Super p-branes

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Abstract

It is shown that the extension of the *spacetime* supersymmetric Green-Schwarz covariant superstring action to p-dimensional extended objects (p-branes) is possible if and only if the on-shell p-dimensional bose and fermi degrees of freedom are equal. This is further evidence for *world-tube* supersymmetry in these models. All the p-brane models are related to superstring actions in $d = 3, 4, 6$ or 10 dimensions by double dimensional reduction, (which we generalise to reduction on arbitrary compact spaces), and we also show how they may be considered as topological defects of supergravity theories.

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The covariant spacetime-supersymmetric action of Green and Schwarz [1] can be generalised [2,3] to an action for a p-dimensional extended object, a "p-brane", moving in a d-dimensional, N-extended superspace, at least for certain values of (p,d,N). The action is

$$S = \int \mathrm{d}^{p+1}\xi \Big\{ \frac{1}{2}\sqrt{g} g^{ij} E_i{}^a E_j{}^b \eta_{ab} - \frac{(p-1)\sqrt{g}}{2} + \frac{1}{(p+1)!}\epsilon^{i_1 i_2 \ldots i_{p+1}} E_{i_1}{}^{A_1} E_{i_2}{}^{A_2} \ldots E_{i_{p+1}}{}^{A_{p+1}} B_{A_{p+1}\cdots A_2 A_1} \Big\}. \tag{1}$$

where ξ^i $(i = 0, 1, \ldots, p)$ are coordinates for the $(p+1)$-dimensional "world-tube" with metric g_{ij}, swept out by the (closed) p-brane in the course of its evolution. The $E_i{}^A$ $(A = a, \alpha)$ are defined by

$$E_i{}^A = \partial_i z^M(\xi) E_M{}^A, \tag{2}$$

where $\partial_i \equiv \partial/\partial\xi^i$, $z^M = (x^m, \theta^\mu)$ are the coordinates of d-dimensional superspace $(m = 0, 1, \ldots, (d-1))$ and $E_M{}^A$ is the supervielbein. The superspace $(p+1)$-form

$$B = \frac{1}{(p+1)!} E^{A_1} \cdots E^{A_{p+1}} B_{A_{p+1}\cdots A_1} \tag{3}$$

is the potential for a closed $(p+2)$-form $H = \mathrm{d}B$. Our metric convention for both $(p+1)$ and d dimensions is "mostly plus", $\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\eta^{ab}$ and $g \equiv |\det g_{ab}|$. Our superspace conventions are those of Howe [4].

As for the superstring action $(p = 1)$, we require of the action (1) that it possess a certain fermionic gauge invariance. In terms of $\delta z^A \equiv \delta z^M E_M{}^A$ the gauge transformation is

$$\delta z^a = 0, \qquad \delta z^\alpha = (1\!\!1 + \Gamma)^\alpha{}_\beta\, \kappa^\beta(\xi),$$
$$\Gamma^\alpha{}_\beta \equiv \frac{(-1)^{(p-1)(p+2)/4}}{(p+1)!\sqrt{g}} \epsilon^{i_1\ldots i_{p+1}} E_{i_1}{}^{a_1} \cdots E_{i_{p+1}}{}^{a_{p+1}} \left(\Gamma_{a_1\ldots a_{p+1}}\right)^\alpha{}_\beta, \tag{4}$$

where the parameter κ^β is a world-tube scalar but spacetime spinor, and $\Gamma_{a_1\ldots a_k}$ is the antisymmetrised product (with "strength one") of k (spacetime) γ-matrices $(\Gamma_a)^\alpha{}_\beta$. For $p > 1$ the equation of motion for g_{ij} is

$$g_{ij} = E_i{}^a E_j{}^b \eta_{ab}. \tag{5}$$

(For $p = 1$ g_{ij} is determined only up to an arbitrary scale factor.) Using this equation one can show that $\Gamma^2 = 1\!\!1$, so that $\frac{1}{2}(1\!\!1 + \Gamma)$ is a projection operator. For this reason only *half* the components of κ^β are effective in the gauge transformation (4) which can be used to gauge away *half* of the components of θ^α.

