D-Branes and Doubled Geometry

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

جامعة نيويورك ابوظبي
NYU | ABU DHABI

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with Vincenzo Emilio Marotta

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Introduction

- Manifest T-duality invariance: Correct description involves algebroids and 'doubled geometry'
- Generalized geometry: $TM \longrightarrow \mathbb{T}M \; = \; TM \oplus T^*M$ with structure of (twisted) Courant algebroid (Hitchin '02; Gualtieri '04)
- Double field theory (DFT): $M \longrightarrow \mathcal{M} \; = \; M \times \widetilde{M}$

  Solving strong constraint (polarisation) reduces DFT structure to standard Courant algebroid (Siegel '93; Hull & Zwiebach '09; Hohm, Hull & Zwiebach '10;. . .)
- In this talk: Global description of DFT provided by para-Hermitian geometry and metric algebroids
- Phenomena described by T-duality: What is a D-brane in this setting?
- Conformal boundary conditions for Born sigma-model: Covariant version of doubled sigma-models for duality-symmetric string theory (Duff '90; Tseytlin '90; Hull '05; Berman, Copland & Thompson '07; Hull & Reid-Edwards '09; Copland '11; Lee & Park '13 . . .)
- Generalize previous treatments of D-branes and doubled geometry (Hull '04; Lawrence, Schulz & Wecht '06; Albertsson, Kimura & Reid-Edwards '08; Hull & Sz '19; Sakatani & Uehara '20)

Double Field Theory and Para-Hermitian Geometry

- Para-Hermitian Geometry: A “real version” of complex Hermitian geometry
- Addresses global issues of doubled geometry, provides simple elegant framework for generalized flux compactifications and non-geometric backgrounds (Hull ’04; Vaisman ’12; Freidel, Rudolph & Svoboda ’17; Chatzistavrakidis, Jonke, Khoo & Sz ’18; Svoboda ’18; Marotta & Sz ’18; Mori, Sasaki & Shiozawa ’19; Hassler, Lüst & Rudolph ’19; Kimura, Sasaki & Shiozawa ’22; ...)
- Other applications of para-Hermitian geometry:
  - Formulation of $\mathcal{N} = 2$ vector multiplets in Euclidean spacetimes (Cortés, Mayer, Mohaupt & Saueressig ’03; Cortés & Mohaupt ’09)
  - Lagrangian and non-Lagrangian dynamical systems (Marotta & Sz ’18)
  - $2D$ ‘twisted’ SUSY sigma-models (Abou-Zeid & Hull ’99; Stojevic ’09; Hu, Moraru & Svoboda ’19)
- Modern perspective: Geometry on $\mathbb{T}M = TM \oplus T^*M \longleftrightarrow T\mathcal{M}$
- Examples: Fibre bundles (T^*M, TM, ...), Doubled Lie groups, Drinfel’d doubles, and quotients (T^{2d}, doubled twisted torus, ...)

Para-Hermitian Manifolds

- Para-complex structure $K : T\mathcal{M} \longrightarrow T\mathcal{M}$ on $2d$-dim manifold $\mathcal{M}$ with $K^2 = +\mathbb{1}$, whose ± 1-eigenbundles $L_\pm$ have same rank d
- Splits $T\mathcal{M} = L_+ \oplus L_-$, integrability of L_+ and L_- independent
- Para-Hermitian structure (K, η): metric η with signature (d, d) satisfying compatibility $K^\top \eta \, K = -\eta$
- Fundamental 2-form $\omega = \eta \, K$, $\mathrm{d}\omega$ = 'generalized fluxes'
  If symplectic ($\mathrm{d}\omega = 0$) then (K, η) para-Kähler structure
- $L_\pm$ maximally isotropic with respect to η and ω
- Example: $\mathcal{M} = T^*M \xrightarrow{\pi} M$ with canonical symplectic 2-form ω_0; para-Hermitian structures correspond to isotropic splittings of

$$0 \longrightarrow \ker(\pi_*) \longrightarrow T(T^*M) \longrightarrow \pi^*(TM) \longrightarrow 0$$

- Para-Hermitian vector bundles: $\mathbb{T}M = TM \oplus T^*M$,
  exact Courant algebroids , ...

