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Sigma models and string topology

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ABSTRACT. Gromov's homology class of J-holomorphic curves is shown to satisfy at the chain level the quantum master equation $\partial S = \Delta S + \frac{1}{2}\{S, S\}$ in a background described by algebraic topology and transversality. Conjecturally, this provides a mathematical interpretation of Witten's discovery of the corresponding Gromov homology periods as the correlations of a quantum field theory. The basic idea is due to Zwiebach. A new concept of unbounded algebraic structures arising from transversality is used.

CONTENTS

1. Short introduction

Associated to any oriented smooth manifold M of even dimension d, the target manifold of the sigma model[1], there is a modular operad structure (C_{ij}, S_{ij}) on one version, and a related BV algebra structure[2] $(\delta, \{\ ,\})$ on another version of an equivariant singular chain complex associated to the mapping spaces $\mathrm{Map}(\Sigma, M)$ of the corresponding sigma model. These structures are in the unbounded sense

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[1] Sigma models are putative quantum field theories where the classical fields are based on all maps of all Riemann surfaces probing a target Riemannian manifold M, and a classical action on these is given by the energy of the map.

[2] Used by Batalin and Vilkovisky [**2**] in an algebraic quantization scheme for classical field theories with constraints. In the sigma model the defining equations of the target M could be considered to be constraints. A BV algebra is a graded commutative algebra with an operator Δ of degree -1 so that $\Delta \cdot \Delta = 0$ and the deviation of Δ from being a derivation, $\{\ ,\}$, is itself a derivation in each variable. It follows $\{\ ,\}$ is a Lie bracket of degree -1 and Δ is a derivation of $\{\ ,\}$.

(see Section 3). Here Σ runs over diffeomorphism types of closed oriented connected surfaces of genus g with n marked points (i.e., there is one Σ for every (g, n)). After an even shift of gradings by $(d-6)(1-g)+2n+c(\beta)$, $\beta \in H_2(M)$, c linear and even, the degrees of the chain complex ∂ operator and transversal gluing operations C_{ij}, S_{ij}, δ, Δ, and $\{\ ,\}$ are all equal to -1. A solution S in degree zero of the quantum master equation, or $\Delta S + \frac{1}{2}\{S, S\} = \partial S$ in the BV algebra, will be treated homologically in a future publication.

When M is almost complex, the dimension shift $(d-6)(1-g)+2n+2c_1(\beta)$ gives the formal real dimension of the piece of the moduli space corresponding to J-holomorphic curves of genus g and n marked points in the homology class of β in $H_2(M)$. A J-holomorphic curve in M is an equivalence class of a pair (complex structure on Σ, map of Σ into M) such that $\bar{\partial}(\text{map}) = 0$.

When M is closed symplectic, one can introduce constraints in the complex structure of Σ and the concentration of energy in the map defining a J-holomorphic curve or a perturbed version. Then for each g, n, and β, the set of perturbed J-holomorphic curves defines a compact set and, assuming enough transversality, an oriented chain. These chains with boundary yield a degree zero solution of the quantum master equation in the BV algebra using the dual pictures of Gromov [**8**] and Sen-Zwiebach [**12**]. The homological treatment will determine Gromov-Witten invariants.

The transversality and parametrized surgery defining the unbounded structure maps are just like those in String Topology [**3, 14**]. Underlying both cases, here and [**3**], is the structure of an unbounded modular operad with Δ and $\{\ ,\}$ added in. See the longer introduction below for the definition of modular operad and more detailed descriptions.

The discussion here for general M may be viewed as a chain level or off-shell background for the closed string A-model or Gromov-Witten theory, defined when the target M is closed and symplectic, but which may be formulated more generally in terms of the quantum master equation of the BV algebra. One can ask if there is also a relation with the B-model defined when M is complex with holomorphic volume form.

Whereas the above discussion uses only closed surfaces Σ with finite subsets I of interior marked points, there is also a construction for surfaces with boundary mapping to M where pieces of the boundary between marked boundary points could be required to land in various submanifolds of M. Using parametrized interior and ∂ connected sum one obtains an (unbounded) generalized structure to the one presented here. This generality is needed to discuss open and closed strings and D-branes, see [**14**] for a beginning.