The invariance of the action under the fermionic gauge transformation requires that the superspace $(p+2)$-form H and the superspace torsion 2-form T^A satisfy certain constraints [3]. For the supermembrane ($p = 2$) in ($N = 1$) 11-dimensional superspace these constraints are in fact equivalent to the equations of motion of 11-dimensional supergravity [3,5]. Similarly, one can show that the constraints for a 5-brane in an ($N = 1$) 10-dimensional superspace are equivalent to the equations of motion of ($N = 1$) 10-dimensional supergravity*. In each case the supergravity model has a $(p+1)$-th rank antisymmetric tensor in its spectrum. (For $d = 10$ the 6th rank tensor is dual to the 2nd rank tensor of the usual formulation of $d = 10$ supergravity.) This suggests a correspondence between the existence of certain supergravity theories and the existence of a fermionic gauge invariant super p-brane action. We shall elaborate on this point at the end of this paper. Our principal result, however, is a classification of those values (p,d,N) for which the *flat* superspace fermionic gauge invariant p-brane action exists.

In flat superspace the constraints required for fermionic gauge invariance reduce to

$$\begin{aligned} H = h &\equiv \frac{1}{2(p!)} i e^{a_p} e^{a_{p-1}} \cdots e^{a_1} \mathrm{d}\bar{\theta}\Gamma_{a_1 \ldots a_p} \mathrm{d}\theta, \\ T^A = t^A &\equiv \left(\frac{i}{2}\mathrm{d}\bar{\theta}\Gamma^a \mathrm{d}\theta, 0\right), \end{aligned} \tag{6}$$

(with analogous expressions for $N > 1$), where

$$e^A = \left(\delta_m{}^a(\mathrm{d}x^m - \frac{i}{2}\bar{\theta}\Gamma^m \mathrm{d}\theta),\ \mathrm{d}\theta^\alpha\right). \tag{7}$$

The closure of h is now equivalent to the validity of the identity

$$\left(\mathrm{d}\bar{\theta}\Gamma_a \mathrm{d}\theta\right)\left(\mathrm{d}\bar{\theta}\Gamma^{ab_1 \ldots b_{p-1}} \mathrm{d}\theta\right) = 0 \tag{8}$$

for a *commuting* spinor $\mathrm{d}\theta$. When $\mathrm{d}\theta$ is a *complex* (possibly chiral) spinor this identity is equivalent to

$$\left(\Gamma_a \mathcal{P}\right)^{(\alpha}{}_{(\beta}\left(\Gamma^{ab_1 \ldots b_{p-1}} \mathcal{P}\right)^{\gamma)}{}_{\delta)} = 0, \tag{9}$$

where $\mathcal{P}$ is the chirality projection operator in the case of Weyl spinors, and the identity matrix otherwise. For Weyl spinors the identity (8) is equivalent to (9) only for p odd

* We thank Paul Howe for discussions on this point.

because for p even the second factor in (8) vanishes by itself, so $h \equiv 0$. If $\mathrm{d}\theta$ is Majorana (or Majorana-Weyl with p odd) then (8) is equivalent to the weaker identity

$$\big(C\Gamma_a\mathcal{P}\big)_{(\alpha\beta}\big(C\Gamma^{ab_1\ldots b_{p-1}}\mathcal{P}\big)_{\gamma\delta)} = 0, \tag{10}$$

except when the second factor of (10) vanishes by itself. (Spinor indices have been lowered here with the charge conjugation matrix C.) As for extended superspace, one can always consider an $N = 2$ Majorana spinor as a complex one. (For $N > 2$ (8) cannot be satisfied.)

A clue as to when these Γ-matrix identities might be satisfied is provided by supersymmetry. In *flat* superspace the action (1) is invariant under the transformations

$$\delta x^m = \frac{i}{2}\bar{\varepsilon}\Gamma^m\theta, \qquad \delta\theta = \varepsilon \tag{11}$$

for constant anticommuting spinor parameter ε. On fixing the fermionic and reparametrisation invariances this supersymmetry must be accompanied by a compensating κ-transformation. The action of this combined transformation on the remaining $(p+1)$-dimensional fields has been worked out for the 11-dimensional supermembrane, where it appears to be that of an $N = 8$ (rigid) *world-tube supersymmetry* [6]. If it is generally true that spacetime supersymmetry and fermionic gauge invariance lead to world-tube supersymmetry (as is the case for $p = 1$) then one would expect that, with (1) regarded as the action of a $(p+1)$-dimensional field theory, the number of (on-shell) bose fields x^μ should equal the number of (on-shell) "fermi" fields θ^α. Although the θ^α are *anticommuting* they are apparently world-tube *scalars*. They are the components of a spacetime spinor, however. For flat superspace the equation of motion for θ in the gauge $\Gamma\theta = -\theta$ is