Generalized Metrics & Born Geometry

- *B*-transformation of (K, η) on $T\mathcal{M} = L_+ \oplus L_-$:

$$e^B = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \in \mathrm{Aut}(T\mathcal{M}) \quad \text{where} \quad B : L_+ \longrightarrow L_- \quad \text{with}$$

$$\eta\big(B(X), Y\big) = -\eta\big(X, B(Y)\big) =: b(X, Y)$$

- $K \longrightarrow K_B = e^B \, K \, e^{-B}$ where (K_B, η) is another para-Hermitian structure with fundamental 2-form $\omega_B = \eta \, K_B = \omega + 2 \, b$

- Generalized metric on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$:
  $I \in \mathrm{Aut}(T\mathcal{M})$ covering id_M with $I^2 = \mathbb{1}$
  Defines Riemannian metric $\mathcal{H} = \eta \, I$ on $\mathcal{M}$

- $\mathcal{H}$ defined by metric on L_+ and B-transformation (g, b)

- If $\mathcal{H} \, \omega^{-1} \, \mathcal{H} = -\omega$ then $(\eta, \omega, \mathcal{H})$ is a Born geometry
  Specified by metric g on L_+

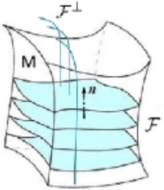
- Generalized T-duality: $O(T\mathcal{M}) \subset \mathrm{Aut}(T\mathcal{M})$ isometries of η, preserve Born geometry structure: $K_\vartheta = \vartheta \, K \, \vartheta^{-1}$, $\mathcal{H}_\vartheta = \vartheta^*(\mathcal{H})$ for $\vartheta \in O(T\mathcal{M})$

Born Sigma-Model

$$S[\mathbb{X}] \ = \ \frac{1}{4}\int_{\Sigma_2} \mathcal{H}_{IJ}\,\mathrm{d}\mathbb{X}^I \wedge \star\,\mathrm{d}\mathbb{X}^J + \frac{1}{4}\int_{\Sigma_2} \mathbb{X}^*(\omega)$$

$$\mathbb{X} : \Sigma_2 \longrightarrow \mathcal{M} \quad , \quad (\eta, \omega, \mathcal{H}) \ = \ \text{Born geometry on } \mathcal{M}$$

- **Strong Constraint:** Assuming $L_- \subset T\mathcal{M}$ involutive selects physical spacetime as a **quotient** $M = \mathcal{M}/\mathcal{F}$ by action on leaves of foliation of $\mathcal{M}$ by $\mathcal{F}$ with $L_- = T\mathcal{F}$ (Hull & Reid-Edwards '09; Vaisman '12; Park '13; Lee, Strickland-Constable & Waldram '15)



Reduces Born sigma-model if there is a Riemannian submersion $q : (\mathcal{M}, \mathcal{H}) \longrightarrow (M, \bar{g})$ such that $g = q^*\bar{g}$ is leaf-invariant (Marotta & Sz '19)

- Killing Lie algebroid: Lie algebroid A over Riemannian target space $(\mathcal{M}, \mathcal{H})$ such that $\nabla^A \mathcal{H} \ = \ 0$ for flat A-connection $\nabla^A : \Gamma(A) \times \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T\mathcal{M})$ (Kotov & Strobl '14; ...)

Gauging the Born Sigma-Model

- Apply to Killing Lie algebroid $\rho : T\mathcal{F} \hookrightarrow T\mathcal{M}$:
Born sigma-model can be gauged along foliation $\mathcal{F} \iff \mathcal{L}_V g = 0$ for all $V \in \Gamma(T\mathcal{F})$, where $\mathcal{H}$ is determined by metric g on L_+
- If $\mathcal{M}/\mathcal{F}$ is smooth, then there is a Riemannian submersion $q : (\mathcal{M}, \mathcal{H}) \longrightarrow (\mathcal{M}/\mathcal{F}, \bar{g})$ such that $g = q^* \bar{g}$
- ω descends to 2-form $\bar{b}$ on $\mathcal{M}/\mathcal{F}$ if L_+ is locally spanned by projectable vector fields V_i: $[V_i, W] \in \Gamma(T\mathcal{F})$ for all $W \in \Gamma(T\mathcal{F})$, and $\mathcal{L}_W \eta = 0$
- $\mathrm{d}\mathbb{X}^I \longrightarrow \mathrm{D}^A\mathbb{X}^I = \mathrm{d}\mathbb{X}^I - \rho^{Ij} A_j$ for $T\mathcal{F}$-valued connection 1-form A
- Euler-Lagrange equation for A gives 'self-duality constraint':

