11

Open and closed string field theory interpreted in classical algebraic topology

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Abstract

There is an interpretation of open–closed string field theory in algebraic topology. The interpretation seems to have much of the expected structure but notably lacks the vacuum expectations. All the operations are defined by classical transversal intersection of ordinary cycles and homologies (derived from chains in path spaces) inside finite-dimensional smooth manifolds. The closed string theory can be deduced from the open string theory by the known equivariant chain or homology construction. One obtains the interpretation of open and closed string field theory combined. The algebraic structures derived from the first layer of open string interactions realize algebraic models discussed in work of Segal and collaborators. For example Corollary 1 of §11.1 says that the homology of the space of paths in any manifold beginning and ending on any submanifold has the structure of an associative dialgebra satisfying the module compatibility (equals Frobenius compatibility). See the appendix for the definition of six kinds of dialgebras. Corollary 2 gives another dialgebra structure which is less known. Corollary 3 gives yet another, the Lie bialgebra of [3].

1 Open string states in M

The open string theory interpretation in topology takes place on the homology or on the chain level–referred to respectively as 'on-shell' and 'off-shell'. On-shell there will be a linear category $[\vartheta M]$ for each ambient space M, a finite dimensional oriented smooth manifold possibly with general singularities. A morphism in this category is called an (on-shell) open string state. In Greg Moore's paper in this volume the idea to connect formal properties of open string states in physics to morphisms of a category is credited to Graeme Segal. Eventually the categories here become dicategories generalizing the dialgebras of the appendix.

The objects in the category $[\vartheta M]$ include the smooth oriented submanifolds (without singularities) $L_a, L_b, L_c, \ldots$ of M. The set of morphisms $[\vartheta_{ab}]$ between two such objects L_a and L_b is the graded homology (with coefficients in $\mathbb{Z}/n$, $\mathbb{Z}$, or $\mathbb{Q}$) of $P(a, b)$, the space of smooth paths starting in L_a and ending in L_b. The composition of morphisms $[\vartheta_{ab}] \otimes [\vartheta_{bc}] \overset{\wedge}{\to} [\vartheta_{ac}]$ is defined as follows. Choose representative cycles x in $P(a, b)$ and y in $P(b, c)$. The endpoints of x and beginning points of y define respectively two cycles (of points) in L_b. These cycles can be intersected transversally in L_b after small perturbation in L_b to obtain a cycle z of dimension equal dim x + dim y − dim L_b.

Now z parametrizes a set of paths from L_a to L_b and a set of paths from L_b to L_c which are composable along z. These are made out of the original paths plus small pieces from the perturbation. After composition (joining and parametrizing) z defines a cycle in the space of paths from L_a to L_c defining an element in $[\vartheta_{ac}]$. The composition is well defined and associative on the level of homology, namely on-shell, using familiar arguments (see the discussion immediately following).

When M is an oriented manifold without singularities each of the submanifolds $L_a, L_b, \ldots$ has an oriented normal bundle and the additional structure of dicategory can be defined. By this we mean for each triple of objects L_a, L_b and L_c there is a cocomposition or cutting operation $[\vartheta_{ac}] \overset{\vee_t}{\to} [\vartheta_{ab}] \otimes [\vartheta_{bc}]$ defined as follows. Choose a $t \epsilon [0, 1]$ and a representative cycle z for an element in $[\vartheta_{ac}]$. Evaluating each path labelled by z at time t yields a cycle (of points) in M. After small perturbation one can transversally intersect in M this cycle (of points in M) with L_b to obtain a cycle w in L_b of dimension equal to (dimension z–codimension L_b). The cycle w labels pairs of paths, one from L_a to L_b (the part of z's path from 0 to t) and one from L_b to L_c (the part of z's path from t to 1). Passing to homology classes and applying the Kunneth property yields an element in $[\vartheta_{ab}] \otimes [\vartheta_{ac}]$ (at least with coefficients in $\mathbb{Z}/n$ or $\mathbb{Q}$).

If t' is a different time in [0,1] we can evaluate z between t and t' to obtain a chain W in M of dimension equal to dimension $(z) + 1$. Assuming z at t and at t' is transversal to L_b, W will also be transversal to L_b near its boundary (at t or t'). A small relative perturbation can make W transversal to L_b without changing it near the boundary. (This kind of argument is used often to show that these transversally defined operations are well defined in homology.)