2. Longer introduction

A singular chain complex of the smooth mapping spaces $\mathrm{Map}(\Sigma, M)$ of closed oriented surfaces Σ into a target manifold M of even dimension d, taken altogether as Σ varies and equivariantly with respect to diffeomorphism groups of the Σ's (see Subsection 3.1) has an algebraic structure called *unbounded modular operad.* This means [**7**] there are non-negative graded chain complexes $(C(I), \partial)$ with an additional direct sum decomposition over the genus $C(I) = \bigoplus_{g=0}^{\infty} C(g, I)$ functorially attached to finite sets $I, J, \ldots$, with unbounded chain maps

$$C(I) \otimes C(J) \xrightarrow{C_{ij}} C(I \cup J - \{i,j\})$$

with $i \in I, j \in J$, and unbounded chain maps $C(I) \xrightarrow{S_{ij}} C(I - \{i,j\})$ with $\{i,j\} \subset I$. C_{ij} is defined for each pair of families of maps of Σ into M as a parametrized connected sum at i and j of different surfaces along the locus where the maps of the family transversally (over the base) identify i and j. S_{ij} is defined for each family of maps as a parametrized self-connected sum at i and j along the locus where the map of the fibre transversally identifies i and j. These structure maps act on bidegrees by

$$(g_1, k) \otimes (g_2, l) \mapsto (g_1 + g_2, k + l - d + 1)$$

and $(g, k) \mapsto (g+1, k-d+1)$ where $d = \dim M$, $g =$ genus Σ, and k, l are the geometric dimensions of the equivariant chains. The structure maps satisfy the relations that any compositions are associative and anticommute (see Subsection 3.3). The structure maps C_{ij} and S_{ij} are unbounded in the sense they are only defined on dense core domains which are subcomplexes of $\otimes_\alpha C(I_\alpha)$. These subcomplexes satisfy inclusion relations and have isomorphic homology to the entire complexes (see Subsection 3.2). These unbounded (or partial) operations are sufficient for our homological or derived category purposes because there are functorial quasi-isomorphic completion constructions (in two senses, see [**15**] motivated by [**9**]). The core domains are defined by transversality conditions, and the structure maps are defined by parametrized surgery both analogous to String Topology [**3**]. The geometric operations C_{ij} and S_{ij} for d odd will not be discussed further here.

There is a further direct sum decomposition of $C(g, I) = \oplus_\beta C(\beta, g, I)$ where β ranges over all elements in $H_2(M)$ which may be represented by maps of a genus g surface into M.

THEOREM 1. *When d is even, the chain complexes $C(I) = \oplus_g \oplus_\beta C(g, \beta, I)$ form a modular operad (in the unbounded sense explained above). After an even grading shift down by $(d-6)(1-g) + 2n + c(\beta)$, where c is linear in β and even, all the operations ∂, C_{ij}, and S_{ij} have degree -1.*

Now we add more structure to the modular operad. We let $C = C(\emptyset)$, and we define $\{\ ,\}$ and δ (Subsection 3.4) by the compositions

$$C \otimes C \xrightarrow{M_1 \otimes M_1} C(\{i\}) \otimes C(\{j\}) \xrightarrow{C_{ij}} C$$

and

$$C \xrightarrow{M_2} C(\{i,j\}) \xrightarrow{S_{ij}} C$$

by adding marked points via operators M_1 and M_2. An anomaly appears in the structure of δ, which is discussed in Remark 3.10.

THEOREM 2. *When d is even and c is even, $(C, \{\ ,\}, \delta)$ is defined and is an unbounded differential Lie algebra of degree -1. (See Subsection 3.4)*

REMARK 2.1. $(C, \delta, \{\ ,\})$ is a Lie algebra of degree -1 means $[a,b] = (-1)^{|a|}\{a,b\}$ is the bracket of a differential Lie algebra (of degree zero) on the graded space C shifted by one.

