpha\beta} = -i \left(\frac{e}{m\gamma^2} \right) \mathbf{B} \cdot \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{1}{m^2} \frac{(\mathbf{k} \cdot \boldsymbol{\sigma}) \mathbf{k}}{\gamma + 1} \right)_{\alpha\beta} \right\} - i \mathcal{A}_{\alpha\beta, i} \dot{k}^i,$$

$$\mathcal{A}_{\alpha\beta, i} = i u_\alpha^\dagger \frac{\partial u_\beta}{\partial k^i}, \quad i = 1, 2, 3$$

and

$$\mathcal{F}_{\alpha\beta} = -\frac{1}{2m^2\gamma^3} \left\{ \left(\boldsymbol{\sigma} + \frac{1}{m^2} \frac{(\mathbf{k} \cdot \boldsymbol{\sigma}) \mathbf{k}}{\gamma + 1} \right)_{\alpha\beta} \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

Non-covariant WKB

With $u_\alpha^\dagger u_\beta = \delta_{\alpha\beta} = v_\alpha^\dagger v_\beta$, have

$$\left\{ \delta_{\alpha\beta} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{1}{2} \operatorname{div} \mathbf{v} \right) + N_{\alpha\beta} \right\} C^\beta(\mathbf{x}, t) = 0,$$

with

$$N_{\alpha\beta} = -i \left(\frac{e}{m\gamma^2} \right) \mathbf{B} \cdot \left\{ \frac{1}{2} \left(\boldsymbol{\sigma} + \frac{1}{m^2} \frac{(\mathbf{k} \cdot \boldsymbol{\sigma}) \mathbf{k}}{\gamma + 1} \right)_{\alpha\beta} \right\} - i \mathcal{A}_{\alpha\beta, i} \dot{k}^i,$$

$$\mathcal{A}_{\alpha\beta, i} = i u_\alpha^\dagger \frac{\partial u_\beta}{\partial k^i}, \quad i = 1, 2, 3$$

and

$$\mathcal{F}_{\alpha\beta} = -\frac{1}{2m^2\gamma^3} \left\{ \left(\boldsymbol{\sigma} + \frac{1}{m^2} \frac{(\mathbf{k} \cdot \boldsymbol{\sigma}) \mathbf{k}}{\gamma + 1} \right)_{\alpha\beta} \right\} \cdot (d\mathbf{k} \times d\mathbf{k}).$$

Berry curvature has opposite sign!

Outline

- 1 Covariant Berry Connection
 - Anomalous Velocity
 - WKB and Berry
 - Berry, Thomas, and Pauli-Lubanski
- 2 Relativistic Mechanics of Spinning Particles
 - Mathisson-Papatrou-Dixon equations
 - Anomalous velocity
 - Meaning of Conditions on Spin Tensor
- 3 Massless Case
 - A Gauge Invariance?
 - Wigner Translations
 - Physical Meaning of Wigner Translations
- 4 Conclusions

Classical action for spinning particle in GR

Let λ be a Lorentz transformation in Dirac representation, e_a^μ and e_μ^{*a} a frame and co-frame, and define

$$\begin{aligned} k_a &= \text{tr} \{ \kappa \lambda^{-1} \gamma_a \lambda \}, & \kappa &= \kappa^a \gamma_a \\ S_{ab} &= \text{tr} \{ \Sigma \lambda^{-1} \sigma_{ab} \lambda \}, & \Sigma &= \frac{1}{2} \Sigma^{ab} \sigma_{ab} \end{aligned}$$

where $[\kappa, \Sigma] = 0$, so that $k^a S_{ab} = 0$ (Tulczyjew-Dixon condition)
Action

$$S[x, \lambda] = \int \{ k_a e_\mu^{*a} dx^\mu - \text{tr} \{ \Sigma \lambda^{-1} (d + \omega) \lambda \} \}.$$

where

$$\omega = \frac{1}{2} \sigma_{ab} \omega^{ab}{}_\mu dx^\mu$$

is spin connection one-form.

