Slice Knots: Knot Theory in the 4th Dimension

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1 The Next Best Thing to the Unknot

Definition 1.1. A *knot* is an oriented, locally flat, embedding of $S^1 \hookrightarrow S^3$.

In traditional knot theory, a knot $S^1 \overset{K}{\hookrightarrow} S^3$ is trivial if it bounds a disc. Now, any knot $S^1 \hookrightarrow S^4$ is trivial. This is because the freedom of the extra dimension allows one to pass strands through one another and thus untie any knot. However this operation requires that the disc which this unknotting operation traces out lives in the D^4 on either side of S^3 ($S^4 = D^4 \cup_{S^3} D^4$) - it is necessary to push one strand into the 4th dimension in one direction and the other strand into the 4th dimension the other way. The interesting question to ask is therefore whether a knot bounds a disc in D^4 on one side only of S^3. First we recall some definitions.

Reminder. An *immersion* is a differentiable map whose derivative is everywhere injective.
An *embedding* is an immersion which is also a homeomorphism onto its image, where the image has the subspace topology.

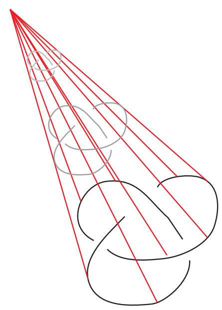
Definition 1.2. An embedding $S^1 \overset{K}{\hookrightarrow} S^3$ is *locally flat* if for each point $x \in S^1$ there is a neighbourhood $U \subset S^1$ of x and a neighbourhood $V \subset S^3$ of $K(x)$ such that the pair $(V, K(U))$ can be mapped homeomorphically onto $(\mathring{D}^3, \mathring{D}^1)$.

Similarly $D^2 \overset{q}{\hookrightarrow} D^4$ is *locally flat* if for each point $x \in D^2$ there is a neighbourhood U of x and a neighbourhood V of $q(x)$ such that the pair $(V, q(U))$ can be mapped homeomorphically onto $(\mathring{D}^4, \mathring{D}^2)$.

Definition 1.3. A knot K is *slice* if it is the boundary of a locally flat disc D^2 embedded into the 4-ball D^4.

We may also think of K as the cross-section of a locally flat 2-sphere S^2 in $\mathbb{R}^4$ by a hyperplane $\mathbb{R}^3$.

Remark 1.4. Flatness is essential. Any knot $K \subset S^3$ is the boundary of a disc D^2 embedded in D^4, which can be seen by taking the cone over the knot (see figure below).



The cone over the knot is homeomorphic to a disc D^2, and if we embed it into 4-space there will be no singularities.

However, there is something displeasing about what is happening at the vertex of the cone, where the knot gets squashed to a point. The problem is that the embedding at this point is not *locally flat*; there is not a neighbourhood around it which looks topologically like the standard embedding of a disc D^2 into D^4.

Slice knots are special kinds of knots where it is possible to find such a disc whilst avoiding these kinds of singularities. The original motivation for the study of slice knots, and the first definition, was made by Fox and Milnor in 1958; they were interested in smoothing PL singularities of surfaces in 4-space, which arise naturally when considering complex hypersurfaces.

Slice knots are also intimately related with the failure of the Whitney trick in 4-dimensions. The Whitney trick is used to remove intersections of submanifolds which cancel algebraically: if there are paths between the intersection points in each submanifold which form a loop, and this loop can be made to bound an embedded disk, then by isotoping across the disk the intersections can be removed. This works in high dimensions, but in dimension 4 the disks can only be immersed generically. The question of improving these to embeddings is like trying to slice a knot.

Finally, slice knots are interesting because they enable us to make the set of all knots into a group, which will be our central object of study:

Example 1.5. If K is a knot which is symmetric with respect to a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ then K is slice because we can spin it through $\mathbb{R}^4_+$ about the axis $\mathbb{R}^2$ to produce the desired locally flat disc. [We can spin a point $x = (x_1, x_2, x_3, 0)$ of $\mathbb{R}^3_+$ about $\mathbb{R}^2$ according to the formula $x_\theta = (x_1, x_2, x_3 \cos\theta, x_3 \sin\theta)$. The spin $K^* = \{x_\theta : x \in K, 0 \leq \theta \leq 2\pi\}$ is a 2-sphere in $\mathbb{R}^4$.]

So if K is a knot in S^3 and $r\colon S^3 \to S^3$ is an orientation-reversing homeomorphism, and if $\overline{K}$ is the same knot but with the string orientation reversed, then $K \# \overline{rK}$ is a slice knot.

Example 1.6. Our main bank of examples in this text will be the *Twist Knots* as shown below. The box with an n in it should really mean $n-1$ full twists, so that the total number of twists is n. (Figure 1):

We will see in due course that K_n is slice if and only if $n = 0$ or $n = 2$ ($n = 0$ is the unknot so is trivially slice).

How to see that a knot is slice We can visualise a slice disc by making movies. If a knot is (smoothly) slice then it bounds a disc $D^2 \subset D^4$ so that concentric 3-spheres move through (intersect) it to produce either

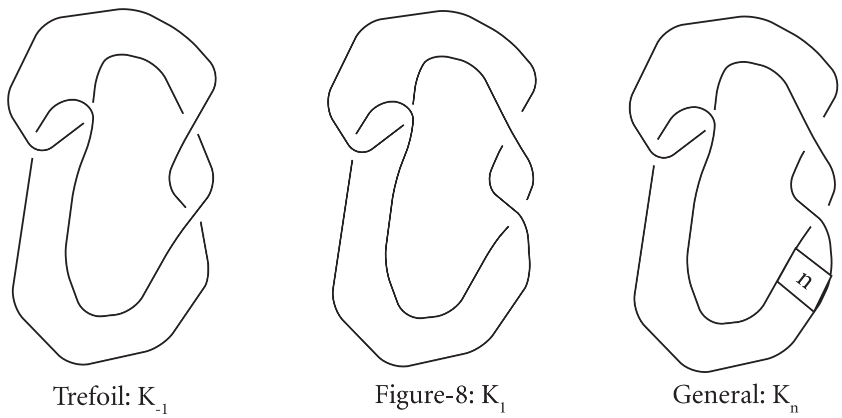
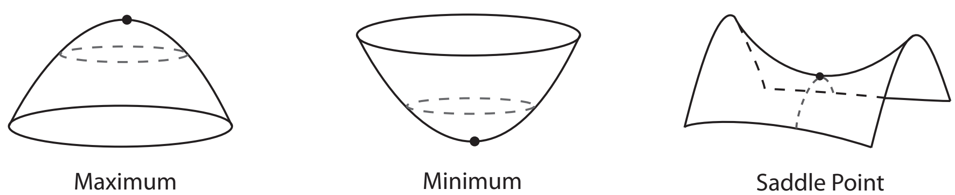
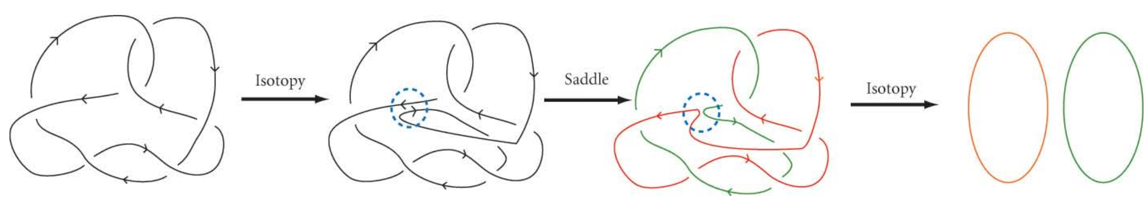


Figure 1: The Twist Knots

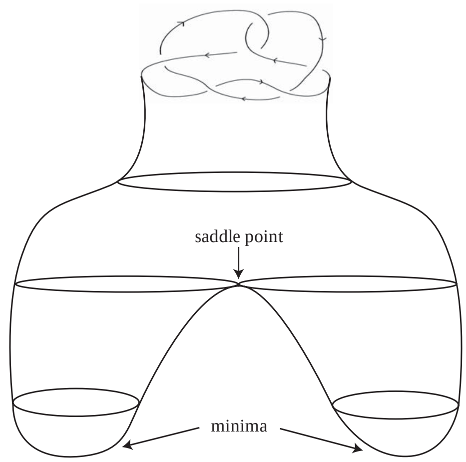
1. An ordinary nonsingular knot or link
2. A knot or link with singularities corresponding to one of
   (a) Simple maximum or minimum
   (b) Saddle point



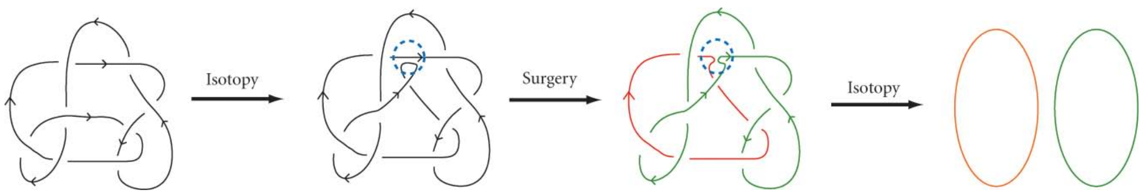
Example 1.7. Stevedore's knot, otherwise known as 6_1 in the standard knot tables, is the simplest slice knot (other than the unknot). The following "movie" shows how 3-spheres move through the slice disc:



The slice disc is shown schematically below - of course in reality this is a knotted disc in 4-space:



Example 1.8. Another example of a slice knot is the 8-crossing knot 8_8. Here is the corresponding slice 'movie':



Definition 1.9. A *ribbon* disc is a slice disc without local maxima.

To construct a ribbon knot, start with an unlink (which corresponds to local minima; we can arrange the disk so that these come first, then the saddles). Now add bands connecting them until the result is a knot/disc. Then local ribbon singularities are the only self-intersections, as opposed to *clasp* singularities.

The ribbon disc is the image $\alpha(D^2)$ of a mapping $\alpha\colon D^2 \to \mathbb{R}^3$ whose only singularities are of the following form. Each component of the singular set is the image of a pair of closed intervals in D^2, one with endpoints on the boundary of D^2 and one entirely interior to D^2.

We can locally resolve a ribbon singularity into 4-space to get back a slice disc. See Figure 2, where the cross-hatched parts can be pushed off S^3 into D^4 to remove the self-intersections. This does not work for clasp singularities.

Fact. Every ribbon knot is a slice knot.

The following is a famous unsolved conjecture of Fox.

Conjecture 1.10. Every slice knot is a ribbon knot.

Note that this is about smooth knots. Any ribbon knot is smoothly slice, since we used Morse theory to get a handle decomposition of the slice disc. As we shall see below there are knots which are topologically slice but not smoothly slice, so in particular these cannot be ribbon.

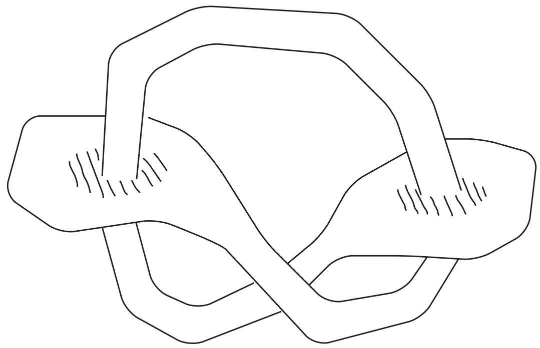


Figure 2: Stevedore's knot 6_1 bounds a singular disc with two arcs of intersection

2 The Knot Concordance Group

Definition 2.1. Knots K_0 and K_1 are called *concordant* if $K_0 \# - K_1$ is slice. (Here $-K$ denotes the mirror image of the knot with reversed orientation, while $\#$ is connected sum.)

Definition 2.2.

$$\mathcal{C} := \left\{\text{oriented knots in } S^3, \#\right\} / \sim$$

where $K_0 \sim K_1$ if they are concordant.

Notice that the group can be constructed by semigroup quotient, i.e. $K_1 \sim K_2$ in $\mathcal{C}$ iff there exist slice knots S_1, S_2 such that $K_1 \# S_1 = K_2 \# S_2$.

There is an easier to visualise picture of concordance which is related to cobordism of manifolds.

Definition 2.3. Two knots K_0 and K_1 are called *concordant* if there is a locally flat embedding of $S^1 \times [0,1]$ into $S^3 \times [0,1]$ having boundary the knots K_0 and $-K_1$ in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively.

So we can think of concordance as when two knots are connected by a cylinder embedded (nicely) into the 4th dimension. Slice knots, of course, are those which are concordant to the unknot.

Theorem 2.4. *$\mathcal{C}$ really is a group.*

Proof. For a knot K the knot $r\overline{K}$ is an inverse. It is possible to glue two null-concordances together, so that $K_1 \sharp K_2$ is slice if both K_1 and K_2 are slice. It is also the case that if $K_1 + K_2$ is slice and K_2 is slice, then K_1 is also slice. To see this glue the null-concordances together according to: $K_1 \sim K_1 + U \sim K_1 + K_2 \sim U$. □

Lemma 2.5. *K is slice if and only if there exists a ribbon knot R such that $K \# R$ is ribbon.*

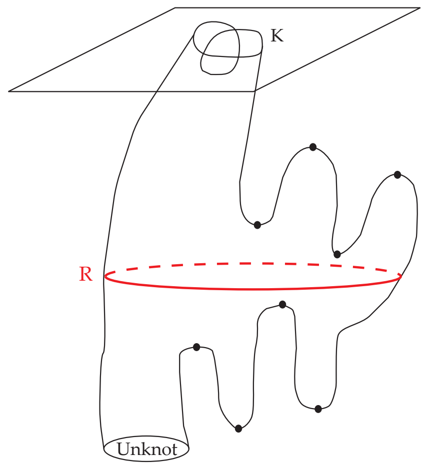


Figure 3: Finding a ribbon knot within a slice cobordism.

Proof. "$\Leftarrow$" Ribbon knots are slice, so R is slice and $K\#R$ is concordant to K. Since $K\#R$ is ribbon and therefore slice, this means that K is concordant to a slice knot, and is therefore slice.

"$\Rightarrow$" Suppose K is slice. Then we can draw a cobordism to the unknot with the maxima first, then saddles then minima

In the middle of this cobordism we can find a knot R which will be ribbon because it is cobordant to the unknot with only minima (Figure 3).

Take the connected sum of R and K along the left-hand boundary of the concordance (Figure 4, left-hand picture). With a little imagination this can turn into the right-hand picture in Figure 4, which is a schematic of a ribbon disc.

□

Proposition 2.6. The group $\{\text{knots}, \#\} / \{\text{ribbon knots}\}$ is isomorphic to $\mathcal{C}$.

Proof. Suppose $K_1 \sim K_2$ in Knots/Ribbon. So there exist R_1 and R_2 ribbon such that $K_1\#R_1 = K_2\#R_2$. But ribbon knots are slice, so this means $K_1 \sim K_2$ in Knots/Slice.

Now suppose $K_1\#S_1 = K_2\#S_2$ for S_i slice. By Lemma 2.5 we can find ribbon knots R_i such that $S_i\#R_i$ is ribbon for $i = 1, 2$. Then

$$K_1\#S_1\#(R_1\#R_2) = K_2\#S_2\#(R_1\#R_2)$$

which we can re-bracket as

$$K_1\#((S_1\#R_1)\#R_2) = K_2\#((S_2\#R_2)\#R_1).$$

Since the addition of two ribbon knots is ribbon, we have the result that $K_1\#\text{ribbon} = K_2\#\text{ribbon}$. Thus $K_1 \sim K_2$ in Knots/Ribbon. □

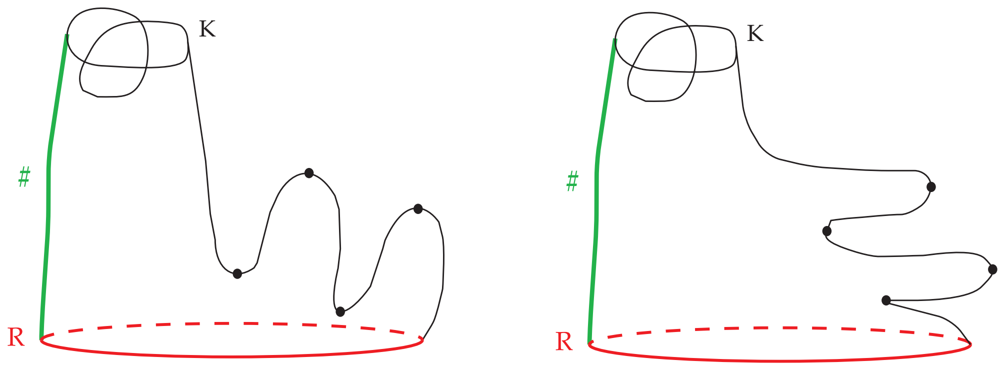


Figure 4: Seeing a ribbon knot in the connected sum.

We want to show that $\mathcal{C}$ is non-trivial. We can do this with additive knot invariants that vanish on ribbon knots (often more convenient to show).

Recall, the linking number of curves l_1, $l_2 \subset S^3$ is defined as $\text{lk}(l_1, l_2) = l_1 \cdot F_2$, the intersection number of l_1 with a Seifert surface F_2 for l_2. To generalise to 4 dimensions, choose Seifert surfaces F_1, F_2 embedded in D^4 transverse to the boundary such that $\partial F_1 = l_1$, $\partial F_2 = l_2$. Orient F_1, F_2 so that they induce the given orientations on l_1 and l_2. Then

$$\text{lk}(l_1, l_2) = F_1 \cdot F_2$$

where this intersection number is taken in D^4. The observation that linking information in S^3 strongly corresponds to intersection data in D^4 is a fundamental one, as we shall see.

We must show that this definition is independent of the Seifert surfaces we chose. Pick alternative surfaces G_1 and G_2, in a different copy of D^4. Then glue the two copies of D^4 together to get closed surfaces $F_1 \cup G_1$ and $F_2 \cup G_2$ in S^2. Then $H_2(S^4) \cong 0$ so any intersections cancel: let R be the 3-chain whose boundary is $F_1 \cup G_1$; then the intersections of $F_2 \cup G_2$ with this are arcs whose endpoints are the intersections of $F_1 \cup G_1$ and $F_2 \cup G_2$. These endpoints have opposite intersection signs, so

$$(F_1 \cup G_1) \cap (F_2 \cup G_2) = 0.$$

From this definition, we see that a link with non-zero linking number cannot be slice (i.e. bound disjoint discs in 4-space).

Let F be a Seifert surface for K. Then we have the Seifert pairing on $H_1(F) \cong \mathbb{Z}^{2g}$

$$S\colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z};$$

$$S(x, y) = \mathrm{lk}(x, y^+),$$

where x, y are representative chains of the homology classes, and y^+ is the push off of y into $S^3 \setminus F$ along a normal vector to F. From S we can get the Alexander polynomial

$$A(K)(t) = \det(S - tS^T) \in \mathbb{Z}[t]/\pm t^n.$$

We can show this is independent of basis choices and the choice of surface: this is the case since any two Seifert surfaces are related by some sequence of isotopies and handle additions/subtractions. By examining the effect of such moves on the Seifert matrix, detailed below, one can see that the various invariants are independent of the choice of surface, without which property it would be hard to justify calling them invariant.

Definition 2.7. The *signature* $\sigma(K)$ is the signature of the symmetric form $S + S^T$, taken over $\mathbb{R}$ so that it can be diagonalised. This is also independent of choice of S.

Remark 2.8. Independence of the choice of S is from Morse theory – the elementary changes are handle additions which have the effect

$$S \longleftrightarrow \left(\begin{array}{ccc|cc} & & & 0 & \vdots \\ & S & & \vdots & \vdots \\ & & & 0 & \vdots \\ \hline 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & 1 & * \end{array}\right)$$

This is called S-equivalence. One also checks that the signature and the Alexander polynomial are invariant under integral congruences corresponding to changing the basis of $H_1(F)$.

Remark 2.9. We can normalise the Alexander polynomial $A(K)(t) \in \mathbb{Z}[t^{\pm 1}]$ by

1. $A(K)(t) = A(K)(t^{-1})$
2. $A(K)(1) = +1$ (notice that $S - S^T$ is the intersection form J and has $\det \pm 1$).
3. Addition of knots corresponds to block addition of Seifert matrices, so that the signature is additive and addition of knots corresponds to multiplication of Alexander polynomials: $A(K_1 \sharp K_2) \doteq A(K_1)A(K_2)$.

The Conway polynomial $C_K(z)$ is the $z = t^{1/2} + t^{-1/2}$ version of this.

Theorem 2.10. If K is slice then there exists a half-rank direct summand L in $H_1(F)$ such that $S|_L = 0$.

Corollary 2.11. (a) K slice $\Rightarrow \sigma(K) = 0$.

(b) K slice $\Rightarrow A(K)$ is of the form $f(t)f(t^{-1})$ (up to $t^{\pm n}$).

Proof. (of corollary)

(a) Let $M = S + S^T = \begin{pmatrix} 0 & L \\ L^T & N \end{pmatrix}$, where $N = C + C^T$. The matrix $S - S^T$ is non-singular so $S + S^T$ is non-singular over $\mathbb{Z}_2$, and therefore also over $\mathbb{Z}$, that is $\det(S + S^T) \neq 0$, which means that $\det(L) \neq 0$ and L is invertible over $\mathbb{Q}$. Let $P = \begin{pmatrix} L^{-1} & 0 \\ CL^{-1} & -I \end{pmatrix}$. Then $PMP^T = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$, so $\sigma(K) = \sigma(M) = \sigma(PMP^T) = 0$.