We can consider the Maurer-Cartan equation $\delta S + \frac{1}{2}\{S, S\} = \partial S$ where S has degree zero. A solution here implies a solution to the equation $\Delta\, e^S = \partial\, e^S$ in the graded symmetric algebra ΛC generated by C. Here Δ is the (second order) derivation on ΛC obtained by adding together both δ and $\{\ ,\}$ extended to be coderivations of ΛC. The equation $\Delta \cdot \Delta = 0$ summarizes the differential Lie algebra properties of $(C, \{\ ,\}, \delta)$. The second order nature and nilpotence of Δ comes by viewing Δ as the extension of δ plus $\{\ ,\}$ to all disconnected surfaces by summing over the possible gluings (i, j). Then since $a \wedge b$ can be viewed in terms of disjoint union of surfaces and maps over the product of the bases for a and for b,

$$\Delta(a \wedge b) = (\Delta a) \wedge b + (-1)^{|a|} a \wedge \Delta b + \{a, b\},$$

where $\{a, b\}$ denotes the extension of $\{\ ,\}$ on C to all of ΛC to a binary operation which is a graded derivation in each variable. The equation for S in terms of Δ is called the quantum master equation and can be written either as $\Delta S + \frac{1}{2}\{S, S\} = \partial S$, or $\Delta\, e^S = \partial e^S$.

THEOREM 3. *When d is even and $c(\beta)$ is even, $(\Lambda C, \Delta)$ is defined and is an unbounded BV algebra attached to the spaces* $\mathrm{Map}(\Sigma, M)$ *modulo diffeomorphisms of the source. Δ is a second order derivation and a coderivation. $(\Lambda C, \Delta, \partial)$ is a BV background for the sigma model in physics*[3] *constructed by algebraic topology and transversality.*

(See Subsection 3.4 for the proof).

2.1. Application. As mentioned above, the even integer $(d - 6)(1 - g) + 2n + c(\beta)$ used to shift the chain complexes $C(g, I)$ above, when $c(\beta) = 2c_1(\beta)$ yields the formal dimension of the moduli space of J-holomorphic maps of Σ into an almost complex manifold (M, J) representing the homology class β in $H_2(M)$, n = cardinality of I.

If the target manifold M is a closed symplectic manifold with 2-form ω, and J is a compatible almost complex structure ($\omega(x, Jy)$ is symmetric and positive definite), Gromov studied the J-holomorphic mappings $\Sigma \xrightarrow{f} M$ of Riemann surfaces into M and analyzed the non-compactness of the space of these in the given homology class β in $H_2(M)$. He found exactly the familiar non-compactness in the complex structure of Σ corresponding to pinching curves, together with Freed-Uhlenbeck bubbling off of 2-spheres in the map into M. Bubbling-off or pinching-off an essential separating curve in the complex structure of Σ will be seen to be inverse to an operation like C_{ij}. Similarly, pinching-off a nonseparating curve is inverse to an operation like S_{ij}. Thus, if one introduces cut-off inequalities to prevent these pinching and bubbling phenomenon from happening (see Section 4) one obtains for each $C(\beta, g)$ a compact family of J-holomorphic curves. Using $\mathbb{Q}$ coefficients and

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[3] In the physics discussions the ordinary maps of surfaces Σ into the target M are augmented by (contractible) odd degree components. A (formal) symplectic manifold of odd degree appears, whose functions define (formally) an odd version of Poisson algebra with bracket derived from a BV operator [**2**]. The functions are closely related to differential forms on the original mapping spaces. New variables are added to form the Borel-Cartan version of equivariant forms relative to diffeomorphisms of Σ, the BV algebra structure extends formally and a nondegenerate solution of the quantum master equation is sought where the leading term is a classical action in the original superfields. The structure is formal. However, see [**1**] for a rigorous finite dimensional analogue.

perturbing the equation the (relative) cycle is defined [**10**], [**6**] and has a boundary essentially described by a sum over the operations $\{\ ,\}$ and δ. In this way (after shifting each $C(\beta, g)$ down by $(d-6)(1-g)+2c_1(\beta)$ we obtain a solution S to the quantum master equation of degree zero, where S assigns to each β and each genus of Σ the oriented Gromov moduli space compactified by cut-off, plus small collars.