This provides a homology between the cycle w defining $\vee_t(z)$ and the cycle w' defining $\vee_{t'}(z)$. Thus $\vee_t$ on homology is independent of t in [0,1]. A similar argument shows $\vee_t$ is coassociative on the level of homology.

The independence of t allows two different computations of $\vee_t(x \wedge y)$. By choosing t in x's time we get on the cycle level $\vee_t(x) \wedge y$ (composing in the right factor of the pair of paths). By choosing t in y's time we get on the cycle level $x \wedge \vee_t(y)$ (composing on the left factor of the pair of paths). Summarizing, we have:

Theorem 1 (on shell Frobenius compatibility). *For each oriented manifold the sets of homology classes $[\vartheta_{ab}]$ of the path spaces between arbitrary oriented submanifolds L_a and L_b are the sets of morphisms of a dicategory satisfying module compatibility (equals Frobenius compatibility). By this phrase we mean we have objects and morphisms and there are associative compositions of morphisms $[\vartheta_{ab}] \otimes [\vartheta_{bc}] \overset{\wedge}{\to} [\vartheta_{ac}]$ and coassociative cocompositions of morphisms $[\vartheta_{ac}] \overset{\vee_t}{\to} [\vartheta_{ab}] \otimes [\vartheta_{bc}]$ satisfying $x \wedge \vee_t(y) = \vee_t(x \wedge y) = \vee_t(x) \wedge y$. (In this formulation the coefficients are $\mathbb{Z}/n$, $\mathbb{Z}$, or $\mathbb{Q}$ for the composition and $\mathbb{Z}/n$ or $\mathbb{Q}$ for the cocomposition.)*

Corollary 1. *For each object $L_a \subset M$, the on shell self morphisms of the object L_a of the dicategory $[\vartheta M]$, the homology of paths in M beginning and ending on L_a with coefficients in $\mathbb{Z}/n$ or $\mathbb{Q}$, forms an associative dialgebra satisfying the module compatibility (equals Frobenius compatibility). See the appendix for a discussion of dialgebras.*

Example 1 (manifolds). If L_a is taken to be all of M, the space of paths from L_a to L_a is homotopy equivalent to M itself. Then $\wedge$ is identified on shell with ordinary intersection of homology in M. Also $\vee_t$ is identified on shell with the diagonal map on homology of M. We recover the classical fact that the homology of an oriented manifold has the structure of an associative dialgebra with module compatibility (equals Frobenius compatibility). If M is also closed the intersection multiplication and the diagonal comultiplication are in hom duality via the Poincaré duality inner product and we have the special case of a (graded) commutative Frobenius algebra.

Example 2 (free loop space). If the ambient space is $M \times M$ and the submanifold is the diagonal M in $M \times M$, then the space of paths in $M \times M$ beginning and ending in M is homotopy equivalent to the free loop space of M defined by smooth maps of the circle into M. The homology of the free loop space receives a product $\wedge$ and a coproduct $\vee_t$. The product agrees (on shell) with the loop product from 'String Topology' [2]. The coproduct is only non-zero if M is a closed manifold with non-zero Euler characteristic. (Otherwise $M = L_a$ can be deformed in $M \times M$ off of itself to $L_{a'}$. We can compute $[\vartheta_{aa}] \overset{\vee_t}{\to} [\vartheta_{aa'}] \otimes [\vartheta_{a'a}]$ for $t = 0$ (or $t = 1$) at the cycle level to see we get zero. Then we identify $[\vartheta_{aa}]$ with $[\vartheta_{aa'}]$ and with $[\vartheta_{a'a}]$ using the deformation of L_a to $L_{a'}$.)

Example 3 (based loop space). If L_a is a point in M the space of paths beginning and ending on L_a is just ΩM the based loop space of M. The product on the homology of ΩM defined above agrees (by definition) with the usual Pontryagin product on the homology of the based loop space. The coproduct defined above (when M is a manifold near L_a) is zero because L_a can be deformed in M off of itself (see example 2). The fact that the Pontryagin algebra is a Hopf algebra (for the diagonal coproduct) and the fact that a (finite dimensional) Hopf algebra is a Frobenius algebra is only suggestive at this point *vis-à-vis* the above theory.

