D-BRANES AND BIVARIANT K-THEORY

Richard Szabo

Heriot–Watt University, Edinburgh

Maxwell Institute for Mathematical Sciences

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D-branes and K-theory

- D-brane charges in string theory are classified by K-theory of spacetime X (Minasian & Moore, Witten, Hořava)
- Ramond-Ramond fields classified by differential K-theory of X (Moore & Witten, Hopkins & Freed, Hopkins & Singer, Bunke & Schick)
- **Explains:** stable non-BPS branes with torsion charges, self-duality/quantization of RR-fields, worldsheet anomalies and RR-field phase factors in string theory path integral
- **Predicts:** instability of D-branes wrapping non-contractible cycles, obstruction to simultaneous measurement of electric/magnetic RR fluxes

Classification of D-branes – Categories

- Problem: Given a closed string background X, find possible states of D-branes in X (consistent boundary conditions in BCFT)

- Many have no geometrical description: Regard D-branes as objects in a suitable category

- Topological strings/homological mirror symmetry: B-model D-branes in derived category of coherent sheaves, A-model D-branes in Fukawa category (Douglas)

- BCFT/K-theory: Open string boundary conditions in category of 2D open/closed TFT (Moore & Segal)

Classification of D-branes – Bivariant K-theory

- Combine worldsheet description with target space classification in terms of Fredholm modules: D-branes are objects in a certain category of separable C^*-algebras

- Category underlying Kasparov's bivariant K-theory (KK-theory), related to open string algebras in SFT

Why bivariant K-theory?

- Unifies K-theory and K-homology descriptions of D-branes
- Intersection product provides correct framework for duality between C^*-algebras (e.g. Poincaré duality)
- Explains equivalence of K-theory and K-homology descriptions of D-brane charge
- K-orientation/Freed-Witten anomaly cancellation, select consistent sets of D-branes from category

Why bivariant K-theory?

- Open string T-duality as categorical KK-equivalence
  (refines/generalizes Morita equivalence)

- Examples of “non-geometric” backgrounds
  e.g. noncommutative spacetimes as globally defined, open string versions of **T-folds**

- Noncommutative version of D-brane charge vector

Noncommutative geometry

Develop more tools for dealing with noncommutative spaces in purely algebraic framework of separable C^*-algebras:

- Noncommutative versions of Poincaré duality, orientation
- Topological invariants of noncommutative spaces
  e.g. Todd genus
- Noncommutative version of Grothendieck–Riemann–Roch theorem (D-brane charge)

J. Brodzki, V. Mathai, J. Rosenberg, RS:
arXiv: hep-th/0607020 , 0708.2648 [hep-th] , 0709.2128 [hep-th]

Review (RS): arXiv:0809.3029 [hep-th]

D-branes and K-cycles

- $X =$ compact spin^c-manifold (no H-flux)

- D-brane in $X =$ Baum–Douglas K-cycle (W, E, f)
  $f : W \hookrightarrow X$ closed spin^c (worldvolume)
  $E \longrightarrow W$ Chan–Paton gauge bundle with connection
  (**stable** element of $\mathrm{K}^0(W)$)

- Quotient by bordism and Baum–Douglas "gauge equivalence"
  $\cong$ K-homology of X, stable homotopy classes of Fredholm modules over commutative C^*-algebra $\mathcal{A} = C(X)$

D-branes and K-cycles

- $(W, E, f) \longmapsto (\mathcal{H}, \rho, \not{D}_E^{(W)})$, where:
  - $\mathcal{H} = L^2(W, S \otimes E)$ (spinors on W)
  - $\rho(\phi) = m_{\phi \circ f}$ ($*$-representation of $\phi \in \mathcal{A}$)
  - $\not{D}_E^{(W)} =$ Dirac operator on W
- D-branes naturally provide K-homology classes on X, dual to K-theory classes $f_!(E) \in \mathrm{K}^d(X)$ ($f_! =$ K-theoretic Gysin map, $d = \dim(X) - \dim(W)$)