THEOREM 4 (Gromov, Eliashberg, Fukaya, Hofer, Kontsevich, Li, McDuff, Ono, Ruan, Salomon, Sullivan, Tian, …, Zwiebach). *For a closed symplectic manifold M the set of all cut-off oriented moduli spaces of perturbed connected J-holomorphic curves defines a Q-chain which provides a degree zero solution to the quantum master equation $\partial S = \Delta S + \frac{1}{2}\{S, S\}$ in an unbounded BV algebra attached to the sigma model of mapping spaces with target M.*

(See Section 4 for the proof.)

REMARK 2.2. The dots in the theorem refer to anyone omitted who has contributed to the formidable task of making the picture of the Gromov chain or homology class into rigorous mathematics.

Let us look at the quantum master equation in Zwiebach's form [**12**], which directly inspired this paper,

$$\partial S = \delta S + \frac{1}{2}\{S, S\}.$$

Note that we can write δS instead of ΔS because S has monomial grading one. We see the geometry of Gromov's virtual cycle where the first term on the right-hand side corresponds to approaching the boundary of S by pinching a nonseparating curve, while the second term on the right-hand side corresponds to approaching the boundary of S by either pinching-off an essential separating curve in the complex structure, or by bubbling off a 2-sphere in the map.

3. The modular operad and BV algebra structure

3.1. The equivariant chain complex. We first define the equivariant chain complexes $C(I)$ for the mapping spaces $\mathrm{Map}(\Sigma, M)$. Let I be a finite set and for each oriented connected pseudomanifold σ (see Remark 3.1), consider bundles η over σ with fibre the pair (Σ, I), where Σ is a closed connected piecewise smooth oriented surface of genus g and $I \subset \Sigma$ is an embedding of I into Σ. We also need piecewise smooth maps $f : \eta \to M$. Two such pairs (η, f) and (η', f') are equivalent iff there is an oriented bundle isomorphism b which is an oriented piecewise diffeomorphism between η and η', sends I to I on each fibre by the identity, and relates f to f', namely $f' \circ b = f$. We take the vector space $C(g, I)$ over $\mathbb{Q}$ on these equivalence classes for σ connected and oriented. We add the relation:

$$-(\eta, f, \mathit{orientation}\ \sigma) = (\eta, f, \mathit{opposite\ orientation}\ \sigma).$$

REMARK 3.1. We work in the piecewise differentiable category of spaces built by gluing together compact manifolds with corners (e.g., curvilinear polyhedra) and piecewise smooth maps. A pseudomanifold of dimension k by definition is an object in this category which admits such a decomposition with each $(k-1)$-dimensional part lying in the face of one or two k-dimensional pieces. The ∂ is the $(k-1)$-dimensional part made of pieces with only one k-dimensional face.

REMARK 3.2. A self-equivalence of (σ, η, f) which is orientation reversing on σ forces that generator to be zero.

The sum of the oriented geometric boundary components of σ defines the boundary operator ∂. The direct sum of all these chain complexes for the equivalence relation above requiring the identity map on I over all genera $g = 0, 1, 2, \ldots,$ is the chain complex $C(I)$ over $\mathbb{Q}$ functorially attached to the finite set I.

3.2. Core domains of unbounded structures. Now we define core domains $D(I) \subset C(I)$ and $D(I_1, I_2, ..., I_n) \subset \bigotimes_{\alpha=1}^{n} C(I_\alpha)$ by transversality. In each case the constraint is imposed on the connected basis elements and then the core domain is closed under $\mathbb{Q}$-linear combinations. For one generator (η, f) the constraint is that the map $\sigma \longrightarrow M^I$ which evaluates f at the subset I of the fibre is in general position relative to all the subdiagonals of M^I. For a collection $(\eta_\alpha, f_\alpha, \sigma_\alpha)$ the constraint is that the product evaluation map is in general position relative to all the subdiagonals of the big product, and this is also true for the restriction to all the natural product strata in the Cartesian product of the $(\sigma_\alpha, \partial\sigma_\alpha)$.