Mathisson-Papapetrou-Dixon equations

■ Varying x^μ gives us

$$\frac{Dk_c}{D\tau} + \frac{1}{2}S_{ab}R^{ab}{}_{cd}\dot{x}^d = 0$$

Mathisson-Papapetrou-Dixon equations

■ Varying x^μ gives us

$$\frac{Dk_c}{D\tau} + \frac{1}{2}S_{ab}R^ab_{cd}\dot{x}^d = 0$$

■ Varying λ gives

$$\frac{DS_{ab}}{D\tau} + \dot{x}_a k_b - k_a \dot{x}_b = 0$$

Mathisson-Papapetrou-Dixon equations

- Varying x^μ gives us

$$\frac{Dk_c}{D\tau} + \frac{1}{2}S_{ab}R^{ab}{}_{cd}\dot{x}^d = 0$$

- Varying λ gives

$$\frac{DS_{ab}}{D\tau} + \dot{x}_a k_b - k_a \dot{x}_b = 0$$

- Need additional condition such as $k^a S_{ab} = 0$ or $n^a S_{ab} = 0$ for closed system.

Anomalous velocity

Use $k^a S_{ab} = 0$ to get

$$-\frac{Dk^a}{D\tau} S_{ab} = k^2 \dot{x}_b - k_b (\dot{x} \cdot k).$$

or

$$\dot{x}_a = \frac{1}{m^2} \left(k_a (\dot{x} \cdot k) + S_{ac} \frac{Dk^c}{D\tau} \right).$$

Anomalous velocity

Use $k^a S_{ab} = 0$ to get

$$-\frac{Dk^a}{D\tau} S_{ab} = k^2 \dot{x}_b - k_b (\dot{x} \cdot k).$$

or

$$\dot{x}_a = \frac{1}{m^2} \left(k_a (\dot{x} \cdot k) + S_{ac} \frac{Dk^c}{D\tau} \right).$$

Chose “time” so that $\dot{x}^0 = 1$, then

$$1 = \frac{1}{m^2} \left\{ (\dot{x} \cdot k) E + S_{0c} \frac{Dk^c}{Dt} \right\},$$

Anomalous velocity

Use $k^a S_{ab} = 0$ to get

$$-\frac{Dk^a}{D\tau} S_{ab} = k^2 \dot{x}_b - k_b (\dot{x} \cdot k).$$

or

$$\dot{x}_a = \frac{1}{m^2} \left(k_a (\dot{x} \cdot k) + S_{ac} \frac{Dk^c}{D\tau} \right).$$

Chose “time” so that $\dot{x}^0 = 1$, then

$$1 = \frac{1}{m^2} \left\{ (\dot{x} \cdot k) E + S_{0c} \frac{Dk^c}{Dt} \right\},$$

Eliminate k^0 , then

$$\dot{x}_i = \frac{k_i}{E} + \frac{1}{m^2} \left(S_{ij} - S_{i0} \frac{k_j}{E} - \frac{k_i}{E} S_{0j} \right) \frac{Dk^j}{Dt}$$

Meaning of conditions on spin tensor

- Lab frame energy centroid X_L^i :

$$\left\{ \int_{x^0=t} T^{00} d^3x \right\} X_L^i = \int_{x^0=t} x^i T^{00} d^3x.$$

Meaning of conditions on spin tensor

- Lab frame energy centroid X_L^i :

$$\left\{ \int_{x^0=t} T^{00} d^3x \right\} X_L^i = \int_{x^0=t} x^i T^{00} d^3x.$$

- Angular momentum about x_A^μ :

$$M_A^{\mu\nu} = \int_{x^0=t} \{ (x^\mu - x_A^\mu) T^{\nu 0} - (x^\nu - x_A^\nu) T^{\mu 0} \} d^3x$$