A simple observation

- Natural bilinear pairing in cohomology (Poincaré duality):

$$(x, y)_{\mathrm{H}} \; = \; \langle x \smile y \, , [X] \rangle \qquad (= \; \int_X \alpha \wedge \beta)$$

- Natural bilinear pairing in K-theory:

$$(E, F)_{\mathrm{K}} \; = \; \mathrm{index}(\not{D}_{E \otimes F})$$

- Chern character isomorphism:

$$\mathrm{ch} \, : \, \mathrm{K}^{\bullet}(X) \otimes \mathbb{Q} \xrightarrow{\approx} \mathrm{H}^{\bullet}(X, \mathbb{Q})$$

doesn't preserve two pairings.

A simple observation

- By Atiyah-Singer index theorem:

$$\mathrm{index}(\not{D}_{E\otimes F}) \;=\; \big\langle \mathrm{Todd}(X) \smile \mathrm{ch}(E\otimes F)\,,\,[X]\big\rangle$$

we get an isometry with the modified Chern character:

$$\mathrm{ch} \;\longrightarrow\; \sqrt{\mathrm{Todd}(X)} \smile \mathrm{ch}$$

- **Ramond-Ramond charge** of D-brane (W, E, f) (Minasian–Moore):

$$Q(W,E,f) \;=\; \mathrm{ch}\big(f_!(E)\big) \smile \sqrt{\mathrm{Todd}(X)} \;\in\; \mathrm{H}^\bullet(X,\mathbb{Q})$$

Zero mode part of boundary state in RR-sector

Worldsheet description of D-branes

- Open strings = relative maps: $(\Sigma, \partial\Sigma) \longrightarrow (X, W)$
  Σ = oriented Riemann surface
- In BCFT on $\Sigma = \mathbb{R} \times [0,1]$, Euler-Lagrange equations require suitable boundary conditions — label by $a, b, \ldots$
- Compatibility with superconformal invariance constrains W
  e.g. in absence of H-flux, W must be spin^c (cancellation of global worldsheet anomalies)
- Problem: What is a **quantum** D-brane?
- Define consistent boundary conditions after quantization of BCFT — look at open string field theory

Algebraic characterization of D-branes

Concatenation of open string vertex operators defines algebras and bimodules:

- a-a open strings: Noncommutative algebra $\mathcal{D}_a$ of open string fields (opposite algebra $\mathcal{D}_a^{\mathrm{o}}$ by reversing orientation)

- a-b open strings: $\mathcal{D}_a$-$\mathcal{D}_b$ bimodule $\mathcal{E}_{ab}$ (dual bimodule $\mathcal{E}_{ab}^\vee = \mathcal{E}_{ba}$ by reversing orientation)
  $\mathcal{E}_{aa} = \mathcal{D}_a$ trivial $\mathcal{D}_a$-bimodule

- "Category of D-branes": Objects = boundary conditions, Morphisms $a \to b = \mathcal{E}_{ab}$, with associative $\mathbb{C}$-bilinear composition law:

$$\mathcal{E}_{ab} \times \mathcal{E}_{bc} \longrightarrow \mathcal{E}_{ac}$$

KK-theory

- In certain instances (e.g. $X = \mathbb{T}^n$ with constant B-field in Seiberg–Witten scaling limit) composition law extends by associativity to:

$$\mathcal{E}_{ab} \otimes_{\mathcal{D}_b} \mathcal{E}_{bc} \longrightarrow \mathcal{E}_{ac}$$

  Natural identifications $\mathcal{D}_a \cong \mathcal{E}_{ab} \otimes_{\mathcal{D}_b} \mathcal{E}_{ba}$, $\mathcal{D}_b \cong \mathcal{E}_{ba} \otimes_{\mathcal{D}_a} \mathcal{E}_{ab}$ mean that $\mathcal{E}_{ab}$ is a Morita equivalence bimodule: T-duality

