DO NOT REMOVE

It is a classical procedure in algebraic topology to triangulate a space and associate the chain complex

$$(*) \quad \ldots \to C_{i+1} \xrightarrow{\partial} C_i \to \ldots$$

of groups or vector spaces generated by the pieces of the triangulations.

This complex only determines the additive homology structures and says little about the intersection theory of cycles in a manifold, the fundamental group, the higher homotopy groups with their Whitehead products, and all the further "algebraic topology" of the space.

We are going to describe in this paper how to enhance the chain construction (*) by enlarging the vector spaces and adding products so that all the "algebraic topology" of the space <u>after tensoring with the rational field becomes readily calculable</u>.

First consider a single n-simplex τ. The points P of τ have natural barycentric coordinates $(x_o, \ldots, x_n)$ defined by

$$P = x_o V_o + x_1 V_1 + \ldots + x_n V_n$$

where $(V_o, \ldots, V_n)$ are the vectices of τ.
and the x_i's are nonnegative real numbers with $x_o + x_1 + \ldots + x_n = 1$.

The plane of τ, A_τ, is defined by dropping the non-negative condition on the barycentric coordinates $\{x_i\}$.

The classical cochain construction associates constants to the simplices τ of the space. In the enhanced construction we associate ω_τ a differential form on the plane of τ, to each simplex τ in the triangulation.

We assume that ω_τ can be expressed as a sum,

$$\omega_\tau = \Sigma\, a_{i_1 .. i_r}\, dx_1 \wedge .. \wedge dx_r,$$

where each $a_{i_1 ... i_r} = a_{i_1 ... i_r}(x_o, x_1, ..., x_n)$ is a polynomial* in the barycentric coordinates of the plane of τ. The collection $\{\omega_\tau\}$ where τ ranges through the pieces of the triangulation of a space X is called a differential form on X if the following coherence is satisfied:

whenever σ is a face of τ, ω_τ restricted to the plane of σ equals ω_σ in the sense of differential forms.

Denote the collection of differential forms on a triangulated space X by $\mathcal{E}_X$.

<u>Theorem A:</u> The rational de Rham complex $\mathcal{E}_X$ of a triangulated space X is a <u>graded</u> <u>commutative</u> differential algebra over Q. The cohomology algebra of $\mathcal{E}_X$ is isomorphic by integration of differential forms to the rational cohomology algebra of X.

* With rational coefficients.

This theorem is proved much in the spirit of the proof of a real de Rham theorem for polyhedra in the wonderful book of Whitney, "Geometric Integration" Princeton, 1956.

The algebra $\mathcal{E}_X$ has several attractive geometric properties (see Appendix G) but we will first describe an algebraic process for differential graded algebras which will exhibit the higher order algebraic topology of a space X when it is applied to $\mathcal{E}_X$.

<u>Construction of the minimal model</u>

Let $\mathcal{E}$ be any differential graded algebra. We will try to construct a simple model of the "homotopy theory" of $\mathcal{E}$ If $(x_1, x_2, \ldots)$ are variables in various dimensions, let $\Lambda(x_1, x_2, \ldots)$ denote the polynomial algebra on those in even degrees tensor the extenior algebra on those in odd degrees. We will construct a map (for certain variables $x_1, x_2, \ldots$)

$$\Lambda(x_1, x_2, \ldots) \overset{\rho}{\to} \mathcal{E}$$

and a differential d in $\Lambda(x_1, x_2, \ldots)$ so that

i) ρ induces an isomorphism on cohomology

ii) each dx_i is a sum of non-trivial products

$x_{j_1} \wedge \ldots \wedge x_{j_r}$ for $j_1, \ldots, j_r < i$.

It turns out that the model $\Lambda(x_1, x_2, \ldots; d)$ is well defined up to isomorphism by i) and ii).

Suppose always that $\mathcal{E}$ is homologically connected and for the moment that the first Betti number of $\mathcal{E}$ is zero.

Step 1: Construct $\Lambda(x_1, \ldots, x_n) \xrightarrow{\rho_1} \mathcal{E}$ where $x_1, \ldots, x_n$ are in dimension 2 and $\rho_1 x_1, \ldots, \rho_1 x_n$ generate $H^2\mathcal{E}$. Define d in $\Lambda(x_1, \ldots, x_n)$ by $dx_i = 0$, $i = 1, \ldots, n$. Note that ρ_1 is an isomorphism on cohomology in dimension 2 and injective in dimension 3.

