Elliptic Curves & Number Theory

R. Sujatha
School of Mathematics
TIFR
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\( \mathbb{Z} \): set of integers \((\ldots, -3, -2, -1, 0, 1, 2, \ldots)\)

\( \mathbb{Q} \): Rational numbers
• Recall that a real number is rational if it can be expressed in the form $\alpha = \frac{m}{n}$, where $m$ and $n$ are in $\mathbb{Z}$.

• Irrational numbers: Those which cannot be expressed in the form $m/n$, $m, n \in \mathbb{Z}$.

Example:

$$\sqrt{2}, \quad \pi = 3.1419, \quad \frac{1 + \sqrt{5}}{2}.$$
Pythagorean Triples:

- $PQR$ is right angled triangle
- Sides have lengths $a, b, c$. 
Pythagoras Theorem: $PQR$ a right angled triangle. Then

$$(PQ)^2 + (QR)^2 = (PR)^2 \quad i.e. \quad a^2 + b^2 = c^2.$$
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**Examples:**

$$(3)^2 + (4)^2 = (5)^2$$
$$(12)^2 + (5)^2 = (13)^2$$
• \((a, b, c)\) is called a Pythagoras triple; \((3, 4, 5), (5, 12, 13)\) are Pythagoras triples.
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- There exist infinitely many Pythagoras triples in \(\mathbb{N}\).
- Note that the Pythagoras Theorem forces one to come to terms with irrational numbers!

\[ a = b = 1, \quad c = \sqrt{2} \]
Fundamental Theorem of Arithmetic

Each integer $n > 1$ can be written uniquely (up to reordering) as a product of powers of primes;

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r},$$

where $p_i$ are distinct prime numbers, $\alpha_i \in \mathbb{N}$. 
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$p_i$ are distinct prime numbers, $\alpha_i \in \mathbb{N}$.

Henceforth: Will consider only right angled triangles.
Let $T$ denote a right angled triangle with sides of length $a, b, c$. 
Simple Exercise: Use the fundamental theorem to show that $\sqrt{2}$ is irrational.
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Of course, usually $T$ has at least one of its sides rational but we want to consider only triangles $T$ having all of its sides with rational length.
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Of course, usually $T$ has at least one of its sides rational but we want to consider only triangles $T$ having all of its sides with *rational* length.

Example:

\[
\begin{align*}
  a &= \frac{40}{6}, \quad b = \frac{9}{6}, \quad c = \frac{41}{6} \\
  \left(\frac{40}{6}\right)^2 + \left(\frac{9}{6}\right)^2 &= \left(\frac{41}{6}\right)^2.
\end{align*}
\]
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- We say $T$ is *primitive* if $a, b, c$ are positive integers and $(a, b, c) = 1$ (i.e. they are *relatively prime* which means $a, b, c$ have no common divisor other than 1).

Can clearly bring any rational triangle to be similar to a *unique* primitive triangle.
First Observation: If $T$ is primitive, then precisely one of its sides $a$ or $b$ is even.

- If $2|a$ and $2|b$, then as $a^2 + b^2 = c^2$, $2|c \Rightarrow 2$ divides $a$, $b$ and $c$; contradicting that $T$ is primitive.
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- If $a$ and $b$ are both odd, say $a = 2a_1 + 1$, $b = 2b_1 + 1$; then $a^2 + b^2 = 4k + 2 = c^2 \Rightarrow 2|c^2 \Rightarrow 2|c \Rightarrow 4|c^2$. 
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Second Observation: If $T$ is primitive, then there exist positive integers $m, n$ with $(m, n) = 1$ such that

$$a = n^2 - m^2, \quad b = 2mn; \quad c = m^2 + n^2$$
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\[ a = n^2 - m^2, \quad b = 2mn; \quad c = m^2 + n^2 \]

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\[ a = 2mn, \quad b = n^2 - m^2; \quad c = m^2 + n^2. \]
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OR

$$a = 2mn, \quad b = n^2 - m^2; \quad c = m^2 + n^2.$$  

• Note that

$$a^2 + b^2 = (n^2 - m)^2 + (2mn)^2 = (n^2 + m^2)^2 = c^2,$$

so we do have a Pythagorean triple.
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• Let us suppose that $a$ is odd and $b$ is even. Then clearly $c$ is odd, hence $(b, c) = 1$.

Also $(a, c) = 1$ because $a^2 + b^2 = c^2$.

• Put $w_1 = \frac{1}{2}(c - a) \quad w_2 = \frac{1}{2}(c + a)$. 
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- Put $w_1 = 1/2(c - a)$, $w_2 = 1/2(c + a)$.