- $\mathcal{E}_{ab} \rightarrow$ Kasparov bimodule $(\mathcal{E}_{ab}, F_{ab})$, generalize Fredholm modules. "Trivial" bimodule $(\mathcal{E}_{ab}, 0)$ when $\mathcal{E}_{ab}$ is Morita equivalence bimodule

- Stable homotopy classes define $\mathbb{Z}_2$-graded KK-theory group $\mathrm{KK}_\bullet(\mathcal{D}_a, \mathcal{D}_b) =$ "generalized" morphisms $\mathcal{D}_a \longrightarrow \mathcal{D}_b$

KK-theory

- $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ homomorphism of separable C^*-algebras, then $[\phi] \in \mathsf{KK}_\bullet(\mathcal{A}, \mathcal{B})$ represented by "Morita-type" bimodule $(\mathcal{B}, \phi, 0)$
- $\mathsf{KK}_\bullet(\mathbb{C}, \mathcal{B}) \ = \ \mathsf{K}_\bullet(\mathcal{B})$ K-theory of $\mathcal{B}$
- $\mathsf{KK}_\bullet(\mathcal{A}, \mathbb{C}) \ = \ \mathsf{K}^\bullet(\mathcal{A})$ K-homology of $\mathcal{A}$ (Kasparov bimodules = Fredholm modules over $\mathcal{A}$)

Intersection product

$$\otimes_{\mathcal{B}} \; : \; \mathsf{KK}_i(\mathcal{A},\mathcal{B}) \times \mathsf{KK}_j(\mathcal{B},\mathcal{C}) \;\longrightarrow\; \mathsf{KK}_{i+j}(\mathcal{A},\mathcal{C})$$

- Bilinear, associative
- $\phi : \mathcal{A} \longrightarrow \mathcal{B}$, $\psi : \mathcal{B} \longrightarrow \mathcal{C}$ then $[\phi] \otimes_{\mathcal{B}} [\psi] \;=\; [\psi \circ \phi]$
- Makes $\mathsf{KK}_0(\mathcal{A},\mathcal{A})$ into unital ring with $1_{\mathcal{A}} \;=\; [\mathrm{id}_{\mathcal{A}}]$
- Defines bilinear, associative **exterior product**:

$$\otimes \; : \; \mathsf{KK}_i(\mathcal{A}_1,\mathcal{B}_1) \times \mathsf{KK}_j(\mathcal{A}_2,\mathcal{B}_2) \;\longrightarrow\; \mathsf{KK}_{i+j}(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$$

D-brane categories

(Higson, Meyer, Nest)

- Additive category: Objects = separable C^*-algebras , Morphisms $\mathcal{A} \to \mathcal{B}$ = $\mathrm{KK}_\bullet(\mathcal{A}, \mathcal{B})$
- **Universal** category: KK = unique bifunctor on category of separable C^*-algebras , $*$-homomorphisms with homotopy invariance, stability and split exactness
- Composition law = intersection product
- Not abelian, but **triangulated**
- "Weak" monoidal category: multiplication = spatial tensor product on objects, external Kasparov product on morphisms, identity = one-dimensional C^*-algebra $\mathbb{C}$

KK-equivalence

- $\alpha \in \mathsf{KK}_d(\mathcal{A},\mathcal{B})$ determines homomorphisms:

$$\otimes_{\mathcal{A}}\alpha \,:\, \mathsf{K}_j(\mathcal{A}) \longrightarrow \mathsf{K}_{j+d}(\mathcal{B}) \quad \text{and} \quad \alpha\otimes_{\mathcal{B}} \,:\, \mathsf{K}^j(\mathcal{B}) \longrightarrow \mathsf{K}^{j+d}(\mathcal{A})$$