$$\mathrm{D}^A\mathbb{X} = \eta^{-1} \mathcal{H} \star \mathrm{d}\mathbb{X}$$

Reduces Born sigma-model to standard string sigma-model into physical spacetime $(\mathcal{M}/\mathcal{F}, \bar{g}, \bar{b})$
- Generalized T-duality $(\mathcal{M}, \eta, K, \mathcal{H}) \longrightarrow (\mathcal{M}, \eta, K_\vartheta, \mathcal{H}_\vartheta)$ with $T\mathcal{M} = L_+^\vartheta \oplus L_-^\vartheta$; if $L_-^\vartheta = T\mathcal{F}^\vartheta$ then sigma-models for $(\mathcal{M}/\mathcal{F}, \bar{g}, \bar{b})$ and $(\mathcal{M}/\mathcal{F}^\vartheta, \bar{g}^\vartheta, \bar{b}^\vartheta)$ are T-dual

Boundary Conditions for the Born Sigma-Model

- (σ, τ) local coordinates for Σ, with boundary $\partial\Sigma$:

$$\left(-\tfrac{1}{2}\,\mathcal{H}_{IJ}\,\partial_\sigma \mathbb{X}^J\,\mathrm{d}\sigma + \omega_{IJ}\,\partial_\tau \mathbb{X}^J\,\mathrm{d}\tau \right)\Big|_{\partial\Sigma} \;=\; 0$$

- Solution given by subbundle $L \subset T\mathcal{M}$ ("tangent vectors"):

$$0 \longrightarrow L \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{M}/L \longrightarrow 0$$

  and orthogonal splitting $TM \;=\; L \oplus L^\perp$ wrt generalized metric $\mathcal{H}$, with orthogonal projectors $\Pi : T\mathcal{M} \longrightarrow L$ and $\Pi^\perp : T\mathcal{M} \longrightarrow L^\perp$

- Together with self-duality constraint, conformal boundary conditions are solved by

$$\eta\big(\Pi(Z_I), \Pi(Z_J)\big) = 0 = \eta\big(\Pi^\perp(Z_I), \Pi^\perp(Z_J)\big) \quad , \quad \omega\big(\Pi(Z_I), \Pi(Z_J)\big) = 0$$

  for a local frame $\{Z_I\}$ of $T\mathcal{M}$

- Thus L is maximally isotropic wrt both η, ω (but not necessarily integrable)

Born D-Branes

- Def.: A Born D-brane is a maximally isotropic subbundle $L_D \subset T\mathcal{M}$ such that $K(L_D) = L_D$
- Examples: Eigenbundles $L_\pm$ of para-complex structure K
- If $W_D = L_+ \cap L_D$ has constant rank, then $L_D = W_D \oplus \eta^\sharp(\mathrm{Ann}(W_D))$ with metric

$$\mathcal{H}_D = \begin{pmatrix} g_D & 0 \\ 0 & \eta\, g_D^{-1} \end{pmatrix} \quad , \quad g_D = g|_{W_D}$$

- Generalized T-duality $\vartheta \in O(T\mathcal{M})$ sends D-brane L_D for Born sigma-model $S(\mathcal{H}, \omega)$ into $(\mathcal{M}, K, \eta)$ to D-brane $L_D^\vartheta = \vartheta(L_D)$ for Born sigma-model $S(\mathcal{H}_\vartheta, \omega_\theta)$ into $(\mathcal{M}, K_\vartheta, \eta)$
- Standard picture of D-branes as submanifolds when $L_D = T\mathcal{F}_D$ is integrable: Each leaf of foliation $\mathcal{F}_D$ of $\mathcal{M}$ is a d-dim submanifold of $\mathcal{M}$ whose tangent vectors satisfy the boundary conditions
- Chan-Paton bundles induced by B-transformations (with suitable integrality)