PROPOSITION 3.3. *The core domains $D(I_1, I_2, ..., I_n)$ are subcomplexes and satisfy the following two properties (compare* [**9**] *and* [**11**]*).*
i) For each partition $\alpha_1, \alpha_2, ..., \alpha_k$ of $\{1, 2, ..., n\}$ with $\alpha_i = \{j_i, j_i + 1, \ldots, j_i + m_i\}$, we have

$$D(I_1, I_2, ..., I_n) \subseteq D(\{I_\alpha\}_{\alpha\in\alpha_1}) \otimes \cdots \otimes D(\{I_\alpha\}_{\alpha\in\alpha_k}) \subset \bigotimes_{\gamma=1}^{n} C(I_\gamma);$$

ii) Each inclusion in i) induces an isomorphism on homology.

PROOF. They are subcomplexes because of the definition of general position. The inclusions of *i*) exist because more constraints are added by the definitions passing from right to left. To see that any such inclusion is onto in homology observe that any cycle in the range may be perturbed slightly to satisfy the constraints or the further constraints in the inclusions of *ii*). Injectivity follows because a homology in the range between cycles in general position is also in general position near its boundary and by a small deformation relative to its boundary may be put in general position everywhere. ☐

REMARK 3.4. The set of generators which are in general position is dense for a natural topology. The "denseness" of the core domain motivates the adjective "unbounded" in describing these structures.

3.3. The operations of the modular operad. Now we define the unbounded operations of the modular operad. First

$$C(I) \otimes C(J) \xrightarrow{C_{ij}} C(I \cup J - \{i, j\}).$$

On the core domain subcomplex (see Subsection 3.2) $D(I, J) \subset C(I) \otimes C(J)$ for each generator $\sigma_I \times \sigma_J$ in $D(I, J)$ the preimage of the diagonal by $(f(i), f(j))$ (called the locus (i, j)) has a normal bundle by transversality. Thus, the normal bundle and then the locus may be oriented when M is oriented to define a relative cycle whose ∂ lies in $\partial(\sigma_I \times \sigma_J)$ in a manner compatible with the natural pieces $\partial\sigma_I \times \sigma_J$ and $\sigma_I \times \partial\sigma_J$. Note this orientation of the normal bundle is independent of the order of i and j because d is even.

Along the locus (i,j) we have a well defined mapping of the one point union of the fibre over σ_I and the fibre over σ_J into M. Now for each surface we replace the marked point by the set of tangent directions at that point and glue these together along the boundary by a rigid map. The ambiguity in such a map is a circle parametrizing the set of orientation reversing isometries between these two boundaries (after choosing metrics on the boundaries). We can map any of the connected sums to the one point union by collapsing the glued up circles to the one common point.

Combining all this we get a fibration of connected sum surfaces over a circle bundle (of all rigid gluings) over the (i,j) locus and a canonical map into M of the total space of the surface bundle. Up to our equivalence the result is a well defined map into M. We combine the natural orientation of the circle (see Remark 3.5) with the orientation of the locus (i,j) to orient the circle bundle. This is the output of the operation between two elements C_{ij} which commutes with the ∂ operator.

The self-connected sum construction of $C(I) \overset{S_{ij}}{\to} C(I-\{i,j\})$ on the domain $D(I) \subset C(I)$ is done essentially the same way as in the case of C_{ij}, and S_{ij} also commutes with ∂.

REMARK 3.5 (Orientation of the circle). The surfaces Σ are oriented. Thus, each small circle around a marked point is oriented. In the gluing an orientation reversing isometry between boundary circles is used so that the glued up surfaces has a natural orientation. There is a circle of such orientation reversing isometries and this circle needs to be given an orientation. The family of all the glued up surfaces is a standard Dehn twist family over the circle of gluings. We orient that base circle by declaring the monodromy around that direction to be a right Dehn twist.