- α **invertible**, i.e., there exists $\beta \in \mathsf{KK}_{-d}(\mathcal{B},\mathcal{A})$ with $\alpha \otimes_{\mathcal{B}} \beta = 1_{\mathcal{A}}$ and $\beta \otimes_{\mathcal{A}} \alpha = 1_{\mathcal{B}}$, then $\beta =: \alpha^{-1}$ and

$$\mathsf{K}_j(\mathcal{A}) \cong \mathsf{K}_{j+d}(\mathcal{B}) \qquad \text{and} \qquad \mathsf{K}^j(\mathcal{B}) \cong \mathsf{K}^{j+d}(\mathcal{A})$$

- Algebras $\mathcal{A},\mathcal{B}$ are **KK-equivalent**

KK-equivalence

- Example: Morita equivalence $\Longrightarrow$ KK-equivalence ($\alpha = \big[(\mathcal{E}_{ab}, 0)\big]$); but KK-equivalence generally **refines** usual T-duality

- Note: Universal coefficient theorem (Rosenberg & Schochet):

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}\big(\mathsf{K}_{\bullet+1}(\mathcal{A})\,,\,\mathsf{K}_\bullet(\mathcal{B})\big) \longrightarrow \mathsf{KK}_\bullet\big(\mathcal{A}\,,\,\mathcal{B}\big) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathsf{K}_\bullet(\mathcal{A})\,,\,\mathsf{K}_\bullet(\mathcal{B})\big) \longrightarrow 0$$

  Holds for class of C^*-algebras KK-equivalent to comm. algs.

Poincaré duality – Definition

(Connes, Kaminker & Putnam, Emerson, Tu)

- $\mathcal{A}$ = separable C^*-algebra, $\mathcal{A}^{\rm o}$ = opposite algebra ($\mathcal{A}$-bimodules = $(\mathcal{A} \otimes \mathcal{A}^{\rm o})$-modules)

- $\mathcal{A}$ is a Poincaré duality (PD) algebra if there is a **fundamental class** $\Delta \in \mathsf{KK}_d(\mathcal{A} \otimes \mathcal{A}^{\rm o}, \mathbb{C}) = \mathsf{K}^d(\mathcal{A} \otimes \mathcal{A}^{\rm o})$ with inverse $\Delta^\vee \in \mathsf{KK}_{-d}(\mathbb{C}, \mathcal{A} \otimes \mathcal{A}^{\rm o}) = \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{A}^{\rm o})$ such that:

$$\Delta^\vee \otimes_{\mathcal{A}^{\rm o}} \Delta = 1_{\mathcal{A}} \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A})$$

$$\Delta^\vee \otimes_{\mathcal{A}} \Delta = (-1)^d \, 1_{\mathcal{A}^{\rm o}} \in \mathsf{KK}_0(\mathcal{A}^{\rm o}, \mathcal{A}^{\rm o})$$

Poincaré duality – Definition

- Determines inverse isomorphisms:

$$\mathsf{K}_i(\mathcal{A}) \xrightarrow{\otimes_{\mathcal{A}}\Delta} \mathsf{K}^{i+d}(\mathcal{A}^{\mathrm{o}}) = \mathsf{K}^{i+d}(\mathcal{A})$$

$$\mathsf{K}^i(\mathcal{A}) = \mathsf{K}^i(\mathcal{A}^{\mathrm{o}}) \xrightarrow{\Delta^\vee\otimes_{\mathcal{A}^{\mathrm{o}}}} \mathsf{K}_{i-d}(\mathcal{A})$$

- More generally: $\mathcal{A}^{\mathrm{o}} \to \mathcal{B} \Longrightarrow$ PD pairs $(\mathcal{A}, \mathcal{B})$

Poincaré duality – Example

- $\mathcal{A} \;=\; C_0(X) \;=\; \mathcal{A}^{\rm o}$, $X \;=\;$ complete oriented manifold
  $\mathcal{B} \;=\; C_0(T^*X)$ or $\mathcal{B} \;=\; C_0\big(X\,,\,{\rm Cliff}(T^*X)\big)$