Meaning of conditions on spin tensor

- Lab frame energy centroid X_L^i :

$$\left\{ \int_{x^0=t} T^{00} d^3x \right\} X_L^i = \int_{x^0=t} x^i T^{00} d^3x.$$

- Angular momentum about x_A^μ :

$$M_A^{\mu\nu} = \int_{x^0=t} \{ (x^\mu - x_A^\mu) T^{\nu 0} - (x^\nu - x_A^\nu) T^{\mu 0} \} d^3x$$

- Therefore

$$\begin{aligned} M_A^{i0} &= \int_{x^0=t} \{ (x^i - x_A^i) T^{00} - (x^0 - x_A^0) T^{i0} \} d^3x \\ &= (X_L^i - x_A^i) E. \end{aligned}$$

Meaning of conditions on spin tensor

- $M^{i0} = 0$ when $x_A^i = X_L^i$, meaning that angular momentum is about lab-frame energy centroid.

Meaning of conditions on spin tensor

- $M^{i0} = 0$ when $x_A^i = X_L^i$, meaning that angular momentum is about lab-frame energy centroid.
- $n_a M^{ab} = 0$ for angular momentum about centroid in frame where $n^a = (1, 0, 0, 0)$.

Meaning of conditions on spin tensor

- $M^{i0} = 0$ when $x_A^i = X_L^i$, meaning that angular momentum is about lab-frame energy centroid.
- $n_a M^{ab} = 0$ for angular momentum about centroid in frame where $n^a = (1, 0, 0, 0)$.
- Thus $k_a S^{ab} = 0$ implies that S^{ab} is the **intrinsic angular momentum**, meaning angular momentum about energy centroid in rest frame where $k^a = (m, 0, 0, 0)$.

Meaning of conditions on spin tensor

- $M^{i0} = 0$ when $x_A^i = X_L^i$, meaning that angular momentum is about lab-frame energy centroid.
- $n_a M^{ab} = 0$ for angular momentum about centroid in frame where $n^a = (1, 0, 0, 0)$.
- Thus $k_a S^{ab} = 0$ implies that S^{ab} is the **intrinsic angular momentum**, meaning angular momentum about energy centroid in rest frame where $k^a = (m, 0, 0, 0)$.
- $k_a S^{ab} = 0$ implies that $x^\mu(\tau)$ in the M-P-D equation is trajectory of “centre of mass” — *i.e.* energy centroid in rest frame.

Meaning of conditions on spin tensor

- $M^{i0} = 0$ when $x_A^i = X_L^i$, meaning that angular momentum is about lab-frame energy centroid.
- $n_a M^{ab} = 0$ for angular momentum about centroid in frame where $n^a = (1, 0, 0, 0)$.
- Thus $k_a S^{ab} = 0$ implies that S^{ab} is the **intrinsic angular momentum**, meaning angular momentum about energy centroid in rest frame where $k^a = (m, 0, 0, 0)$.
- $k_a S^{ab} = 0$ implies that $x^\mu(\tau)$ in the M-P-D equation is trajectory of “centre of mass” — *i.e.* energy centroid in rest frame.
- Also see that Pauli-Lubansky “Berry curvature”

$$S_{\mu\nu} - S_{\mu 0} \frac{k_\nu}{E} - \frac{k_\mu}{E} S_{0\nu}$$

is angular momentum about lab-frame centroid.



Myron Mathisson explaining Spin

Outline

- 1 Covariant Berry Connection
 - Anomalous Velocity
 - WKB and Berry
 - Berry, Thomas, and Pauli-Lubanski
- 2 Relativistic Mechanics of Spinning Particles
 - Mathisson-Papatrou-Dixon equations
 - Anomalous velocity
 - Meaning of Conditions on Spin Tensor
- 3 Massless Case
 - A Gauge Invariance?
 - Wigner Translations
 - Physical Meaning of Wigner Translations
- 4 Conclusions

Massless case

When $m^2 = 0$ bad things happen!