Dirac Structures

- Generalised submanifold $(\mathcal{W}, L)$ for an exact Courant algebroid $E \longrightarrow \mathcal{M}$ with anchor ρ: $\mathcal{W} \subset \mathcal{M}$, $L \subset E$ maximally isotropic integrable with $\rho(L) = T\mathcal{W}$ (Gualtieri '04; Zambon '07)
- Generalized para-complex D-brane supported on $\mathcal{W} \subseteq \mathcal{M}$ for an exact Courant algebroid $E \longrightarrow \mathcal{M}$ with anchor ρ and generalized para-complex structure $\mathcal{K}$: Generalized submanifold $(\mathcal{W}, L)$ such that $\mathcal{K}(L) = L$
- Born sigma-model corresponds (up to B-transformations) to the 'large Courant algebroid' $\mathbb{T}\mathcal{M} = T\mathcal{M} \oplus T^*\mathcal{M}$ with generalized metric determined by $\mathcal{H}$ (Alekseev & Strobl '04; Ševera '15)
- (η, K) gives generalized para-complex structure $\mathcal{K}_K = \begin{pmatrix} K & 0 \\ 0 & -K^\top \end{pmatrix}$ preserving splitting on $\mathbb{T}\mathcal{M}$ (Hu, Moraru & Svoboda '19)
- Born D-brane L_D defines Dirac structure $D = L_D \oplus \mathrm{Ann}(L_D)$ on $\mathbb{T}\mathcal{M}$
- For each leaf $\mathcal{W}_D$ of L_D, $(\mathcal{W}_D, D|_{\mathcal{W}_D})$ is a generalized para-complex brane: $\mathcal{K}_K(D|_{\mathcal{W}_D}) = D|_{\mathcal{W}_D}$ (since $K(L_D) = L_D$)

Metric Algebroids

- Metric algebroid: Anchored pseudo-Euclidean vector bundle $(E, \langle -,- \rangle_E, \rho)$ with bracket $[-,-]_E : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E)$:
  - $\rho(e) \cdot \langle e_1, e_2 \rangle_E \ = \ \langle [e, e_2]_E, e_2 \rangle_E + \langle e_2, [e, e_2]_E \rangle_E$
  - $\langle [e, e]_E, e_1 \rangle_E \ = \ \frac{1}{2}\, \rho(e_1) \cdot \langle e, e \rangle_E$

- Any anchored pseudo-Euclidean vector bundle admits infinitely many metric algebroid stuctures (Vaisman '12)

- Metric algebroids $\longleftrightarrow$ symplectic 2-algebroids

  (del Carpio-Marek '15; Marotta & Sz '21)

  (aka 'symplectic nearly Lie 2-algebroids' (Bruce & Grabowski '16)
  'symplectic pre-NQ-manifolds of degree 2' (Deser & Sämann '16)
  'pre-QP-manifolds' (Heller, Ikeda & Watamura '16))

- Any para-Hermitian manifold $(\mathcal{M}, K, \eta)$ admits a unique 'canonical' metric algebroid bracket $[-,-]_{T\mathcal{M}}$ preserving K, with anchor $\mathbb{1}_{T\mathcal{M}}$:

  $L_\pm$ are involutive wrt $[-,-]_{T\mathcal{M}}$, and $[-,-]_{T\mathcal{M}}$ is compatible with Lie algebra of vector fields on $T\mathcal{M}$ (Freidel, Rudolph & Svoboda '17))

Related Algebroids

- Pre-Courant algebroids: $\rho : E \longrightarrow T\mathcal{M}$ bracket morphism (symplectic almost Lie 2-algebroids (Bruce & Grabowski '16))
- Courant algebroids: Jacobi identity for $[-,-]_E$ (symplectic Lie 2-algebroids (Ševera '98; Roytenberg '99))
- DFT algebroid on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$: $\rho : (E, \langle -,- \rangle_E) \longrightarrow (T\mathcal{M}, \eta)$ isomorphism of pseudo-Euclidean vector bundles with $\rho\,\rho^* = \eta^{-1}$ (Chatzistavrakidis, Jonke, Khoo & Sz '18; Svoboda '18; Hu, Moraru & Svoboda '19; Grewcoe & Jonke '20; Marotta & Sz '21)
- Example: Splitting and projection of large Courant algebroid $\mathbb{T}\mathcal{M}$ is a DFT algebroid isomorphic to canonical metric algebroid, reduces to standard Courant algebroid on physical spacetime $\mathcal{M}/\mathcal{F}$ when $L_- = T\mathcal{F}$ — DFT algebroids lie "in between" two Courant algebroids
- **Note:** Generalised para-complex branes make sense for exact pre-Courant algebroids — extension to metric algebroids?