Remark. There is an algebra homomorphism (on shell) of the free loop space algebra to the base loop space algebra of degree − dim (ambient space). It is defined by transversally intersecting the cycle of marked points from a cycle of loops with the base point (compare [2]).

Remark. The readers comfortable with homology intersection defined by geometric cycles and transversality will be able to add details they feel are needed above and below to the proofs of Theorems 1 and 2 and Corollaries 1 and 2 except possibly for orientations. Orientations are discussed in [3] along with specific information about the proof of Theorem 3 and Corollary 3 to be expanded elsewhere. Otherwise the text should be regarded as an outline or sketch of the proof of these results.

2 On-shell and off-shell

In the above constructions at the cycle or chain level the conclusions were stated at the homology level. We refer to these two levels respectively as ‘off-shell’ and ‘on-shell’. A remark about these expressions in terms of familiar topology may be useful. If one tries to lift a homological structure to the geometric level discrepancies often show up. For example one can associate harmonic forms to cohomology classes. Harmonic forms are ‘on-shell’ for physicists because they satisfy the critical point equations associated to the energy action. The cohomological product is represented by the wedge product of these harmonic forms which is (almost always) not harmonic and therefore ‘off-shell’.

One knows that putting in chain homotopies resolving this discrepancy of the product (and continuing) constructs algebraic models of the real (or rational) homotopy type [13] [15]. Recently a remarkable result [10] of Michael Mandell shows similar chain homotopies for integral cochains and their cup product when suitably organized determines the entire homotopy type for simply connected spaces.

Bearing this in mind it seems worthwhile to also study the above string theory at the cycle and chain level–namely off-shell. The idea is that an algebraic structure on-shell will be reflected in a more elaborate structure off-shell made out of a hierarchy of chain homotopies. A further idea is that the off-shell structure may be easier to work with in certain respects than the on-shell structure. For example Quillen's model of rational homotopy theory is a *differential* on a *free Lie algebra* which (we now know) is organizing the off-shell strong homotopy commutative associative cup product structure. In some sense the on-shell structure 'graded commutative algebra' is harder to classify and understand than the off-shell structure, which is 'free differential graded Lie algebra', because of the freeness.

Another idea comes up here. The notion of these hierarchical homotopies or strong homotopy structures due to Stasheff is very intuitive but combinatorially complicated. However in a number of cases this complexity is absorbed in a single operator of square zero on a free object for a dual structure (see Ginzburg–Kapranov [7] for the definition and explanation of this property of Koszul dual pairs of structures over operads and see [6] for algebras over dioperads).

Let us return to the rational homotopy example and the graded Lie algebra of homotopy groups of a space. The off-shell version of the Lie algebra of homotopy groups would be a strong homotopy Lie algebra which can be described by a differential on a *free* graded commutative algebra (by Koszul duality between commutative algebra and Lie algebra). The latter differential may be computed [15] inside the differential forms starting from harmonic forms (or any other lift) and iteratively correcting the off-shell wedge products by chain homotopies.

An early example of this on-shell–off-shell discussion in topology (and the first exactly in this vein) was Stasheff's notion of a strong homotopy associative algebra, or A_∞ algebra. The latter may be described by a differential on the free tensor algebra (of the dual space). An analogous notion of A_∞ category is also defined where composition is only associative up to homotopy etc. (see [1]).

3 Open strings off shell

Now we work off-shell with the set of cycles and chains ϑ_{ab} in the path spaces $P(a, b)$. For example, we can take ϑ_{ab} to be linear combinations (over $\mathbb{Z}/n$, $\mathbb{Z}$ or $\mathbb{Q}$) of smooth maps of standard simplices into $P(a, b)$ (namely, (simplex) x [0,1]$\rightarrow$ M is smooth).

The discussion in §1 of operations defined by transversality can now be considered off-shell at the chain level. Intersection of ordinary chains in M was developed as an A_∞ structure [1]. A similar discussion should show composition or joining of off-shell open string states (chains in $P(a, b)$, ...) will generate an A_∞ category. Going further the off-shell analogue of Theorem 1 becomes.