$$(\Gamma^i)^\alpha{}_\beta\partial_i\theta^\beta + \mathrm{O}(\theta^2) = 0, \tag{12}$$

where $\Gamma^i \equiv g^{ij}E_i{}^a\Gamma_a$ satisfies $\{\Gamma^i,\Gamma^j\} = 2g^{ij}$ (using $g_{ij} = E_i{}^aE_j{}^b\eta_{ab}$). As $(p+1)$-dimensional Γ-matrices the Γ^i are reducible so that (12) can be interpreted as a number of $(p+1)$-dimensional Dirac equations, and hence the θ^α as a number of world-tube *spinors*. (Effectively, we are now taking the $(p+1)$-dimensional Lorentz group to be a diagonal subgroup of the original Lorentz group and an $SO(1,p)$ subgroup of $SO(1,d-1)$ determined by the embedding $x^m(\xi)$.)

It is easy to count the number of bose and fermi fields for any given (p,d,N). One has to take into account the $(p+1)$ reparametrisations, the fermionic κ-symmetry (which removes half of the θ's) and the fact that the θ-equation is first order (which removes half

	$p =$ 1	2	3	4	5	6	7	8	9	10
$d =$ 3	✓									
4	✓	✓								
5		✓								
6	✓		✓							
7		✓								
8			✓							
9				✓						
10	✓				✓					
11		✓								
12										

Table 1. Equality of number of bosons and fermions of the super p-brane action, (regarded as a $(p+1)$-dimensional field theory) is denoted by a tick. The wavy line along the diagonal passes through the limiting cases of a p-brane in $(p+1)$-dimensional spacetimes which we ignore in this paper. For $p > 1$ one must take $N = 1$ while for $p = 1$ one can take $N = 2$ (type II string) or $N = 1$ (heterotic string).

again if $p \geq 2$). One finds in this way that the boson and fermion fields are equal in number if and only if

$$d - p - 1 = \frac{n_{\rm min} N}{4}, \qquad p \geq 2 \qquad (13a)$$

where $n_{\rm min}$ is the dimension of the "minimal" spinor in d-dimensions. This equation is satisfied only for $N = 1$ and then only for certain values of p and d. The $p = 1$ case is special because the modes can be divided into left-movers and right-movers. If one requires that the sum of both left- and right-moving bosons and fermions be equal in number then one again finds eqn. (13a), but now this is satisfied only if $N = 2$ and then only if $d = 3, 4, 6$ and 10. If one requires only that the left- (or right-) moving bosons and fermions match then one finds that

$$d - 2 = \frac{n_{\rm min} N}{2} \qquad (13b),$$

which is satisfied for $N = 1$ ($d = 3, 4, 6, 10$). This possibility is of course relevant for the heterotic string. The results for all $p \geq 1$ are shown in the table.

Observe that for those special cases where we know that the super p-brane action exists the number of bose and fermi fields is indeed equal. The 3-brane in $d = 6$ was discussed in [2], for example, and the existence of a closed 7-form in $d = 10$ superspace (which guarantees the existence of a 5-brane action in $d = 10$) is known from studies of off-shell $d = 10$ supergravity [7]. Moreover, for the supermembrane ($p = 2$) it has been shown [6] (using Fierz identities) that the Γ-matrix identities necessary for the closure of h hold for $d = 4, 5, 7$ and 11.

These observations led to the conjecture [6] that *the super p-brane action exists if and only if the bose and fermi degrees of freedom match.* (We exclude here the degenerate case of a p-brane in $d = p + 1$ dimensions.) We shall now show that this conjecture is correct. We will first show that closure of h implies equality of the number of bosons and fermions. We define a spinor Ψ, symmetric matrices Σ, $\tilde{\Sigma}$ satisfying

$$\tilde{\Sigma}^a \Sigma^b + \tilde{\Sigma}^b \Sigma^a = 2\eta^{ab}, \tag{14}$$

and a matrix X such that (8) can be rewritten as

$$\Psi^T \Sigma_a \Psi \Psi^T X \Sigma^{ab_1 \ldots b_{p-1}} \Psi = 0. \tag{15}$$