(b) $S = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$ so $\det(S - tS^T) = \det \begin{pmatrix} 0 & A - tB^T \\ B - tA^T & C - tC^T \end{pmatrix} = \det(A - tB^T)\det(B - tA^T) = f(t)f(t^{-1})$ up to units.

□

Remark 2.12. The knot signature is not to be confused with the signature of a 4-manifold, which is the signature of the middle-dimensional intersection form on the 2nd homology of (some cover of) the manifold. (Although as we shall see for a judicious choice of 4-manifold the two signatures can coincide.)

Example 2.13. RH Trefoil: $S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, so $M = S + S^T = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. Eigenvalues are both negative, so $\sigma = -2$.
Figure-8: $S = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ so $M = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. Eigenvalues are $\pm\sqrt{5}$ so $\sigma = 0$.

The fact that $\sigma = 0$ for the Figure-8 agrees with the fact that σ is additive and the Figure-8 is amphichiral (so $2\sigma(K) = \sigma(K\#\overline{K}) = 0$, since $K\#\overline{K}$ is slice).

However, $A(\text{Fig-8}) = \det \begin{pmatrix} 1-t & -t \\ 1 & -1+t \end{pmatrix} = -(1-t)^2 + t = -t^2 + 3t - 1 \cong -t + 3 - t^{-1}$, and this isn't $f(t)f(t^{-1})$ (since $A(-1)$ is not a square). Therefore the Figure-8 is a 2-torsion element in $\mathcal{C}$, and the Trefoil generates a free summand of $\mathcal{C}$.

Example 2.14. We return to the twist knots. We can get a Seifert surface by resolving the obvious disc with clasp singularity into two half twisted bands. Then, for the twist knot K_n we get a Seifert matrix

$$S = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}.$$

We constructed a homomorphism $\sigma\colon \mathcal{C} \to \mathbb{Z}$, the signature, by using the theorem that any slice knot K with a Seifert surface F has a Lagrangian $L \subset H_1(F)$ (i.e. a half-rank direct summand for which $S|_L = 0$). Now we need to prove this.

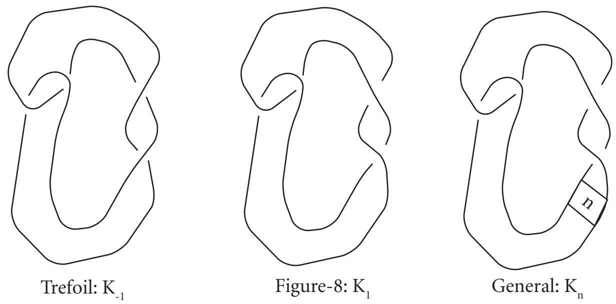


Figure 5: The Twist Knots again.

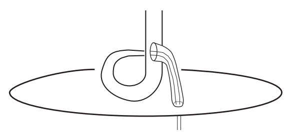


Figure 6: Add a tube and feed the band through. The knot remains the same!

Proof of Theorem 2.10. **Warmup exercise for ribbon knots:** Prove for ribbon knots that σ is well-defined. Actually, it is enough to show that σ vanishes on ribbon knots for *some* Seifert surface, because we know $\mathcal{C} =$ knots/slice $\cong$ knots/ribbon and σ is independent of choice of surface.

Ribbon singularities can be resolved (see Figure 6).

Desingularising the whole ribbon like this gives a disc and a tube for each ribbon singularity, and with obvious Lagrangian: circles surrounding the ribbon cuts; a basis for the Lagrangian is given by an *unlink* of circles.

Proof of Theorem:
Step 1: There exists an oriented submanifold $M^3 \subset D^4$ with boundary $F \cup \Delta$, where F is a Seifert surface of K and Δ is a slice disc.

Proof of Step 1. This elementary application of obstruction theory was lifted from the textbook of Lickorish. Let X be the exterior of K. We want to define a map $\phi\colon X \to S^1$ so that $\phi_*\colon H_1(X) \to H_1(S^1)$ is an isomorphism and $\phi^{-1}(\mathrm{pt}) = F$. On a product neighbourhood of F in X, define ϕ to be the projection $F \times [-1, 1] \to [-1, 1]$ followed by the map $t \mapsto e^{i\pi t} \in S^1$. Let ϕ map the remainder of X to $-1 \in S^1$.

Let $N = \Delta \times I^2$, a neighbourhood of Δ. We extend ϕ to the rest of $\partial(\overline{D^4 - N})$ so that the inverse image of $1 \in S^1$ is $F \cup (\Delta \times *)$ for some point $* \in \partial I^2$ (note: $\partial D \times *$ is a longitude of K). We now need to extend the map over all of $\overline{D^4 - N}$.

Consider the simplices of some triangulation of $\overline{D^4 - N}$. Let T be a tree in the 1-skeleton containing all the vertices of this triangulation, that contains a similar maximal tree of $\partial(\overline{D^4 - N})$. Extend ϕ over all of T in an arbitrary way. Then on a 1-simplex σ not in T, define ϕ so that if c is a 1-cycle of $H_1(\overline{D^4 - N})$ consisting of σ summed with a 1-chain in T (joining up the ends of σ), then $[\phi c] \in H_1(S^1)$ is the image of $[c]$ under the isomorphism

$$H_1(\overline{D^4 - N}) \overset{\cong}{\longleftarrow} H_1(X) \xrightarrow{\phi_*} H_1(S^1).$$

Trivially, the boundary of a 2-simplex τ of $\overline{D^4 - N}$ represents zero in $H_1(\overline{D^4 - N})$, so $[\phi(\partial\tau)] = 0 \in H_1(S^1)$. Hence ϕ is null-homotopic on $\partial\tau$ and so extends over τ. Finally, ϕ extends over the 3 & 4-simplices, as any map from the boundary of an n-simplex to S^1 is null-homotopic when $n \geq 3$.

Now regard $\phi\colon \overline{D^4 - N} \to S^1$ as a simplicial map to some triangulation of S^1 in which 1 is not a vertex. Then $\phi^{-1}(1)$ is a 3-manifold M^3, and ϕ was constructed so that $\partial M^3 = F \cup (\Delta \times *)$.

Step 2: $P := \ker[H_1(\partial M;\mathbb{Q}) \to H_1(M,\mathbb{Q})]$ is a Lagrangian subspace of dimension g (where ∂M has genus g).

Proof of Step 2. A Lagrangian subspace is a vector subspace P of half rank which satisfies $P = P^{\perp}$. Look at the homology exact sequence of $(M, \partial M)$ (over $\mathbb{Q}$):

$$\begin{tikzcd}
0 \arrow[r] & H_3(M,\partial M) \arrow[r] & H_2(\partial M) \arrow[r] & H_2(M) \arrow[r] & H_2(M,\partial M) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
 & P \arrow[rr] \arrow[d] & & 0 & & \\
H_1(\partial M) \arrow[ur, "k"] \arrow[r] & H_1(M) \arrow[r] & H_1(M,\partial M) \arrow[r] & H_0(\partial M) \arrow[r] & H_0(M) \arrow[r] & 0
\end{tikzcd}$$

Lefschetz duality says $H_1(M,\partial M) \cong H_2(M)$ (since $H^k(M,\partial M) \cong H_{n-k}(M)$ and $H^k(M,\partial M) \cong H_k(M,\partial M)$ by the Universal Coefficient Theorem, as we are working over a field). Poincaré duality says $H^k(M) \cong H_{n-k}(M)$, which again implies $\dim(H_k(M)) = \dim(H_{n-k}(M))$.

So let

$a = \dim(H_3(M,\partial M)) = \dim(H_0(M))$

$b = \dim(H_2(\partial M)) = \dim(H_0(\partial M))$

$c = \dim(H_2(M)) = \dim(H_1(M,\partial M))$

$d = \dim(H_2(M,\partial M)) = \dim(H_1(M))$

$e = \dim(H_1(\partial M))$

Then the exactness of the sequence implies that

$$a - b + c - d + e - d + c - b + a = 0 \Rightarrow 2(a - b + c - d) + e = 0$$

But $\dim P = e - d + c - b + a = e + (a - b + c - d)$.

Therefore $2(\dim P - e) + e = 0 \Rightarrow 2\dim P = e$, and thus $\dim P = \frac{1}{2}\dim(H_1(\partial M))$.

Now note that $H_1(\partial M) = H_1(F)$ (recall $\partial M = F \cup \Delta$ and $H_1(\Delta) \cong 0$). Suppose we have α, $\beta \in \ker[H_1(\partial M) \to H_1(M)]$. There exist surfaces $A, B \subset M$ with $\partial A = \alpha$, $\partial B = \beta$. When α is moved to α^+, the surface A can also be moved off M to $M \times \{1\}$ (so $\partial A^+ = \alpha^+$), and then the intersection of A^+ and B is empty. Thus $\mathrm{lk}(\alpha^+, \beta) = A^+ \cdot B = 0$.

To move from $\mathbb{Q}$ to $\mathbb{Z}$ coefficients is no problem, although the $\mathbb{Z}$-kernel of $H_1(F) \to H_1(M)$ might not be a direct summand. Use instead $L = \{a \in H_1(F) \colon \exists\, n \in \mathbb{Z} \setminus \{0\} \text{ s.t. } na \in P\}$. Its rank over $\mathbb{Q}$ is the same as that of the kernel and it is a direct summand because $H_1(F)/L$ is torsion-free - any torsion is automatically in L. Note also that $S(na, b) = 0 \Longrightarrow S(a, b) = 0$ by linearity, so $S|_L = 0$, as required. □

Now we can use the twist knots K_n to show that $\mathcal{C}$ is not finitely-generated.

Definition 2.15. For $\omega \in S^1 \setminus \{1\} \subseteq \mathbb{C}$, we define the *twisted signatures* $\sigma_\omega \colon \mathcal{C} \to \mathbb{Z}$ by

$$\sigma_\omega(K) := \sigma((1 - \omega)S + (1 - \overline{\omega})S^T)$$

where S is a Seifert matrix for K.

Notice that the matrix $Q := (1 - \omega)S + (1 - \overline{\omega})S^T$ is hermitian so the eigenvalues are real and the signature is well-defined. Note also that $Q = (1 - \omega)(S - \overline{\omega}S^T)$ and $\det Q = (1 - \omega)^{2g} A_K(\overline{\omega})$ where A_K is the Alexander polynomial. The finitely many zeros of the Alexander polynomial mean finitely many places at which the form becomes degenerate. In fact, $\sigma_\omega(K)$ is continuous as a function of ω (K fixed) except at zeros of the Alexander polynomial. Since signature is an integer, this means that the signature is constant between roots and jumps around the unit circle.

Theorem 2.16. The signature $\sigma_\omega \colon \mathcal{C} \to \mathbb{Z}$ is a homomorphism (i.e. σ_ω vanishes on slice knots and is additive).

Proof. K slice implies that $Q = \left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right)$. The proof is thus the same as before provided that $A_K(\omega) \neq 0$. If $A_K(\omega) = 0$ then we define σ_ω to be the average of the two limits, extending the definition to the bad points. □

The following Theorem is the central concrete aim of the rest of these notes.

Theorem 2.17. **(a)** K_n slice $\Leftrightarrow n = 0, 2$ (This was first proved by Casson-Gordon)

(b) $\{K_n\}$ are independent in $\mathcal{C}$ for $n<0$ and for $(4n+1)=l^2$ $(n>0)$

(c) If $n>0$ and $4n+1$ is not a square, then $\{K_n\}$ are $\mathbb{Z}_2$-independent in $\mathcal{C}$.

Corollary 2.18. All K_n are distinct in $\mathcal{C}$ except for the unknot $K_0=K_2$.

Conjecture 2.19. All K_n are $\mathbb{Z}$-independent (other than $K_0=K_2=0$) in $\mathcal{C}$.

Proof of Theorem 2.17. Developing the tools to prove the whole of this Theorem will be the ultimate aim of these notes. For now we give the proofs for which we currently have the technology. We have $\sigma_\omega\colon \mathcal{C}\to\mathbb{Z}$ defined by $\sigma((1-\omega)S+(1-\overline{\omega})S^T)$, where for twist knots K_n, $S=\begin{pmatrix}-1 & 1\\ 0 & n\end{pmatrix}$.

Compute $\sigma_\omega(K_n)\colon \mathcal{C}\to\mathbb{Z}$; this will jump at the roots of the Alexander polynomial.

$$\begin{aligned} A_{K_n}(t) &= \det\begin{pmatrix}-1+t & 1\\ -t & n-nt\end{pmatrix} = n(-1+t)(1-t)+t = -nt^2+(2n+1)t-n\\ &= -n(t^2-\tfrac{2n+1}{n}t+1)\end{aligned}$$

This has roots

$$\frac{2n+1}{n}\pm\frac{\sqrt{\frac{4n^2+4n+1}{n^2}-4}}{2}=\frac{2n+1\pm\sqrt{4n+1}}{2n}.$$

So A_{K_n} has *real* roots if and only if $n>0$, in which case σ is constant ($\sigma\equiv 0$) around the unit circle.

For $n<0$, the roots are on the unit circle, since $\omega_n\cdot\overline{\omega_n}=1$, and they are all distinct from one another.

Computing at $t=-1$ we get $\sigma\begin{pmatrix}-2 & 1\\ 1 & 2n\end{pmatrix}=-2$.

(N.B. If A is a symmetric $n\times n$ matrix and the principal minors $A_1,A_2,\dots,A_n$ are all non-zero then the signature of A is $n-2\times$(the number of changes of sign in the sequence $1,A_1,\dots,A_n$).)

Therefore the $\{K_n\}$ are independent in $\mathcal{C}$ for $n<0$.

(We have now proved half of (b)).

Now consider $n>0$. The Alexander polynomial is reducible over $\mathbb{Q}$ if and only if $4n+1$ is a square. So assume it is not a square.

For any symmetric irreducible polynomial $p\in\mathbb{Q}[t^{\pm1}]$, define a homomorphism $h_p\colon\mathcal{C}\to\mathbb{Z}_2$ by $K\mapsto\{\text{exponent of } p \text{ in } A_k,\ \ \text{mod } 2\}$ (factorised over P.I.D. $\mathbb{Q}[t^{\pm1}]$).

This is additive under $\#$ because Alexander polynomials multiply. We need to show that it vanishes on slice knots. But for these, $A_k(t)=f(t)f(t^{-1})$, so the exponent of $p(t)=p(t^{-1})$ is even in it. For case (c) we have $A_{K_n}=-nt+(2n+1)-nt^{-1}$, which are distinct irreducible symmetric polynomials. By applying the appropriate h_p function we can show that K_n are $\mathbb{Z}_2$-independent.

This proves (c).

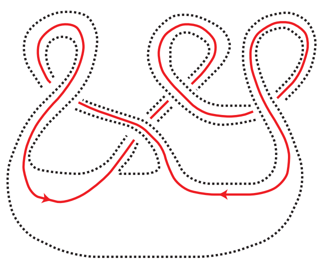
The hard case to do is "$4n+1 =$ square", in (b). The conjecture that all $\{K_n\}_{\neq 0,2}$ are $\mathbb{Z}$-independent in $\mathcal{C}$ is partially known by work of Livingston and Naik.

Suppose $4n+1 = l^2$. Then $4n = (l-1)(l+1)$, $l = 2m+1$, so $n = m(m+1)$, $m > 0$.
Notice that K_n has a genus 1 Seifert surface F, so $H_1(F)$ has two basis vectors, say s and l.

For $m = 1$ we have K_2, called by mathematicians the *Stevedore's knot*, which is actually slice. The claim is proved by finding that $\gamma = -s + l$ is a vector with square 0:

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

The curve on the Seifert surface is the unknot - visibly it is a 0-framed (i.e. self-linking zero) unknot.



K_2 is therefore ribbon: we see this by cutting out an annulus around our curve $\gamma = -s + l$ to get two parallel copies of γ. This is possible since γ has zero self-linking. We then attach slice discs for γ and γ', and this will introduce only ribbon singularities.

To do the surgery and get two non-intersecting discs we need a 0-framing on the unknotted curve, and we also need the curve γ to be itself slice.

Lemma 2.20. If $n = m(m+1)$ then there are precisely two curves with self-linking zero. These are $\gamma = (-m)s + l$ and $\gamma = (m+1)s + l$.

Proof. Let $\gamma = \alpha s + \beta l$, and suppose $\operatorname{lk}(\gamma, \gamma^+) = 0$. Then we have:

$$0 = \operatorname{lk}(\gamma, \gamma^+) = \alpha^2 \operatorname{lk}(s, s^+) + \alpha\beta(\operatorname{lk}(s, l^+) + \operatorname{lk}(l, s^+)) + \beta^2 \operatorname{lk}(l, l^+)$$

$$= -\alpha^2 + \alpha\beta + m(m+1)\beta^2$$

Completing the square with respect to α, we get that

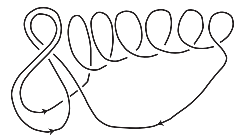
$$\alpha = \frac{\beta \pm \beta(2m+1)}{2}$$

so that $\gamma = (m+1)s + l$ or $\gamma = (-m)s + l$. $\square$

Definition 2.21. K is called *algebraically slice* if there exists a Lagrangian in $H_1(F)$.

Remark 2.22. (a) K slice $\Rightarrow$ K algebraically slice, by Theorem 2.10.

(b) The $K_{m(m+1)}$ are algebraically slice by Lemma 2.20.

For $m = 2$ ($n = 6$) we consider $\gamma = -2s + l$. 

This is in fact the trefoil, which is not slice; we can calculate its signature or show that its Alexander polynomial does not factorise as it should.

Similarly for $K_{m(m+1)}$ we get $\gamma = T(m, m+1)$, a torus knot which twists m times round the meridian of a standardly embedded torus in S^3, and $m+1$ times around the longitude. Such torus knots are slice if and only if $m = 1$; again one can calculate their signatures. This gives us a good reason to believe that these twist knots are not slice, though we will need the following theorem before we can actually prove it.

Theorem 2.23. (Cochran-Orr-Teichner) If K is a genus 1 knot with $A_k \neq 1$ then K slice $\Rightarrow$ there exists $\gamma \subset F$ such that $S(\gamma, \gamma) = 0$ and γ is a generator of $H_1(F)$ such that $\int_{\omega \in S^1} \sigma_\omega(\gamma) = 0$.

So by integrating the twisted signatures of the torus knots $T(m, m+1)$ around the circle we can obstruct the sliceness of the twist knots. We will prove this theorem in Section 6, once we have developed the necessary machinery. This is our motivating example, although of course the new theory has a far wider reach than merely reproving the results of Casson and Gordon.

It is not known whether every slice knot has a (collection of) slice γ-curve(s), but this result is a substitute.

The original proofs of the parts of this Theorem were done by Casson-Gordon for part (a), while for (b) Tristram did the case $n < 0$ and Jiang the case $4n + 1$ prime. The remaining statements are contained in the work of Cochran-Orr-Teichner.

□

3 A Survey of Homology, Intersection Forms and Linking Forms in Low Dimensions

3.1 Homology

Definition 3.1. Let X be any space. $\Omega_i(X)$ is the group of oriented bordism classes of manifolds of dimension i. Elements of $\Omega_i(X)$ have the form (M^i, f), where M^i is a closed oriented manifold and $f \colon M^i \to X$ is a continuous map. We say that two such pairs (M_0, f_0), (M, f) are *bordant* if the disjoint union $M_0 \sqcup -M$ is the boundary of an oriented $(i+1)$-manifold W, and there exists a continuous mapping $h \colon W \to X$ such that $h|_{M_0} = f_0$, $h|_M = f$. The group operation is disjoint union, and inverses are by reversing orientation.

Proposition 3.2. Let X be any space; $i = 0,1,2,3$. Then $\Omega_i(X)$ is isomorphic to $H_i(X)$, with the isomorphism given by

$$(M, f) \mapsto f_*[M]$$

Proof. The first 'essential" singularity in an oriented manifold is a cone on $\mathbb{C}P^2$. (With $\mathbb{Z}_2$, a cone on $\mathbb{R}P^2$) $\square$

Proposition 3.3. Let X^n be a manifold of dimension $n \leq 4$. Then homology classes $H_i(X)$ are represented by (embedded) submanifolds

3.2 Intersection and Linking Pairings

Let X^n be a closed oriented manifold. Then the Universal Coefficient Theorem gives:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{p-1}(X), \mathbb{Z}) \to H^p(X) \to \mathrm{Hom}(H_p(X), \mathbb{Z}) \to 0.$$

By Poincaré Duality we have

$$H_{n-p}(X) \cong H^p(X).$$

We also have the isomorphisms[1]

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{p-1}(X), \mathbb{Z}) \cong \mathrm{Hom}(TH_{p-1}(X), \mathbb{Q}/\mathbb{Z}) \cong TH^p(X)$$

where $TA = \{x \in A \,|\, sx = 0 \text{ for some } s \neq 0 \in \mathbb{Z}\}$.

So rewrite as:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(TH_{p-1}(X), \mathbb{Q}/\mathbb{Z}) \arrow[r] & H^p(X) \arrow[r] & \mathrm{Hom}(H_p(X)/TH_p(x), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & TH_{n-p}(X) \arrow[r] \arrow[u] & H_{n-p}(X) \arrow[r] \arrow[u] & H_{n-p}(X)/TH_{n-p}(X) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Thus

(a) We get an *intersection form* $I_X \colon H_{n-p}(X)/TH_{n-p}(X) \;\times\; H_p(X)/TH_p(X) \to \mathbb{Z}$

(b) We get a *linking pairing* $\mathrm{lk}_X \colon TH_{n-p}(X) \;\times\; TH_{p-1}(X) \to \mathbb{Q}/\mathbb{Z}$

which are both non-singular.