Conjecture 1. *The off-shell open string states, the chains ϑ_{ab} on path spaces ($P(a, b)$, ...) form the morphisms of a strong homotopy dicategory, satisfying the module compatibility (equals Frobenius compatibility) between composition and cocomposition.*

A special case of Conjecture 1 is that the chains on paths in M from L_a back to L_a has the structure of a strong homotopy dialgebra satisfying the module compatibility (equals Frobenius compatibility).

There is another structure beyond this we could mention now. Consider cutting paths at any time and then use Eilenberg Zilber relating chains in a Cartesian product to tensor product of chains to define a new cocomposition $\vartheta_{ac} \overset{\vee}{\rightarrow} \vartheta_{ab} \otimes \vartheta_{bc}$ of one degree higher than $\vee_t$. This operator does not commute with the ∂ operator in general. In fact (as proved above) $\partial \vee + \vee \partial = \vee_1 - \vee_0$.

This operator satisfies a new compatibility with $\wedge$ called derivation compatibility.

Theorem 2 (off-shell derivation compatibility). *For appropriately transversal chains $\wedge$ and $\vee$ are defined and satisfy $\vee(x \wedge y) = \vee(x) \wedge y \pm x \wedge \vee(y)$.*

Corollary 2. *When L_a is deformable in M off of itself the homology of the space of paths in M beginning and ending on L_a has the structure of an associative dialgebra satisfying derivation compatibility.*

Remark. In the papers of Aguilar dialgebras with derivation compatibility are called 'infinitesimal bialgebras'. Aguilar attributes the concept to Gian Carlo-Rota *et al.* who introduced it in the 1960s to study certain combinatorial problems.

Proof of Theorem 2. $x \wedge y$ is represented by paths of x joined to paths of y (where the appropriate endpoints transversally intersect). Cutting along some L transversally we get two terms where the cut belongs to the x part or to the y part. This is the right-hand side of compatibility.

Proof of Corollary 2. As mentioned above when L_a is deformable off of itself to $L_{a'}$ then with regard to cutting paths from L_a back to itself along $L_{a'}$ $\vee_0$ and $\vee_1$ are zero at the chain level. Then $\vee$ defined transversally commutes with ∂ and passes to homology. The required identity on-shell follows from Theorem 2.

Remark. The structure of Corollary 2 may depend on the isotopy class of the push off.

Corresponding to Theorem 2 there is:

Conjecture 2. *The off-shell open strings states have a strong homotopy structure involving $\wedge$, $\vee_0$, $\vee_1$, and $\vee$, the various associativities, the two compatibilities Frobenius and derivation, and $\partial \vee + \vee \partial = \vee_1 - \vee_0$.*

Adding to the fun of formulating conjecture 2 exactly we note that in [6] it is asserted that the algebraic structures associated to $(\wedge, \vee_t)$ and to $(\wedge, \vee)$ on-shell in Corollary 1 and Corollary 2 are Koszul dual (for more see Appendix).

4 Closed strings

We have seen in example 2 of §1 above that open strings beginning and ending on the diagonal in $M \times M$ gives the free loop space of M, namely smooth maps of the circle into M. Now the free loop space also has a circle action by rotating the domain. *The closed string states in M on-shell or off-shell will be defined as the equivariant homology or chains relative to this circle action.* There are several models for the equivariant theories. We will employ here a geometric one called ‘closed string space’.

A point in closed string space $S(M)$ is a pair (L, f) where L is a complex line in $\mathbb{C}^\infty$ = {finite sequences of complex numbers} and f is a smooth map of the unit circle in L into M.

Remark. Note that:

(1) $S(M)$ fibres over $\mathbb{C}P^\infty$ with fibre the free loop space of M.
(2) $S(M)$ is the base of a circle fibration with total space equivariantly homotopy equivalent to the free loop space of M.

Proof. Projection onto the first factor of the pair (L, f) proves 1. For 2 let the total space be triples (v, L, f) where v is a unit vector in a complex line L in $\mathbb{C}^\infty$ and f is a smooth map of the unit circle in L into M. Note the set of (v, L) is contractible. □

Definition. The homology classes of closed string space $S(M)$ are the *on-shell closed string states*. The chains on $S(M)$ are the *off-shell closed string states*.