Step 2: Adjoin three-dimensional variables to $\Lambda(x_1, \ldots, x_n)$ to make ρ_1 surjective on cohomology in dimension 3 and injective on cohomology in dimension 4.

More explicitly, we form

$$\Lambda(x_i; y_j)\ ,\ \rho_2, \text{ and } d.$$

d is defined by

$dy_1 = 0, \ldots, dy_\ell = 0$

$dy_{\ell+k} = \Sigma\, a^k_{ij}\, x_i \Lambda x_j \qquad k = 1, \ldots, m$

where ℓ is the rank of $H^3\mathcal{E}$ and $\{\Sigma a^k_{ij} x_i \Lambda x_j\}$ is a basis for kernel ρ_1 on 4-dimensional cohomology.

ρ_2 is defined by choosing $\rho_2 y_1, \ldots, \rho_2 y_\ell$ to generate $H^3\mathcal{E}$ and $\rho_2(y_{\ell+k})$, $k = 1, \ldots, m$, to satisfy

$$d(\rho_2(y_{\ell+k})) = \sum_{i,j} a^k_{ij}\, \rho_1(x_i)\Lambda\rho_1(x_j).$$

(Now ρ_2 is an isomorphism on H^3 and injective on H^4.

At this point one can show by a standard Postnikov argument how this construction relates to the homotopy theory of a simply connected space X. If $\mathcal{E}$ is the de Rham algebra of X (in any sense), then one finds the second homotopy $\pi_2 X$ has rank n, $\pi_3 X$ has rank $\ell+m$, the Hurewicz map $\pi_3 X \to H_3 X$ has rank ℓ, and the symmetric pairing given by Whitehead product

$$\pi_2 X \otimes \pi_2 X \xrightarrow{[,]} \pi_3 X$$

is described over the rationals by the i,j symmetric tensor (a_{ij}^k).)

Now suppose by induction that $\Lambda(x_1, \ldots, x_n; y_1, y_2, \ldots; z_1, z_2, \ldots : d) \xrightarrow{\rho_n} \mathcal{E}$ has been constructed so that d of any generator is a polynomical in previous generators while ρ_n is an isomorphism on cohomology up through dimension n and injective in dimension n + 1.

Then as in step 2 we adjoin variables in dimension n+1 to make ρ onto in dimension n+1 and injective in dimension n+2. In this general case we add variables $\omega_1, \ldots, \omega_d, \omega_{d+1}, \ldots, \omega_r$ and define d and $\rho = \rho_{n+1}$ as above. Now $d\omega_1, \ldots, d\omega_d$ are zero and $\rho\omega_1, \ldots, \rho\omega_d$ give generators of the cokernel of ρ_n on H^{n+1}, $d\omega_{d+1}, \ldots, d\omega_r$ are polynomials in $x_1, x_2, \ldots; y_1, y_2, \ldots; z_1, z_2, \ldots$ which generate the kernel of ρ_n on $H^{n+2}(\Lambda(x_1, x_2, \ldots; y_1, y_2, \ldots; z_1, z_2, \ldots : d))$ and $\rho\omega_{d+1}, \ldots, \rho\omega_r$ are elements in $\mathcal{E}$ whose differentials are these same polynomials in

$$\rho x_1, \rho x_2, \ldots; \rho y_1, \rho y_2, \ldots; \rho z_1, \rho z_2, \ldots .$$

Continuing in this way we construct the minimal model

$$\Lambda(x_1, x_2, \ldots : d) \underset{\rho}{\to} \mathcal{E}$$

satisfying properties i) and ii) above.

Theorem B: Let X be a triangulated space which is simply connected. If

$$\Lambda(x_1, x_2, \ldots) \underset{\rho}{\to} \mathcal{E}$$

is a minimal model of the de Rham algebra then

i) the rank of $\pi_i X$ is the number of variables in degree i.

ii) the rank of the Hurewicz homomorphism $\pi_i X \to H_i X$ is the number of d-closed generators in degree i.

iii) the Whitehead products on homotopy are described by the quadratic terms of the d-formulae of generators.