- $(\mathcal{A},\mathcal{B}) \;=\;$ PD pair: $\Delta \;=\;$ Dirac operator on ${\rm Cliff}(T^*X)$

- $X \;=\;$ spin^c $\Longrightarrow$ $\mathcal{A} \;=\;$ PD algebra:
  $\Delta \;=\; \not{D}$ on diagonal of $X\times X$ (image of Dirac class under $m^* : {\rm K}^\bullet(\mathcal{A}) \longrightarrow {\rm K}^\bullet(\mathcal{A}\otimes\mathcal{A})$ induced by product $m : \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}$)
  $\Delta^\vee \;=\;$ Bott element

K-orientation and Gysin homomorphisms

(Connes & Skandalis)

- $f : \mathcal{A} \longrightarrow \mathcal{B}$ $*$-homomorphism of separable C^*-algebras in suitable category

- K-orientation for $f =$ functorial way of associating $f! \in \mathrm{KK}_d(\mathcal{B}, \mathcal{A})$

- Determines Gysin "wrong way" homomorphism:

$$f_! = \otimes_{\mathcal{B}}(f!) \; : \; \mathrm{K}_\bullet(\mathcal{B}) \longrightarrow \mathrm{K}_{\bullet+d}(\mathcal{A})$$

K-orientation and Gysin homomorphisms

- $\mathcal{A}$, $\mathcal{B}$ PD algebras, any $f : \mathcal{A} \longrightarrow \mathcal{B}$ K-oriented with K-orientation:

$$f! \;=\; (-1)^{d_{\mathcal{A}}} \; \Delta_{\mathcal{A}}^{\vee} \otimes_{\mathcal{A}^{\mathrm{o}}} [f^{\mathrm{o}}] \otimes_{\mathcal{B}^{\mathrm{o}}} \Delta_{\mathcal{B}}$$

  $d \;=\; d_{\mathcal{A}} - d_{\mathcal{B}}$

- Functoriality $g! \otimes_{\mathcal{B}} f! = (g \circ f)!$ for $g : \mathcal{B} \to \mathcal{C}$ by associativity of Kasparov intersection product

K-orientation – Example

Any D-brane (W, E, f) in X determines canonical KK-theory class $f! \in \mathrm{KK}_d\big(C(W)\,,\,C(X)\big)$:

- Normal bundle $\nu \;=\; f^*(TX)/TW$ spin^c

- $i^W! := \big[(\mathcal{E}, F)\big] \in \mathrm{KK}_d\big(C(W)\,,\,C_0(\nu)\big)$ invertible element associated to ABS rep. of Thom class of zero section $i^W : W \hookrightarrow \nu$

- $j! \in \mathrm{KK}_0\big(C_0(\nu)\,,\,C(X)\big)$ extension by zero

- K-orientation for f:

$$f! \;=\; i^W! \otimes_{C_0(\nu)} j!$$

- K-orientation $\equiv$ Freed-Witten anomaly cancellation condition

Local cyclic cohomology – Definition

- $\mathcal{A}$ unital; noncommutative differential forms on $T\mathcal{A}$:

$$\Omega^n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)} \oplus \mathcal{A}^{\otimes n}, \qquad \mathrm{d} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\cong \mathrm{Span}_{\mathbb{C}}\left\{ a_0 \, \mathrm{d}a_1 \cdots \mathrm{d}a_n \;\middle|\; a_0, a_1, \dots, a_n \in \mathcal{A} \right\}$$

- Completion of $\Omega^\bullet(\mathcal{A}) = \mathbb{Z}_2$-graded X-complex:

$$X(T\mathcal{A}) : \Omega^0(T\mathcal{A}) = T\mathcal{A} \underset{\mathrm{b}}{\overset{\natural \circ \mathrm{d}}{\rightleftarrows}} \Omega^1(T\mathcal{A})_\natural = \frac{\Omega^1(T\mathcal{A})}{\left[\Omega^1(T\mathcal{A}), \Omega^1(T\mathcal{A})\right]}$$

$$\mathrm{b} : \omega_0 \, \mathrm{d}\omega_1 \longmapsto [\omega_0, \omega_1], \quad \mathrm{b}^2 = 0$$