If $\mathrm{d}\theta$ is a complex spinor Σ, $\tilde{\Sigma}$, Ψ and X are defined by

$$\begin{gathered}\Sigma^a = \begin{pmatrix} 0 & (\Gamma^a \mathcal{P})^T \\ \Gamma^a \mathcal{P} & 0 \end{pmatrix}, \qquad \tilde{\Sigma}^a = \begin{pmatrix} 0 & \Gamma^a \mathcal{P} \\ (\Gamma^a \mathcal{P})^T & 0 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \mathcal{P}\mathrm{d}\theta \\ \overline{(\mathcal{P}\mathrm{d}\theta)}^T \end{pmatrix} \\ \text{and} \qquad X = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix},\end{gathered} \tag{16}$$

for $p = 1 \bmod 4$, $p = 2 \bmod 4$, $p = 3 \bmod 4$ and $p = 4 \bmod 4$ respectively. Once again $\mathcal{P}$ is a chirality projection operator if $\mathrm{d}\theta$ is a Weyl spinor and the identity matrix otherwise. If $\mathrm{d}\theta$ is Majorana (or Majorana-Weyl) we define

$$\Sigma^a = C\Gamma^a \mathcal{P}, \qquad \tilde{\Sigma}^a = \mathcal{P}\Gamma^a C^{-1}, \qquad \Psi = \mathrm{d}\theta \quad \text{and} \quad X = 1\!\!1. \tag{17}$$

Since we have doubled the dimension for the case of the complex spinor, we can always move to a *real basis* by a similarity transformation. Eqn. (15) is therefore equivalent to

$$\big(\Sigma^a\big)_{(\alpha\beta}\big(X\Sigma_{ab_1 \ldots b_{p-1}}\big)_{\gamma\delta)} = 0. \tag{18}$$

Contracting (18) with $\big(\tilde{\Sigma}^b\big)^{\gamma\delta}$ one obtains

$$\begin{aligned} n_{\min}\big(X\Sigma_{bb_1 \ldots b_{p-1}}\big)_{\alpha\beta} &+ 4\big(X\Sigma_{ab_1 \ldots b_{p-1}}\tilde{\Sigma}_b \Sigma^a\big)_{(\alpha\beta)} \\ &+ \mathrm{Tr}\big(\tilde{\Sigma}_b X \Sigma_{ab_1 \ldots b_{p-1}}\big)\big(\Sigma^a\big)_{(\alpha\beta)} = 0. \end{aligned} \tag{19}$$

For $1 < p < d-1$ the last term vanishes and using (14) and the symmetry of the second term we obtain

$$[n_{\min} - 4(d-p-1)] \left(X\Sigma_{ab_1\ldots b_{p-1}}\right)_{(\alpha\beta)} = 0 \tag{20}$$

and so

$$d - p - 1 = n_{\min}/4 \tag{21a}$$

as required. If $p = 1$ the first and last terms contribute equally to give

$$d - 2 = n_{\min}/2. \tag{21b}$$

Thus the Γ-matrix identity relevant to the closure of the $(p+2)$-form h can hold only in the *finite* number of cases given in the table. To show that it does hold in all these cases we use "double dimensional reduction". The idea is to take spacetime to be the product of $(d-1)$-dimensional Minkoswki space and a circle, to wrap the p-brane around the circle and then discard masssive modes. Because the p-brane can still contract freely in the remaining $p-1$ spatial directions to zero p-volume, the lowest energy classical configuration is still zero, and in fact the action reduces to that of a $(p-1)$-brane in $(d-1)$ dimensions. In [5] this was applied to the $d = 11$ supermembrane action of [3] to derive the type IIa superstring action in a background supergravity field, but it is of general applicability. In fact, by inspection of the table one sees that all super p-brane actions lie on one of four sequences in which each member of a sequence is related to the next by double dimensional reduction. The four sequences start at ($p = 2$, $d = 4$, $N = 1$), ($p = 3$, $d = 6$, $N = 1$), ($p = 5$, $d = 10$, $N = 1$) and ($p = 2$, $d = 11$, $N = 1$), and terminate at the four superstring actions in $d = 3, 4, 6, 10$. Since we know of the existence of super p-brane actions at the higher-dimensional end of each sequence the existence of all the others is guaranteed.