Remark. The projection of the circle bundle (or the inclusion of the fibre of S(M) over $\mathbb{C}P^{\infty}$) defines E a degree zero chain or homology mapping from the free loop space of M to the closed string space (E for erase the isometric parametrization (or mark) of the circle). Taking the pre-image of the circle bundle projection defines a degree one chain or homology mapping M in the opposite direction (M for add a mark or isometric parametrization to a closed string in all ways to get a circle of loops.) We hope the double use of 'M' here does not cause a problem.

The composition EM produces a degenerate chain and may be regarded as zero by working in the quotient by degenerate chains. The composition ME is usually denoted $\triangle$. It is the operator of degree $+1$ on chains or homology of the free loop space associated to the circle action. Since $\triangle \cdot \triangle = (ME)(ME) = M(EM)E$, we have $\triangle^2 = 0$ on-shell and even off-shell mod degenerate chains.

In [2] it was shown the operator $\triangle$ on the homology of the free loops space with the open string product defined a BV or Batalin Vilkovisky algebra. Namely, the deviation of $\triangle$ from being a derivation of the open string product is a Lie bracket (of degree $+\ 1$) compatible with the open string product via the Leibniz identity.

Remark. (1) This bracket was also defined [2] from an off-shell operation $*$ by skew symmetrization just as Gerstenhaber did in the Hochshild complex of an associative algebra. This fits with the idea that the Hochshild complex $\oplus_k$ Hom $(A^{\otimes k}, A)$ of the intersection algebra A of chains models the free loop space of a simply connected closed manifold (cf [5][16]).

(2) We will discuss below a Lie product or bracket on the closed string states which is compatible via the mapping M (adding a mark) with the BV or Gerstenhaber Lie bracket mentioned in 1.

(3) This closed string product or bracket generalizes to all manifolds the Goldman bracket on the vector space generated by conjugacy classes in the fundamental group of oriented surfaces. The Goldman bracket is a universal version of the Poisson structure on the moduli space of flat bundles over a surface. We suppose the off-shell string bracket for $S(M)$ bears a similar relation to general bundles with general connections over M (compare Cattaneo Frohlich et al.).

The string product on closed string states satisfying Jacobi (on the transversal chain level [3]) may be defined by the formula $[\alpha, \beta] = E(M\alpha \wedge M\beta)$ where $\wedge$ is the open string product. Other closed string operations c_n can be defined by $c_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = E(M\alpha_1 \wedge M\alpha_2 \wedge \ldots \wedge M\alpha_n)$. These all commute with the ∂ operator and satisfy commutation identities transversally [2].

The collision operators c_n pass to the reduced equivariant complex or *reduced closed string* states which is defined to be the equivariant homology for the S^1 pair (free loop space, constant loops). This passage follows from the formulae for c_n because the marking operator M takes a chain of constant loops to a degenerate chain of constant loops.

We can define a closed string cobracket s_2 by the formula $s_2(\alpha) = (E \otimes E)(\vee(M\alpha))$. In the reduced complex s_2 commutes with ∂ and passes to homology (but not so in the unreduced complex [3]).

Theorem 3 (closed string bracket and cobracket). *The closed string bracket $c_2(\alpha, \beta) = E(M\alpha \wedge M\beta)$ where $x \wedge y = \wedge(x \otimes y)$ and the closed string cobracket $s_2(\alpha) = (E \otimes E)(\vee M\alpha)$ satisfy respectively jacobi, cojacobi, and derivation compatibility (equals Drinfeld compatibility). The term satisfy means either on the level of $\mathbb{Z}/n$ or $\mathbb{Q}$ homology or for transversal chains on the chain level (see appendix for discussion of compatibilities).*

Proof. These formulae in terms of open strings are reinterpretations as in [2] of the definitions given in 'Closed string operators in topology leading to Lie bialgebras and higher string algebra' [3]. In [3] the identities at the transversal chain level were considered. □

Corollary 3. *Homology of reduced closed string states forms a Lie bialgebra,* [3].

Remark. The corollary generalized Turaev's discovery [17] of a Lie bialgebra for surfaces to all manifolds. Questions in [17] motivated this work. See [4] for some answers and further developments.