Geometrically these pairings are given by counting intersections (with sign). The linking number is got by taking multiples of one class, getting a bounded guy, intersecting and dividing:

$$\mathrm{lk}_X(A_{n-p}, B_{p-1}) = \#\frac{A_{n-p} \cap \beta_p}{m} \in \mathbb{Q}/\mathbb{Z}$$

[1] See Appendix 6 for proof.

where $\partial\beta_p = mB_{p-1} (m \neq 0)$. (This is independent of m.)

These are particularly interesting in 'middle' dimensions.
i.e.

(a) $p = n - p$, so $n = 2p$ is even

(b) $n - p = p - 1$, so $n = 2p - 1$ is odd.

Then symmetry is that

(a) $I_X(A_p, B_p) = (-1)^p I_X(B, A)$

(b) $\mathrm{lk}_X(a_p, b_p) = (-1)^{p-1} \mathrm{lk}_X(a, b)$.

Thus, for $n = 4$ there is a symmetric intersection form; for $n = 3$ there is a symmetric linking form. However, for $n = 4$ there is also an interesting linking form for manifolds with boundary which we shall also make use of later.

For manifolds with boundary, which are compact and oriented, we look at the relative (co)homology groups in the upper short exact sequence above. Thus:

$$I_X \colon H_{n-p}(X)/TH_{n-p}(X) \ \times \ H_p(X, \partial X)/TH_p(X, \partial X) \to \mathbb{Z}$$

$$\mathrm{lk}_X \colon TH_{n-p}(X) \ \times \ TH_{p-1}(X, \partial X) \to \mathbb{Q}/\mathbb{Z}$$

We we cannot deal with symmetry here. But we can at least use $H_p(X) \to H_p(X, \partial X)$ to get a nicer pairing.

Remark 3.4. Relative homology classes are representable by proper submanifolds of $(M, \partial M)$.

Example 3.5. Let W^4 be a compact oriented 4-manifold with boundary $\partial W = M$.
Then long exact sequence of a pair in homology gives us:

$$H_2(W) \to H_2(W, M) \xrightarrow{(\twoheadrightarrow)} H_1(M) \to H_1(W)$$

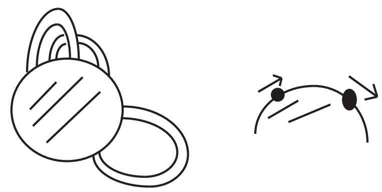
whilst Lefschetz duality and the Universal Coefficient Theorem give us

$$H_2(W, M) \cong H^2(W) \xrightarrow{(*)} \mathrm{Hom}(H_2(W), \mathbb{Z}) \to 0.$$

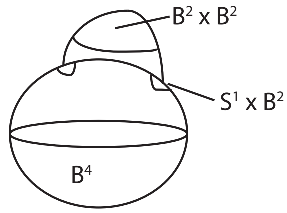
Assume for now that $H_1(W) = 0$; then $(*)$ is an isomorphism, as the relevant Ext group from the Universal Coefficient Theorem vanishes, and $(\twoheadrightarrow)$ is a surjection. We have the exact sequence

$$H_2(W) \xrightarrow{I_W} (H_2(W))^* \to H_1(M) \to 0.$$

Concrete example: $W = B^4 \cup_{\text{framed link } L}$ 2-handles.



Think of 2-manifolds: 1 handles attached
along framed links S^0 in boundary S^1.



Attach 2-handles $B^2 \times B^2$ to framed links in $S^3 = \partial B^4$. Use $\mathbb{Z}$ to give the framing of each component of the link (write a number next to it in a diagram) , i.e. to specify the thickening of each $S^1 \hookrightarrow S^3$ to an embedding $S^1 \times B^2 \hookrightarrow S^3$. The framing measures the self-linking of the boundaries of the core $B^2 \times \{0\}$ of each 2-handle by measuring the linking of two circles $\partial B^2 \times \{p_i\} = S^1 \times \{p_i\} \subset S^1 \times B^2$, $i = 1, 2$.

We get a linking matrix V. $W \simeq$ 0-cell $\cup$ n 2-cells, so $H_2(W) \cong \mathbb{Z}^n$, $\pi_1(W) \cong 0 \cong H_1(W)$. We have:

$$\mathbb{Z}^n \xrightarrow{V=I_W} \mathbb{Z}^n \to H_1(M) \to 0$$

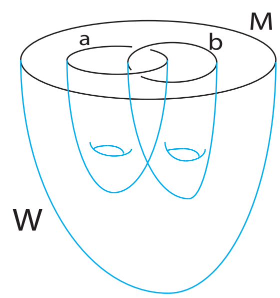
A basis for $H_2(W)$ can be represented by the cores $B^2 \times \{0\}$ of the 2-handles, union with cones on link components, or Seifert surfaces for the components. Use cones, intersected with the Seifert surfaces, to compute linking numbers as intersections.

Now, return to relating the forms algebraically.

The linking form λ on M is determined by I_W. Take a torsion element a in $H_1(M)$ (λ is defined on the torsion part of $H_1(M)$, so we may as well assume that I_W has finite cokernel, so all of $H_1(M)$ is torsion). We can lift a, b to elements $A, B \in H_2(W, M)$, since ($\twoheadrightarrow$) above is a surjection. Now find an integer m such that $mB \mapsto mb = 0 \in H_1(M)$, and so mB comes from an element C of $H_2(W)$. Then take the intersection of C and A using the image of A under $(*)$, and divide by m. We thus obtain $\lambda(a, b)$

If we have:

$$\mathbb{Z}^n \xrightarrow{I_W} \mathbb{Z}^n \to \text{cokernel} = H_1(M) \to 0$$



we get a pairing:

$$(a,b) \in H_1(M) \mapsto (\widetilde{a}, I_W^{-1}(\widetilde{b}))$$

where I_W is inverted over the rationals, tilde denotes a lift into $\mathbb{Z}^n$, and the brackets denote the Kronecker pairing i.e. dot product of vectors. Compare with the localisation exact sequence in L-theory.

Let A be a Principal Ideal Domain. Then the group $L^n(A)$ is the group of algebraic cobordism classes of chain complexes over the ring A. $L^n(A, A - \{0\})$ is the group of cobordism classes of such chain complexes which become contractible upon inverting all non-zero elements of A, which is the for $n = 4$ is the group of linking forms on the 1st homology of the chain complexes. $L^4(A)$ is the group of non-singular intersection forms on 4-dimensional chain complexes. The L-theory localisation exact sequence is:

$$\cdots \to L^4(\mathbb{Z}) \to L^4(\mathbb{Q}) \xrightarrow{\partial} L^4(\mathbb{Z}, \mathbb{Z} - \{0\}) \to L^3(\mathbb{Z}) \to \cdots$$

Every 3-manifold is null-cobordant (Theorem 3.6), so $M \in L^3(\mathbb{Z}, \mathbb{Z} - \{0\})$ maps to zero in $L^3(\mathbb{Z})$. It therefore lifts to $L^4(\mathbb{Q})$; the linking form on M is given by intersections in a 4-manifold which it bounds, with a non-singular form *after localisation*, in other words allowing rational numbers to invert the matrix. This corresponds to the homology of M being $\mathbb{Z}$-torsion.

Recall: given a framed link $L \subseteq S^3$, we attach handles and denote the resulting 4-manifold by W^4_L, and $M^3_L := \partial W^4_L$. M^3_L comes from doing the surgery on S^3 specified by the link, and W is a null-cobordism for M.

Theorem 3.6 (Bing, Rochlin, Lickorish, Wallace, Rourke)**.** Any closed oriented 3-manifold is the boundary of a 4-manifold, i.e. $\Omega_3 = 0$.

Proof. We use the Pontrjagin-Thom construction to get the result that

$$\Omega_3 \cong \pi_3(MSO) := \varinjlim \pi_{3+k}(MSO(k))$$

where $MSO(k)$ is the Thom space of the universal bundle over the classifying space $BSO(k)$: it is the Thom spectrum for the generalised homology theory of oriented bordism. Ω_1 and Ω_2 are zero by classification of 1- and 2-manifolds, so therefore $\pi_{1+k}(MSO(k))$ and $\pi_{2+k}(MSO(k))$ are zero for k sufficiently large. We also know that $MSO(k)$ is a $(k-1)$-connected space since it is the Thom space of a CW-complex, so that $\pi_m(MSO(k)) = 0$ for $1 \leq m < k$. Also $\pi_k(MSO(k)) \cong H_k(MSO(k))$ by the Hurewicz Theorem, so that $\pi_3(MSO) \cong H_3(MSO)$.

Now by the Thom isomorphism theorem, $H_{i+k}(MSO(k)) \cong H_i(BSO(k))$. By definition of $BSO(k)$ we have $\pi_n(BSO(k)) \cong \pi_{n-1}(SO(k))$.

Therefore: $\pi_1(BSO(k)) = \pi_0(SO(k)) = 0$ since $SO(k)$ is connected. Then $\pi_2(BSO(k)) \cong H_2(BSO(k))$ by Hurewicz. Next $\pi_3(BSO(k)) \cong \pi_2(SO(k)) = 0$ since $SO(k)$ is a Lie group. Applying Hurewicz again tells us that the map $\pi_3(BSO(k)) \to H_3(BSO(k))$ is a surjection, which means $H_3(BSO(k)) = 0$. □

Remark 3.7. We can continue to get $\Omega_4 \cong \mathbb{Z}$. Then, step 2 is to surger the 4-manifold on circles to kill π_1. (The circles are embedded, with neighbourhoods which are trivial bundles.) This is equivalent to exchanging the dots for zeros in a Kirby diagram; turning the handle decomposition upside-down and repeating with the 3-handles we get only 0-handles and 2-handles. The cobordism class of a 4-manifold is given by the signature of the intersection form on $H_2(W;\mathbb{Z})$ - for a proof of this see for example Kirby's Topology of 4-manifolds

Addendum: we can assume all framings are even. This comes from $\Omega_3^{\text{spin}} = 0$, some handle-slides, and the Wu formula. e.g. $L = \bigcirc^{+1}$; $W_L = \mathbb{CP}^2 -$ ball (which is the Hopf disk bundle over S^2. See Gompf and Stipsicz p106), is not spin, but it can be closed off with a 4-ball, so it isn't a counterexample.

The Wu formula says that the second Stiefel-Whitney class $w_2(W)$ is characterised by $w_2 \cup x = x \cup x \mod 2$. Since every 3-manifold is parallelisable, every 3-manifold can be given a spin structure, and every spin 3-manifold is the boundary of a spin 4-manifold, such a spin 4-manifold must have $x \cup x = 0 \mod 2$ for all $x \in H^2(W;\mathbb{Z})$. There must therefore be, by Kirby's theorem, a sequence of handle slides which leaves each component of the link having even framing. (See Section 4.2 below.)

Corollary 3.8 (Kervaire, in higher dimensions)**.** Any knot $S^2 \subseteq S^4$ is slice; i.e. extends over a 3-ball in B^5.

Proof. Pick a Seifert surface $F^3 \subseteq S^4, \partial F^3 = K$. (Use the standard construction of a Seifert surface using a map $S^4 - K \to S^1$ representing a generator of $H^1(S^4 - K)$: Alexander Duality $\Rightarrow H^1(S^4 - S^2) \cong \mathbb{Z}$). By Theorem 3.6, F is obtained from surgery on B^3 (cap it off with another B^3 to get S^3, use the theorem, and remove at the end.) Therefore the converse is also true; we can surger F to become B^3 along some link $L \subseteq F$. Now it's no problem: we need only to surger along these by adding 2-disks in B^5. Each circle of L bounds a disk in B^5, and they are disjoint, purely by a general position dimension count.

We now need to discuss framings: thicken each $B^2 \subseteq B^5$ up to a $B^2 \times B^2$. We need *even* framing on the link to do it right. Recall that the normal bundle of a disc B^2 in B^5 is trivial, but then can always find in any trivial $\mathbb{R}^3$-bundle on S^2 a 2-plane sub-bundle with any even Euler characteristic. Thus, surgeries are possible *ambiently*, if and only if we have even framings, but this is always possible so we can surger F^3 *ambiently* in B^5 to become B^3.

$\square$

3.3 High dimensional Concordance

Recall:

$$\mathcal{C}_1 = \frac{\{S^1 \subseteq S^3\}}{\partial\{B^2 \subseteq B^4\}}.$$

There is an analogous definition of high dimensional concordance groups:

$$\mathcal{C}_n := \frac{\{S^n \subseteq S^{n+2}\}}{\partial\{B^{n+1} \subseteq B^{n+3}\}}$$

Remark 3.9. Codimension 2 is the only interesting case to define such concordance groups, at least in Piecewise-Linear theory: we have the Schönflies theorem for codimension 1.

The difficulty for $\mathcal{C}_1$ corresponds to π_1 being non-trivial.

The proof of unknotted-ness in codimension 3 starts by observing that the complement has the same π_1 as an unknot complement.

Theorem 3.10. For $n \geq 4$, the groups $\mathcal{C}_n$ are 4-periodic in n, in the PL category;

(a) $\mathcal{C}_0 = \mathcal{C}_2 = 0$;

(b) $\mathcal{C}_1$ is unknown;

(c) $\mathcal{C}_{2k} = 0 \;\; k \geq 2$;

(d) $\mathcal{C}_{4k+1} \cong \mathcal{AC}^- \;\; k \geq 1$

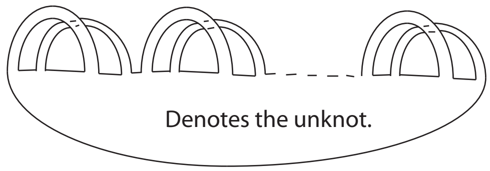
(e) $\mathcal{C}_{4k-1} \cong \mathcal{AC}^+ \;\; k \geq 2$

In addition:

$$0 \to \mathcal{C}_3 \to \mathcal{AC}^+ \to \mathbb{Z}_2 \to 0$$

with the $\mathbb{Z}_2$ term coming from Rochlin's theorem, as $\frac{\sigma(S+S^T)}{8} \pmod{16}$. Recall the definition of the algebraic concordance groups:

$$\mathcal{AC}^{\pm} = \frac{\{\text{Seifert forms } S\colon \mathbb{Z}^n \to (\mathbb{Z}^n)^* \text{ with } S \pm S^T \text{ an isomorphism}\}}{\{\text{metabolic forms: those with a Lagrangian summand}\}}$$



Remark 3.11. $\mathcal{AC}^-$ was original group, $(k=0)$; $\mathcal{AC}$ is the appropriate thing in the other dimensions.
The map: $\mathcal{C}_n \to \mathcal{AC}^\pm$ given by the Seifert form is clear.
$\mathcal{AC}^\pm$ are in fact isomorphic to $\bigoplus_\infty \mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}_2 \oplus \bigoplus_\infty \mathbb{Z}_4$.

Lemma 3.12. $\mathcal{C}_1 \to \mathcal{AC}^-$ is onto.

Proof. Create a knot by making a banded Seifert surface for it: just twist the bands and link them appropriately. Given any matrix S such that $S - S^T = J$, where J is the standard symplectic matrix with block diagonals of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we can do it. As J is the only skew isomorphism (of a given rank, up to congruence), we are done. (In higher dimensions, use plumbing: plumbing of disks creates linking of their boundary spheres.)

□

Sketch proof of theorem 3.10 on $\mathcal{C}_{n\geq 4}$. We have already proved $\mathcal{C}_2 = 0$, while $\mathcal{C}_0 = 0$ is clear. Pick a Seifert surface for the knot. By ambient surgeries below the middle dimension, we can assume that F is alterable to a slice disk $B^{n+1} \subseteq B^{n+3}$ for n even. This finishes case (c).

If instead $S^{2k-1} \subseteq S^{2k+1}$, with $S^{2k-1} \subseteq F \subseteq S^{2k+1}$, then we may assume that F is a connected sum:

$$\sharp(S^k \times S^k) - \dot{B}^{2k}$$

by again performing surgery below the middle dimension. Note that the first stage of this is to kill $\pi_1(F)$; so that the fundamental group of the knot complement is infinite cyclic; as intimated above this is one of the places the argument goes catastrophically wrong in the classical dimension. After this concordance we have an analogue of the $\sharp(S^1 \times S^1) - \dot{B}^2$ which occurs in the dimension 3 case. Now S is again the linking form on the middle homology of this subset F inside S^{2k+1}. Arrange as before to get the onto case $\twoheadrightarrow \mathcal{AC}^\pm$ as appropriate. Final step: if S has a Lagrangian in $H_k(F^{2k})$, then the original knot was slice: this is true when $k > 1$. In order to see this, represent the Lagrangian by a link $L = \amalg S^k \subseteq F^2k \subseteq S^{2k+1}$. For $k \geq 3$ use the Whitney trick to make them disjoint. For $k = 2$ we need the *stable* Whitney trick, and the power of transverse spheres. We need a lemma to complete the proof:

Lemma 3.13. If $L\colon \amalg\, S^k \to S^{2k+1}$ has trivial linking numbers and $k > 1$, then L is slice (in fact trivial).

We are then done as normal by surgery. The classical dimension is different because the dimension does not drop with this progression: e.g. when trying to slice a knot $S^3 \subseteq S^5$, at this stage have to slice $S^2 \subseteq S^5$.

Proof of Lemma:
We have to remove intersections of discs $B^{k+1} \subseteq B^{2k+2}$. Since the linking numbers of the boundaries vanish, we know that any intersections pair up, so we can use the Whitney trick to cancel them. The ambient dimension is at least 6, so this works admirably. $\square$

Remark 3.14. Addendum to high dimensional concordance theorem. It is very important to specify the category in we which we work, either: Smooth, Piecewise Linear, or Topologically Locally Flat. *Theorem 3.10 applied to PL version.* For $n \geq 4$,

$$\mathcal{C}_n^{\mathcal{PL}} = \mathcal{C}_n^{\mathcal{TOP}} = \begin{cases} 0 & n \text{ even} \\ \mathcal{AC}^+ & n = 4k-1 \\ \mathcal{AC}^- & n = 4k+1, \end{cases}$$

whereas in the smooth case there is a problem with realising *symmetric* forms using plumbing: can get exotic spheres knotted. So we have the same result for $\mathcal{C}_n^{\mathcal{DIFF}}$ for $n \neq 4k \pm 1$, *but*:

$$0 \to \mathcal{C}_{4k-1}^{\mathcal{DIFF}} \to \mathcal{AC}^+ \to \mathbb{Z}_{n_k} \to 0$$

The map $\mathcal{AC}^+ \to \mathbb{Z}_{n_k}$ is given by:

$$S \mapsto \frac{\sigma(S + S^T)}{8}.$$

n_k is the order of the Kervaire-Milnor exotic sphere Σ_{4k-1} in the group of exotic spheres Θ_{4k-1}. Recall that:

$$\Theta_{4k-1} := \frac{\{\text{manifolds homeomorphic to } S^{4k-1}\}}{\{\text{those diffeomorphic to } S^{4k-1}\}},$$

with the group operation of connected sum is a finite group. In order to prove that $\Sigma \sharp \overline{\Sigma}$ is diffeomorphic to S^{4k-1}, use the h-cobordism $\Sigma \times I$ minus a tube (the connected sum) and a ball (so we have a cobordism to the standard sphere).

Remark 3.15. In the $4k+1$ case there is a similar Arf invariant problem, better known in this case as the Kervaire invariant problem.[2].

$$0 \to \mathcal{C}_{4k+1}^{\mathcal{DIFF}} \to \mathcal{AC}^- \to \{\mathbb{Z}_2 \text{ or } 0\} \to 0$$

[2] It is now known, by recent work of Hopkins, Hill and Ravenal, that no manifold exists with Kervaire invariant 1 in dimensions greater than 126, while in some dimensions below there is such a manifold; 126 is still open.

The problem in the previous proof was trying to realise an even unimodular form stabilised by hyperbolics geometrically.

Theorem 3.16 (Serre, 1962)**.**

$$(S + S^T) \oplus \text{hyperbolic} \cong \bigoplus_{i=1}^{k} E_8 \oplus \text{hyperbolic}$$

with $8k = \sigma(S + S^T)$.

To realise this as an intersection form of a Seifert surface $W^{4k} \subseteq S^{4k+1}$ do plumbing. By Serre's theorem, it is sufficient to be able to realise the E_8 form. This is equivalent to taking a link L of $(2k-1)$-spheres in S^{4k-1} (framed, with a $\pi_{2k-1}(SO(2k)) \cong \mathbb{Z}$ worth of framings to choose from) whose linking matrix corresponds to the E_8 form's intersection matrix, and using this to attach handles to B^{4k} as usual to make

$$W_L^{4k} \cup \text{ 2-handles } \simeq \bigvee S^{2k}.$$

Note that $\pi_1(W_L) = 1$, and the middle dimensional homology is $H_{2k}(W_L^{4k}) \cong \mathbb{Z}^{\sharp L}$, with the intersection form:

$$H_{2k}(W_L) \to H_{2k}(W_L)^* \cong H_{2k}(W_L, M_L)$$

given by the linking matrix. The isomorphism here is a combination of Universal coefficient theorem and Poincaré-Lefschetz duality. The boundary M_L of W_L also has trivial fundamental group, and is a homology sphere if and only if the linking form is non-singular. This can be seen from the long exact sequence of the pair (W_L, M_L). The non-trivial part of this sequence is

$$H_{2k+1}(W_L, M_L) \to H_{2k}(M_L) \to H_{2k}(W_L) \to H_{2k}(W_L, M_L) \to H_{2k-1}(M_L) \to H_{2k-1}(W_L)$$

which yields

$$0 \to H_{2k}(M_L) \to \mathbb{Z}^{\sharp L} \xrightarrow{\phi} (\mathbb{Z}^{\sharp L})^* \to H_{2k-1}(M_L) \to 0.$$

If the linking form is non-singular then the central arrow ϕ is an isomorphism which implies the vanishing of the middle-dimensional homology groups of M shown. For $k > 1$, the h-cobordism theorem, which implies the high-dimensional Poincaré hypothesis, then implies that we have a homeomorphism $\partial W_L \approx S^{4k-1}$. Note that W_L is not an h-cobordism which would show this; one is however certain to exist. ∂W_L is Σ_{4k-1}, the Kervaire-Milnor sphere $\partial(E_8)$. (This is independent of the choice of link: it is classified up to isotopy by the linking form.) Also in this construction W^{4k} always embeds in S^{4k+1} (proof below), and is a Seifert surface and knot pair. This shows that

$$\mathcal{C}_{4k-1}^{\mathcal{CAT}} \twoheadrightarrow \mathcal{AC}^+$$

for $\mathcal{CAT} = \mathcal{PL}$ or $\mathcal{TOP}$, but not in the smooth case, because Σ_{4k-1} is not diffeomorphic to S^{4k-1}.