Massless case

When $m^2 = 0$ bad things happen!

- Suppose that $k^2 = 0$, and S_{ab} satisfies $S_{ab}k^b = 0$, then

$$\tilde{S}_{ab} = S_{ab} + (k_a S_{pb} - k_b S_{pa}) \Theta^p$$

still satisfies $\tilde{S}_{ab}k^b = 0$.

Massless case

When $m^2 = 0$ bad things happen!

- Suppose that $k^2 = 0$, and S_{ab} satisfies $S_{ab}k^b = 0$, then

$$\tilde{S}_{ab} = S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p$$

still satisfies $\tilde{S}_{ab}k^b = 0$.

- If S_{ab} and x_a satisfy M-P-D equation for $\dot{k}^a = 0$, and

$$\tilde{x}_a = x_a + S_{pa}\Theta^p,$$

then $\tilde{S}_{ab}$, $\tilde{x}_a$ are also a solution of M-P-D for any time-dependent $\Theta^p(\tau)$.

Massless case

When $m^2 = 0$ bad things happen!

- Suppose that $k^2 = 0$, and S_{ab} satisfies $S_{ab}k^b = 0$, then

$$\tilde{S}_{ab} = S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p$$

still satisfies $\tilde{S}_{ab}k^b = 0$.

- If S_{ab} and x_a satisfy M-P-D equation for $\dot{k}^a = 0$, and

$$\tilde{x}_a = x_a + S_{pa}\Theta^p,$$

then $\tilde{S}_{ab}$, $\tilde{x}_a$ are also a solution of M-P-D for any time-dependent $\Theta^p(\tau)$.

A gauge invariance?

Wigner Translations

- Massless reference momentum $\kappa^a = (1, 0, 0, \dots, 0, 1)$.
- Little group: σ_{ab} with $0 < a, b < d - 1$. Generate $\text{SO}(d - 2)$, together with “translations”

$$\pi_a = \kappa^b \sigma_{ba} \equiv \sigma_{0a} + \sigma_{(d-1)a}, \quad 0 < a < d - 1.$$

$$[\pi_a, \pi_b] = 0, \quad [\sigma_{ab}, \pi_c] = \eta_{bc}\pi_a - \eta_{ac}\pi_b.$$

Wigner Translations

- Massless reference momentum $\kappa^a = (1, 0, 0, \dots, 0, 1)$.
- Little group: σ_{ab} with $0 < a, b < d - 1$. Generate $\text{SO}(d - 2)$, together with “translations”

$$\pi_a = \kappa^b \sigma_{ba} \equiv \sigma_{0a} + \sigma_{(d-1)a}, \quad 0 < a < d - 1.$$

$$[\pi_a, \pi_b] = 0, \quad [\sigma_{ab}, \pi_c] = \eta_{bc}\pi_a - \eta_{ac}\pi_b.$$

- Wigner says that the π_a **must have no physical effect...**

Wigner Translations

- Massless reference momentum $\kappa^a = (1, 0, 0, \dots, 0, 1)$.
- Little group: σ_{ab} with $0 < a, b, < d - 1$. Generate $\text{SO}(d - 2)$, together with “translations”

$$\pi_a = \kappa^b \sigma_{ba} \equiv \sigma_{0a} + \sigma_{(d-1)a}, \quad 0 < a < d - 1.$$

$$[\pi_a, \pi_b] = 0, \quad [\sigma_{ab}, \pi_c] = \eta_{bc}\pi_a - \eta_{ac}\pi_b.$$