Clearly both $w_1$ and $w_2$ are positive integers.
• We prove that \( w_1 \) and \( w_2 \) are relatively prime i.e. \((w_1, w_2) = 1\).
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Suppose $d|w_1$ and $d|w_2$; then $d|w_1 + w_2$ and $d|w + 2 = w_1$.

Now $w_1 + w_2 = c_1$, $w_2 - w_1 = a \Rightarrow d|c$ and $d|a$. 
We prove that \( w_1 \) and \( w_2 \) are relatively prime i.e. \((w_1, w_2) = 1\).

Suppose \( d \mid w_1 \) and \( d \mid w_2 \); then \( d \mid w_1 + w_2 \) and \( d \mid w + 2 = w_1 \).

Now \( w_1 + w_2 = c_1, \ w_2 - w_1 = a \Rightarrow d \mid c \) and \( d \mid a \).

Contradicts \((a, c) = 1\)!
• We prove that \( w_1 \) and \( w_2 \) are relatively prime i.e. 
\((w_1, w_2) = 1\).

Suppose \( d | w_1 \) and \( d | w_2 \); then \( d | w_1 + w_2 \) and 
\( d | w + 2 = w_1 \).

Now \( w_1 + w_2 = c_1, \ w_2 - w_1 = a \Rightarrow d | c \) and \( d | a \).

Contradicts \( (a, c) = 1 \)!

\( a^2 + b^2 + c^2 \) takes the form

\[
b^2 = (c^2 - a^2); \quad \frac{b^2}{4} = \frac{c^2 - a^2}{4} \Rightarrow (b/2)^2 = \left(\frac{c - a}{2}\right)\left(\frac{c + a}{2}\right) = w_1 w_2.\]
What We Get: $w_1$ and $w_2$ are relatively prime and their product is a square.
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**Conclusion** (By the fundamental theorem of arithmetic):
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Conclusion (By the fundamental theorem of arithmetic): Both $w_1$ and $w_2$ are squares.

$$w_1 = m_1^2, \; w_2 = n^2 \text{ and } (m, n) = 1.$$ 

But

$$a = w_1 - w_2 = m^2 - n^2$$
$$c = w_1 + w_2 = m^2 + n^2$$
$$b^2 = c^2 - a^2 \Rightarrow b = 2mn.$$
This concludes the proof of second observation.
Areas

Area of $T = \frac{1}{2}ab$
Let $N$ be a positive integer. There exist infinitely many $T$'s such that $\text{Area } (T) = N$ (choose positive rational numbers $a, b$ such that $ab = 2N$).
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**Key Question:** Does there exist a rational $T$ with area $(T) = N$?
• Let $N$ be a **positive integer**. There exist infinitely many $T$’s such that $\text{Area}(T) = N$ (choose positive rational numbers $a, b$ such that $ab = 2N$).

**Key Question:** Does there exist a *rational* $T$ with area $(T) = N$?

(i.e. We want a right angled $\triangle$ with all its sides having rational length and area equal to $N$).
**Definition:** We say $N$ is *congruent* if there exists a rational $T$ with Area $(T) = N$. 
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Example:

- $N = 5$ is congruent $(a, b, c) = (9/6, 40/6, 41/6)$.

\[
\text{Area} = \frac{1}{2} \times 9/6 \times 40/6 = 5.
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Definition: We say $N$ is congruent if there exists a rational $T$ with $\text{Area}(T) = N$.

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  \text{Area} = \frac{1}{2} \times \frac{9}{6} \times \frac{40}{6} = 5.
  \]

- $N = 6$ is congruent $(a, b, c) = (3, 4, 5)$.

Arab Mathematicians (and Indian Mathematicians) made tables of congruent numbers (10th century AD).
Example:

5, 6, 7, 13, 14, 15, 21, 22, 23, 29, 30, 31, ... are all congruent.
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are all congruent.

• If $N$ is congruent, so is $N' = d^2 N$, where $d \in \mathbb{Z}$ (If $N = 1/2(ab)$ and $(a, b, c)$ is a Pythagoras triple, consider

$$(a', b', c') = (da, db, dc); \frac{1}{2}(a'b')\frac{1}{2}(da)(db) = d^2 N = N'.$$
Therefore we may restrict attention to square free natural numbers (i.e. those elements in $\mathbb{N}$ which are not divisible by $p^2$ for any prime $p$).
First Obvious Question: Is 1 a congruent number?

- As it was difficult to find a $Y$ with area 1, the ancients tried to show that 1 was not congruent number with many false proofs.