Remark 3.17. $n_2 = 28$; Σ_7 generates Θ_7. This is the famous initial example of Milnor.

Proposition 3.18. $W_L \subseteq S^{4k+1}$.

Corollary 3.19. The proposition, with $k = 1$, implies that any closed oriented 3-manifold embeds in S^5. (Note that Whitney's best case would give an embedding in S^6 here.)

Proof of Proposition 3.18. Consider $W_L \times I$; each handle has its thickening increased by a dimension, so this is of the form $B^{4k+1} \cup (2k - \text{handles})$. They now attach along a *trivial link* because of the dimension shift. Then, taking B^{4k+1} as one half of S^{4k+1}, we can attach the $2k$-handles ambiently in the other half, using the fact that the link is slice. We then have to make sure the framing is right to be sure that the slice disks can be thickened to $4k$-dimensional $2k$-handles $B^{2k} \times B^{2k}$. However, at present we have trivially framed $4k+1$-dimensional $2k$-handles $B^{2k} \times B^{2k+1}$. We claim that within any $4k+1$-plane bundle, it is possible to find $4k$-bundles with any even Euler class.

To see this, examine the exact sequence of homotopy groups associated to the fibration $SO(2k) \to SO(2k+1) \to S^{2k}$:

$$\pi_{2k}(S^{2k}) \to \pi_{2k-1}(SO(2k)) \to \pi_{2k-1}(SO(2k+1)) \to 0.$$

The framing of the link $L \cong \amalg S^{2k-1} \subseteq S^{4k-1}$ yields elements of $\pi_{2k-1}(SO(2k)) \cong \mathbb{Z}$ as given by the Euler class of the vector bundle, while after allowing an extra dimension the, now trivial, link component's framing gives elements of $\pi_{2k-1}(SO(2k+1))$. From the exact sequence, every element of $\pi_{2k-1}(SO(2k+1))$ lifts to $\pi_{2k-1}(SO(2k))$. It is trivial precisely when the element of $\pi_{2k-1}(SO(2k))$ comes from an element of $\pi_{2k}(S^{2k})$. Since this last group is generated by the identity map, and maps to $TS^{2k} \in \pi_{2k-1}(SO(2k))$, which has Euler number 2, we see that trivial framings lift to even framings when we reduce the ambient dimension by one.

It is precisely even framings on spheres which can be extended across disks; furthermore as we discussed above any 3-manifold can be represented by a link which has only even framings. This completes the proof. □

Theorem 3.20. If a 3-manifold $M \cong M_L$, for L a 0-framed link which is the union of two slice links, then M^3 embeds in S^4.

Example 3.21. Two unknots linked n times via n twists, 0-framed. In other words, two fibres of $n \in \mathbb{Z} \cong \pi_3(S^2)$.

$$H_1(M_L) \cong \operatorname{coker} \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$$

Actually M_L is $L(n,1) \sharp L(n,1)$, *but* a single lens space does not embed in S^4.

Proof of Theorem 3.20. If the whole link were slice, it would be easy: $W_L \subseteq S^4$ as before, but with the ambient dimension one fewer. We need the zero framing here to be able to thicken the slice discs to 2-handles inside B^4. Now if the individual components are slice, attach one 2-handle in one B^4 half of S^4, and the other handle in the other half of S^4. W_L does not embed in S^4 but its boundary 3-manifold does.

(Note that W_L with one handle surgered to be a 1-handle does embed; i.e. slice knot $\cup$ dotted unknotted circle. This is because the 1-handle denotes something to be scooped out of the first B^4, and the 2-handle is then attached into the other half.) $\square$

Next, we must consider 4-manifolds with 1-handles, because $\pi_1(B^4 \setminus \text{ slice disc}) \cong \mathbb{Z}$. These can still be drawn with link pictures, where we use dotted circles to indicate 1-handles.

Theorem 3.22 (Laudenbach)**.** A closed 4-manifold N^4 with a handle decomposition is determined by its 2-skeleton. This follows from the fact that any diffeomorphism of $\sharp\, S^1 \times S^2$ extends over $\sharp\, S^1 \times B^3$.

Remark 3.23. His proof is by the classification of the mapping class group of $\sharp\, S^1 \times S^2$. A particular element is the Glück twist:

$$S^1 \times S^2 \to S^1 \times S^2$$

$$(x, v) \mapsto (x, x.v) \quad x \in SO(2) \subset SO(3) \text{ which acts on } S^2$$

We can cut out $S^2 \times B^2$ from a 4-manifold and reglue it in this way, which *might* be a way to get an exotic smooth structure on S^4.

4 Back to Classical Knot Concordance

In this section we have knots $S^1 \subseteq S^3$, which are slice if they bound properly embedded, locally flat discs $D^2 \subseteq D^4$, such that $\partial D^2 = S^1, \partial D^4 = S^3$.

4.1 The Cappell-Shaneson way to slice a knot

Lemma 4.1. If K is slice then $M^3_K =$ 0-framed surgery on K bounds a 4-manifold W such that

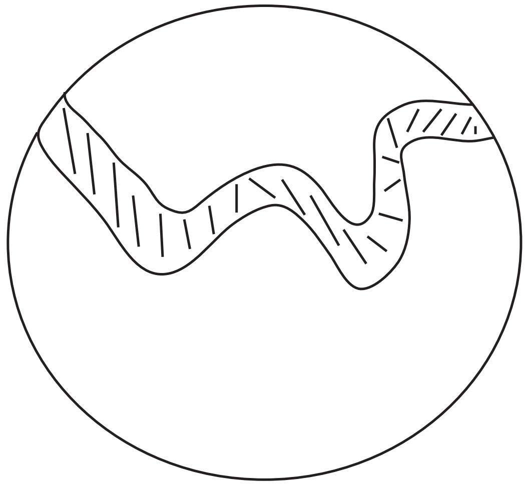
(i) $H_1(M) \cong H_1(W) \cong \mathbb{Z}$ with the isomorphism induced by the inclusion $M \subseteq W$;

(ii) $H_2(W) = 0$;

(iii) $\pi_1(W)$ is normally generated by the meridian of the knot.

Theorem 4.2 (Freedman)**.** The converse holds in the topological locally flat category.

Remark 4.3. If (i) and (ii) are satisfied then K is slice in a homology 4-ball. We call such a knot *homologically slice*. However, there are no known obstructions which are able to tell the difference between homologically slice and actually slice.



Proof of Lemma 4.1. Define $W^4 = B^4 \setminus (D^2 \times D^2)$, a 4-ball minus a thickened slice disk for the knot. Then $\partial W = M_K$. It is the 0-framed surgery because the push-off of K along the slice disk, towards its centre, does not link with K. Then (i) and (ii) follow from Alexander duality, or a Mayer-Vietoris calculation. Decomposing $M_K = X \cup_{(S^1 \times S^1)} (D^2 \times S^1)$, where $X = \overline{S^3 \setminus (S^1 \times D^2)}$ is the knot exterior, the Mayer-Vietoris sequence yields that $H_i(M_K;\mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 1, 2, 3$, and is of course zero otherwise. Decomposing $B^4 = W \cup_{D^2 \times S^1} (D^2 \times D^2)$, Mayer-Vietoris implies (i) and (ii).

(iii) follows from the geometric fact that adding a handle which kills the meridian makes the group zero. This can be seen with the Seifert-Van Kampen theorem: since $D^4 = W \cup_{D^2 \times S^1} D^2 \times D^2$. The boundary of $D^2 \times D^2$ is $D^2 \times S^1 \cup S^1 \times D^2$, but the $S^1 \times D^2$ represents the neighbourhood of the knot: W and $D^2 \times D^2$ are just glued together along $D^2 \times S^1$ which is properly embedded in D^4. Therefore:

$$\pi_1(D^4) \cong 1 \cong \frac{\pi_1(W) * \pi_1(D^2 \times D^2)}{\pi_1(D^2 \times S^1)} \cong \frac{\pi_1(W)}{\pi_1(D^2 \times S^1)}$$

which shows that the image of the meridian generates $\pi_1(W)$ as claimed. If B^4 were just a homology 4-ball, then (i) and (ii) still hold, but (iii) might not. $\square$

Proof of Remark 4.3. Suppose there is a W such that (i) and (ii) hold. Start with this W, and attach a 2-handle to W^4 along the meridian - do the reverse surgery to that which made M_K from S^3. Then $W \cup_{M_K} (D^2 \times D^2)$ is a homology 4-ball. Its boundary is S^3, and the knot lives in S^3 as the belt sphere of the attached 2-handle $\{0\} \times S^1 \subseteq D^2 \times D^2$, and the cocore of the handle is a homology slice disk for the knot. $\square$

Proof of Theorem 4.2. Start as above, making a homology 4-ball B with a slice disk. If we also have condition (iii), then $\pi_1(B) = 0$ so we have a *homotopy* 4-ball. We can then apply Freedman's topological Poincaré hypothesis show that B is homeomorphic to B^4. This is a strange homeomorphism: the same strangeness which means that it may not be smoothly a 4-ball, and which made embedding Casson handles so hard. The slice disk looks strange but it is locally flat because the homeomorphism carries a regular neighbourhood with it. The smooth case of this is an open question: is a homotopy 4-ball diffeomorphic to B^4? □

Remark 4.4. There are examples of knots which are topologically locally-flat but not smoothly slice: for example, the Whitehead double of a 0-framed trefoil had Alexander Polynomial 1 and so is topologically slice, however results of gauge theory which estimate the slice genus can show that it is not smoothly slice.

The Cappell-Shaneson way to slice a knot is, instead of looking for the slice disk, to start with a slice disk in a 4-manifold with boundary M_K and try to change the 4-manifold so that the conditions of Lemma 4.1 are satisfied. In this way the machinery of surgery theory can be employed to build an obstruction theory.

4.2 Drawing 4-manifolds with 0, 1, and 2-handles

We have already discussed the case where there are no 1-handles. To attach a 1-handle to B^4, we attach a copy of $B^1 \times B^3$ to a pair of embedded 3-balls in the boundary S^3. Since 1- and 2-handle pairs cancel, this is the same as removing a 2-handle from the interior of B^4 (it is helpful to first imagine the picture in 1 dimension fewer, and then cross everything with I). For all the 1-handles to be attached, we indicate the belt spheres $\{0\} \times S^1$ of the 2-handles to be removed from the interior by drawing unknotted, unlinked dotted circles in S^3 (in the 3-dimensional case we would indicate an S^0 in the boundary S^2). The 1-handles generate the fundamental group.

We can then draw a framed link in S^3, which may link with the dotted circles, to indicate the attaching spheres for the 2-handles with their framing.

Theorem 4.5. The diffeomorphism type of W_L is unchanged under the following moves on L:

(a) Isotopy of L.

(b) Handle slides of one handle over another. This means taking connected sum of one component of L with a push off of another, using the framing to perform the push off (1-handles have zero framing for this purpose). This is legal for 1-handles on 1 handles, 2 on 1, and 2 on 2.

(c) After first pushing them both off everything else, cancellation of a 1- and 2-handle pair. They will be represented in L by a Hopf link in which one of the components is dotted.

Remark 4.6. We can compute $\pi_1(W_L)$ by writing relations corresponding to the way in which the 2-handle attaching maps link the dotted circles, which represent generators of the fundamental group. We can compute $H_*(W_L)$ using the handle chain complex. Actually, we can compute the handle chain complex and homology of the universal cover of W: replace each $\mathbb{Z}$ in the chain complex with a copy of $\mathbb{Z}[\pi_1(W)]$, and read off the intersection matrix with $\mathbb{Z}[\pi_1(W)]$ coefficients.

These can be defined if each of the handles is attached to a base point with some choice of path; each intersection point between two handles comes with an element of $\pi_1(W)$ given by concatenating the paths belonging to each of the handles.

$\pi_2(W_L) = \pi_2(\widetilde{W_L}) = H_2(\widetilde{W_L};\mathbb{Z}) = H_2(W_L;\mathbb{Z}[\pi_1(W)])$ can be computed using the handle chain complex as described.

Remark 4.7. When sliding handles over one another, we cannot do a band sum going through the spanning discs for the 1-handles, since the spanning disk has been removed from B^4; recall that this is what the dotted circle denotes.

Remark 4.8. If $L = L_1 \amalg L_2$, then $W_L = W_{L_1} \sharp W_{L_2}$; $\partial W_L = \partial W_{L_1} \sharp \partial W_{L_2}$. This means that adding a ± 1-framed unknot to L does not change the boundary of W_L: it changes W_L to $W_L \sharp \mathbb{CP}^2$. This is called blowing up, and the reverse procedure is called blowing down. The name comes from algebraic geometry, where the technique is used to blow up unwanted singularities of complex surfaces by adding the freedom of an extra copy of $\mathbb{CP}^2$. (See Gompf and Stipsicz, Section 2.2)

Theorem 4.9 (Kirby)**.** Two 3-manifolds $M^3_{L_1} \cong M^3_{L_2}$ are diffeomorphic if and *only if* L_2 can be obtained from L_1 by a finite sequence of isotopies, handle slides, births/deaths, and blowing up and down.

Remark 4.10. For any smooth manifold, handle decompositions exist, since there exists a Morse function and a gradient-like vector field on the tangent space. They are unique up to handle slides and cancellation (Cerf theory - it has to be shown that the handles can always be arranged in ascending or descending index with respect to the Morse function.)

4.3 Which 4-Manifolds does M_K bound?

First attempt:

$$M_K = \partial(B^4 \cup_K \text{ 2-handle})$$

The trouble with this is that it is simply connected but has second homology non-zero. We want to get one with fundamental group equal to $\mathbb{Z}$, so that the first homology is also $\mathbb{Z}$.

Theorem 4.11. (a) There always exists a W^4 with $\partial W = M_K$ and $H_1(M) \xrightarrow{i_*} H_1(W) \cong \mathbb{Z}$ an isomorphism. (and $\pi_1(W) \cong \mathbb{Z}$). The $H_2(W) = 0$ condition of Lemma 4.1 then gives rise to the obstruction to slicing the knot. $\pi_2(W) = H_2(\widetilde{W})$ is a free $\mathbb{Z}[\mathbb{Z}]$-module of the same rank as the rank of $H_2(W)$ over $\mathbb{Z}$.

(b) Let $\lambda = I_{\widetilde{W}}$ be the intersection form on the universal cover of W. Then representing the intersection form as a matrix over $\mathbb{Z}[\pi_1(W)] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[m.m^{-1}]$ we have that $\det(\lambda)(m) = \Delta_K(m)$, the Alexander polynomial of K.

(c) Our W^4 from (a) has a twisted signature $\sigma(I_{\widetilde{W}} \otimes \mathbb{C}(z))$, which coincide with the original definition for a knot signature $\sigma_z(K)$, for $z \in S^1 \subseteq \mathbb{C}$. $\mathbb{C}(z)$ is the field of fractions (so that a signature is well-defined) of the ring of polynomials in z with complex coefficients, and z is a character on $\mathbb{Z} \cong \pi_1(W)$, $n \mapsto z^n \in S^1 \subset \mathbb{C}$.

Remark 4.12. Recall that if a knot K is slice then it is homologically slice, so M_K is the boundary of N with $H_1(N) \cong H_1(M_K), H_2(N) = 0$. This last condition, however, would force the intersection form to be trivial and hence the Alexander polynomial to be 1. Since such knots are already known to be slice, we are interested in those without this property, and so will need 4-manifolds with more complicated groups than $\mathbb{Z}$ in order to apply the Cappell-Shaneson method to the classical dimension.

Proof of Theorem 4.11. (a) Pick a Seifert surface F for the knot, and draw it in band form. Note that the linking and framing of the bands define the Seifert matrix of linking numbers, which will be of the form

$$V = \begin{pmatrix} A & C \\ I + C^T & B \end{pmatrix},$$

where $A = A^T$, $B = B^T$, and $V - V^T$ is symplectic.

We can use this presentation of the Seifert surface to draw a link in S^3, and thus to define a 4-manifold with the desired properties. The central disc of the Seifert surface becomes a dotted circle, and the bands become 2-handle attaching maps, whose embeddings depend on the knotting and linking of the bands. Figure 7 shows the procedure.

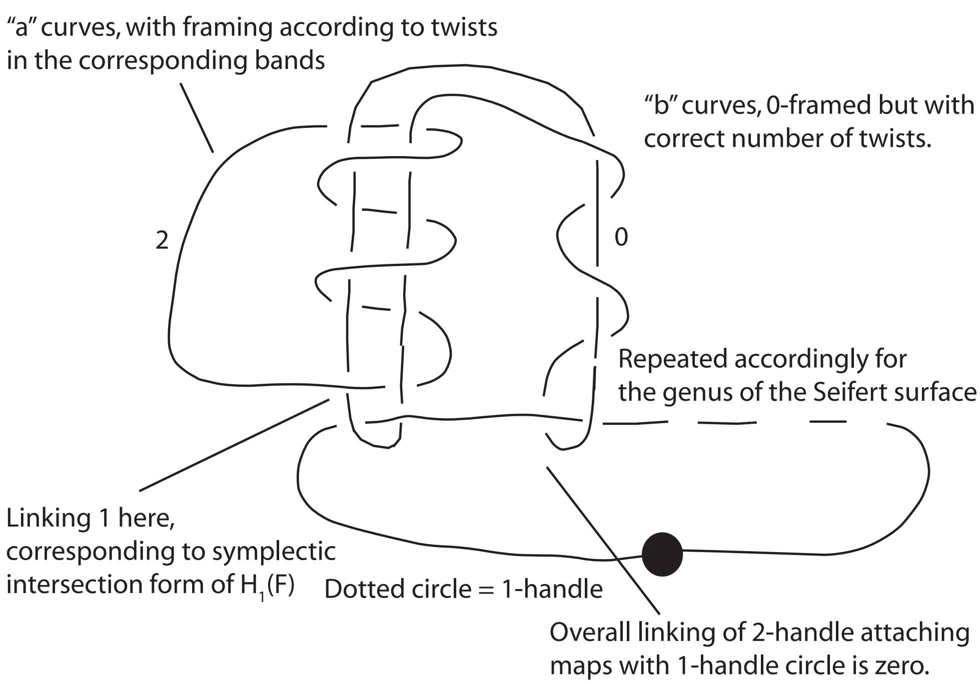


Figure 7: Drawing a 4-manifold W_0 with $\pi_1(W_0) \cong \mathbb{Z}$ and $\partial W_0 = M_K$.

So W_0 is comprised of a 0-handle, a 1-handle, and $2g$ 2-handles. $\pi_1(W_0) \cong \mathbb{Z}$ since there is a single 1-handle which has zero algebraic linking numbers with each of the 1-handles. $W_0 \simeq S^1 \vee \bigvee_{2g} S^2$. Thus $\pi_2(W_0) \cong H_2(\widetilde{W_0}) \cong \bigoplus_{2g} \mathbb{Z}[\pi_1(W_0)] \cong \bigoplus_{2g} \mathbb{Z}[\mathbb{Z}]$, while $H_2(W_0) \cong \mathbb{Z}^{2g}$.

To see that this 4-manifold has boundary M_K, we have to apply some Kirby moves to change this link into the knot: change the 1-handle to a 2-handle and slide it twice over each of the "a"-curves, using a band which follows the "b"-curves to perform the connected sum for one of the slides, and the obvious small band for the other one. This leaves the a and b curves as geometric duals which can be cancelled, and leaves the old 1-handle zero framed and in the shape of K.

(b) To compute the intersection form λ_W on $H_2(\widetilde{W}) \cong \pi_2(W)$, we take a basis of immersed spheres $f_i \colon S^2 \looparrowright W$, with a threading. A *threading* is a path from the basepoint of W to the basepoint of $f_i(s)$ where s is a basepoint of S^2. Such spheres arise as the unions of the cores of the 2-handles with immersed discs which form null-homotopies of the attaching maps of the 2-handles inside $S^1 \times B^3$, which is the 4-manifold created by attaching a single 1-handle to B^4. These null-homotopies exist because each of the 2-handle attaching maps link the dotted circle zero times. We count intersections of our spheres

$$(f_i, f_j) = \sum_{x \in f_i(S^2) \cap f_j(S^2)} \varepsilon(x) \cdot g(x)$$

where $\varepsilon(x)$ is the sign which arises from comparing orientations of the tangent bundles at a transverse intersection, and $g(x)$ is an element of $\pi_1(W)$ which is the composite of the threading of f_i, paths from $f_i(s)$ to x and then from x to $f_j(s)$, followed by the inverse of the threading for f_j. The self intersections are counted by taking a parallel copy of each f_i and counting as usual. (Not to be confused with the quadratic self-intersection μ-form which we have not introduced.) Note that we must have spheres and not surfaces (we could guarantee embedded surfaces to represent each homology class of W), unless there are conditions on the fundamental groups of the surfaces, since otherwise they will not lift to the covering space due to their non-trivial fundamental groups. The intersection form satisfies

$$\lambda(f_1, f_2) = \overline{\lambda(f_2, f_1)}$$

where the involution used is induced from $g \mapsto g^{-1}$ on $\pi_1(W_0)$. There is no sign change when switching variables since the dimensions are even.