Local cyclic cohomology – Definition

- Puschnigg's completion of $X(T\mathcal{A})$:

$$\widehat{X}(T\mathcal{A}) \;:\; \prod_{n\geq 0} \Omega^{2n}(\mathcal{A}) \;\underset{\longleftarrow}{\longrightarrow}\; \prod_{n\geq 0} \Omega^{2n+1}(\mathcal{A})$$

- $\mathbb{Z}_2$-graded bivariant local cyclic cohomology:

$$\mathrm{HL}_\bullet(\mathcal{A},\mathcal{B}) \;=\; \mathsf{H}_\bullet\left(\mathrm{Hom}_{\mathbb{C}}(\widehat{X}(T\mathcal{A}),\widehat{X}(T\mathcal{B}))\,,\,\partial\right)$$

Local cyclic cohomology – Properties

Cyclic theory "closest" to KK-theory; encompasses other cyclic theories (analytic, periodic, ...):

- Defined on large classes of topological/bornological algebras, **and** for separable C^*-algebras

- Bifunctor homotopy invariant, split exact and satisfies excision

- Bilinear, associative composition product:

$$\otimes_{\mathcal{B}} \; : \; \mathrm{HL}_i(\mathcal{A}, \mathcal{B}) \times \mathrm{HL}_j(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{HL}_{i+j}(\mathcal{A}, \mathcal{C})$$

- Bilinear, associative exterior product:

$$\otimes \; : \; \mathrm{HL}_i(\mathcal{A}_1, \mathcal{B}_1) \times \mathrm{HL}_j(\mathcal{A}_2, \mathcal{B}_2) \longrightarrow \mathrm{HL}_{i+j}(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$$

Local cyclic cohomology – Example

- X compact oriented manifold, $\dim(X) = d$
- $C^\infty(X) \hookrightarrow C(X) \Longrightarrow \mathrm{HL}\big(C(X)\big) \cong \mathrm{HL}\big(C^\infty(X)\big) \cong \mathsf{HP}\left(C^\infty(X)\right)$
- Puschnigg complex $=$ de Rham complex $\big(\Omega^\bullet(X)\,,\,\mathrm{d}\big)$
- Connes–Hochschild–Kostant–Rosenberg theorem:

$$f^0\,\mathrm{d}f^1\cdots\mathrm{d}f^n \ \longmapsto\ \tfrac{1}{n!}\ f^0\,\mathrm{d}f^1\wedge\cdots\wedge\mathrm{d}f^n\ , \quad f^i\in C^\infty(X)$$

- Putting everything together:

$$\mathrm{HL}_\bullet\big(C(X)\big)\ \cong\ \mathsf{H}^\bullet_{\mathrm{dR}}(X) \qquad (\mathbb{Z}_2\text{-graded})$$

Local cyclic cohomology – Example

Cyclic d-cocycle induces orientation fundamental class $\Xi = m^*[\varphi] \in \mathrm{HL}^d\big(C(X) \otimes C(X)\big)$:

$$\varphi\big(f^0, f^1, \ldots, f^d\big) = \frac{1}{d!} \int_X f^0 \, \mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^d$$

Chern character

There is a natural bivariant $\mathbb{Z}_2$-graded Chern character homomorphism:

$$\mathrm{ch} \, : \, \mathsf{KK}_\bullet(\mathcal{A},\mathcal{B}) \, \longrightarrow \, \mathrm{HL}_\bullet(\mathcal{A},\mathcal{B})$$