Adding a Wess-Zumino Term

- Difference between any two metric algebroid brackets on $(\mathcal{M}, K, \eta)$ is a 3-form H_D on $\mathcal{M}$
- Canonical 3-form $H_{\rm can}$: Choose canonical metric algebroid and reference bracket induced by Levi-Civita connection of η ($H_{\rm can} = 0$ iff $\mathrm{d}\omega = 0$)
- $(\mathcal{M}, K, \eta)$ is admissible if $H_2(\mathcal{M}) = 0$ and $\frac{1}{4\pi}\,[H_{\rm can}] \in H^3(\mathcal{M};\mathbb{Z})$
- Defines Wess-Zumino term $\displaystyle \frac{1}{2}\int_V \mathbb{X}^*(H_{\rm can})$, $\partial V = \Sigma$ for Born sigma-model, $H_{\rm can}$ represents Ševera class of associated Courant algebroid
- For open strings, consider relative maps $\mathbb{X} : (\Sigma, \partial\Sigma) \longrightarrow (\mathcal{M}, \mathcal{W})$ and relative admissibility: $H_2(\mathcal{M}, \mathcal{W}) = 0$, $\frac{1}{4\pi}\,[(H_{\rm can}, B_{\rm can})] \in H^3(\mathcal{M}, \mathcal{W};\mathbb{Z})$ for some 2-form $B_{\rm can}$ on $\mathcal{W}$
- $L_{\mathcal{W}} := \mathrm{im}(T\mathcal{W} \longrightarrow T\mathcal{M})$ is a Born D-brane iff $\mathcal{W} \subset \mathcal{M}$ Lagrangian submanifold, $B_{\rm can} = 0$ and $H_{\rm can}\big|_{\mathcal{W}} = 0$ (orientation condition)
- Can only couple to *flat* Chan-Paton bundles — analogous to A-branes

Generalized Para-Complex Branes

- Generalized submanifolds $(\mathcal{W}, L)$ on an exact Courant algebroid correspond to subbundles

$$L \;=\; L^F := \big\{X + \alpha \in T\mathcal{W} \oplus T^*\mathcal{M}|_{\mathcal{W}} \;\big|\; \alpha|_{\mathcal{W}} = \iota_X F\big\} \subset \mathbb{T}\mathcal{M}$$

  for some 2-form F on $\mathcal{W}$ with $\mathrm{d}F + H_{\rm can}\big|_{\mathcal{W}} \;=\; 0$

- Example: For a Born D-brane L_D and its Dirac structure $D \;=\; L_D \oplus \mathrm{Ann}(L_D)$, $(\mathcal{W}_D, D|_{\mathcal{W}_D})$ is a generalized para-complex D-brane iff $H_{\rm can}\big|_{\mathcal{W}_D} \;=\; 0$ (since $F \;=\; 0$)

- Example: For a Born D-brane L_D and given $F \in \Omega^2(\mathcal{W})$, $(\mathcal{W}_D, L^F)$ is a generalized para-complex D-brane iff

$$K^\top(\iota_X F) + \iota_{K(X)}F \;\in\; \mathrm{Ann}(T\mathcal{W}_D) \qquad \forall X \in \Gamma(T\mathcal{W}_D)$$

  If $H_{\rm can}\big|_{\mathcal{W}_D} \;=\; 0$ (integrability), $F \in \Omega^2_{\mathbb{Z}}(\mathcal{W}_D)$ and K integrable, then F is the curvature of a para-holomorphic Chan-Paton bundle (C, ∇^C) on $\mathcal{W}_D$ — analogous to B-branes (Lawn & Schäfer '05)

D-Branes on the Physical Spacetime

- D-branes are defined by "tangent vectors" – distributions on tangent bundle of target space, need to be integral to interpret leaves as D-brane worldvolumes
- When $L_- = T\mathcal{F}$ and $q : (\mathcal{M}, \mathcal{H}) \longrightarrow (M = \mathcal{M}/\mathcal{F}, \bar{g})$ is a Riemmannian submersion, $\mathrm{d}q|_{L_+} : L_+ \longrightarrow TM$ fibrewise isomorphism
- Born D-brane $L_D = T\mathcal{F}_D \subset T\mathcal{M}$ induces $\mathrm{d}q(L_D) = T\mathcal{F}_D^q \subseteq TM$, leaves of foliation $\mathcal{F}_D^q$ supported by physical D-branes in $(M, \bar{g}, \bar{b})$
- Example: $L_- = T\mathcal{F} \implies$ 0-branes on M (fully Dirichlet)