In order to relate the intersection form to the Alexander polynomial, consider the long exact sequence of the pair $(W_0, \partial W_0 = M_K)$ with $\mathbb{Z}[\mathbb{Z}]$ coefficients:

$$H_2(W_0; \mathbb{Z}[\mathbb{Z}]) \xrightarrow{\lambda} H_2(W_0, \partial W_0; \mathbb{Z}[\mathbb{Z}]) \to H_1(M_K; \mathbb{Z}[\mathbb{Z}]) \to H_1(W_0; \mathbb{Z}[\mathbb{Z}])$$

$\pi_1(W_0) \cong \mathbb{Z}$, so $H_1(W_0; \mathbb{Z}[\mathbb{Z}]) \cong 0$. Poincaré-Lefschetz duality and universal coefficients mean that $H_2(W_0, \partial W_0; \mathbb{Z}[\mathbb{Z}]) \cong H^2(W_0; \mathbb{Z}[\mathbb{Z}]) \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(H_2(W_0; \mathbb{Z}[\mathbb{Z}]), \mathbb{Z}[\mathbb{Z}])$. This justifies the labelling of the arrow λ; it is indeed the intersection form. $H_2(W_0; \mathbb{Z}[\mathbb{Z}])$ is free since there are no 3-handles of W_0. We therefore have $H_1(M_K; \mathbb{Z}[\mathbb{Z}]) \cong \mathrm{coker}(\lambda)$, and since the Alexander polynomial is the characteristic polynomial of this $\mathbb{Z}[\mathbb{Z}]$-module, we have that $\Delta_K = \det(\lambda)$. Note that the determinant of λ depends on a choice of basis, which could vary by units in $\mathbb{Z}[\mathbb{Z}]$; however the Alexander polynomial is also only ever defined up to a unit.

(c) Now, to compute the signature of the intersection form, we give an explicit matrix for it, using the explicit 4-manifold W_0 depicted in Figure 7. The immersed 2-spheres which form our basis of $H_2(W_0; \mathbb{Z}[\mathbb{Z}])$ are given, as usual, by the cores of the 2-handles combined with immersed discs in $S^1 \times B^3$. These discs will have self intersections governed by the framing of the link components, and will intersect one another depending on the way the components link one another. We count these intersections with coefficients in $\langle m \rangle$ governed by linking with the dotted circle which denotes the 1-handle. We claim that:

$$\lambda_{\widetilde{W_0}} = \begin{pmatrix} A & I + (m-1)C \\ I + (m^{-1} - 1)C^T & (m-1)(m^{-1}-1)B \end{pmatrix}$$

where each of A, B, C are block $g \times g$ matrices from above, so $A = A^T$ and $B = B^T$. For the off diagonal entries, we try to homotope the a curves away from the b curves (look again at Figure 7). Since the b curves imitate the bands of a Seifert surface, we get *pairs* of intersection points, and an extra intersection at the end (the $+1$) due to the single linking of the a and b curves. Since the two strands of the b curves differ by a single passage around the 1-handle i.e. through the dotted circle, this explains the $(m-1)$ coefficient of C in the top right. The Hermitian property of λ fills in the bottom left square for us. For the b-b pairs, each linking of the bands induces *four* intersection points: hence $(m-1)(m^{-1}-1)B$. Similarly λ-self intersections, that is homological self intersections, so intersections with a pushed off copy of the sphere, produces 4 intersection points for each twist in the band. Note here the difference with the μ-form which counts actual self intersection points. The intersections of the a curve spheres depend on the framing of the a curves which were taken from the linking of the corresponding bands in the Seifert surface and do not depend on m. Now, tensor with $\mathbb{C}(z)$, so that we can invert polynomials. We have

$$\lambda_{\widetilde{W_0}} = \begin{pmatrix} A & I + (z-1)C \\ I + (z^{-1} - 1)C^T & (z-1)(z^{-1}-1)B \end{pmatrix}$$

Make a change of basis with change of basis matrix

$$P = \begin{pmatrix} (z-1)I & 0 \\ 0 & I \end{pmatrix}$$

In particular note that $z-1$ can be inverted now, so P is an invertible matrix. Also, in the calculation below, the key step is that $q = z - 1$ is a quadratic element, so that $q\overline{q} = q + \overline{q}$. The matrix of λ_{W_0} becomes:

$$\begin{aligned}
\lambda_{W_0} &= \overline{P}^T \begin{pmatrix} A & I+(z-1)C \\ I+(z^{-1}-1)C^T & (z-1)(z^{-1}-1)B \end{pmatrix} P \\
&= \begin{pmatrix} (z^{-1}-1)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & I+(z-1)C \\ I+(z^{-1}-1)C^T & (z-1)(z^{-1}-1)B \end{pmatrix} \begin{pmatrix} (z-1)I & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} (z-1)(z^{-1}-1)A & (z^{-1}-1)I+(z-1)(z^{-1}-1)C \\ (z-1)I+(z-1)(z^{-1}-1)C^T & (z-1)(z^{-1})B \end{pmatrix} \\
&= \begin{pmatrix} ((z-1)+(z^{-1}-1))A & (z^{-1}-1)I+((z-1)+(z^{-1}-1))C \\ (z-1)I+((z-1)+(z^{-1}-1))C^T & ((z-1)+(z^{-1}))B \end{pmatrix} \\
&= (z-1)\begin{pmatrix} A & C \\ I+C^T & B \end{pmatrix} + (z^{-1}-1)\begin{pmatrix} A^T & C+I \\ C^T & B^T \end{pmatrix} \text{ since } A=A^T \text{ and } B=B^T \\
&= (z-1)V+(z^{-1}-1)V^T
\end{aligned}$$

the signature of which was defined earlier to be the signature of the knot $\sigma_z(K)$. If $z=1$, this change of basis is not possible; however, we then have $\sigma_1(K)=0$, and the matrix $\begin{pmatrix} A & I \\ I & 0 \end{pmatrix}$, so we still have agreement.

□

Example 4.13. Consider the link shown in Figure 8. This is a Kirby diagram for a 4-manifold W with $\pi_1(W) \cong \mathbb{Z} \cong \langle m \rangle$, $\pi_2(W) \cong \mathbb{Z}[\pi_1(W)] \cong \mathbb{Z}[m, m^{-1}]$ (since W has just one 2 handle it is homotopy equivalent to $S^1 \vee S^2$) and $H_2(W) \cong \mathbb{Z}$. The Hurewicz homomorphism is given by augmentation $m \mapsto 1$. The intersection form $I_W = (1)$, as given by the framing on the 2-handle attaching map.

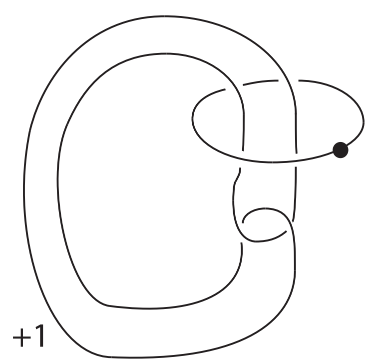


Figure 8: The Whitehead link with one component dotted as an example of W^4 with $\pi_1(W) \cong \mathbb{Z}$ and $\partial W = M_K$.

We can also compute the intersection form $I_{\widetilde{W}}$ of the universal cover, which here is the infinite cyclic cover. We can generate the second homology $H_2(\widetilde{W};\mathbb{Z}) \cong H_2(W;\mathbb{Z}[m,m^{-1}])$ with an immersed 2-sphere constructed from the union of the core of the 2-handle with a disc in $S^1 \times B^3$ with one self intersection. We cannot use the core union a Seifert surface here because the Seifert surface isn't a disc, and so doesn't extend to the universal cover as a surface because it has non-trivial fundamental group. Passing through the dotted circle takes us to a different sheet of the covering. The attaching circle of the 2-handle links adjacent copies of itself with linking number -1 and links itself with linking number 1. This corresponds to analogous intersections of the corresponding 2-spheres. Therefore

$$I_{\widetilde{W}} = (-m - m^{-1} + 1)$$

which is the Alexander polynomial of the trefoil. Furthermore, we can work out the boundary of W as follows. First, change the 1-handle to a zero framed 2-handle. Then exploit the symmetry of the Whitehead link to obtain the same picture as in Figure 8 but with the dot replaced with a $+1$ and the $+1$ replaced with a 0 (See Figure 9, where the link components are labelled x and y). Slide the handle labelled x twice over the handle labelled y, once for each strand of x which passes through y. This should be done with orientations opposite on each handle slide. The effect of this is to move y away from x, and to introduce a full twist in x, without changing the framing. We can then blow down y to leave a trefoil knot with 0 framing. Therefore W has boundary M_{3_1}, the zero-surgery on the trefoil.

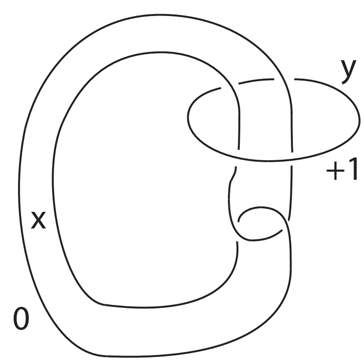


Figure 9: Finding the boundary of W.

5 L^2-homology

5.1 Introduction

So far we do not have any new concordance invariants beyond the "high-dimensional" invariants which detect non-algebraically slice knots: we have only looked at 4-manifolds with fundamental group $\mathbb{Z}$. In order to obtain more sophisticated invariants we need to look at 4-manifolds with larger groups, and their equivariant intersection forms. The group rings will be non-commutative rings; our tools (to be defined below) for detecting the signatures of forms over non-commutative rings will be the L^2-homology and the L^2-signature. We need a generalisation of the idea of dimension for submodules after embedding the group ring in its Von Neumann algebra. This enables us to look at the difference between the dimensions of the positive and negative eigenspaces of a Hermitian operator on a Hilbert space corresponding to a symmetric sesquilinear form, and so to define a real number which is the L^2-signature, giving a homomorphism from the Witt group of such forms to $\mathbb{R}$. When defining this for a knot, to get a concordance invariant, we look at the L^2-signature of the equivariant intersection form of a 4-manifold whose boundary is M_K. We then define the reduced L^2-signature which is an invariant of the zero surgery, so does not depend on the choice of 4-manifold. In order to calculate these signatures we are therefore able to make judicious choices of 4-manifold, whose intersection forms we can exhibit explicitly. In the case of W_0 above, it turns out that the L^2-signature is given by integrating the twisted signatures of the knot around S^1.

5.2 Atiyah's original definition of L^2 Betti Numbers

Suppose M is a compact Riemannian manifold, and $\overline{M} \to M$ is a regular cover with an infinite group as deck transformations, so $\overline{M}$ is non-compact. The idea is to measure the space of smooth L^2-integrable p-forms on $\overline{M}$ (if they exist - this is not immediately clear). It will be either zero or infinite dimensional, so we want a measurement of dimension which is zero if and only if there are no such forms. Thus, the analytic L^2-Betti number is

$$b_p^{(2)}(M) = \lim_{t\to\infty} \int_{\text{a fund. domain for } \overline{M}} \operatorname{tr}_{\mathbb{C}}(e^{t\triangle_p}(x,x))\, dx$$

- $\triangle_p \colon \Omega^p(\overline{M}) \to \Omega^p(\overline{M})$ is the Laplacian on p-forms (lift the metric).
- $e^{t\triangle_p}(x,x)$ is the heat kernel.
- $\operatorname{tr}_{\mathbb{C}}$ means the normal matrix trace of the section of $\operatorname{End}(\Lambda^p(T\overline{M}))$ *i.e.* fibre-wise trace.
- The group Γ of the covering is brought in by the choice of fundamental domain *i.e.* it cannot just be an invariant of the covering.

Remark 5.1. These Betti numbers are zero for circle and tori, but are non-zero for surfaces and $\mathbb{H}^2$ covering spaces.

5.3 Alternative Approach

If $\widetilde{X} \to X$ is the universal covering of a finite CW-complex X, and $\mathcal{M}$ is a $\pi_1(X)$ module, then we can define

$$H_p(X;\mathcal{M}) = H_p(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathcal{M}).$$

For example, suppose we have a surjection $\pi_1(X) \twoheadrightarrow \Gamma$. Then we can define $H_p(X;\mathbb{Z}[\Gamma]) = H_p(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\Gamma]) = H_p(\overline{X})$, the homology of the cover corresponding to Γ. $H_p(\overline{X})$ is a $\mathbb{Z}[\Gamma]$ module, but algebraically it is not at all well understood - it might not even be finitely generated, since the ring $\mathbb{Z}[\Gamma]$ is non-Noetherian which means that submodules of finitely generated modules may not be finitely generated.

The idea is then to embed $\mathbb{Z}[\Gamma] \hookrightarrow \mathcal{N}\Gamma$, the *Von Neumann Algebra* of Γ, a ring with a much simpler representation theory. Then:

Definition 5.2.

$$H_p(\overline{X}) = H_p(X;\mathcal{N}\Gamma) \text{ is the } L^2 \text{ homology; and}$$

$$b_p^{(2)}(\overline{X}) = \dim_{\mathcal{N}\Gamma} H_p(X;\mathcal{N}\Gamma) \in [0,\infty)\,.$$

We now define the Von Neumann Algebra.

Let Γ be a countable discrete group and consider the Hilbert space $l^2\Gamma$ of square-summable sequences of group elements with complex coefficients.

$$\{\Sigma a_g g \,|\, a_g \in \mathbb{C}, \Sigma |a_g|^2 < \infty\}.$$

The group Γ acts by left- and right- multiplication on $l^2\Gamma$. These operators are obviously isometries and we can consider the embedding

$$\mathbb{C}\Gamma \hookrightarrow \mathcal{B}(l^2\Gamma)$$

corresponding to (sums of) *right* multiplications into the space of bounded operators on $l^2\Gamma$.

Definition 5.3. The (reduced) C^*-algebra $C^*\Gamma$ is the completion of $\mathbb{C}\Gamma$ with respect to the operator norm on $\mathcal{B}(l^2\Gamma)$. The von Neumann algebra $\mathcal{N}\Gamma$ is the completion of $\mathbb{C}\Gamma$ with respect to pointwise convergence in $\mathcal{B}(l^2\Gamma)$. In particular we have $\mathbb{C}\Gamma \subseteq C^*\Gamma \subseteq \mathcal{N}\Gamma$.

Von Neumann's double commutant theorem says that $\mathcal{N}\Gamma$ is equal to the set of bounded operators which commute with the *left* Γ-action on $l^2\Gamma$:

$$\mathcal{N}\Gamma = \mathcal{B}(l^2\Gamma)^\Gamma$$

Example 5.4. **(a)** $|\Gamma| < \infty \Rightarrow \mathcal{N}\Gamma = \mathbb{C}[\Gamma]$ which is semisimple, so the algebra is manageable.

(b) $\Gamma = \mathbb{Z}$: then $\mathcal{N}\Gamma = L^\infty(S^1;\mathbb{C})$, the space of (almost everywhere) bounded functions $S^1 \to \mathbb{C}$. $\mathbb{C}[\Gamma]$ sits inside this as Laurent polynomials.

5.4 Trace

We define a trace operator in order to measure dimension of subspaces: recall that if we have a subspace U of a finite dimensional vector space V, then choosing a basis and representing the projection $pr\colon V \to U$ as a matrix, and taking its trace tells us the dimension of U; this follows easily from the fact that the trace is invariant under change of basis, and so in a sense the trace tells us the definition of the projection operator. We need to generalise this to the setting of Hilbert spaces operators and thus to Von Neumann algebras; we now get a *real valued* idea of dimension. We define:

$$\mathrm{tr}\colon \mathcal{N}\Gamma \to \mathbb{C} \text{ by:}$$
$$a \mapsto \langle a(e), e\rangle_{l^2\Gamma},$$

where $e \in l^2(\Gamma)$ is the identity element, so that for $a \in \mathbb{C}\Gamma$, $\mathrm{tr}(a) = a_e =$ coefficient of e. (This is not the usual augmentation.) It satisfies:

$$\text{(i)}\quad \mathrm{tr}(ab) = \mathrm{tr}(ba).$$

To see this, it is enough to show that it is true for $a \in \mathcal{N}\Gamma$ and $g \in \Gamma$ by linearity. But

$$\langle ag(e), e\rangle = \langle (ae)g, e\rangle = \langle ae, g^{-1}e\rangle = \langle g(ae), e\rangle$$

using the *bi-module* property and that the left and right Γ-actions are isometries.

(ii) tr is faithful, i.e. $\mathrm{tr}(a^*a) = 0$ if and only if $a = 0$. This is because

$$\langle a^*a(e), e\rangle = \langle ae, ae\rangle = 0 \Leftrightarrow ae = 0.$$

But then $ag = 0$ for all $g \in \Gamma$ by right multiplication and then by continuity $a \cdot \overline{\mathbb{C}\Gamma} = 0$ so $a = 0$.

Definition 5.5. Let P be a f.g. projective $\mathcal{N}\Gamma$ module; i.e. $P = \mathrm{im}(p)$ for some $p \in M_n(\mathcal{N}\Gamma)$ with $p^2 = p = p^*$. Then we can use the trace operator to define the Γ-dimension:

$$\dim_{\mathcal{N}\Gamma} P = \mathrm{tr}_\Gamma(p),$$

where the right hand side is given by the sum of the Γ-traces of the diagonal entries of p. Since $p = p^*$ and $\mathrm{tr}(a^*) = \overline{\mathrm{tr}(a)}$, this is a real number, in fact a positive real number: $\mathrm{tr}(p) \in [0, \infty)$ since:

$$\mathrm{tr}(p) = \mathrm{tr}(p^2) = \mathrm{tr}(p^*p) = \langle pe, pe\rangle \geq 0.$$

We therefore have a map:

$$K_0(\mathcal{N}\Gamma) \to \mathbb{R}.$$

Theorem 5.6 (Farber,Lück)**.** $\mathcal{N}\Gamma$ is a semi-hereditary ring: any f.g. submodule of a projective module is projective.

Corollary 5.7. Given $\overline{X} \to X$, there are finitely generated projective modules P_i, Q_i such that the following is exact:

$$0 \to P_i \to Q_i \to H_i^{(2)}(\overline{X}) \to 0$$

(The L^2 homology modules have homological dimension 1.) Thus we can define, for $H_i^{(2)}(\overline{X})$, which is not necessarily projective, the Betti numbers:

$$b_i^{(2)}(\overline{X}) = \dim_{\mathcal{N}\Gamma} Q_i - \dim_{\mathcal{N}\Gamma} P_i.$$

Remark 5.8. According to W. Lück we can actually extend the dimension to all modules over $\mathcal{N}\Gamma$.

Remark 5.9. Specialise again to $\Gamma = \mathbb{Z}$: $l^2(\mathbb{Z})$ is the Fourier expansion $L^2(S^1;\mathbb{C})$ but taking closure in bounded operators gets us as above to $L^\infty(S^1;\mathbb{C})$ which is all bounded functions modulo differences on sets of measure zero; trace becomes integration over S^1, since in $\langle f.e, e\rangle$, e is the constant function 1 and inner product is integral.

Let A be a C^* algebra with involution $*$. Pick $a \in A$ which is normal, that is $aa^* = a^*a$, and consider $\mathrm{spec}(a)$. Since A is a C^*-algebra, $\mathrm{spec}(a)$ is compact, so there is a $*$-ring homomorphism.

$$C^0(\mathrm{spec}(a);\mathbb{C}) \to A; \;\; f \mapsto f(a)$$

which sends $\mathrm{Id}_{\mathrm{spec}(a)} \mapsto a$, and the constant $1 \mapsto 1$. We send functions to S^1 to units, polynomials to polynomials and real valued functions to hermitian operators. a generates an *abelian subalgebra* which equals $C^0(\mathrm{spec}(a))$ by Gelfand-Naimark. If $\mathcal{A}$ is a Von-Neumann Algebra, then the above applies, and moreover it also applies with $L^\infty(\mathrm{spec}(a);\mathbb{C}) \to \mathcal{A}$, i.e. with bounded almost everywhere functions rather than just with continuous functions. This map is the functional calculus - starting with a normal element of our Von Neumann algebra we obtain a new element for each function on the spectrum. For $\mathcal{A} = M_n(\mathcal{N}\Gamma)$, this element can be used to define projective submodules of $\mathcal{A}$.

Given $h \in M_n(\mathcal{N}\Gamma)$ which satisfies $h = h^*$, we want to define the L^2-signature $\sigma^{(2)}_\Gamma(h) \in \mathbb{R}$. To define the L^2-signature, consider a hermitian $(n \times n)$-matrix over $\mathcal{N}\Gamma$, $h \in \mathrm{Herm}_n(\mathcal{N}\Gamma)$, as a bounded, hermitian Γ-equivariant operator on the Hilbert space $(l^2\Gamma)^n$. Its spectrum $\mathrm{spec}(h)$ is a (compact) subset of the real line and for any bounded measurable function f on $\mathrm{spec}(h)$ we may define the bounded Γ-equivariant operator $f(h) \in M_n(\mathcal{N}\Gamma)$ by functional calculus. In particular, consider the characteristic functions $p_+, p_-, p_0 : \mathbb{R} \to \mathbb{R}$ of $(0, +\infty)$ respectively $(-\infty, 0)$, $\{0\}$.