- Multiplicative: $\mathrm{ch}(\alpha \otimes_{\mathcal{B}} \beta) \, = \, \mathrm{ch}(\alpha) \otimes_{\mathcal{B}} \mathrm{ch}(\beta)$
- Compatible with exterior product
- $\mathrm{ch}\big([\phi]_{\mathsf{KK}}\big) \, = \, [\phi]_{\mathrm{HL}}$ for any $\phi : \mathcal{A} \longrightarrow \mathcal{B}$
- If $\mathcal{A},\mathcal{B}$ obey UCT for KK-theory and $\mathsf{K}_\bullet(\mathcal{A})$ finitely generated:

$$\mathrm{HL}_\bullet(\mathcal{A},\mathcal{B}) \, \cong \, \mathsf{KK}_\bullet(\mathcal{A},\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{C}$$

- Every PD pair for KK is also a PD pair for HL (but $\Xi \neq \mathrm{ch}(\Delta)$).

Todd classes

- $\mathcal{A}$ PD algebra with fund. K-homology class $\Delta \in \mathsf{K}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$, fund. cyclic cohomology class $\Xi \in \mathrm{HL}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$
- **Todd class** of $\mathcal{A}$:

$$\mathrm{Todd}(\mathcal{A}) \;:=\; \Xi^\vee \otimes_{\mathcal{A}^{\mathrm{o}}} \mathrm{ch}(\Delta) \;\in\; \mathrm{HL}_0(\mathcal{A},\mathcal{A})$$

- Invertible: $\mathrm{Todd}(\mathcal{A})^{-1} \;=\; (-1)^d \;\mathrm{ch}\big(\Delta^\vee\big) \otimes_{\mathcal{A}^{\mathrm{o}}} \Xi$
- $\mathcal{A} \;=\; C(X)$, $X \;=\;$ compact complex manifold, is a PD alg.:
  $\Delta \;=\;$ Dolbeault op. ∂ on $X \times X$, $\;\Xi \;=\;$ orientation cycle $[X]$
  By UCT, $\mathrm{HL}_0(\mathcal{A},\mathcal{A}) \cong \mathsf{End}\,\big(\mathsf{H}^\bullet(X,\mathbb{Q})\big)$
  Then $\mathrm{Todd}(\mathcal{A}) \;=\; \smile\; \mathrm{Todd}(X)$ with $\mathrm{Todd}(X) \in \mathsf{H}^\bullet(X,\mathbb{Q})$

Grothendieck–Riemann–Roch theorem

- $f : \mathcal{A} \longrightarrow \mathcal{B}$ K-oriented — compare $\mathrm{ch}(f!)$ with HL orientation class $f*$ in $\mathrm{HL}_d(\mathcal{B}, \mathcal{A})$. If $\mathcal{A}$, $\mathcal{B}$ PD algs., then $d = d_{\mathcal{A}} - d_{\mathcal{B}}$ and:

$$\mathrm{ch}(f!) \ = \ (-1)^{d_{\mathcal{B}}} \ \mathrm{Todd}(\mathcal{B}) \otimes_{\mathcal{B}} (f*) \otimes_{\mathcal{A}} \mathrm{Todd}(\mathcal{A})^{-1}$$

- Commutative diagram:

$$\begin{array}{ccc} \mathsf{K}_\bullet(\mathcal{B}) & \xrightarrow{f_!} & \mathsf{K}_{\bullet+d}(\mathcal{A}) \\ \downarrow {\scriptstyle \mathrm{ch}\otimes_{\mathcal{B}}\mathrm{Todd}(\mathcal{B})} & & \downarrow {\scriptstyle \mathrm{ch}\otimes_{\mathcal{A}}\mathrm{Todd}(\mathcal{A})} \\ \mathrm{HL}_\bullet(\mathcal{B}) & \xrightarrow[f_*]{} & \mathrm{HL}_{\bullet+d}(\mathcal{A}) \end{array}$$