- The first proof that 1 is not a congruent number was given by Fermat, a 17th century French lawyer and government official by profession, but a polymath of great erudition.
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• Basic Idea: (each time the triangles are primitive) Start with $\triangle T_1$ such that $c_1 = \text{Area } (T_1)$ is a square; then produce a $T_2$ such that $c_2 = \text{Area } T_2$ is again a square and $c_2 < c_1$. 
Repeating this step, we can construct an infinite sequence of primitive triangles $T_i$ whose area $c_i$ is always a square and

$$c_1 > c_2 > c_3 > \cdots$$

But this cannot go on forever as one cannot have an infinite strictly decreasing sequence of positive integers!
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\[ c_1 > c_2 > c_3 > \cdots \]

But this cannot go on forever as one cannot have an infinite strictly decreasing sequence of positive integers!

• Heart of the argument uses the second observation we made before.
Corollary: The equation \( x^4 - y^4 = z^2 \) has no solution in integers \( x, y, z \) with \( xyz \neq 0 \).
**Corollary:** The equation $x^4 - y^4 = z^2$ has no solution in integers $x, y, z$ with $xyz \neq 0$.

**Proof:** Suppose a solution exists. Put

\[
\begin{align*}
n &= x^2, & m &= y^2 \\
a &= n^2 - m^2, & b &= 2mn, & c &= n^2 + m^2
\end{align*}
\]

so that $(a, b, c)$ is a Pythagoras triple with

\[
\text{Area} = (1/2)ab = nm(n^2 - m^2) = x^2y^2(x - y) = x^2y^2z^2.
\]
Define $T'$ with sides of length $(a', b', c')$ where

\[
a' = \frac{a}{\lambda}, \quad b' = \frac{b}{\lambda}, \quad c' = \frac{c}{\lambda}, \quad \lambda = xyz (\neq 0)
\]

\[
\text{Area}(T') = 1/2 \cdot \frac{ab}{\lambda \lambda} = \frac{ab}{2\lambda^2} = \frac{2x^2y^2z^2}{2x^2y^2z^2} = 1.
\]

$1$ is a congruent number, contradiction!
In particular, this shows that \( x^4 - y^4 = w^4 \) has no solution in integers with \( x y w \neq 0 \). This would have led Fermat to conjecture his famous **Last Theorem** that for any integer \( n \geq 3 \), the equation

\[
x^n = y^n + z^n.
\]

has no solution in integers \( x, y, z \) with \( xyz \neq 0 \! \). This is now a celebrated **Theorem of Andrew Wiles**.
non-congruent \( N: 1, 2, 3, 10, 11, 17, 19, 26, 33, 35 \ldots \)
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**Ancient Question I:** Is there an algorithm for deciding in a *finite* number of steps whether a given positive integer $N$ is congruent or not.
non-congruent $N$: 1, 2, 3, 10, 11, 17, 19, 26, 33, 35 . . .

Ancient Question I: Is there an algorithm for deciding in a \textit{finite} number of steps whether a given positive integer $N$ is congruent or not.

Ancient Question II: Prove that every square free integer of the form $8n + 5$ or $8n + 6$ or $8n + 7$ ($n = 1, 2, \ldots$) is congruent.
Conjecturally, there is a very simple answer to both these Questions, but the conjecture is now a Millennium Problem worth a million dollars!
Elliptic Curves

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This is an area of study which has connections with different branches of mathematics like complex manifolds, algebraic geometry, arithmetic geometry modular forms and of course in recent years in cryptography.

Perhaps it would be no exaggeration to say that it is an area where endless mining for problems, research and applications is possible!
For our purposes today, we shall satisfy ourselves with considering elliptic curves over \( \mathbb{Q} \). They can then be studied as solutions of equations of the form

\[
E : y^2 = f(x)
\]

where \( f(x) \) is a polynomial over \( \mathbb{Q} \) of degree 3. One can even assume that

\[
f(x) = ax^3 + cx + d, \quad a, c, d \in \mathbb{Q}, \quad a \neq 0.
\]
Its set of *real* points looks like
Why are they relevant to the study of congruent numbers?
Why are they relevant to the study of congruent numbers?

A natural number \( n \) is congruent if and only if the elliptic curve over \( \mathbb{Q} \) defined by

\[
E_n : y^2 = x^3 - n^2 x
\]

has \textit{infinitely} many solutions over \( \mathbb{Q} \).
To see this equivalence is not too difficult. It follows from Pythagoras identity and transferring one curve to another by ‘birational isomorphisms’.
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• Another crucial property that is useful here is that the set of solutions $E(\mathbb{Q})$ for any elliptic curve $E/\mathbb{Q}$ has the structure of an abelian group.
• To see this equivalence is not too difficult. It follows from Pythagoras identity and transferring one curve to another by ‘birational isomorphisms’.