Theorem 5.10 (Functional Calculus)**.** There are finitely generated projective $\mathcal{N}\Gamma$-modules H_+, H_-, H_0 such that after a base change $(h \mapsto aha^*)$, there is an orthogonal decomposition

$$H := (l^2\Gamma)^n = H_0 \oplus H_+ \oplus H_-$$

such that

$$h = \begin{pmatrix} 0 & & \\ & +1 & \\ & & -1 \end{pmatrix}$$

Proof. h is a Hermitian operator in $H := \mathcal{B}((l^2\Gamma)^n)$. Then $\mathrm{spec}(h)$, the set of $\lambda \in \mathbb{C}$ such that $h - \lambda.\mathrm{Id}$ is not invertible, is a compact subset of $\mathbb{R}$. (Bounded operators have compact spectra; Hermitian operators have real elements in their spectra.)

Then, given any bounded function $f\colon \mathrm{spec}(h) \to \mathbb{R}$, there is a Hermitian operator $f(h)$ with $\mathrm{spec}(f(h)) = f(\mathrm{spec}(h))$. The proof of this is that for polynomial f, $f(h)$ is defined uniquely. In general, f is the point-wise limit of polynomials f_i: define $f(h)$ = point-wise limit of $f_i(h)$, which by definition lives in $M_n(\mathcal{N}\Gamma)$. Note that, as above, the functional calculus

$$L^\infty(\operatorname{spec} h, \mathbb{C}) \to M_n(\mathcal{N}\Gamma);$$
$$f \mapsto f(h)$$

is a C^*-algebra homomorphism. Now define f_+, f_-, f_0 by Heaviside type functions, so that we use the characteristic functions of the intervals $(0,\infty)$, $(-\infty,0)$ and the singleton $\{0\}$ in $\mathbb{R}$. $1 = f_+ + f_- + f_0$, so $h = f_+(h) + f_-(h) + f_0(h)$ giving an orthogonal decomposition of the Hilbert space as $H_+ := f_+(h)(H)$, $H_- := f_-(h)(H)$ and $H_0 := f_0(h)(H)$. For example, $f_+(h) \in M_n(\mathcal{N}\Gamma)$, so its image is a projective submodule of H. Since the functional calculus is a homomorphism, the maps f_+, f_-, f_0 are commuting projections, and so we get an orthogonal decomposition as claimed. $\square$

Definition 5.11. The signature map $\sigma_\Gamma : \mathrm{Herm}_n(\mathcal{N}\Gamma) \to K_0(\mathcal{N}\Gamma)$ is defined by sending h to the formal difference $p_+(h) - p_-(h)$ of projections in $M_n(\mathcal{N}\Gamma)$. The L^2-signature of $h \in \mathrm{Herm}_n(\mathcal{N}\Gamma)$ is defined to be the real number

$$\sigma_\Gamma^{(2)}(h) := \mathrm{tr}_\Gamma(p_+(h)) - \mathrm{tr}_\Gamma(p_-(h)) = \dim_{\mathcal{N}\Gamma} P_+ - \dim_{\mathcal{N}\Gamma} P_-.$$

Remark 5.12. $|\sigma_\Gamma^{(2)}(h)| < n$. To see this note that $\dim_{\mathcal{N}\Gamma}(\mathcal{N}\Gamma) = 1$; since $p = p^2 = p^*$ acts on one $\mathcal{N}\Gamma$. Also $0 \leq \mathrm{tr}(p) \leq 1$: just use $(1-p)^* = (1-p)$ as well, so $\mathrm{tr}\, p \geq 0$, $\mathrm{tr}(1-p) \geq 0$.

Lemma 5.13. The L^2-signature only depends on the Γ-isometry class of $h \in \mathrm{Herm}_n(\mathcal{N}\Gamma)$, i.e. it is unchanged under $h \mapsto a^*ha$ for $a \in \mathrm{GL}_n(\mathcal{N}\Gamma)$.

Proof. Consider the Hilbert space $H := (l^2\Gamma)^n$ with the bounded Γ-equivariant operators h and a. We have an orthogonal decomposition of Hilbert spaces

$$H = H_0 \perp H_+ \perp H_-$$

as above. For a vector v in one of the three summands above, one has by definition that

$$\langle h(v), v\rangle = 0, > 0 \text{ respectively } < 0.$$

It follows that depending on whether v is in $a^{-1}H_0$, $a^{-1}H_+$ respectively $a^{-1}H_-$ one has

$$\langle (a^*ha)(v), v\rangle = \langle h(av), av\rangle = 0, > 0 \text{ respectively } < 0.$$

Therefore, the three orthogonal projections

$$a^{-1}H_\dagger \to p_\dagger(a^*ha)H \quad \text{ for } \dagger \in \{0,+,-\}$$

are monomorphisms and thus

$$\dim_\Gamma H_\dagger = \dim_\Gamma a^{-1}H_\dagger \leq \dim_\Gamma p_\dagger(a^*ha)H \quad \text{ for } \dagger \in \{0,+,-\}.$$

But the three dimensions on both sides must sum up to the total dimension n of H and therefore the inequalities are actually equalities. $\square$

Example 5.14. We recapitulate the crucial example, and consider once more the case $\Gamma = \langle t \rangle \cong \mathbb{Z}$. Then $\mathbb{C}\Gamma$ consists of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ which embed naturally into the space of complex valued continuous functions on the circle S^1. Indeed, Fourier transformation gives an isomorphism of Hilbert-spaces $l^2\mathbb{Z} \cong L^2(S^1;\mathbb{C})$ and point-wise multiplication by a function induces the isomorphism $C^*\Gamma \cong C(S^1;\mathbb{C})$. This is a consequence of the Stone-Weierstraß theorem on the density of polynomials in the space of all continuous functions in the supremum norm. Completing in the topology of pointwise convergence then leads to the von Neumann algebra $\mathcal{N}\Gamma$ which turns out to be the space $L^\infty(S^1;\mathbb{C})$ of complex valued, bounded, measurable functions on the circle, defined almost everywhere. Finally, the standard Γ-trace is just given by integration.

Now consider $h \in \mathrm{Herm}_n(C(S^1;\mathbb{C}))$ and think of it as a continuous map from S^1 to $\mathrm{Herm}_n(\mathbb{C})$. The ordinary signature $\sigma : \mathrm{Herm}_n(\mathbb{C}) \longrightarrow \mathbb{Z}$ counts the number of positive Eigenvalues minus the number of negative Eigenvalues.

Definition 5.15. The *twisted signature* of h is the step function $\sigma(h) : S^1 \longrightarrow \mathbb{Z}$ which assigns to each $s \in S^1$ the signature $\sigma(h(s))$. Moreover, the real number $\sigma^{(2)}(h)$ is defined to be the integral of this function $\sigma(h)$ over the circle (normalized to have total measure 1).

Thus $\sigma^{(2)}(h)$ is the average of all the twisted signatures. It is clear that $\sigma(h)$ makes sense almost everywhere for $h \in \mathrm{Herm}_n(L^\infty(S^1;\mathbb{C}))$ and therefore $\sigma^{(2)}(h)$ is well defined even in this case. As an example, consider the following element in $\mathrm{Herm}_2(\mathbb{C}[t^{\pm 1}])$:

$$h := \begin{pmatrix} t + t^{-1} - 2 & t - 1 \\ t^{-1} - 1 & t + t^{-1} - 2 \end{pmatrix}$$

Notice that $\sigma(h)$ is a step function with jumps at most at the zeroes of the "Alexander polynomial" $\det(h) \in \mathcal{C}[t, t^{-1}]$. We have, up to multiplication by a unit $\pm t^n$, $\det(h) = (t + t^{-1} - 2)^2 - (t - 1)(t^{-1} - 1) = (t-1)^2(t + t^{-1} - 1) = (t-1)^{2\,\mathrm{genus}(3_1)}\Delta_{3_1}(t)$ which has roots on S^1 exactly for the two primitive 6-th roots of unity $\pm e^{i\pi/3}$, and for $t = 1$. For $t = 1$ we have the zero matrix and so this does not contribute to the signature. We only need to calculate $\sigma(h)$ at two points on the circle which interlace with the two roots of $\Delta_{3_1}(t)$, e.g. at -1 and $\sqrt{3}/2 + i/2$. One easily checks that the ordinary signature of $h(-1)$ is -2, and of $h(\sqrt{3}/2 + i/2)$ is zero. Since these roots are of distance $1/3$ and $2/3$ apart, depending on which way round of the circle we measure, we therefore have:

$$\sigma^{(2)}(h) = (1/3) \cdot 0 + (2/3) \cdot (-2) = -4/3 \neq 0.$$

Of course this Hermitian operator is equivalent to the equivariant intersection form of our 4-manifold W_0 from Theorem 4.11 for the trefoil knot, so has the twisted signatures of the trefoil for each $s \in S^1 \subseteq \mathbb{C}$. It is possible to similarly work this out for all twist knots and for torus knots (recall that the torus knots arise as the self-linking zero curves on the obvious Seifert Surfaces for the twist knots).

Lemma 5.16. The average $\sigma^{(2)}(h)$ equals the L^2-signature $\sigma^{(2)}_\Gamma(h)$ for $\Gamma = \mathbb{Z}$.

Proof. Notice that $\mathrm{tr}_\Gamma(p_+(h))$ is the Γ-dimension of the "positive Eigenspace" of h. In the functional calculus one approximates (say) p_+ by a sequence of real polynomials p_i into which any operator can easily be substituted. Then one takes the pointwise limit to define

$$p_+(h) := \lim_i p_i(h) \quad \text{for } h \in \mathcal{N}\mathbb{Z}.$$

For example, if h is a finite dimensional matrix, then one checks that p_+ is just the projection onto the $(+1)$-Eigenspace of h. This implies that for a point $s \in S^1$, a fancy way to count the number of positive Eigenvalues of $h(s) \in \mathrm{Herm}_n(\mathbb{C})$ is to take the ordinary trace of $p_+(h(s)) := \lim_i p_i(h(s))$. But now one clearly sees that the integral of the function $\sigma(h)$ which associates for each $s \in S^1$ the difference $p_+(h(s)) - p_-(h(s))$ is almost everywhere the same as $\sigma^{(2)}_\Gamma(h)$. $\square$

Corollary 5.17.

$$\sigma^{(2)}_{\mathbb{Z}}(h) = \int_{S^1} \sigma(h(z))\, dz$$

Remark 5.18. Note that the orthogonal projections ($p = p^* = p^2$) in $\mathcal{N}\mathbb{Z}$ correspond one to one with measurable sets in S^1.

$$p = p^* \Rightarrow p \in L^\infty(S^1, \mathbb{R});$$

$p^2 = p \Rightarrow p$ has values 0 or 1 almost everywhere, so we can view it as a characteristic function. A consequence of this is that any positive real number occurs as a dimension of a projective module over $\mathcal{N}\mathbb{Z}$: just use the function p to define $p\mathcal{N}\mathbb{Z}$ with dimension $p, 0 \leq p \leq 1$. Then add copies of $\mathcal{N}\mathbb{Z}$ to make larger numbers.

Also $K_0(\mathcal{N}\mathbb{Z}) \cong L^\infty(S^1; \mathbb{Z})$. This follows since a projection operator can be diagonalised without changing the trace, with each diagonal element satisfying $p_i = p_i^* = p_i^2$ as above, which means it picks out a measurable subset of S^1 as the characteristic function. Taking the matrix trace of the tr_Γ's yields the sum of these characteristic functions which is a bounded measurable function $S^1 \to \mathbb{Z}$. Negative numbers are associated to the projective modules in $K_0(\mathcal{N}\mathbb{Z})$ which are included as formal inverses in the Grothendieck completion which defines the projective class group. Integrating the elements of $L^\infty(S^1; \mathbb{Z})$ yields an epimorphism to $\mathbb{R}$.

5.5 Properties of L^2 Betti Numbers

Recall $H_i^{(2)}(\overline{X}) = H_i(X; \mathcal{N}\Gamma)$, where $\mathbb{Z}\Gamma \subseteq \mathbb{C}\Gamma \subseteq \mathcal{N}\Gamma = \mathcal{B}(l^2\Gamma)^{\Gamma_{\text{right}}}$, which is a completion of $\mathbb{C}\Gamma$ under point-wise convergence of operators. We can actually think of the chain complex:

$$C_i(\overline{X}) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}\Gamma = (\mathcal{N}\Gamma)^{\sharp\, i \text{ cells}}$$

with the boundary maps being matrices $d_i \in M_n(\mathcal{N}\Gamma)$.

Property 1. Homotopy Invariance If $f\colon X \simeq Y$ is a homotopy equivalence which lifts to a homotopy equivalence of the Γ-cover, then $H_i^{(2)}(\overline{X}) = H_i^{(2)}(\overline{Y})$. (This would be surprising using the analytic integral of the heat kernel definition of L^2-Betti numbers.)

Recall that $H_i(\overline{X})$ is not quite a projective module: it can be given by

$$0 \to P_i \to Q_i \to H_i^{(2)}(\overline{X}) \to 0$$

where $H_i^{(2)}(\overline{X})$ decomposes as a f.g. projective part, as measured by the dimension, and a torsion part, as measured by the Novikov-Shubin invariants.

Property 2. Euler-Poincaré Formula For a finite complex:

$$\chi(X) = \sum(-1)^i b_i(X) = \sum(-1)^i b_i^{(2)}(\overline{X})$$

Property 3. Multiplicative under finite covers Consider the tower of covering spaces

$$\begin{tikzcd} \overline{X} \arrow[r, "\Gamma_0"] \arrow[d, "\Gamma"'] & X' \arrow[dl, "n"] \\ X & \end{tikzcd}$$

where Γ_0 has index n in Γ. Then

$$b_i^{(2)}(X')_{\Gamma_0} = n \cdot b_i^{(2)}(X)_{\Gamma}.$$

Proof of Property 2.

$$\sum(-1)^i b_i = \sum(-1)^i \dim_{\mathbb{Q}}(C_i(\overline{X}) \otimes \mathbb{Q}) = \sum(-1)^i \dim_{\mathcal{N}\Gamma}(C_i(\overline{X}) \otimes \mathcal{N}\Gamma) = \sum(-1)^i b_i^{(2)}(\overline{X})$$

just from additivity of rational dimension under short exact sequences and similarly for the Von Neumann dimension. Both chain complexes consist of free modules whose rank in each dimension is equal to the number of cells of that dimension. Since X is a finite complex, this is legal. $\square$

Proof of Property 3. The algebraic statement necessary is that $\Gamma_0 \leq \Gamma$ induces $\mathcal{N}\Gamma_0 \to \mathcal{N}\Gamma$; if $\mathcal{M}$ is an $\mathcal{N}\Gamma$-module then we can restrict to $\mathcal{N}\Gamma_0$ with dimension given by

$$\dim_{\mathcal{N}\Gamma_0}(\mathrm{res}_{\mathcal{N}\Gamma_0}\mathcal{M}) = n \cdot \dim_{\mathcal{N}\Gamma} \mathcal{M}.$$

$\square$

Conjecture 5.19 (Atiyah Conjecture)**.** Γ torsion free implies that $b_i^{(2)}(\overline{X})_\Gamma \in \mathbb{N}$. This is known for elementary amenable groups, free groups, class closed under certain operations (not hyperbolic yet).

Property 4. Poincaré Duality If X is a closed oriented manifold of dimension n, then $b_i^{(2)}(X)_\Gamma = b_{n-i}^{(2)}(X)_\Gamma$. The proof is that the usual chain equivalence between homology and cohomology complexes, tensored with $\mathcal{N}\Gamma$, remains a chain equivalence. There is a subtlety, namely that we have to use L^2-cohomology, but it is very close to L^2-homology.

Property 5. Künneth formula

$$b_n^{(2)}(X_1 \times X_2)_{\Gamma_1 \times \Gamma_2} = \sum_{p+q=n} b_p^{(2)}(X_1)_{\Gamma_1} b_q^{(2)}(X_2)_{\Gamma_2}.$$

Property 6. If X is connected, then:

$$b_0^{(2)}(X)_\Gamma = \begin{cases} \frac{1}{|\Gamma|} & |\Gamma| \text{ finite; and} \\ 0 & \text{otherwise.} \end{cases}$$

Look at

$$C_1(\overline{X}) = \mathbb{Z}\Gamma^n \to C_0(\overline{X}) = \mathbb{Z}\Gamma \to H_0(\overline{X}) \to 0$$

and use homotopy invariance. 1-cells have $e_i^1 \mapsto e^0(g_i - 1)$. Then for the finite group case, $H_0(\overline{X}) = \mathbb{Z}$, which as a $\mathbb{Z}\Gamma$-module has dimension $1/|\Gamma|$. In the infinite case we get

$$\mathcal{N}\Gamma^n \xrightarrow{d} \mathcal{N}\Gamma \to H_0(\overline{X}) \to 0$$

and we claim that the dimension of the image of d is 1.

Remark 5.20. Also a limiting process for residually finite groups can express $b_i^{(2)}$ as weighted limit of the usual Betti numbers.

Example 5.21. These properties allow us to deduce the L^2-Betti numbers of some spaces.

- S^1 with group $\mathbb{Z}$: $b_0^{(2)} = b_1^{(2)} = 0$ from connectedness, infinite group and χ formula (or duality).
- This does the torus too by Property 5.
- For a surface Σ_g and universal cover, $b_0^{(2)} = b_2^{(2)} = 0$, $b_1^{(2)} = 2g - 2$.
- For the free group F_n (so the space $\bigvee_n S^1$; we talk of the Betti numbers of a group, by which we mean the Betti numbers of a corresponding $K(\pi, 1)$), we have $b_0 = 0$, $b_1 = n - 1$
- For a knot complement, we need to find out, but guess that they are all zero. L^2-torsion should be the Alexander polynomial. Also, for a 3-manifold we guess all Betti numbers are zero. L^2-torsion is hyperbolic volume or Gromov norm.

5.6 Application to Manifolds

Definition 5.22. Let X^{4k} be a compact oriented manifold, and $\rho\colon \pi_1(X) \to \Gamma$ a representation. Then define

$$\sigma^{(2)}_\Gamma(X,\rho) = \sigma^{(2)}_\Gamma(\lambda^{(2)}_X) \in \mathbb{R}$$

where $\lambda^{(2)}_X$ is a Hermitian form defined as the following intersection form:

$$\lambda^{(2)}_X\colon H_{2k}(X;\mathcal{N}\Gamma) \to H_{2k}(X,\partial X;\mathcal{N}\Gamma) \cong H^{2k}(X,\mathcal{N}\Gamma) \to \mathrm{Hom}_{\mathcal{N}\Gamma}(H_{2k}(X;\mathcal{N}\Gamma),\mathcal{N}\Gamma).$$

The last map may not be an isomorphism: instead of Universal Coefficient Theorem there is a spectral sequence. The definition of the map is the usual one.

Remark 5.23. To define σ for hermitian form h on a projective module P which is not a free module, use the inverse Q in K_0, so that $P \oplus Q = (\mathcal{N}\Gamma)^n$ and take $\sigma(h_P \oplus 0_Q)$ as the definition of the signature.

Definition 5.24 (Lück)**.** Given M a finitely generated module over $\mathcal{N}\Gamma$, define the torsion of M

$$T(M) = \{m \in M\colon f(m) = 0 \text{ for all linear } f\colon M \to \mathcal{N}\Gamma\}.$$

The usual definition of the torsion submodule fails for non-commutative rings. Now, $\lambda^{(2)}_X$ is viewed as a pairing on $H_{2k}(X;\mathcal{N}\Gamma)/T(H_{2k}(X;\mathcal{N}\Gamma))$.

Theorem 5.25 (Lück)**.** M/Torsion is projective. (Like f.g. abelian groups.) That is, any M is of the form $M/T(M) \oplus T(M)$.

Theorem 5.26 (L^2-signature theorem (Atiyah))**.** If Y^{4k} is a closed manifold then for any representation ρ:

$$\sigma^{(2)}_\Gamma(Y,\rho) = \sigma(Y)$$

Actually, this is true for any elliptic operator. e.g. lift a twisted Dirac operator to the Γ-cover, compute the index via $\dim_{\mathcal{N}\Gamma}$. The above theorem is also true for Poincaré complexes.

Lemma 5.27. If $\pi_1(X_i^{4k}) \xrightarrow{\rho_i} \Gamma$ are given (additively) such that $\partial X_1 \cong_\phi \partial X_2$ and

$$\begin{tikzcd}
\pi_1(\partial X_1) \arrow[r, "\simeq"] \arrow[d, "\rho_1"] & \pi_1(\partial X_2) \arrow[dl, "\rho_2"'] \\
\Gamma &
\end{tikzcd}$$

then

$$\sigma^{(2)}_\Gamma(X_1,\rho_1) + \sigma^{(2)}_\Gamma(X_2,\rho_2) = \sigma^{(2)}_\Gamma(X_1 \cup X_2, \rho_1 \cup \rho_2).$$

Proof. The proof is exactly the same as the usual proof of Novikov Signature additivity: decompose the module as a projective and torsion part, ignore the torsion and follow through the definitions. $\square$

Corollary 5.28. The *reduced L^2-signature* $\widetilde{\sigma}^{(2)}_\Gamma(X,\rho) = \sigma^{(2)}_\Gamma - \sigma(X)$ is an invariant of the boundary ∂X and the representation $\rho|_{\partial X}$.

Remark 5.29. Given (M^{4k-1}, g, ρ), where g is a metric we can define the η invariant, which is independent of the choice of metric:

$$\widetilde{\eta}^{(2)}_\Gamma(M,g,\rho) = \eta^{(2)}_\Gamma(M,g,\rho) - \eta(M,g)$$

(Theorem of Gromov and Cheeger). By Atiyah-Patodi-Singer,

$$\widetilde{\eta}^{(2)}_\Gamma(M,\rho) = \widetilde{\sigma}^{(2)}_\Gamma(X,\rho)$$

if $M = \partial X$.