Double dimensional reduction also provides a means for understanding the necessity of having equal numbers of bosons and fermions on the world-tube. Given a p-brane in d dimensions, which we shall call a (p, d)-brane, we obtain a $(p-1, d-1)$-brane by double dimensional reduction. This may be one with extended $(N > 1)$ spacetime supersymmetry, but it can always be truncated to one with minimal supersymmetry (or the least value of N consistent with the non-vanishing of h). If h is a closed form in an N-extended superspace, then it will also be closed in the $(N-1)$-extended subsuperspace; one has to check that it does not vanish. Therefore, if we know of the *non-existence* of a (p, d)-brane (with minimal spacetime supersymmetry) we can deduce the non-existence of $(p+k, d+k)$-branes (with arbitrarily extended spacetime supersymmetry), for the existence of the latter would (by double dimensional reduction and subsequent truncation) imply the existence of the former. For example, by showing that the Γ-matrix identity (10) *fails* for a 6-brane in eleven dimensions we establish that it fails for all $(p,d) = (6+k, 11+k)$.

As a check on these results we have verified by computer, for $p \le d-2$, $d \le 12$, that h is closed if and only if the $(p+1)$-dimensional boson and fermion degrees of freedom are equal.

This is strong evidence for the conjecture that spacetime supersymmetry and fermionic gauge invariance *always* implies world-tube supersymmetry. If true, one would expect the cancellation of vacuum energies found for the 11-dimensional supermembrane [8] to persist for all super p-branes.

Perhaps the most interesting case after the 11-dimensional supermembrane is the 10-dimensional super 5-brane because of the chirality of $N = 1$ $d = 10$ supergravity and, presumably, of the 6-dimensional world-tube supersymmetric action that results from fixing the fermionic gauge invariance. In order to obtain the action for a superstring one would have to double dimensionally reduce it by four dimensions to $d = 6$. So far this process has been applied only for *flat* extra dimensions, i.e. tori. However, we know that *any* solution of the $(N = 1)$ 10-dimensional supergravity equations is a background that allows the existence of the 5-brane action. In particular, one could take the solution for which the 10-dimensional spacetime is a product of a 6-dimensional Minkowski space and a compact 4-dimensional Ricci flat space such as K_3 [9]. This particular space has the advantage of preserving chirality on reduction to $d = 6$ [10]. There is a generalisation of double dimensional reduction that might allow us to take advantage of this possiblity. One simply identifies the "extra" dimensions of spacetime with the "extra" dimensions of the p-brane, equating both the coordinates, ($\xi^{\bar{\imath}}$, $\bar{\imath} = 2,3,4,5$ and $x^{\bar{m}}$, $\bar{m} = 6,7,8,9$), and the metrics, ($g_{\bar{\imath}\bar{\jmath}}$ and $g_{\bar{m}\bar{n}}$), i.e. the 5-brane world-tube is taken to be a product of K_3 with a world-sheet swept out by a string moving in the remaining six dimensions. To illustrate why this works consider the equation

$$g^{ij}\left\{\partial_i\left(\partial_j x^m\right) - \Gamma_{ij}{}^k \partial_k x^m + \partial_i x^p \partial_j x^q \Gamma_{pq}{}^m\right\} = 0 \qquad (22)$$

for the bosonic p-brane in an arbitrary background metric g_{mn}, (which follows from the action (1) with $\theta = 0$ and $H = 0$). We assume that g_{ij} is given by $\partial_i x^m \partial_j x^n g_{mn}$, (i.e. we use the g_{ij} equation as a constraint here). With the ansatz

$$g_{mn} = \begin{pmatrix} g_{\hat{m}\hat{n}} & 0 \\ 0 & g_{\bar{m}\bar{n}} \end{pmatrix} \qquad (23)$$

where $g_{\bar{m}\bar{n}}$ is the metric for a compact n-dimensional space ($n \leq p$), the $m = \hat{m}$ part of the equation clearly reduces to that of a $(p-n)$-brane, while the $m = \bar{m}$ part is satisfied because $\partial_i \partial_j x^{\bar{m}} = 0$ (by hypothesis) and the connection terms cancel. This generalisation of double dimensional reduction is analogous to certain compactifications of σ-models coupled to supergravity [11]. The extension to the supersymmetric case will be discussed elsewhere.