Theorem B is proved by a not so standard Postnikov argument using Theorem A and the commutativity of the wedge product multiplication in $\mathcal{E}$. The argument is based on the beautifully simple Guy Hirsh method of computing the cohomology of a principal G-bundle over a manifold using differential forms (see appendix H). The Hirsch method was suggested by Rene Thom in the study of Postnikov systems in the Cartan Seminar 1954.

The proof of theorem B shows the model can be used to directly construct the rational Postnikov system of X. These constructions yield

Theorem C: There is a one to one correspondance between simply connected rational homotopy types and strict isomorphism classes of differential graded algebras

$$\Lambda(x_1, x_2, \ldots ; d), \dim x_i > 1.$$

where each dx_i is a decomposable polynomal in previous generators.[1]

The statement for non-simply connected spaces is given in Appendix N.

The Topology of Smooth Manifolds

Theorem C (and its generalization) show that we can identify the rational homotopy theory of a space X with a certain rational differential graded algebra, namely the model

$$\Lambda(x_1, x_2, \ldots; d).$$

If M is a smooth manifold we have the de Rham algebra of C^∞ forms, A_M. This is an algebra over the real numbers, R, which should fit into the above picture. Here's how.

We can perform the algebraic construction of a model for the smooth de Rham complex A_M. We obtain a real algebra, well defined up to isomorphism by i) and ii)

---

1 See appendix F for discussion of finiteness conditions.

above,

$$\Lambda^M(x_1, x_2, \ldots : d) \to A_M.$$

Theorem D: (Generalized de Rham) The algebraic model of the C^∞ de Rham complex $\Lambda^M(x_1, x_2, \ldots : d)$ is isomorphic to the model of the rational homotopy type $\Lambda(x_1, x_2, \ldots : d)$ tensored by the real numbers.

This theorem is proved by choosing a smooth triangulation and considering the diagram of algebras.

$$\begin{array}{ccc} & & \{\text{polynomial forms}\} \otimes R \\ & & \downarrow \\ \{\text{smooth forms}\} & \to & \{\text{piecewise smooth forms}\} \end{array}$$

Each of these algebras computes the cohomology of M so the three models are isomorphic by uniqueness.

So we can compute the homotopy groups of a simply connected manifold tensored by the reals, the Whitehead products and so on. For non-simply connected manifolds we can determine from the smooth de Rham complex in dimensions 1 and 2 (as described in Appendix N) a tower of real nilpotent Lie groups $\{\ldots N_k \to N_{k-1} \to \ldots \to R^n \to e\}$ which are related to the fundamental group as follows: if Γ_k denotes π_1(modulo k-fold commutators then modulo torsion), then the geometry of M determines an embedding of Γ_k in N_k as a discrete subgroup with compact quotient. N_k is determined up to isomorphism by this property.

Non-Abelian periods on smooth manifolds

We can find from the d-formulae of the model a set of integrals defining a determining set of periods for elements of the nilpotent π_1 quotients $\{\Gamma_k\}$. There is an analogous discussion for the higher homotopy groups.

For example suppose the model begins as $\Lambda(x_1,x_2,y)$ with $dy = x_1 \Lambda x_2$. Let ω_1, ω_2, η denote the 1-forms in the manifold which are the ρ-images of these model generators. If γ is a loop in M based at p we can form the three ingegrals

$$\int_\gamma \omega_1,\ \int_\gamma \omega_2,\ \int_\gamma \eta - \int_\gamma \left(\int_{\gamma_t} \omega_1\right)\omega_2 {}^{1}$$

The first two periods are homology invariants of since ω_1 and ω_2 are closed. The third integral is a homotopy invariant of γ because $d\eta = \omega_1 \Lambda \omega_2$. If we traverse γ_1 and then γ_2 the periods multiply like the matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

---

1 These iterated integrals have been considered in a general way, by Chen. Our theory seems to guide their efficient application to homotopy theory.

The map ϕ of M to the compact nil-manifold H/Γ also generalizes and we obtain a tower of maps of M into a tower of nil-manifolds beginning with the Jacobian $H_1(M,R)/H_1(M,Z)$.