• Another crucial property that is useful here is that the set of solutions $E(\mathbb{Q})$ for any elliptic curve $E/\mathbb{Q}$ has the structure of an abelian group.

• The law of addition on $E(\mathbb{Q})$ is not naive coordinate addition; it involves beautiful geometric ideas.
• One knows more about $E(\mathbb{Q})$; in fact it is a finitely generated abelian group which in simple words means that a finite set suffices to construct all the elements in $E(\mathbb{Q})$. This is known as Mordell’s Theorem.
• One knows more about $E(\mathbb{Q})$; in fact it is a \textit{finitely generated} abelian group which in simple words means that a \textit{finite set} suffices to construct all the elements in $E(\mathbb{Q})$. This is known as \textit{Mordell’s Theorem}.

• In particular, we have

$$E_{n}(\mathbb{Q}) = \text{“free infinite part”} \oplus \text{“finite part”}$$
Caution: It might happen that a given elliptic curve $E$ has only one trivial point, i.e. $E(\mathbb{Q})$ is a singleton set; this point necessarily lies in the Finite torsion part.
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\[ n \text{ is a congruent number } \iff E_n : y^2 = x^3 - n^2x \text{ has infinitely many rational points.} \]
**Known:** The “finite” (i.e. torsion) part of $E_n(\mathbb{Q})$ consists of 4 elements.

Thus we are now faced with the

**Question:** When is $E_n(\mathbb{Q})$ infinite?
• It is at this step that one of the most famous conjectures of the last century intervenes. This is the so-called **Birch & Swinnerton-Dyer** Conjecture (B-SD) which relates the nature of $E(\mathbb{Q})$ to something completely different!
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**Hasse-Weil $L$-Functions:** We shall not go into the technical definition of this. Suffice it to say that it is a vast ingenious generalisation of the classical **Riemann Zeta function**.
• It is “built” using information about the number of points that $E$ has over finite fields $\mathbb{F}_p$, as $p$ varies over all primes.
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Given an elliptic curve \( E/\mathbb{Q} \), its Hasse-Weil \( L \)-function is denoted by \( L(E, s) \), \( s \) a complex variable.
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Given an elliptic curve $E/\mathbb{Q}$, its Hasse-Weil $L$-function is denoted by $L(E, s)$, $s$ a complex variable.

• It has an expansion into an infinite product, the product varying over all primes.

• $L(E, s) = \prod_p (1 - 2a_pp^{-s} + p^{1-2s})^{-1}$. 
Can expand this ‘Euler product’ to get a ‘Dirichlet Series’, i.e. an infinite sum:

\[ L(E, s) = \sum_{n=0}^{\infty} a_n n^{-s} = \sum_{n=0}^{\infty} a_n / n^s. \]
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• Deep and important conjectures on convergence properties of this function on the complex plane.

**B-SD Conjecture:** \( E(\mathbb{Q}) \) is infinite if and only if \( L(E, s) \) vanishes at \( s = 1 \) (i.e. \( L(E, 1) = 0 \)).
So we can now reformulate our original Question and ask:

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- **Coates-Wiles (1970’s):** $L(E_n, 1) \neq 0 \Rightarrow E_n(\mathbb{Q})$ is finite.
- **Finding $E(\mathbb{Q})$** for an elliptic curve $E$ is in general very difficult, even with computers! On the other hand, computations with $L$-functions are more amenable to calculations!
\( n \) is a congruent number \( \iff \) \( E_n : y^2 = x^3 - n^2x \); \( E_n(\mathbb{Q}) \) is infinite \( \Rightarrow L(E_n, 1) = 0 \).

Conjecturally (BSD Conjecture) \( L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q}) \) is infinite.
We will connect the vanishing of $L(E_n, 1)$ now to
We will connect the vanishing of $L(E_n, 1)$ now to **Modular forms**

This is the next area in mathematics from which we need to draw our artillery now!