PROPOSITION 3.6. *When d is even, S_{ij} only depends on the unordered pair $\{i,j\}$, and C_{ij} is graded symmetric. The gluing operations have odd degree, and compositions are associative and anticommute.*

PROOF. This follows from the construction, Remark 3.5, and Remark 3.2. □

Now let us shift down the grading in the (g,β) component of $C(I)$, $n =$ cardinality of I, by $(d-6)(1-g)+2n+c(\beta)$, where c is linear in β. In the new grading we have

PROPOSITION 3.7. *When d is even and $c(\beta)$ is even, the dimension shift is even and ∂, C_{ij} and S_{ij} each has degree -1.*

PROOF. We check the degrees. Before C_{ij} sent $(g_1,n_1,k_1)\otimes(g_2,n_2,k_2) \mapsto (g_1+g_2,n_1+n_2-2,k_1+k_2-d+1)$ and S_{ij} sent $(g,n,k)\mapsto(g+1,n-2,k-d+1)$. Using $c(\beta_1+\beta_2)=c(\beta_1)+c(\beta_2)$ the degree of C_{ij} in the new grading is thus

$$\begin{gathered}((d-6)(1-g_1)+2n_1+(d-6)(1-g_2)+2n_2)+(-d+1)\\ -((d-6)(1-g_1-g_2)+2(n_1+n_2-2))=-1\end{gathered}$$

by direct calculation.

Similarly, the degree of S_{ij} in the new grading is $((d-6)(1-g)+2n)+(-d+1)-((d-6)(1-g-1)+2(n-2))=-1$. □

REMARK 3.8. The number $(d-6)(1-g)+2n+c(\beta)$ used to shift the grading in the above proposition when $c(\beta)=2c_1(\beta)$ gives the formal (real) dimension of the moduli space of J-holomorphic curves of genus g with n marked points in the homology class β of any almost complex manifold (M, J) of real dimension d. In other words, the formal complex dimension is $(d_{\mathbb{C}} - 3)\ (1 - g) + n + c_1(\beta)$ where $d_{\mathbb{C}}$ is the complex dimension of (M, J). Thus, if $c_1 = 0$ (e.g., in the Calabi-Yau case), $d_{\mathbb{C}} = 3$ (or $d = 6$) is the critical dimension when there is a discrete number of robust J-holomorphic curves of every genus.

3.4. The operations of the BV algebra. We define δ and $\{\ ,\}$ on the chain complex C of Subsection 3.1 and on an enlargement NC which includes maps of surfaces with nodes. There is a map $C \overset{M_1}{\rightarrow} C(\{*\})$ which adds a marked point in all possible positions to the surfaces in the family defining an element in C (that is, the bundle $\eta \rightarrow \sigma$ is pulled back to η and η becomes the new parameter space). M_1 has degree $+2$ because the base of $M_1(x)$ in $C(\{*\})$ is the total space of x in C. The bracket $\{\ ,\}$ in C is the composition

$$C \otimes C \xrightarrow{M_1 \otimes M_1} C(\{i\}) \otimes C(\{j\}) \xrightarrow{C_{ij}} C$$

where i is the marked point in the first factor and j in the second. This bracket has degree $4 - d + 1 = -d + 5$ and is analogous to the generalized Goldman bracket, or string bracket in [**3**] of degree $-d + 2$.

We will define δ analogously as the composition $C \overset{M_2}{\rightarrow} C(\{i, j\}) \overset{S_{ij}}{\rightarrow} C$, where M_2 replaces a family of surfaces with no marked points with the enlarged family with a pair of distinguished distinct points in all possible positions. The base of $M_2(x)$ is the total space of the fibrewise 2-point configuration space over the base of x. There is a compactness issue which is discussed in Remark 3.10. Since M_2 has degree 4, the composition δ also has degree $4 - d + 1 = -d + 5$.

Now form ΛC, the free graded commutative algebra generated by C, which is also a graded cocommutative coalgebra and a bialgebra with the diagonal uniquely defined by $x \mapsto x \otimes 1 + 1 \otimes x$, $x \in C$. Extend $-\partial$, δ, and $\{\ ,\}$ to coderivations of ΛC. The sum of the extensions of δ and $\{\ ,\}$ is called Δ. Extend $\{\ ,\}$ to a bracket on ΛC so that the Leibniz rule holds, and denote it $\{\ ,\}$.