Conjecture 3. *The off-shell closed string states (reduced) have the structure of a strong homotopy Lie bialgebra.*

Remark. Other cobracket or splitting operations $s_3, s_4, \ldots$ can be defined similarly by iterations of $\vee$, $s_n(\alpha) = E \otimes \ldots \otimes E(\ldots \vee \otimes 1 \cdot \vee(M\alpha))$. These also commute with ∂ and pass to homology in the reduced equivariant theory. A conjecture about $c_2, c_3, \ldots; s_2, s_3, \ldots$ generating genus zero closed string operators and the algebraic form of this structure was proposed in [3] and relates to [9].

5 Interplay between open and closed string states

Let $\mathcal{C}$ denote the closed string states in M, a manifold of dimension d, and let ϑ denote any of the complexes ϑ_{ab} of open string states. Transversality yields an action of closed strings on open strings

$$\mathcal{C} \otimes \vartheta \to \vartheta \qquad \text{degree} = (-d+2)$$

and a coaction of closed strings on open strings

$$\vartheta \to \mathcal{C} \otimes \vartheta \qquad \text{degree} = (-d+2).$$

The operations are defined off-shell for transversal chains. In the coaction we let the open string hit itself transversally inside M of dimension d at any two times and split the event into a closed string and an open string. In the action we let a closed string combine with an open string to yield an open string. We lose d dimensions by the intersection in M and gain two from the possible positions on each string of the attaching points.

The action is a Lie action of the Lie algebra of closed strings by derivations on the algebra of open strings. This is seen by looking directly at the construction at the transversal chain level. Both the action and the coaction have a non-trivial commutator with the boundary operator on chains. These boundary terms are expressed by interactions between the closed string and the open string at the endpoints of the open string.

For the action the individual boundary terms commute with the boundary operator and pass to homology. For the coaction the individual boundary terms have themselves additional boundary terms to be elucidated.

Problem and Conjecture 4. *The action and coaction between open strings and closed strings and their boundary interaction terms are described by a strong homotopy structure to be elucidated.*

6 Connection to work of Segal and collaborators

Dialgebras satisfying the module or Frobenius compatibility give examples of $1+1$ TQFT's without vacuum expectations. In the commutative case we associate the underlying vector space to a directed circle, its tensor products to a disjoint union of directed circles and to a connected 2D oriented bordism between two non-empty collections the morphism obtained by decomposing the bordism into pants and composing accordingly the algebra or coalgebra map. The module compatibility (equals Frobenius compatibility) is just what is required for the result to be independent of the choice of pants decomposition.

N.B. this description differs from the usual one because we do not have disks to close up either end of the bordism. One knows these discs at both ends would force the algebra to be finite dimensional and the algebra and coalgebra to be related by a non-degenerate inner product. We refer to these generalizations of the Atiyah–Segal concepts as the positive boundary version of TQFT (a name due to Ralph Cohen). The editor notes that Segal refers to the underlying algebras of positive boundary 2D TQFT as non compact Frobenius algebras.

An exactly similar discussion with non-commutative associative dialgebras satisfying the Frobenius compatibility leads to a positive boundary version of a TQFT using open intervals. Now the algebra and coalgebra are associated to 1/2 pants (a disc with ∂ divided into six intervals–three (1/2 seams) alternating with two (1/2 cuffs) and one (1/2 waist)). Any planar connected bordism between two non-empty collections of intervals determines a mapping between inputs and outputs.

The structures we have found (including ∂ labels $L_a, L_b, \ldots$) for open strings using the composition $\wedge$ and fixed time cutting $\vee_t$ satisfies this Frobenius compatiblity up to a chain homotopy and we can apply it at the homology level in the relative TQFT scheme just mentioned.

Remark. One can show the on-shell structure of open and closed strings gives an example of the structure described in Moore's article of these proceedings. This follows by showing cycles and homologies on the moduli space of open closed string Riemann surfaces acts on open closed string states. For example, there is an operation on pairs of open strings which combine and reconnect at arbitrary interior points. A general operation is essentially a composition of the latter with all the above.

7 Summary

We have described the part of the interpretation of open and closed string field theory in topology associated to the basic product and coproduct (and in the equivariant setting certain implied n-variable splitting and collision operators as in [3]). The coproduct discussion has two levels involving a coproduct $\vee_t$ and an associated chain homotopy coproduct $\vee$.