Again, this is a vast, fascinating and technical subject in its own right with **beautiful connections to elliptic curves**.
At its very simplest, a **modular form** is a **holomorphic function** $f(z)$ on the upper half plane (which is the part of the complex plane with imaginary part $> 0$), such that it has a Fourier expansion (called the $q$-expansion)

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad q = e^{2\pi i z}, \quad a_n \in \mathbb{C}.$$
We will consider special modular forms, called "Cusp forms"; these have an expansion

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\[ f(z) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i z}, \quad a_n \in \mathbb{C}. \]

**Example**

\[ \Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \ldots \]

(\(\tau \sim \) Ramanujan’s Tau function).
• Given a modular form (cusp form), it has an $L$-series associated to it:

\[ L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]
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L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.
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Let \( E'_1 : y^2 = x^3 - x \).

\( L(E'_1, s) \) related to \( L(f, s) \) for some \( f \).
Deep work of Shimura, Waldspurger and Tunnell then allows us to relate $L(E_n, s)$ and $L(f, s)$; the bridge being $L(E'_1, s)$. More precisely:
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- There exist modular forms $g_1, g_2$ which are obtained via $f$:

$$g_1 = \sum_{n=1}^{\infty} a(n)q^n, \quad g_2 = \sum_{n=1}^{\infty} b(n)q^n$$

such that

$L(E_n, 1)$ related to coefficients of $g_1$ & $g_2$. 
Thus we can connect this to our original problem by the following theorem:

**Theorem (Tunnell, 1983):** \( L(E_n, 1) = 0 \) if and only if \( a(n) = 0 \) for \( n \) odd, or \( b(n/2) = 0 \) for \( n \) even. Moreover,

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a(n) + b(n/2) \neq 0 \Rightarrow L(E_n, 1) \neq 0.
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a(n) + b(n/2) \neq 0 \Rightarrow L(E_n, 1) \neq 0.
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In particular, if \( a(n) + b(n/2) \neq 0 \), then \( n \) is not congruent.
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Conjecturally (BSD Conjecture) \[ L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q}) \text{ is infinite.} \]
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Conjecturally (BSD Conjecture) \[ L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q}) \text{ is infinite.} \]

Finally, \[ L(E_n, 1) = 0 \iff a(n) = 0 \text{ for } n \text{ odd or } b(n/2) = 0 \text{ for } n \text{ even.} \]
If $a(n) + b(n/2) = 0$, then $n$ is congruent.

$$a(n) + b(n/2) = 0 \Rightarrow L(E_n, 1) = 0.$$ 

**B – SD conjecture:** $L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q})$ is infinite.
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**B - SD conjecture:** $L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q})$ is infinite.

Beauty of this result:
If \( a(n) + b(n/2) = 0 \), then \( n \) is congruent.

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a(n) + b(n/2) = 0 \Rightarrow L(E_n, 1) = 0.
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\textbf{B - SD conjecture: } \( L(E_n, 1) = 0 \Rightarrow E_n(\mathbb{Q}) \) is infinite.

Beauty of this result:

\textit{Conjecturally} it reduces the problem of determining if \( n \) is congruent to an algebraic computation involving in finitely many steps (\( \sim n^{3/2} \) steps).
Unconditional Results

$p$ prime.

- $p \equiv 3 \mod 8$, then $p$ is not congruent.

\[(a(p) \neq 0; \ e.g. \ 43, 443, \ldots)\]
Unconditional Results

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- \( p \equiv 3 \mod 8 \), then \( p \) is not congruent.
  
  \[(a(p) \neq 0; \text{ e.g. } 43, 443, \ldots)\]

- \( n \equiv 1 \mod 8 \), some \( n \) of this form not congruent.

E.g.: 57, 489 \((a(n) \neq 0)\).
• $p, q$ primes $\equiv 5 \mod 8$; then $2pq$ is not congruent.

$$(b(pq) \equiv 4 \mod 8). \quad \text{eg. } 754; \ (754 = 2.13.29).$$

• Eg: 157 is a congruent number.

Simplest Rational triangle with Area 157 (Computed by D. Zagier).
\[ x = \frac{6803298487826435051217540}{411340519227716149383203} \]
\[ y = \frac{411340519227716143983203}{21666555693714761309610} \]
\[ z = \frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830} \]
I hope I have succeeded in convincing you that deep, intricate and mysterious connections exist in number theory between simply stated problems and areas at the frontier of Modern Research.
Iwasawa theory

This is a theory which provides us an effective tool towards attacking the Birch-Swinnerton Dyer Conjecture.
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- Unfortunately, even the basic philosophy and ideas in this theory require a sophisticated knowledge and background of mathematics.

- One of its spectacular applications is in the work of Coates-Wiles stated above.
• My own work focuses on **Non-commutative Iwasawa theory**. This is a relatively young area of research, classical Iwasawa theory mainly dealt with commutative structures.