THEOREM 5. *ΛC with its multiplication and the operator Δ is a BV algebra with derived Lie bracket $\{\ ,\}$ of odd degree. Namely, $\Delta \cdot \Delta = 0$ and the deviation of Δ from being a derivation is $\{\ ,\}$. Furthermore, Δ is a coderivation of the natural coalgebra structure on ΛC. (All in the unbounded sense).*

PROOF. A general point on the locus where each of two pairs is identified in M contributes two points (by definition) to the chain representing the output of $\Delta \cdot \Delta$. Since the circles of gluings for each pair is taken in opposite orders for the two points, the orientations at the two points are opposite. Thus, the involution interchanging the two points is orientation reversing and this output chain is equivalent to zero by Remark 3.2. The rest follows directly from the definitions and the gluing picture. ☐

REMARK 3.9. We may extend all the above to the chain complex NC corresponding to maps of connected nodal surfaces into M — namely, disconnected surfaces glued together at distinct pairs of points to make the entire collection connected. These are called *garlands* in Chernov-Rudyak [**4**] that studied a factor of $\{\ ,\}$. Now there is also a compactness issue for $\{\ ,\}$, when a point of a gluing pair we add comes close to a nodal point. This also concerns δ as well as the issue for δ of the two points of the gluing pair approaching each other in the surface.

REMARK 3.10. The locus for the compactified definition of δ will be a subspace of the union $\mathcal{C}$ over the base of the direction blow-up of the diagonal in the 2-point configuration spaces of the fibres. The locus intersected with the interior of $\mathcal{C}$ consists of distinct pairs on the fibre transversally (over the base) identified by the map.

For two points x and y in a surface Σ the operation $g(x, y)$ of forming a self-connected sum between x and y extends continuously to the diagonal blow-up of the 2-point configuration space. Now blow-up each point in an antipodal pair of boundary points which separate the boundary into antipodal arcs. This adds two new arcs to the boundary. Identify the mentioned antipodal arcs (by the old antipodal map) to obtain a surface with two boundary components. These circles may be identified by orientation reversing maps to extend $g(x, y)$.

We define the domain of δ by: i) at an interior point (x, y) of $\mathcal{C}$ we have the transversal coincidence of the map at x and y, a codimension d locus; ii) on the boundary of $\mathcal{C}$ (assume the base has no boundary for simplicity now) the points of i) tend generically to pairs $(x = y, l)$ where the differential of the map at x transversally drops rank by one with kernel l; iii) this picture is completed by the transversal locus when the rank of differential of the map drops by two of codimension $2d$, pulled back to the boundary of $\mathcal{C}$ by the projection.

Then δ is defined on this closed locus using the gluing extension of $g(x, y)$ above. Note that even though (in this special case) the base has no boundary, the output of δ does have boundary. In the general case, where the base has boundary, the output of δ will have extra boundary, called the *anomalous* boundary.

In the next section when we apply δ to a family of perturbed J-holomorphic curves, we will trim away a collar neighborhood of the anomalous boundary corresponding to these pairs (x, y) which are so close that the gluing operation creates a Riemann surface violating the cut-off constraints imposed in the definition of the cut-off Gromov chain.

4. The Gromov chain of perturbed J-holomorphic curves

4.1. Perturbed J-holomorphic curves. We will refer to the formalism of [**6**], especially Chapter 1, Section 6, for discussing the Gromov chain associated to all J-holomorphic curves in a symplectic almost complex manifold (M, J, ω). Fixing a stable combinatorics and homology data (Σ, β), there is a set of J-holomorphic curves with this type which is compact if we add the J-holomorphic curves of all finitely many stable combinatorial types (Σ', β') obtained by degenerating. (see Subsection 4.2 and Subsection 4.3 below). This compactness is the achievement of Gromov to which many have added. See [**10**], [**6**], and the introduction and references. This compact set of J-holomorphic curves admits a finite stratification into types but the dimension and the regularity of the various strata is not necessarily what we want because the solution set $\{(\text{complex structure, map}) : \overline{\partial}(\text{map}) = 0\}$ may not be defined transversally.