Thus, the reduced L^2 signature is *defined* when there does not exist a null-bordism.

The η-invariant is obtained using the eigenvalues of the σ-operator.

Definition 5.30. Given a knot $K \subseteq S^3$, and a representation $\rho\colon \pi_1(M_K) \to \Gamma$, define $\widetilde{\sigma}^{(2)}_\Gamma(K,\rho) = \widetilde{\eta}^{(2)}_\Gamma(M_K,\rho)$.

Example 5.31. $\Gamma = \mathbb{Z}$, $\rho =$ abelianisation $\pi_1 \to H_1 \cong \mathbb{Z}$. Then we may as well choose an expedient 4-manifold with which to make our calculations:

$$\widetilde{\sigma}^{(2)}_\Gamma(K,\rho) = \sigma^{(2)}_\Gamma(W_0,\rho)$$

where W_0 is the nice 4-manifold we constructed earlier with $\pi_1(W) \cong \mathbb{Z}$ which is bounded by M_K. This is just $\widetilde{\sigma}^{(2)}_\Gamma(\lambda^{(2)}_W) = \widetilde{\sigma}^{(2)}_\mathbb{Z}(\lambda^{(2)}_W)$ which by Corollary 5.17 is

$$\int_{S^1} \sigma_\mathbb{Z}(K)\, dz.$$

6 Casson-Gordon Invariants and the Cochran-Orr-Teichner Filtration

To summarise the previous section, we defined the reduced L^2-signature $\sigma^{(2)}_\Gamma(K,\rho) \in \mathbb{R}$, given $\rho\colon \pi_1(S^3_K) \to \Gamma$, such that, if $\Gamma = \mathbb{Z}$ and ρ is the abelianisation, we have that

$$\widetilde{\sigma}^{(2)}_\mathbb{Z}(K,\rho) = \int_{S^1} \sigma_z(K)\, dz.$$

What we really want are more interesting choices of Γ, ρ, which are such that these are concordance invariants. We already know that for $\Gamma = \mathbb{Z}$ we can detect the non-algebraically slice knots. Recall that we are aiming to reproduce the famous result of Casson-Gordon that the twist knots are not slice, using the L^2 signatures to detect the intersection forms on the 4 manifold. We will choose a 4-manifold with a more complicated fundamental group, and correspondingly more complicated representations; these will be related to quotients $\pi/\pi^{(n)}$ of π by its derived subgroups.

Definition 6.1. Define

$$\Gamma = \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z}$$

where $\mathbb{Z} = \langle t \rangle$; define $A := \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$, so $\mathbb{Z}$ acts on A via right multiplication by t. Γ is a metabelian group. A is an infinite dimensional $\mathbb{Q}$-vector space. The construction of A is analogous to the situation where $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, and the $\mathbb{Q}/\mathbb{Z}$ is formed to be the value ring of linking forms. $\mathbb{Q}/\mathbb{Z}$ was for torsion abelian groups, whereas A is for torsion $\mathbb{Q}[t, t^{-1}]$ modules. There are no finite dimensional representations of Γ, except for 1-dimensional ones which factor through $\mathbb{Z}$, so we are going to have to use $l^2\Gamma$ to make our interesting representations.

Now, define a representation variety

$$\mathrm{Rep}^*(\pi, \Gamma) := \{\rho \colon \pi \to \Gamma \mid \rho(\text{meridian}) = t\}.$$

Note that since Γ is metabelian they will all factor through $\pi/\pi^{(2)}$, where the (2) here denotes the 2nd derived subgroup. (In fact, $\Gamma^{(1)} = [\Gamma, \Gamma] = A$.) Further, we know that:

$$\frac{\pi}{\pi^{(2)}} = \frac{\pi^{(1)}}{\pi^{(2)}} \rtimes \frac{\pi}{\pi^{(1)}},$$

since the sequence

$$0 \to \frac{\pi^{(1)}}{\pi^{(2)}} \to \frac{\pi}{\pi^{(2)}} \to \frac{\pi}{\pi^{(1)}} \cong \mathbb{Z} \to 0$$

is exact and splits since $\mathbb{Z}$ is free. Therefore, Rep^* is given by $\overline{\rho} \rtimes \mathrm{id} \colon \frac{\pi^{(1)}}{\pi^{(2)}} \rtimes \mathbb{Z} \to A \rtimes \mathbb{Z}$, where $\overline{\rho}$ must be a homomorphism of abelian groups $\frac{\pi^{(1)}}{\pi^{(2)}} \to A$ commuting with the $\mathbb{Z}$-action.

Definition 6.2.

$$A_K = \frac{\pi^{(1)}}{\pi^{(2)}} \otimes \mathbb{Q}$$

the rational Alexander module of K.

Thus, $\mathrm{Rep}^*(\pi, \Gamma) = \mathrm{Hom}_{\mathbb{Q}[t,t^{-1}]}(A_K, A)$ is a finite dimensional $\mathbb{Q}$-vector space. (We could work with integers here, but it is easier not to.) The dimension is finite because A_K is a finitely generated torsion module over $\mathbb{Q}[t, t^{-1}]$, as we will prove shortly. It should be reasonably clear, however, from the fact that the Alexander polynomial is always monic over $\mathbb{Q}$, and this represents the zero element in A_K. Therefore any high powers of t can always be replaced with lower ones using the relation imposed on A_K by the Alexander polynomial.

Theorem 6.3 (Cochran-Orr-Teichner)**.** If K is slice then there is a vector subspace $R \subseteq \mathrm{Rep}^*(\pi, \Gamma)$ of half dimension such that $\sigma^{(2)}_{\Gamma}(K, \rho) = 0$ for all $\rho \in R$. In fact, R comes from a Lagrangian of the Blanchfield form.

Proof. We denote by M the 0-framed surgery along $K \subseteq S^3$, and by $\overline{M}$ its infinite cyclic cover. We also abbreviate $\pi = \pi_1(M)$. Then $\pi_1(\overline{M}) = \pi^{(1)}$, and $H_1(\overline{M}) = \frac{\pi^{(1)}}{\pi^{(2)}}$. By definition $A_K = \frac{\pi^{(1)}}{\pi^{(2)}} \otimes \mathbb{Q} \cong H_1(\overline{M}; \mathbb{Q})$. To prove that it is a f.g. torsion module over $\mathbb{Q}[t, t^{-1}]$, we use the Gysin sequence, with $\mathbb{Q}$ coefficients. (As always from now on, we would like to not have to use $\mathbb{Q}$ but it makes life a lot easier.) The sequence which arises, where we use rational coefficients, is:

$$H_1(\overline{M}) \xrightarrow{1-t} H_1(\overline{M}) \to H_1(M) \xrightarrow{b} H_0(\overline{M}) \xrightarrow{1-t} H_0(\overline{M})$$

$\overline{M}$ is connected so $H_0(\overline{M}) \cong \mathbb{Q}$, and the map $(1-t)$ here is the zero map. Also $H_1(M) \cong \mathbb{Q}$, and the map labelled b is an isomorphism, since any surjective homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$ is an isomorphism. Therefore, by exactness of the sequence, the map $(1-t)\colon H_1(\overline{M}) \to H_1(\overline{M})$ is a surjection. This cannot be the case if there is a free part of $H_1(\overline{M})$ over $\mathbb{Q}[t, t^{-1}]$, since $1-t$ is not invertible. Therefore it is indeed a torsion module. It is finitely generated because 1-chains are f.g. free, $\mathbb{Q}[t, t^{-1}]$ is a Principal Ideal Domain, and so is Noetherian, so a sub-quotient is finitely generated.

Note that the same argument would apply to show that for any CW complex with $H_1(\cdot; \mathbb{Q}) \cong \mathbb{Q}$, the 1st homology of the infinite cyclic cover with $\mathbb{Q}$ coefficients is a f.g. torsion $\mathbb{Q}[t, t^{-1}]$ module.

We now recall the algebraic approach to the torsion linking form on the homology $A_K = H_1(\overline{M}; \mathbb{Q}) = H_1(M; \mathbb{Q}[t, t^{-1}])$ of the infinite cyclic cover: the Blanchfield form.

First, apply Poincaré duality with twisted coefficients to get $A_K \cong H^2(M; \mathbb{Q}[t, t^{-1}])$. From the coefficient sequence

$$0 \to \mathbb{Q}[t, t^{-1}] \to \mathbb{Q}(t) \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]} \to 0$$

we get the Bockstein homomorphism, which fits into the long exact sequence which arises from this short exact sequence:

$$H^1(M; \mathbb{Q}(t)) \to H^1(M; \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]}) \to H^2(M; \mathbb{Q}[t, t^{-1}]) \to H^2(M; \mathbb{Q}(t)).$$

Now, $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain, so its quotient field $\mathbb{Q}(t))$ is flat over $\mathbb{Q}[t, t^{-1}]$. Therefore, $H_1(M; \mathbb{Q}(t)) \cong H_1(M; \mathbb{Q}[t, t^{-1}]) \otimes \mathbb{Q}(t)$ by the universal coefficient theorem for homology. Since $\mathbb{Q}(t)$ is a field, $H_1(M; \mathbb{Q}(t)) \cong H^1(M; \mathbb{Q}(t))$. With Poincaré duality, and the fact that A_K is $\mathbb{Q}[t, t^{-1}]$-torsion, we see that both of the end groups displayed in the previous sequence are zero, meaning that $A_K \cong H^1(M; \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]})$. Another application of a universal coefficient theorem yields:

$$A_K \cong A_K^{\wedge} := \mathrm{Hom}_{\mathbb{Q}[t, t^{-1}]}(H_1(\overline{M}; \mathbb{Q}); \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]})$$

which isomorphism is the adjoint of the Blanchfield form:

$$Bl\colon A_K \times A_K \to A$$

a unimodular hermitian linking form. ($A_K^\wedge$ is sometimes called the Pontrjagin dual.) Geometrically, do the usual linking form construction, multiplying an element of $H_1(\overline{M};\mathbb{Q})$ by polynomials until it is zero, choosing a surface whose boundary is this chain, intersecting this with the other element, and then dividing out by whatever we needed to make the 1st 1-cycle bound. Geometrically, it is then clear that this is a Hermitian form. The theorem constructs a Lagrangian

$$R \leq \mathrm{Rep}^*(\pi, \Gamma) = \mathrm{Hom}_{\mathbb{Q}[t,t^{-1}]}(A_K, A) = A_K^\wedge \cong A_K$$

where the last isomorphism is by the Blanchfield isomorphism as above.

We now show that there exists a Lagrangian R. Let W be B^4 minus a slice disk for K, so that $\partial W = S^3_K = M$, the zero surgery on S^3 along K, and $H_1(W) \cong H_1(M)$ and $H_2(W) = 0$. (So homologically slice will be enough in this case to prove the theorem, i.e. a knot which is slice in a homology 4-ball.) Consider the following sequence of homology of covering spaces:

$$\to H_2(\overline{W}) \hookrightarrow H_2(\overline{W}, \overline{M}) \to H_1(\overline{M}) \to H_1(\overline{W}) \twoheadrightarrow H_1(\overline{W}, \overline{M}) \xrightarrow{0} H_0(\overline{M}) \xrightarrow{\cong} H_0(\overline{W})$$

Define $R := \mathrm{im}(H_2(\overline{W}, \overline{M})) \to H_1(\overline{M}) = \ker(H_1(\overline{M}) \to H_1(\overline{W}))$ Then we claim:

(1.)

$$\dim_{\mathbb{Q}}(R) = \frac{1}{2} \dim_{\mathbb{Q}} H_1(\overline{M})$$

(2.) If $\rho\colon \pi \to \Gamma$ comes from R then it extends to $\widetilde{\rho}\colon \pi_1(W) \to \Gamma$.

The point here is that we can calculate the L^2 signature of (M, ρ) via the 4-manifold with its extended representation ρ: $\widetilde{\sigma}^{(2)}_\Gamma(K, \rho) = \sigma^{(2)}_\Gamma(W, \widetilde{\rho}) - \sigma_1(W)$. The final step will then be to show that the Betti numbers vanish: $b_2^{(2)}(W, \widetilde{\rho}) = 0 \Rightarrow \sigma^{(2)}_\Gamma(W, \widetilde{\rho}) = 0$. This is Lemma 6.5 below.

To see (1.), we need to look at linking forms on W. They are constructed just like the linking form above:

$$H_i(\overline{W}) = H_i(W; \mathbb{Q}[t^{\pm 1}]) \cong H^{4-i}(W, M; \mathbb{Q}[t^{\pm 1}]) \cong H^{3-i}(W, M; \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]})$$

$$\cong \mathrm{Hom}_{\mathbb{Q}[t^{\pm 1}]}(H_{3-i}(W, M; \mathbb{Q}[t^{\pm 1}]), \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]})$$

Again this uses Poincaré Duality, the Bockstein homomorphism, which is again an isomorphism since all are torsion modules again. The universal coefficient isomorphism follows again since there are no higher Ext groups; we therefore have isomorphisms $H_i(\overline{W}) \cong H_{3-i}(\overline{W}, \overline{M})^\wedge$. This yields the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_2(\overline{W}) \arrow[r] \arrow[d, "\cong"] & H_2(\overline{W}, \overline{M}) \arrow[r, "\partial_*"] \arrow[d, "\cong"] & H_1(\overline{M}) \arrow[r, "i_*"] \arrow[d, "\cong"] & H_1(\overline{W}) \arrow[r] \arrow[d, "\cong"] & H_1(\overline{W}, \overline{M}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_1(\overline{W}, \overline{M})^\wedge \arrow[r] & H_1(\overline{W})^\wedge \arrow[r, "i^\wedge"'] & H_1(\overline{M})^\wedge \arrow[r, "\partial^\wedge"'] & H_2(\overline{W}, \overline{M})^\wedge \arrow[r] & H_2(\overline{W})^\wedge \arrow[r] & 0
\end{tikzcd}$$

Noting that $\dim_{\mathbb{Q}} N^\wedge = \dim_{\mathbb{Q}} N$, we see that the dimensions on the top and bottom sequences agree, so the standard argument for Lagrangian dimensions applies.

To see the extension of the representation, $x \in R$ maps to a function $Bl(x, \bullet)$, which comes from an element of $y \in H_1(\overline{W})^\wedge$ which we can use to extend ρ:

$$\begin{tikzcd} \frac{\pi_1(W)}{\pi_1(W)^{(2)}} = H_1(\overline{W}) \rtimes \mathbb{Z} \arrow[d, "y \rtimes \mathrm{id}"'] & H_1(\overline{M}) \rtimes \mathbb{Z} \arrow[l] \arrow[dl, "\rho"'] \\ \Gamma = A \rtimes \mathbb{Z} & \end{tikzcd}$$

R is a Lagrangian of the Blanchfield form. To see this, take $x, x' \in R$. Now, we can lift x to an element $w \in H_2(\overline{W}, \overline{M})$ such that linking with x, which is an element $Bl(x, \bullet) \in H_1(\overline{M})^\wedge$ coincides with linking with w, which is an element $y \in H_1(\overline{W})^\wedge$. We can think of $x' \in H_1(\overline{M})$ as an element of $H_1(\overline{W})$ under the inclusion induced map i_*. We have $Bl(x, \bullet) = Bl(\partial_*(w), \bullet) = i^\wedge(Bl_W(w, \bullet)$ by commutativity of the diagram above. Then by definition of $i^\wedge$, we get that $Bl(x, \bullet) = Bl_W(w, \bullet) \circ i_*$, so that $Bl(x, x') = Bl_W(w, i_*(x'))$. However, x' also belongs to R, so $i_*(x') = 0 \in H_1(\overline{W})$ which implies $Bl(x, x') = 0$ by linearity. The proof is then finished modulo Lemma 6.5. $\square$

Remark 6.4. The basic idea is that the infinite cyclic cover of M behaves like a surface, and the slice disk complement W behaves like a null-cobordism of the surface. This is true for fibred knots, while for other knots it is true homologically.

Lemma 6.5. Let Γ be a Poly-Torsion-Free-Abelian (PTFA) group. This means a group which can be constructed with a finite number of iterated extensions by torsion-free abelian groups. If $H_1(W^4; \mathbb{Q}) \cong H_1(\partial W; \mathbb{Q}) \cong \mathbb{Q}$ and $\rho \colon \pi_1(W) \to \Gamma$ then $b_\Gamma^{(2)} = b_2(W)$.

Remark 6.6. If $\pi = \pi_1(S^3 \setminus K)$ is a knot group, then $\pi/\pi^{(n)}$ is PTFA; Strebel showed that the quotients are torsion free. Also, if Γ is torsion free, then $\mathbb{Z}\Gamma$ has an Ore skew-field of fractions. $\mathcal{K}\Gamma$: it has no zero divisors and all pairs as^{-1} can be reversed to pairs of the form $c^{-1}d$. $\mathbb{Z}\Gamma$ injects into $\mathcal{K}\Gamma$. An example of a sequence of PTFA groups is as follows. These groups are universal with respect to maps of knot groups into solvable groups.

$$\Gamma_0 = \mathbb{Z} \leftarrow \Gamma_1 = \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z} = \frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]} \rtimes \Gamma_0 \leftarrow \Gamma_2 = \frac{\mathcal{K}\Gamma_1}{\mathbb{Q}[\Gamma_1]} \rtimes \Gamma_1 ?$$

Note that A is torsion-free abelian. The final group is a first attempt at an extension by a value ring for a secondary Blanchfield pairing. It is not quite right yet however. Note that we used above the fact that $\mathbb{Q}[\Gamma_0]$ is a PID. This is not true for $\mathbb{Q}[\Gamma_1]$, so we make a small alteration: we include $\mathbb{Q}[\Gamma_1]$, which is actually a twisted polynomial ring, into a larger ring where the coefficients of the original polynomial ring are inverted.

$$\mathbb{Q}\Gamma_1 = \mathbb{Q}\left[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]}\right]_\alpha [t^{\pm 1}] \subseteq \mathcal{K}\left(\mathbb{Q}\left[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]}\right]\right)_\alpha [t^{\pm 1}]$$

We have here twisted polynomial rings: α is a choice of automorphism of the ring of which it is a subscript, which occurs when commuting coefficients and the indeterminate t. Due to this choice, the identifications here with twisted polynomial rings are non-canonical. We now have a Principal Ideal Domain. Γ_2 is therefore given by:

$$\frac{\mathcal{K}\Gamma_1}{\mathcal{K}[\Gamma_1^{(1)}]_\alpha[t^{\pm 1}]} \rtimes \Gamma_1$$

Note that the denominator here is the same as above i.e. $\Gamma_1^{(1)} = [\Gamma_1, \Gamma_1]$, the commutator subgroup of Γ_1, is isomorphic to $\mathbb{Q}[\frac{\mathcal{K}\Gamma_0}{\mathbb{Q}[\Gamma_0]}]$.

We don't really want to have to do this, but having a Principal Ideal Domain at certain points turns out to be very useful, particularly when defining non-singular higher order Blanchfield pairings, but also in order to sidestep some slice-ribbon difficulties to do with $R \subseteq R^\perp$ versus $R = R^\perp$, by making the functor $\bullet^\wedge$ right exact. Iterating the process, we define:

$$\Gamma_{n+1} = \frac{\mathcal{K}\Gamma_n}{\mathcal{K}[\Gamma_n^{(1)}][t^{\pm 1}]} \rtimes \Gamma_n$$

the "universal rationally solvable groups" based on $\mathbb{Z} = \Gamma_0$, corresponding to H_1 of a knot complement (we could try taking $\Gamma_0 = \mathbb{Z}^n$ to deal with link complements).

Theorem 6.7 (Main Theorem of Cochran-Orr-Teichner)**.** If K is slice then it is h-solvable for all $h \in \frac{1}{2}\mathbb{N}$.

Definition 6.8. A knot is (0)-solvable if and only if its Arf invariant is zero. This implies the existence of a well-defined obstruction to being slice:

$$B_0 \in L^0(\mathcal{K}\Gamma_0)/L^0(\mathbb{Z}\Gamma_0)$$

which can be detected using L^2 signatures (different use of the letter L) and therefore corresponds to integrating the signature $\sigma_z(K)$ over the circle $z \in S^1$. This is the usual obstruction to a knot being algebraically slice: the Blanchfield or Seifert form. This gives the normal twisted signature: B_0 is essentially of the form $(1-z)S + (1-z^{-1})S^T$, a hermitian matrix over the group ring $\mathbb{Z}\Gamma_0$, which becomes invertible upon localisation, that is, when we consider it as a matrix over the quotient field $\mathcal{K}\Gamma_0$. This then gives us an element of L-theory: a symmetric non-degenerate form. Quotienting by $L^0(\mathbb{Z}\Gamma_0)$ corresponds to a lack of signature at $z = 0$.

A knot is (0.5)-solvable if and only if K is algebraically slice $\Rightarrow$ $B_0 = 0$. (the actual definition of h-solvable is in terms of gropes or in terms of intersection forms on the middle dimensional homology of derived covers of a 4-manifold whose boundary is M.

A knot is (1)-solvable implies there exists a Lagrangian P_0 inside B_0 such that for all $f_0 \in P_0$ there is a well defined obstruction

$$B_1 \in \frac{L^0(\mathcal{K}\Gamma_1)}{L^0(\mathbb{Z}\Gamma_1)} \xrightarrow{\tilde{\sigma}_\Gamma^{(2)}} \mathbb{R}$$

That is, there is a representation extending to a 4-manifold which defines an intersection form. This doesn't depend on the choice of extensions. If a knot is (1.5)-solvable, then the Casson-Gordon invariants vanish. (1.5) solvable implies that there exists $P_0 \in B_0$ such that $B_1(p_0) = 0$ for all $p_0 \in P_0$. This implies there is a Lagrangian $P_1 \in B_1(p_0)$.