Isometric pairing formula

- $\mathcal{A}$ PD alg. with **symmetric** fund. classes Δ, Ξ, i.e., $\sigma(\Delta)^{\mathrm{o}} = \Delta$ in $\mathsf{K}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$, where $\sigma : \mathcal{A} \otimes \mathcal{A}^{\mathrm{o}} \longrightarrow \mathcal{A}^{\mathrm{o}} \otimes \mathcal{A}$, $x \otimes y^{\mathrm{o}} \longmapsto y^{\mathrm{o}} \otimes x$

- Symmetric bilinear pairing on K-theory of $\mathcal{A}$:

$$(\alpha, \beta)_{\mathsf{K}} = (\alpha \otimes \beta^{\mathrm{o}}) \otimes_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}} \Delta \in \mathsf{KK}_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$$

  Index pairing when $\mathcal{A} = C(X)$, X spin^c ($\Delta = \not{D} \otimes \not{D}$):

$$(\alpha, \beta)_{\mathsf{K}} = \not{D}_\alpha \otimes_{C(X)} \beta = \mathrm{index}(\not{D}_{\alpha \otimes \beta})$$

- Symmetric bilinear pairing on local cyclic homology:

$$(x, y)_{\mathrm{HL}} = (x \otimes y^{\mathrm{o}}) \otimes_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}} \Xi \in \mathrm{HL}_0(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

Isometric pairing formula

- If $\mathcal{A}$ satisfies UCT then $\mathrm{HL}_\bullet(\mathcal{A},\mathcal{A}) \cong \mathrm{End}\big(\mathrm{HL}_\bullet(\mathcal{A})\big)$
  If $n := \dim_{\mathbb{C}}\big(\mathrm{HL}_\bullet(\mathcal{A})\big) < \infty$, then $\mathrm{Todd}(\mathcal{A}) \in GL(n,\mathbb{C})$ and $\sqrt{\mathrm{Todd}(\mathcal{A})}$ defined using Jordan normal form

- Then **modified Chern character**:

$$\mathrm{ch} \otimes_{\mathcal{A}} \sqrt{\mathrm{Todd}(\mathcal{A})} \,:\, \mathsf{K}_\bullet(\mathcal{A}) \longrightarrow \mathrm{HL}_\bullet(\mathcal{A})$$

  isometry of inner products

Noncommutative Minasian–Moore formula

- $\mathcal{A}$, $\mathcal{D}$ noncommutative D-branes with $\mathcal{A}$ as before, $f : \mathcal{A} \longrightarrow \mathcal{D}$ K-oriented, and Chan–Paton bundle $\xi \in \mathsf{K}_\bullet(\mathcal{D})$:

$$Q(\mathcal{D}, \xi, f) \ = \ \mathrm{ch}\big(f_!(\xi)\big) \otimes_{\mathcal{A}} \sqrt{\mathrm{Todd}(\mathcal{A})} \ \in \ \mathrm{HL}_\bullet(\mathcal{A})$$

- D-brane in noncommutative spacetime $\mathcal{A}$ described by Fredholm module representing class $\mu \in \mathsf{K}^\bullet(\mathcal{A})$, has "dual" charge:

$$Q(\mu) \ = \ \sqrt{\mathrm{Todd}(\mathcal{A})}^{\,-1} \otimes_{\mathcal{A}} \mathrm{ch}(\mu) \ \in \ \mathrm{HL}^\bullet(\mathcal{A})$$

  Satisfies isometry rule:

$$\Xi^\vee \otimes_{\mathcal{A}\otimes\mathcal{A}^{\rm o}} \big(Q(\mu) \otimes Q(\nu)^{\rm o}\big) \ = \ \Delta^\vee \otimes_{\mathcal{A}\otimes\mathcal{A}^{\rm o}} \big(\mu \otimes \nu^{\rm o}\big)$$

  Minasian–Moore formula when $\mu \ = \ f_!(\xi) \otimes_{\mathcal{A}} \Delta$