The next section shows how this construction becomes canonical if M is Riemannian

The Model of a Riemannian Manifold

If M is Riemannian we have the *-operator on forms, the adjoint of d, $d^* = *d*$, and the Hodge decomposition

$$\omega = dx_\omega + d^* y_\omega + h_\omega$$

where x_ω, y_ω, and h_ω are unique subject to the conditions $x_\omega \varepsilon$ image of d^*, $y_\omega \varepsilon$ image d, $h_\omega \varepsilon \ker d \cap \ker d^*$.

This follows on the formal level from the "disjointness" of d and d^*:

$$dd^* x = 0 \Rightarrow d^* x = 0$$

$$d^* d\, y = 0 \Rightarrow dy = 0.$$

Thus ω is closed iff $y_\omega = 0$. Then ω is cohomologous to a canonical form, the harmonic representative h_ω. And finally if ω is exact y_ω and h_ω are zero and ω is the differential of a canonical form x_ω.

We can use the Hodge decomposition to rigidify the model of a Riemannian manifold. In the construction of ρ

$$\Lambda(x_1, x_2, \ldots; d) \xrightarrow{\rho} \{\text{smooth forms}\}$$

we have to make various choices. For example we have to choose

i) representatives of certain cohomology classes in the inductively constructed model

ii) form representatives of classes in quotients of the cohomology of the manifold

iii) solutions of the equation $dx = \omega$ in the manifold with ω given.

Now the forms of M have a natural metric defined from the metric on M by exterior algebra and integration. The cohomology groups of M inherit the metric of the harmonic subspaces. This makes the choices in ii) canonical - we take the harmonic representative. Further use of the Hodge decomposition discussed above makes the choices in iii) canonical. To rigidify the choices of i) we again use the metric on harmonic forms to inductively construct a metric on the model (so that d restricted to the orthogonal complement of its kernel is an isometry).

<u>Theorem E:</u> A Riemannian manifold has a canonical model

$$\Lambda(x_1, x_2, \ldots) \xrightarrow{\rho} \{\text{forms}\}$$

defined using harmonic and co-closed forms.

This point has several corollaries.

First an isometry of M will act on the canonical model. A simple inductive argument shows that this action is determined by its action on the harmonic forms. Thus we obtain,

<u>Corollary 1:</u> The real homotopy theory of an isometry is determined by its action on the real cohomology.

For example the action of an isometry on H^1 determines its effect on all the nilpotent quotients $\{\Gamma_k\}$. (and analogously for higher homotopy.) One interesting case is that of an orientable surface M^2. Here we can deduce that the isotopy class of the diffeomorphism underlying an isometry is determined by its action on H^1M^2. This is very far from true for non-isometries.

Another point is the following:

The canonical model determines for each k, a canonical real nilpotent Lie group associated to the manifold.[1] We also have canonical 1-forms in the manifold which are algebraically like the invariant forms on N^k. We can form the non-abelian period integrals as above and for each starting point p in M we obtain a uniform discrete subgroup $\Gamma_p \subset N^k$ by integrating around closed paths at p. As p varies continuously this discrete subgroup varies continuously. Thus we have a bundle of lattices, $\{\Gamma_p \subset N^k\}$.

Forming integrals of non-closed paths starting at p leads to natural <u>non-abelian</u> <u>Jacobians</u>

$$M \xrightarrow{\phi_{k,p}} N^k/\Gamma_p.$$

---

1 Thus we can define the real form of π_1 by choosing a metric rather than a base point.

If M is complex and the metric is Kaehler this bundle of lattices and the non-abelian Jacobians only depend on the complex structure. This is clear for surfaces where * on 1-forms is rotation by $\pi/2$ and has holomorphic meaning.

Now consider the wedge product of harmonic forms. On some manifolds for example symmetric spaces, the product of harmonic forms is harmonic. In this case the construction of the model becomes formal. When a polynomial in harmonic forms is exact it is actually zero. So we can write it as d(zero) and send the corresponding generator of the model to zero.

The entire construction is determined by the structure of the cohomology ring.

<u>Corollary 2:</u> There are topological obstructions for M to admit a metric in which the product of harmonic forms is harmonic. If M does the real homotopy theory of M is a formal consequence of the cohomology ring.

More generally, the structure of the harmonic forms with respect to products is mirrored in the structure of the real homotopy theory.