- Wigner says that the π_a **must have no physical effect...**
- **...but**

$$\lambda \rightarrow \lambda \exp \left(\sum_{i=1}^{d-2} \theta^i \pi_i \right), \quad \text{in} \quad S_{ab} = \text{tr}\{\Sigma \lambda^{-1} \sigma_{ab} \lambda\},$$

takes

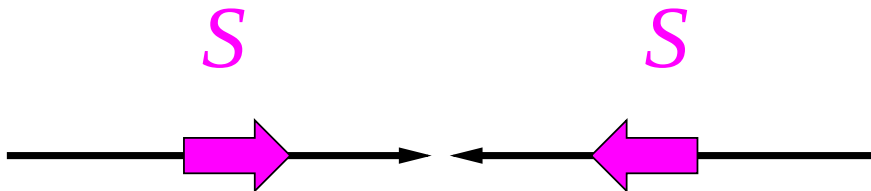
$$S_{ab} \rightarrow S_{ab} + (k_a S_{pb} - k_b S_{pa}) \Theta^p, \quad \Theta^p = \Lambda^p_i \theta^i.$$

Heisenberg, Wigner



Heisenberg and Eugene Wigner

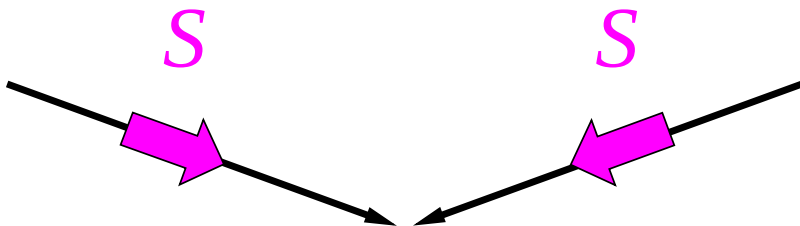
Physical Meaning of Wigner Translations



Head-on collision of massless spinning particles.

$$L = 0, S = 0 \Rightarrow J = 0.$$

Physical Meaning of Wigner Translations

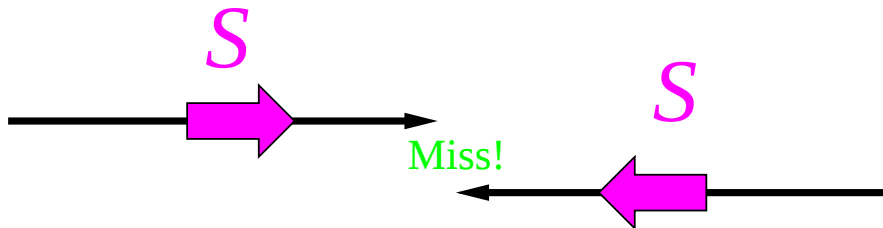


Run towards collision, top view.

$$J = 0, S \neq 0$$

$$\Rightarrow L \neq 0.$$

Physical Meaning of Wigner Translations



Boost towards collision, front view.

Miss by $\delta x = L/k$

Huh!

It's not that weird:

- Any interaction that occurs in one frame still occurs when viewed from another frame.
- Cross-sections depend on $J = L + S$.
- For massless particles, cannot separate L from S .
- Means that particle “position” is **frame dependent**.
- A serious problem for any covariant mechanics!

Show some MathematicaTM plots to prove that frame dependence is a real phenomenon

Outline

- 1 Covariant Berry Connection
 - Anomalous Velocity
 - WKB and Berry
 - Berry, Thomas, and Pauli-Lubanski
- 2 Relativistic Mechanics of Spinning Particles
 - Mathisson-Papatrou-Dixon equations
 - Anomalous velocity
 - Meaning of Conditions on Spin Tensor
- 3 Massless Case
 - A Gauge Invariance?
 - Wigner Translations
 - Physical Meaning of Wigner Translations
- 4 Conclusions

Conclusions

- For massive particles, the Berry-phase equations of motion for relativistic spinning particles are the 3-dimensional reduction of 3+1 Lorentz covariant equations
- The Berry phase equations of motion for massless particles are **not** the $m \rightarrow 0$ limit of the massive-particle equations
- The Berry phase equations of motion for massless particles are **not** the 3-dimensional reduction of covariant equations
- The lack of covariance arises because the position ascribed to a massless particle is the lab-frame centroid, and is **frame-dependent**