We found the open string product and the coproduct $\vee_t$ satisfied the module compatibility (equals Frobenius compatibility) on the level of homology namely on-shell. In a setting where $\vee_0$ and $\vee_1$ were zero or even deformable to zero, $\vee$ emerges as or can be deformed to a coproduct commuting with ∂ and thus a coproduct $\vee$ on homology of one higher degree. Then a new compatibility with the product is observed – the derivation or infinitesimal bialgebra compatibility (also true at the transversal chain level and therefore suggesting a corresponding strong homotopy structure which was Conjecture 2).

Remark. The submanifolds which are the objects of the open string categories here are called D-branes in the math physics literature. We are currently considering more general boundary conditions forced on us by 3D computations which lead us to flat bundles along submanifolds and more general sheaves.

For closed strings in M we considered the equivariant theory associated to open strings on the diagonal in $M \times M$.

The higher genus interpretation of open closed string field theory in topology involves full families of arbitrary cutting and reconnecting operations of a string in an ambient space M. For closed curves some full families of these operators were labelled combinatorially by decorated even valence ribbon graphs obtained by collapsing chords in general chord diagrams in [3]. There is a compactness issue for the full families discussed there for realizing these in algebraic topology. The issue is a correct computation of the boundary. The problem has a parallel with renormalization in Feynman graphs (see the Bott–Taubes [18] treatment of configuration space integrals).

In both cases algebraic topology transversality and Feynman graphs the loops in collapsing subgraphs cause the problems. We hope to address this issue using Penner's intriguing paper [12].

Appendix: (dialgebras and compatibilities)

Let us call a linear space V with two maps $V \otimes V \overset{\wedge}{\to} V$ and $V \overset{\vee}{\to} V \otimes V$ a dialgebra. *Associative dialgebra* means $\wedge$ is associative and $\vee$ is coassociative. *Commutative dialgebra* means besides being associative $\wedge$ and $\vee$ are symmetric. *Lie dialgebra* means both maps are skew symmetric and that jacobi and cojacobi hold.

In all these cases V and $V \otimes V$ have module structures over V and there are two kinds of compatibilities between $\wedge$ and $\vee$ relative to these.

The compatibilities we consider here are:

derivation compatibility $\quad \vee(a \cdot b) = (\vee a) \cdot b + a \cdot \vee(b)$and
module compatibility $\quad \vee(a \cdot b) = \vee(a) \cdot b = a \cdot \vee(b)$

Where the $\cdot$ refers to the algebra structure or the module structure (which means in the associative case $a \cdot (b \otimes c) = (a \cdot b) \otimes c$, $(a \otimes b) \cdot c = a \otimes (b \cdot c)$

and in the Lie case $a \cdot (b \otimes c) = -(b \otimes c) \cdot a = [a, b] \otimes c + b \otimes [a, c]$ where $[x, y] = \wedge(x \otimes y)$).

We get six kinds of structures (five appear in this paper, see table below) which are examples of definitions of algebras over dioperads [6]. Algebras over dioperads are structures whose generators and relations are described diagrammatically by trees.

The familiar example of a compatibility studied by Hopf that $\vee$ is a map of algebras (associative or commutative case but not Lie) is described by a non-tree diagram and is not an algebra over a dioperad.

Table with names of compatibility and/or structure and/or examples.

	Module compatibility	Derivation compatibility
Associative dialgebra	Frobenius compatibility Special case = Frobenius algebra = associative algebra with non-degenerate invariant inner product	These are called infinitesimal bialgebras by Aguilar
Commutative dialgebra	Frobenius compatibility Special case = Commutative Frobenius algebra	commutative cocommutative infinitesimal bialgebra
Lie dialgebra	Frobenius compatibility Special case = Lie algebra with non-degenerate invariant inner product	Drinfeld compatibility These are called Lie bialgebras in the literature

In [6] Koszul dual pairs are defined and there it is proved that upper left and upper right are Koszul dual pairs and that middle left and lower right are Koszul dual pairs. We suppose that the lower left and middle right are also Koszul dual pairs.

We emphasize these Koszul relations because in several important situations a strong homotopy algebraic structure of one kind is very naturally expressed by freely generated diagrams decorated with tensors labeled by the Koszul dual structure. Our main conjecture in the above discussion is that *all the structures that are true transversally will lead to strong homotopy versions on the entire space of states.* These might be usefully expressed in this graphical Koszul dual way.

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