---

1 This is joint work with J. Morgan, P. Griffiths and P. deLigne.

Kaehler Manifolds

If M^{2n} admits a Kaehler matric, for example if M is an algebraic submanifold of complex projective space, then certain facts are known about the cohomology ring of M. For example, the metric defines a 2-dimensional cohomology class $\omega \varepsilon H^2(M,R)$ so that cupping with ω i-times defines an isomorphism

$$H^{n-i}(M) \xrightarrow{\sim} H^{n+i}(M).$$

These theorems are proved using the Hodge decomposition of forms although they were first found by Lefshetz using geometric methods.

We can add to this structure theory of Kaehler manifolds by pursuing further the Hodge method and combining it with the generalized de Rham theory above.

The argument is simple to state and goes as follows:

On any complex manifold M we have the J-operator on real forms and we can form a new differential

$$d_c = J^{-1} d J.$$

Now we can form the natural diagram associated to M,

$$\left\{\begin{matrix} d_c \text{ closed forms} \\ \text{modulo } d_c \\ \text{exact forms} \end{matrix}\right\} \xleftarrow{\text{projection}} \left\{\begin{matrix} d_c \text{ closed} \\ \text{forms} \end{matrix}\right\} \xrightarrow{\text{inclusion}} \left\{\begin{matrix} \text{all} \\ \text{smooth forms} \end{matrix}\right.$$

On the left we have an algebra isomorphic to the cohomology ring and on the right we have the de Rham algebra. Because of the integrability condition that d and d_c anti-commute this is a diagram of <u>differential</u> algebras.

If the complex structure admits a Kaehler metric, the induced differential on the left is identically zero and the two maps induce isomorphisms of cohomology. This is easy to

which states that a closed form of type (p,q) is exact iff it can be written as $\partial\bar{\partial}$ of a form of type (p-1, q-1).
So we have

<u>Theorem K</u>: There is a homotopy equivalence in the sense of differential algebras between the real cohomology ring of a Kaehler manifold and its real de Rham algebra. The equivalence is natural for holomorphic maps between Kaehler manifolds.

<u>Corollary 1:</u> The minimal model for the real homotopy theory of a Kaehler manifold can be deduced formally from the cohomology ring. For example the real form of the lower central series of π_1 can be deduced from $H^1(M)$ and cup products $H^1 \otimes H^1 \to H^2$.

<u>Corollary 2:</u> The real (or rational) homotopy theory of a holomorphic map between Kaehler manifolds is determined by the induced map on real cohomology. For example, a holomorphic map between Riemann surfaces is completely determined up to homotopy by the map on first cohomology.

<u>Remarks</u> i) One should point out that although the isomorphism class of the minimal model of a Kaehler manifold is determined by the cohomology ring, the map of the model into the forms

$$\Lambda(x_1, x_2, \ldots; d) \xrightarrow{\rho} A_M$$

is still very interesting.

As remarked above, a metric makes ρ canonical and the structure of ρ on the subalgebra generated by degree 1 <u>only depends on the complex structure</u>. These ρ-forms should have geometric applications generalzing those of the abelian differentials or holomorphic 1-forms.

ii) The formality of the model is a very precise form of the statement that all higher order cup products (i.e. Massey products) vanish in a Kaehler manifold. This statement in its less precise form was suggested to P. Griffiths, J. Morgan, and myself by P. Deligne for algebraic manifolds as following from the unproved Weil conjecture about the size of the eigenvalues of Frobenius on ℓ-adic cohomology.

This remark certainly spurred the final form of these results which had languished in an embryonic state for nearly a year before Deligne's suggestion. Atthis conference Atiyah informed me that ten years ago Serre had observed that certain triple Massey products of 1-dimensional cohomology classes had to vanish because of the (p,q) decomposition on 1 and 2 dimensional cohomology.

The Space of Closed Curves on A Manifold

If M is a Riemannian manifold, it is interesting in the study of closed goedesics on M to understand the cohomology of the function space of all maps of the circle into M, $\Lambda(M)$.

If we denote the ordinary loop space of M, i.e. the space of based maps of the circle into M by ΩM, then we have the fibration

$$\begin{array}{ccc} \Omega M & \to & \Lambda(M) \\ & & \downarrow \uparrow \\ & & M \end{array}$$

with a canonical section.