The idea is to locally perturb the equation $\overline{\partial}(\text{map}) = 0$ near the compact set and consider the transversal zeros which will be in some neighborhood of the compact set. Because of the presence of finite automorphism groups, orbifolds, orbibundles, and multisections must be used (see [**6**] Chapter 1, Section 6) to construct the

Gromov chain with $\mathbb{Q}$-coefficients. It is a tour de force challenge and achievement. All we need to know here is that the objects are produced by local perturbations and transversality, that *all* perturbed J-holomorphic curves are included, and the non-compactness is described by degeneration as in Subsection 4.3 below.

4.2. Combinatorics. Let us discuss the combinatorics of (Σ, β). By Σ we mean a finite collection of connected closed oriented surfaces together with nodal data — a finite subset of Σ (up to isotopy) with a fixed point free involution so that gluing related points yields a connected "nodal surface". By β we mean an integral homology class assigned to each component of Σ. It is important that the set of homology classes realized by "perturbed J-holomorphic" curves lies in a sharp cone[4] in $H_2(M, \mathbb{R})$. This follows from the hypothesis that M is a closed symplectic manifold, see [**13**] which contains a more general result characterizing symplectic and contact manifolds. Thus, when representations of a realizable class degenerate into components, there are only finitely many possibilities for their homology classes.

4.3. Stable combinatorics and degeneration. The combinatorics is stable if each component with $\beta = 0$ has Euler characteristic strictly less than the number of nodes on that component. Now let us study the stratum or part of the Gromov chain of perturbed J-holomorphic curves corresponding to a certain combinatorics (Σ, β) of "nodal" curves and homological position.

This part of the Gromov chain has an open part with the above combinatorics, and a codimension-two part where more degeneration takes place. If we impose inequalities on the complex structure and on the local energy of the maps, we can carve out a thin (generically tubular) neighborhood of the codimension two part where degeneracies happen. The homological boundary of what is left after carving out is a union of boundaries of tubes around strata where one degeneration happens. Fix one of those terms in the boundary of the cut-off Gromov chain. Along that stratum one sees new combinatorics.

One component of the nodal surface we started with has either had a handle pinched-off or been pinched into two components. The homology class β in the latter case splits into two parts and the genus splits into two parts. Zero homology class and zero genus can occur but the Euler characteristic condition above still holds (partially by definition of the compactification due to Kontsevich).

4.4. The main result. Now we can present the main result. Let S' denote the cut-off Gromov chain (see Subsection 4.3) associated to all stable combinatorial types of connected surfaces with k nodes. So S' defines an element in NC (see Remark 3.9). Set $NC = N$. We will modify S' to S by adding small collars as corrections in the argument below. k fixed.

THEOREM 6. *In ΛN, the free graded commutative algebra generated by N,*

$$\partial S = \delta S + \frac{1}{2}\{S, S\}.$$

PROOF. The ∂ of the total Gromov chain for connected surfaces with k nodes is made out of the boundaries of tubular neighborhoods of the strata corresponding to one additional degeneration. These are approximately described (if the degeneration does not disconnect) by applying the operation δ to the piece of the Gromov

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[4] I.e., a cone over a compact convex set.

chain corresponding to the new combinatorics obtained by pulling apart the new node and erasing the points (see the last paragraph of Remark 3.10). If the degeneration disconnects, the boundary of the tube is approximately described by applying $\{\ ,\}$ to the two pieces obtained by pulling apart the created node and erasing the points.

To replace approximate by exact we add small collar homologies to fill in the gaps. Note we are assuming the families of perturbed J-holomorphic curves satisfy all the transversality required for our operations δ and $\{\ ,\}$ to be defined. Thus, a generic perturbation of the $\bar{\partial}(\text{map}) = 0$ equation is required.

We find the equation $\partial S = \delta S + \frac{1}{2}\{S, S\}$ at the chain level in our background.

□

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