Finally, we return to the question of the relationship between super p-branes and supergravity theories alluded to earlier. By inspection one can verify that for each super p-brane there is a supergravity multiplet (possibly reducible) containing a $(p+1)$-form gauge field B with a $(p+2)$-form field strength $F = \mathrm{d}B$. In the simplest cases the field equation for B is

$$\mathrm{d}\tilde{F} = 0, \tag{24}$$

where $\tilde{F}$ is the $(d-p-2)$-form dual to F. Therefore locally $\tilde{F} = \mathrm{d}A$ for some $(d-p-3)$-form A. If A exists globally then the integral $\int \tilde{F}$ over any $(d-p-2)$-cycle vanishes. However, if $\tilde{F}$ is not an exact form this integral may not vanish, indicating the presence of a topological defect of dimension p. At long wavelengths one expects this defect to behave like a super p-brane, the coordinates z^M being the Goldstone bosons/fermions of partially broken translational and supersymmetry invariance [2]. In this limit the super p-brane would appear as a δ-function source for the gauge potential B through the coupling

$$\int \mathrm{d}^{p+1}\xi \; \epsilon^{i_1 \ldots i_{p+1}} \partial_{i_1} x^{m_1} \cdots \partial_{i_{p+1}} x^{m_{p+1}} B_{m_1 \cdots m_{p+1}}, \tag{25}$$

which is in fact contained in the *curved space* Wess-Zumino term of the action (1).

As an example consider the supermembrane in $d = 4$ and suppose the membrane lies in the x, y-plane at $z = 0$, i.e. a static configuration of infinite extent. It acts as a source for the gauge field B_{mnp}, for which the kinetic Lagrangian is

$$\frac{1}{2.4!} F_{mnpq} F^{mnpq}, \tag{26}$$

with field equation $\partial^m F_{mnpq} = 0$ and solution $F_{mnpq} = \epsilon_{mnpq} \times \mathit{const.}$ However, when one includes the membrane as a source the value of the constant differs for z positive and z negative, i.e. the membrane is a domain wall separating regions of differing vacua, as has been discussed previously [12]. The antisymmetric tensor field B_{mnp} has no propagating modes in $d = 4$ and so can be considered as a special type of auxiliary field. As such it appears in one version of the minimal $d = 4$ supergravity multiplet [13].

As another example, consider a super 3-brane in $d = 6$ lying in the x^1, x^2, x^3-plane. It acts as a source for a 4-form potential B with 5-form field strength F and 1-form dual $\tilde{F}^{(1)}$. Spatial infinity is $(x^4)^2 + (x^5)^2 = R^2$ for $R \to \infty$, i.e. S^1, and because of the 3-brane one has

$$\int_{S^1} \tilde{F}^{(1)} \neq 0. \tag{27}$$

Again, the antisymmetric tensor B appears as part of a (reducible) $d = 6$ supergravity multiplet (with bosonic field content g_{mn}, B_{mn}, ϕ). These examples do not show, of course, that the supergravity theories actually allow *non-singular* p-brane solutions (although the *gauge theory* model of [2] does have such a solution), but they show how such solutions can be interpreted as topological defects. (It is likely that one would have to add higher order "R^2" terms to the action in order to construct them.)

With this in mind, one can consider as topological defects the super p-branes of the other two sequences in the table, namely those which contain the $d = 10$ 5-brane and the $d = 11$ membrane, for which the dual field strength $\tilde{F}$ is a 3-form and 7-form respectively. In the absence of Yang-Mills fields (and for constant dilaton field) the field equation in the former case is $\mathrm{d}\tilde{F}^{(3)} = 0$ and the presence of the 5-brane (for $d = 10$) is signalled by the non-vanishing of

$$\int_{S^3} \tilde{F}^{(3)} \tag{28}$$

for some S^3 encircling the defect. In the latter case the field equation is $\mathrm{d}\tilde{F}^{(7)} = F \wedge F$ or $\mathrm{d}(\tilde{F}^{(7)} - B \wedge F) = 0$, so that the membrane (for $d = 11$) is signalled by the non-vanishing of

$$\int_{S^7} \left(\tilde{F}^{(7)} - B \wedge F\right) \tag{29}$$

for some S^7 encircling the defect. (A similar modification is required in (28) in the presence of Yang-Mills fields.) It is interesting to observe that the four sequences of super p-branes correspond to defects that are signalled by the non-vanishing of an integral of a closed form over S^0, S^1, S^3 or S^7, the parallelizable spheres.

This fact reinforces the idea of a connection with the composition-division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Such a connection has been established for the superstring in $d = 3, 4, 6$ and 10, where the identity (10) (or (9) as applicable) can be understood in terms of the four, degree three, Jordan algebras [14], or in terms of "triality" properties of the transverse Lorentz group [15]. As we have shown here, each of the superstring actions is itself one in a series of super p-brane actions, each of which can be thought of as a $(p+1)$-dimensional field theory with $(1+1)$, $(2+2)$, $(4+4)$ or $(8+8)$ bose + fermi fields.

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