In rational homotopy theory ΩM is fairly easy to understand – its minimal model will have one generator for each generator of the model for M in one lower dimension and zero differential: if model for $M = \Lambda(x_1, x_2, \ldots; d)$ the model for $\Omega M = \Lambda(\bar{x}_1, \bar{x}_2, \ldots;$ with $(\dim \bar{x}_i) = (\dim x_i) - 1$.

The homotopy structure of $\Lambda(M)$ is more subtle and uses all of the information in the model of M. Because of fibration with section, the generators of the model for $\Lambda(M)$ will be the x_i and the $\bar{x}_i$. The differential in the model is defined as follows

dx_i is the same as in the model of M, and

$d\bar{x}_i$ is $\overline{dx_i}$

where the bar operation on polynomials in the x_i is defined to be the unique derivation[1] extending the operation $x_i \to \bar{x}_i$.

1. acting on right to get the signs correct

So the model for the space of all closed curves on M is

$\Lambda(x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots; d)$

$$dx_i = dx_i, \; d\bar{x}_i = \overline{dx}_i.$$

For example let $M = S^2$ with model $\Lambda(x,y,d)$ with dim $x = 2$ and $dy = x^2$. The model for the space of closed curves on S^2 is $\Lambda(x, \bar{x}, y, \bar{y}, d)$ with $dx = d\bar{x} = 0$, $dy = x^2$, $d\bar{y} = 2x\bar{x}$.

We claim this formula is valid for simply connected manifolds (or even nilpotent spaces) and immediately implies the following:

Theorem: For all closed simply connected manifolds M, the space of closed paths on M has infinitely many non-zero Betti numbers in an arithmetic sequence of dimensions.

Proof: Let x denote an odd dimensional generator of the model for M of lowest dimension. (Such an x exists for otherwise the model would have only even dim'l polynomial generators, d would have to be zero, and the cohomology of M would be infinite dim'l.) Suppose dx is a polynomial in the even generators $e_1, e_2, \ldots, e_n$. Then consider in the model for the space of all closed curves on M, the family of elements

$$\{\bar{e}_1 \wedge \bar{e}_2 \ldots \bar{e}_n \wedge \bar{x}^j\} \qquad j = 1, 2, \ldots$$

By the choice of x, de_i is zero, so $d\bar{e}_i = 0$. Also $d\bar{x}$ is in the ideal generated by $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ by our formula. Since these elements are odd dim'l and have square zero we see that the infinite sequence of elements described above are all closed. None of these are exact because the formula shows the ideal of boundaries is contained in the ideal generated by $x_1, x_2, \ldots$. QED

This result gives some information on closed geodesics using the work of Gromall, Mayer, and Klingenberg. One would like to know however that the Betti numbers of the space of all closed curves grow arbitrarily large. In this case one knows that M has infinitely many distinct closed geodesics for any metric.

To have any hope for this one needs to know the Betti numbers of the ordinary loop space ΩM are unbounded. This question can be analyzed.

<u>Theorem</u>: The Betti numbers of the loop space of a simply connected finite complex X are bounded iff the cohomology ring of X is generated by one element.

<u>Problem</u>: Is this theorem true for the space of all closed curves?

We close this section by noting that the formula is proved by an induction argument over the Postnikov system. The final formula can be motivated by the following algebraic problem:

Given a differential algebra Λ find a new differential algebra Λ^- so that the maps of Λ^- into an arbitrary algebra C are in one to one correspondence with the maps of Λ into C with one variable adjoined in degree one.

This is the algebraic analogue of the relationship between a space X and the space of all closed curves on X, ΛX. Namely,

$$[K, \Lambda X] \simeq [K \times S^1, X].$$

The solution of the algebraic problem is just the solution of the homotopy theoretical problem. This idea is used in the proof and allows one to give analogous formulae for the topology of other function spaces.

i) If X has a finite triangulation with vertices $x_1, \ldots, x_n$, the rational de Rham algebra has an explicit presentation

$$\mathcal{E}_X = \Lambda(x_1, \ldots, x_n; dx_1, \ldots, dx_n)/I$$

where I is the ideal generated by

$$x_1 + \cdots + x_n - 1$$
$$dx_1 + \cdots + dx_n$$
$$\{x_{i_1} \cdots x_{i_r} \, dx_{j_1} \wedge \ldots \wedge dx_{j_s}\}$$