  L_+ integrable $\implies$ spacetime-filling D-branes (fully Neumann)
- D-branes are associated with Dirac structures on Courant algebroid for corresponding sigma-model (Zabzine '04; Asakawa, Sasa & Watamura '12)
- Consider reduction of Born D-branes as reduction of Dirac structures, using techniques of Courant algebroid reduction

  (Bursztyn, Cavalcanti & Gualtieri '05; Zambon '07)

Dirac Reduction of Born D-Branes

- For $A = T\mathcal{F} \oplus \{0\} \subset \mathbb{T}\mathcal{M}$, $A^\perp$ spanned by $Y + \mathrm{d}(q^* f)$ for projectable $Y \in \Gamma(T\mathcal{M})$ and $f \in C^\infty(M)$, which are 'basic'
- Hence large Courant algebroid $\mathbb{T}\mathcal{M}$ reduces to standard Courant algebroid $\mathbb{T}M$ through pullback diagram

$$\begin{array}{ccc} \frac{(T\mathcal{F}\oplus\{0\})^\perp}{T\mathcal{F}\oplus\{0\}} & \longrightarrow & \mathbb{T}M \\ \big\downarrow & & \big\downarrow \\ \mathcal{M} & \xrightarrow{\quad q \quad} & M \end{array}$$

- For a Born D-brane $L_D = T\mathcal{F}_D \subset T\mathcal{M}$, $D = L_D \oplus \mathrm{Ann}(L_D)$ is a Dirac structure for large Courant algebroid $\mathbb{T}\mathcal{M}$ such that $D \cap A^\perp$ still spanned by $Y + \mathrm{d}(q^* f)$
- Hence if L_D admits a sub-bundle spanned by projectable vector fields, then D descends to a Dirac structure D_{red} on $M = \mathcal{M}/\mathcal{F}$

Example: D-Branes on Doubled Nilmanifolds

- H $=$ 3d Heisenberg group with Drinfel'd double $T^*\mathrm{H} = \mathrm{H} \ltimes \mathbb{R}^3$, basis $\{Z_i, \tilde{Z}^i\}_{i=x,y,z}$ of left-invariant vector fields on $T(T^*\mathrm{H})$
- $(\mathcal{M}, K, \eta)$: $\mathcal{M} = \Gamma_m \setminus T^*\mathrm{H}$ for discrete cocompact subgroup Γ_m with $m \in \mathbb{Z}$, $K(Z_i) = +Z_i \quad K(\tilde{Z}^i) = -\tilde{Z}^i$, and η induced from duality pairing between Lie(H) and $\mathbb{R}^3$
- Nilmanifold: Principal T^3-bundle $\mathcal{M} \longrightarrow N_m =$ nilmanifold of degree m
  - $L_+ =$ D3-brane filling N_m, Dirac structure $TN_m \subset \mathbb{T}N_m$
  - $L_D = \mathrm{Span}(Z_x, Z_y, \tilde{Z}^z)$ reduces to Dirac structure associated with foliation of N_m with T^2 leaves wrapped by D2-branes
- T^3 with H-flux: T^3-fibration $\mathcal{M} \longrightarrow T^3$, $m =$ DD class of gerbe on T^3 B-transformation sends $\{Z_i, \tilde{Z}^i\} \longrightarrow \{Z_i', \tilde{Z}'^i\}$, $K \longrightarrow K'$
  - $L_D = \mathrm{Span}(Z_x', \tilde{Z}'^y, \tilde{Z}'^z)$ yields Dirac structure associated with foliation of T^3 with S^1 leaves wrapped by D1-branes, T-dual to D0-branes on N_m from reducing Born D-brane L_-
  - $H_{\mathrm{can}} = -\frac{3}{2}\, m\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, $H_{\mathrm{can}}(Z_x', Z_y', Z_z') \neq 0$ forbids D3-branes wrapping T^3 for $m \neq 0$