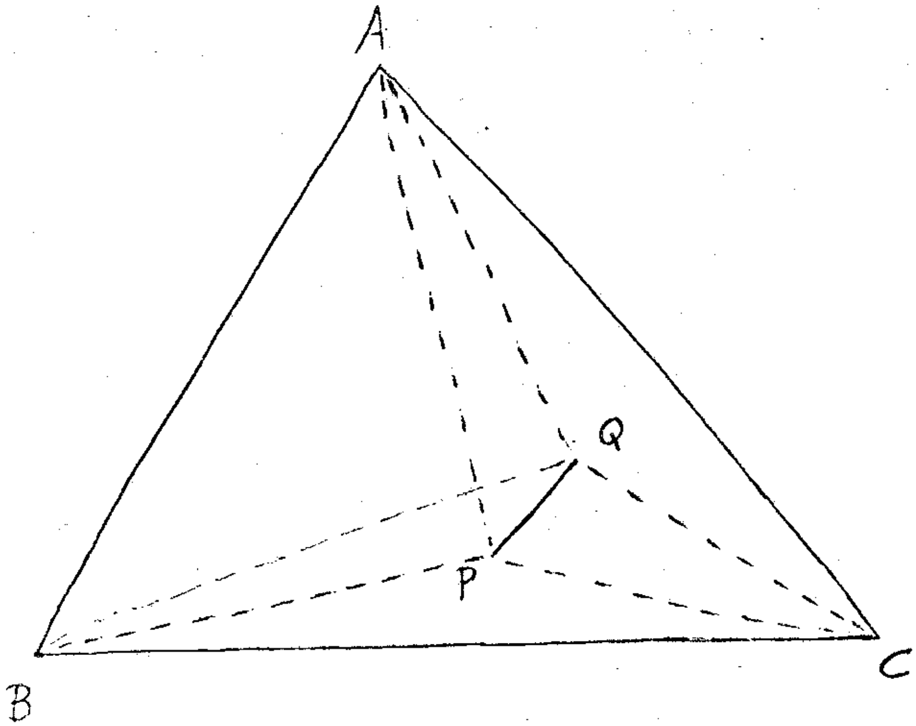
where $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, x_{j_1}, \ldots x_{j_s}\}$ is not spanned by a simplex.

ii) Whitney has defined a canonical chain inverse to integration

$$\text{cochains on } X \xrightarrow{\text{Whitney forms}} \mathcal{E}_X \xrightarrow{\text{Integration}} \text{cochains on } X$$

These Whitney forms will probably figure in certain geometrical questions on triangulated manifolds.

For example using the Whitney forms one can construct a very appealing chain map (over Q or R) from a subdivision (over Q or R) up to a coarser triangulation. In the special case of the figure



this chain mapping carries the one simplex PQ to a one chain whose coefficient on BC is the ratio of the area of APQ to ABC.

This construction can in turn be used to define a natural intersection for chains in the dual subdivision of a manifold. For this one uses the composition

dual subdivision → 1^{st} barycentric subdivision

↓ Whitney

initial triangulation

to push a chain on the dual subdivision back to the original triangulation. Then intersection with another chain in the dual subdivision can be calculated.

The motivation for this entire work on combinatorial forms and de Rham theory came at this point because it was hoped a sufficiently beautiful star operator could be constructed combinatorially to find a formula for the Pontryagin classes.

To find these formula for the Pontryagin classes it is sufficient to find a vertex formula for the signature of a triangulated 4k-manifold. This vertex formula can in turn be found if a symmetric transformation $C^{4k-i} \overset{*}{\to} C^{i}$ can be constructed so that

i) $* d = \pm \partial *$

ii) $* \partial = \pm d *$

iii) * is local in character

iv) * is a matrix for cup product followed by evaluation on the orientation class.

v) the eigenvalues of * are contained in $\{0, +1, -1\}$.

APPENDIX H (Hirsch Method)

If $G \to E \to M$ is a principal G-bundle over a manifold M, and $\mathcal{A}$ is a differential algebra mapping to the forms on M which computes the cohomology of M, then

$$\mathcal{A} \otimes \Lambda(x_1, \ldots, x_n)$$

can be supplied with a differential in such a way that the cohomology of E results. Here $\Lambda(x_1, \ldots, x_n)$ is the cohomology of the fibre G and $d(1 \otimes x_i)$ is $c_i \otimes 1$ where c_i is a representative of the characteristic class of E.

This same method is applied inductively to Postnikov systems to prove the theorems of the paper.

Now we are inductively considering principal fibrations

$$\begin{array}{ccl} K & \to & X_{n+1} \\ & & \downarrow \pi_n \\ & & X_n \end{array}$$

where K has one non-zero Abelian homotopy group in dimension $i_n \geq 1$.

Using the Hirsch method and the existence of the rational de Rham algebra which is commutative and correctly computes cohomology we can inductively construct differential algebras of the minimal model type which compute the cohomology of the spaces $\{X_{n+1}\}$ in any countable system of maps like the π_n.

At each stage in the construction we are adding a vector space of generators of the same rank and dimension of the homotopy group of K to the algebra for X_n to obtain the algebra for X_{n+1}.

This fact and the uniqueness property of the model which is an algebraic analogue of the corresponding properties for Postnikov systems provide the skeleton of the proof of theorem B.

APPENDIX N (Non-Simply Connected Spaces and Nilpotent groups)

When the first Betti number is non-zero, the construction of the minimal model of the de Rham algebra has possibly infinitely many steps in each dimension.

For example, the subalgebra of the model generated by the generators in dimension one is an increasing union of exterior algebras with differential

$$\Lambda_1 = \bigcup_{n=0}^{\infty} \Lambda_{1,n}.$$

$\Lambda_{1,0}$ is the exterior algebra on the first cohomology with $d = 0$, and $\Lambda_{1,n}$ is obtained from $\Lambda_{1,n-1}$ by adjoining a vector space of generators in degree one of the same rank as the Abelian group, C_{n-1}/C_n where $\{C_n\}$ are subgroups of the lower central series of π_1 defined by

$$C_0 = \pi_1, \text{ and } C_{n+1} = [C_n, \pi_1], \ n \geq 0.$$

A differential is defined which determines the structure of the extension of nilpotent groups

$$0 \to C_{n-1}/C_n \to \pi_1/C_n \to \pi_1/C_{n-1} \to 1$$

tensored by Q or R whichever is relevant.

This determination of structure uses two facts

i) a nilpotent group over Q or R determines and is determined by its Lie algebra (over Q or R) using the expontential isomorphism.

ii) the Lie algebra over Q or R when dualized is nothing other than an exterior algebra on a vector space in degree one with a differential.

mines the nilpotent quotient π_1/C_n tensored by Q or R.

It is perhaps important to note that the minimal model Λ of the de Rham algebra is constructed in a purely algebriac computational construction as indicated in the text above. The structure of the model then determines the form of the lower central series of π_1 (over Q and R) and so on for the further rational or real homotopy theory.

The subalgebra Λ_1 of Λ mapping to the de Rham algebra is constructed so we have an isomorphism of H^1 and an injection on H^2. Then 2 dim'l generators are added in a possibly infinite sequence of steps. The first step creates an isomorphism on H^2 and the later steps produce an injection on H^3. Then 3 dim'l generators are added in a possibly infinite sequence of steps, 4 dim'l, and so on to build the complete model.

The structure of model over Q for the rational de Rham algebra of a space corresponds (via the Hirsch calculation described in Appendix H) to a tower of spaces and maps. Each map in the tower is a principal fibration where the fibre has one non-zero homotopy group which is a rational vector space.

This tower of spaces is the rational homotopy type of the space which is defined in general for all spaces regardless of the fundamental group.

We have described above the relationship between the Λ_1 of the model (and therefore the tower) and the lower central series of the fundamental group of the space. The significance of the higher dimensional structure of the model and the tower for the general non-simply connected spaces is somewhat complicated and not yet completely understood except in a philosophical way.

APPENDIX F

Strictly speaking the theorem is correctly stated for spaces with finite Betti numbers. It is easy to treat the general case however. One only needs to observe that any space is a union of its finite subcomplexes and the Q-cohomology of the union is the inverse limit of these finite dimensional cohomologies.

Thus in the theorem we really need inverse limits of models for spaces with finite Betti numbers. Alternatively we could dualize these to coalgebras.