In general, a knot being (n)-solvable implies that there exists P_0 such that for all $p_0 \in P_0$, there exists $P_1(p_0) \in B_1(p_0)$ such that for all $p_1 \in P_1$, there exists $P_2(p_0, p_1) \in B_2(p_0, p_1)$ such that for all $p_2 \in P_2$, ... there exists a well-defined obstruction $B_n(p_0, .., p_{n-1})$ in $L^0(\mathcal{K}\Gamma_n)/L^0(\mathbb{Z}\Gamma_n) \xrightarrow{\widetilde{\sigma}^{(2)}_{\Gamma_n}} \mathbb{R}$.

$(n.5)$-solvable means that the knot is (n)-solvable and $B_n = 0$.

Theorem 6.9 (Cochran-Orr-Teichner, Cochran-Harvey-Leidy)**.** There exist knots which are (n)-solvable but not $(n.5)$-solvable for all $n \in \mathbb{N}$.

Remark 6.10. Before embarking on the proof of Lemma 6.5, we make some remarks. We will actually prove that $\dim_{\mathcal{K}\Gamma}(H_2(W; \mathcal{K}\Gamma)) = b_2(W))$ in a homological algebra sense rather than in the L^2 way. Why is this?

Recall that Γ is PTFA: we can do a completion $\mathbb{Q}\Gamma \subseteq \mathcal{N}\Gamma$. Any Von Neumann algebra satisfies the Ore condition, with no zero-divisors, so has an embedding $\mathcal{N}\Gamma \subseteq U\Gamma$. This last is not a skew field, but any finitely presented module over it is projective, and quite nice. For example, if $\Gamma = \mathbb{Z}$ then $\mathcal{N}\Gamma = L^\infty(S^1; \mathbb{C})$ is all bounded functions and $U\Gamma$ is all measurable functions (e.g. $1/x$ for $x \neq 0$ is now allowed). We can then do a division closure of $\mathbb{Q}\Gamma \subseteq \mathcal{K}\Gamma$ by closing in $U\Gamma$. This closure actually is the Ore localisation if that exists. Now, the Ore localisation is *flat*, so that:

$$\dim_\Gamma(H_2(W; \mathcal{N}\Gamma)) = \dim_\Gamma(H_2(W; \mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} U\Gamma).$$

We can extend the dimension function to $U\Gamma$ but not the trace function. However, $\dim = \mathrm{tr}_\Gamma \colon K_0(\mathcal{N}\Gamma) \to \mathbb{R}$ does extend to $K_0(U\Gamma) \to \mathbb{R}$ (K_0 denotes the projective class group of stable isomorphism classes of modules over a certain ring) because any projection $p \colon (U\Gamma)^n \twoheadrightarrow Q$ satisfies $p^2 = p \cdot p^*$ which makes it bounded because of normal algebra.

By flatness, $\dim_\Gamma(H_2(W; \mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} U\Gamma) = \dim_\Gamma(H_2(W; U\Gamma))$. Also, $H_2(W; \mathcal{K}\Gamma) \otimes_{\mathcal{K}\Gamma} U\Gamma = H_2(W; U\Gamma)$. The dimension is an integer over a skew-field. (This shows the Atiyah conjecture for PTFA groups: $\mathcal{K}\Gamma$ is the Ore localisation, and this shows that L^2 Betti numbers are integers.) Thus $H_2(W; \mathcal{K}\Gamma)$ is free (over a skew field), which implies that so is $H_2(W; U\Gamma)$, and so $\dim_\Gamma(H_2(W; U\Gamma)) \in \mathbb{Z}$.

Proof of Lemma 6.5. We aim to prove that $b_2^\Gamma(W) := \dim_{\mathcal{K}\Gamma}(H_2(W; \mathcal{K}\Gamma)) = b_2(W)$. We claim that $b_0^\Gamma = b_1^\Gamma = b_3^\Gamma = 0$ for W. Then this implies the lemma since the Euler characteristics agree $\chi^\Gamma = \chi$. Firstly, note that with a skew-field the universal coefficient spectral sequence collapses and we have a universal coefficients isomorphism:

$$H^i(X; \mathcal{K}\Gamma) \cong \mathrm{Hom}_{\mathcal{K}\Gamma}(H_i(X; \mathcal{K}\Gamma), \mathcal{K}\Gamma)$$

Poincaré duality therefore means that it is enough to show that $b_0^\Gamma = b_1^\Gamma = 0$.

First,

$$H_0(W;\mathcal{K}\Gamma) = \frac{\mathcal{K}\Gamma}{\{(1-g)k \mid g \in \Gamma, k \in \mathcal{K}\Gamma\}}$$

the set of co-invariants for Γ-actions on $\mathcal{K}\Gamma$, as usual for a connected complex. For $g \neq 1$, $1-g$ is invertible so this is zero. Now, in order to show the remaining fact, which is that $b_1^\Gamma = 0$, we need another Lemma. $\square$

Lemma 6.11. Let X be a connected finite CW complex, ρ a representation $\pi_1(X) \to \Gamma$, which is non-trivial, to a PTFA group. If $b_1^{\mathbb{Q}}(X) = 1$ then $b_1^{\mathcal{K}\Gamma}(X) = 0$. (Recall that for a PTFA group $\mathcal{K}\Gamma$ exists and $\mathbb{Z}\Gamma \subseteq \mathcal{K}\Gamma$.)

Proof. Let $f\colon S^1 \to X$ be a map which induces an isomorphism on $H_1(X;\mathbb{Q})$. Then $H_i(X, S^1;\mathbb{Q}) = 0$ for $i = 0, 1$. Let $\widehat{X} \xrightarrow{\Gamma} X$ be the covering space with deck group Γ, and form the pull back covering: $\widehat{S^1} \to S^1$. Then look at the chain complex $C_*(\widehat{X}, \widehat{S^1};\mathbb{Q})$. It is a free $\mathbb{Q}\Gamma$ chain complex such that $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}) = 0$ for $i = 0, 1$, because the latter is just $H_i(X, S^1;\mathbb{Q})$.

We claim that this implies that $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathcal{K}\Gamma) = 0$ for $i = 0, 1$. From this claim, the lemma follows because then the relative homology groups being zero imply that the absolute groups $H_i(X;\mathcal{K}\Gamma) = H_i(\widehat{X};\mathbb{Q}) = H_i(\widehat{S^1};\mathbb{Q}) = H_i(S^1;\mathcal{K}\Gamma)$, and the chain complex of S^1 is given by $\mathbb{Q}\Gamma \xrightarrow{z-1} \mathbb{Q}\Gamma$ where z is the image of $1 \in \pi_1(S^1)$ under the map $\pi_1(S^1) \xrightarrow{f_*} \pi_1(X) \xrightarrow{\rho} \Gamma$. Now, $z-1$ is invertible in $\mathcal{K}\Gamma$ and so the complex is contractible and so has $H_i(S^1;\mathcal{K}\Gamma) = 0$ for $i = 0, 1$, proving the lemma.

Now, to see the claim we shall use the following theorem of Strebel:

Theorem 6.12. If Γ is a PTFA group and $f\colon F \to F'$ is a homomorphism of free $\mathbb{Q}\Gamma$-modules such that $f \otimes \mathbb{Q}$ is injective, then f is injective.

Example 6.13. $\Gamma = \mathbb{Z}$, $\mathbb{Q}[t,t^{-1}] \xrightarrow{f} \mathbb{Q}[t,t^{-1}]$; if $f(1) \neq 0$ in $\mathbb{Q}$ i.e. $f \otimes \mathbb{Q} \neq 0$, then multiplication by f is injective. But $\mathbb{Q}[t,t^{-1}]^n \xrightarrow{f} \mathbb{Q}[t,t^{-1}]$; just need to show that $\det \neq 0$ but this is the case because $\det(f(1)) \neq 0$. In the general case with a non-commutative Γ, we must iterate this kind of argument.

The consequence of this theorem for our purposes is that, if the rank of F and F' are equal, then $f \otimes \mathbb{Q}$ is an isomorphism, and therefore so is $f \otimes \mathcal{K}\Gamma$. This is because $\mathcal{K}\Gamma$ is *flat* over $\mathbb{Q}\Gamma$, so the tensor product preserves injectivity of maps. With $\mathbb{Q}$ or $\mathcal{K}\Gamma$ coefficients we then have vector spaces of the same dimension, so injective maps are isomorphisms. (This is the same as saying that $\mathbb{Q}\Gamma$ embeds in its Cohn localisation, which is the universal localisation with respect to the diagram

$$\begin{tikzcd}
\mathbb{Q}\Gamma \arrow[r] \arrow[d] & \mathcal{K}\Gamma \\
\mathrm{Cohn}(\mathbb{Q}\Gamma) \arrow[ur] &
\end{tikzcd}$$

where both the maps originating from $\mathbb{Q}\Gamma$ are injections.

Continuing with the proof of the claim, let C_* be a free chain complex over $\mathbb{Q}\Gamma$ such that $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}) = 0$ for $i = 0, .., n$ ($n = 1$ in our case). Since each of the modules in $C_* \otimes \mathbb{Q}$ are f.g. projective (in fact free), there are partial chain homotopies $h \colon C_i \otimes \mathbb{Q} \to C_{i+1} \otimes \mathbb{Q}$ which are chain contractions for $C_* \otimes \mathbb{Q}$ up to dimension n, that is as far as the complex is acyclic. Since C_* is free, these partial homotopies can be lifted. Furthermore $\partial h + h\partial$ is the identity up to dimension n, and so the lifted homotopies are also isomorphisms, by the consequence of Strebel's result above. This means that $C_* \otimes \mathcal{K}\Gamma$ is also chain contractible up to dimension n, which proves the claim as required and finishes the proof. $\square$

Example 6.14. Now let's give some example calculations and an application. Recall the twist knots K_n, with Seifert matrices:

$$\begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

Recall that they are algebraically slice if and only if $4n + 1$ is a square which is if and only if $n = m(m + 1) \geq 0$. Also K_0, K_2 are actually slice. Let s, l be the curves on the Seifert surface which go round each of the generators of $H_1(F)$. The curves $\gamma_1 = -m.s + l$, $\gamma_2 = (m + 1).s + l$ are the only primitive curves on F with $\mathrm{lk}(\gamma_i^+, \gamma_i) = 0$. See Kauffman p.223 "On Knots" where these curves are seen to be torus knots $T(m, m + 1)$. Thus $m = 0, 1$ cases are unknots and so K_0, K_2 are slice. The other cases are not unknots: we use this to show that the knots are not actually slice (though they are algebraically slice).

Theorem 6.15. The knots K_n, $n = m(m + 1), m \geq 2$ are $\mathbb{Z}$-independent in the group $\mathcal{C}_1$.

Theorem 6.16 (Cochran-Orr-Teichner, but see also Casson-Gordon and Gilmer)**.** Let K be an algebraically slice genus one knot with non-trivial Alexander polynomial. If K is slice then $\sigma^{(2)}_{\mathbb{Z}}(\gamma_i) = 0$ for one of the two curves γ_i on a Seifert surface which have $\mathrm{lk}(\gamma_i^+, \gamma_i) = 0$

Remark 6.17. The second theorem implies that the knots of the first are not slice by checking that the signatures $\sigma^{(2)}_{\mathbb{Z}}(T(m, m + 1)) \neq 0$ for the torus knots $T(m, m + 1)$. There is a bit more work to show independence.

Proof of COT/CG. Let F be a genus 1 Seifert surface for K. The curves γ_i in $H_1(F)$ are Lagrangians of the Seifert form. They don't generate $H_1(F)$ over $\mathbb{Z}$ though. Recall that:

$$H_1(F) \otimes \mathbb{Z}[t, t^{-1}] \xrightarrow{\lambda} H_1(F)^* \otimes \mathbb{Z}[t, t^{-1}] \twoheadrightarrow \mathcal{A} \to 0$$

where $\mathcal{A} = H_1(M_K; \mathbb{Z}[t, t^{-1}])$ is the Alexander module of K, and $\lambda = (1 - t)S + (1 - t^{-1})S^T$ is a hermitian restriction of the Blanchfield form. (i.e. $Bl(a, b) = \lambda^{-1}(\overline{a})(\overline{b})$ where λ is invertible over $\mathbb{Q}(t)$ and we take the result modulo $\mathbb{Q}[t, t^{-1}]$.) So a Lagrangian in S gives a Lagrangian of the Blanchfield form; and vice versa actually: $(\gamma_1, \gamma_2) \leftrightarrow (l_1, l_2) \subseteq L_1 \times L_2$. Define, for each $i = 1, 2$:

$$\rho(l_i) \colon \pi_1(M_K) = \pi \twoheadrightarrow \frac{\pi}{\pi^{(2)}} = A_K \rtimes \mathbb{Z} \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]} \rtimes \mathbb{Z} = \Gamma_1$$

$$(x, n) \mapsto (Bl(l_i, x), n)$$

Remember that $A_K \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$, and $0 \neq l_i \in L_i \otimes \mathbb{Q} = \mathbb{Q}$ We have a lemma:

Lemma 6.18.

$$\widetilde{\sigma}^{(2)}_{\Gamma_1}(K, \rho(l_i)) = \sigma^{(2)}_{\mathbb{Z}}(\gamma_i)$$

These signatures are independent of the choice of $l_i \in L_i$.

In this setting, the main theorem, here the fact that K is (1.5)-solvable, now says that K being slice implies that there exists a Lagrangian $L \subseteq Bl_0$ such that for all $l \in L$, $\widetilde{\sigma}^{(2)}_{\Gamma_1}(K, \rho(l)) = 0$. Here there are just two Lagrangians, so K slice implies that $\sigma^{(2)}_{\mathbb{Z}}(\gamma_i) = 0$ for one of the γ_i. This can be calculated not to be the case for the torus knots $T(m, m+1)$, obstructing sliceness of the twist knots as claimed. The proof just requires the proof of Lemma 6.18. □

Proof of Lemma 6.18. Recall the picture of W_0, with boundary M_K in Figure 7. We define a new 4-manifold W with the same boundary: Let K be our knot and let k be the (primitive, so it can be embedded) curve on the Seifert surface F which has self linking number zero. Since it is primitive, which means that it is not a multiple in $H_1(F; \mathbb{Z})$, we can think of it as being one of the bands in the Seifert surface - it may well be necessary to make an isotopy of the knot and the Seifert surface in order for this to be clear. We call W_0 from Theorem 4.11 with respect to a knot J the manifold $W(J)$. We define W by drawing $W(K)$, and then letting the curve for k, one of the "a" curves from our Kirby diagram Figure 7 for W_0, be the 1-handle for the drawing of $W(k)$. See Figure 10. The construction is reminiscent of gropes: we are trying to test whether the secondary knot k is slice. To show that this has the correct boundary, first go through the Kirby moves for the secondary Seifert surface handles for k, to give the original k to attach a 2-handle to. Now, let l be the linking of k and a dual band γ, and let B be the twisting of γ. Then the Seifert matrix of K is:

$$S = \begin{pmatrix} 0 & l+1 \\ l & B \end{pmatrix}$$

The Alexander polynomial of K is then $\pm((l+1) - tl)(l - t(l+1))$. We now compute $\pi_1(W)$. It has generators which we call M, m, the meridians of K, k respectively. ($\langle M \rangle$ will be the $\mathbb{Z}$ in the representation to $\Gamma_1 = \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}] \rtimes \mathbb{Z}$ now.)

$$\pi_1(W) = \langle M, m \,|\, m[M, m^l] \rangle$$

This is computed because of the curve in the Kirby diagram of W corresponding to the γ band of the Seifert surface with self linking B; the "b" curve in Figure 7 and Figure 10. Choosing a base-point and following it around we can see that it links the 1-handles in the word claimed as the relation of $\pi_1(W)$; the first m corresponds to the linking which we always have in order to get the right intersection form. The m^l terms appear from linking which is occurring in the box; we know the linking number from the Seifert matrix and this is the only information we need.

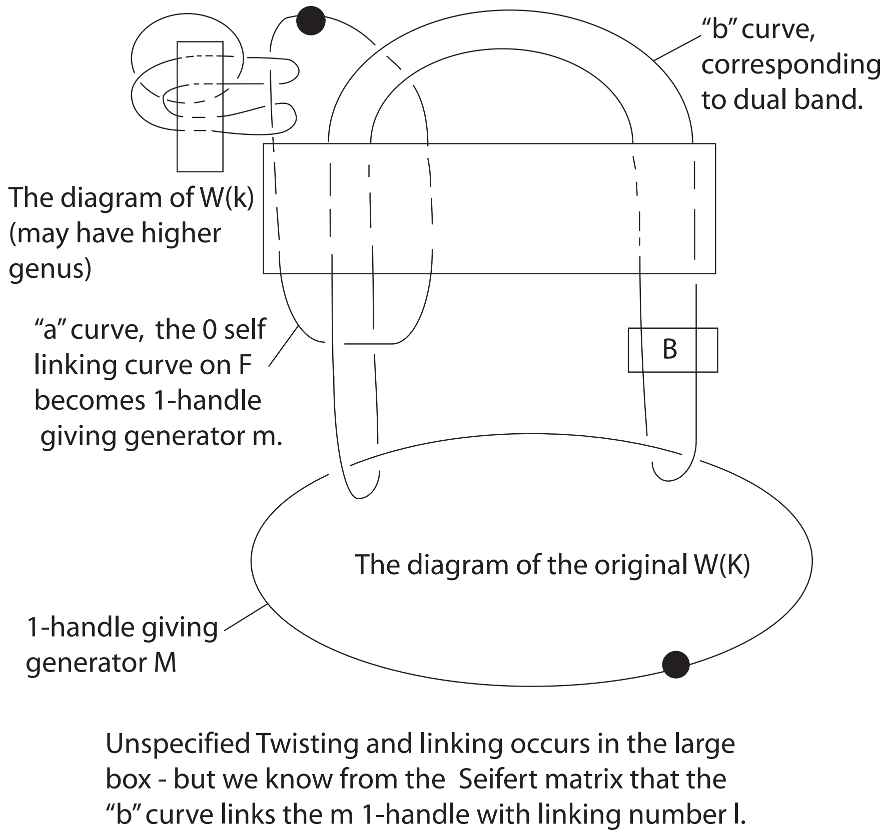


Figure 10: Drawing another four manifold W with more complicated fundamental group; more sophisticated 4-manifolds are required to obstruct sliceness at the higher levels of the Cochran-Orr-Teichner filtration.

It links twice with the M 1-handle in opposite directions. Hence $m^{-1} \in [\pi_1(W), \pi_1(W)]$ and so $m \in [\pi_1(W), \pi_1(W)]$.

There's a short exact sequence of groups

$$\pi_1(W)^{(1)} \rightarrowtail \pi_1(W) \twoheadrightarrow \pi_1(W)/\pi_1(W)^{(1)} \cong \mathbb{Z}$$

where the fact that $m \in \pi_1(W)^{(1)}$ means that $\pi_1(W)/\pi_1(W)^{(1)} \cong \mathbb{Z}$ as generated by M. Since $\mathbb{Z}$ is free, this sequence splits and we see that

$$\pi_1(W) \cong \pi_1(W)^{(1)} \rtimes \mathbb{Z} \cong (\text{Normal closure of } m) \rtimes \langle M \rangle.$$

Now, we work modulo the second commutator group, so all conjugates of m commute. We claim that:

$$\frac{\pi_1(W)}{\pi_1(W)^{(2)}} \cong \frac{\mathbb{Z}[t,t^{-1}]}{lt-(l+1)} \rtimes \langle M \rangle$$

To see this, modifying the above short exact sequence yields

$$\pi_1(W)^{(1)}/\pi_1(W)^{(2)} \rightarrowtail \pi_1(W)/\pi_1(W)^{(2)} \twoheadrightarrow \pi_1(W)/\pi_1(W)^{(1)} \cong \mathbb{Z}$$

which means that

$$\pi_1(W) \cong H_1(W;\mathbb{Z}[\mathbb{Z}]) \rtimes \langle M \rangle$$

where $H_1(W;\mathbb{Z}[\mathbb{Z}]) = H_1(W;\mathbb{Z}[t,t^{-1}])$ is the homology of the universal abelian cover of W, with M acting on it as deck transformations: conjugation by M translates as multiplication (on the right) by t. This homology is generated by m; this becomes the generator of $\mathbb{Z}[t,t^{-1}]$, as M generates the deck group of the $\mathbb{Z}$-covering of W, there is one copy of the 1-handle m for each power of t. Again the b curve yields the relation: it links m l times, passes round M, and then links m a total of $l+1$ times, here with coefficient t since it passed round M, before passing back round M to the original sheet of the covering, completing the loop, and completing the derivation of the claim for $\pi_1(W)/\pi_1(W)^{(2)}$.

So there's a commutative diagram:

$$\begin{tikzcd}
\pi_1(M_K) \arrow[r] \arrow[d] & \Gamma_1 \\
\pi_1(W) \arrow[r] \arrow[ur, "\rho"] & \frac{\mathbb{Z}[t^{\pm 1}]}{\{lt-(l+1)\}} \rtimes \mathbb{Z} \arrow[u]
\end{tikzcd}$$

Note that $m \in \pi_1(W)/\pi_1(W)^{(2)}$ maps to a non-zero element of Γ_1. Therefore:

$$\widetilde{\sigma}^{(2)}_{\Gamma_1}(K,\rho) = \widetilde{\sigma}^{(2)}_{\Gamma_1}(W^4,\rho)$$

We will now calculate the right hand side via computation of $H_2(W;\mathcal{N}\Gamma)$. Let $\Gamma' := \operatorname{im}(\rho) \leq \Gamma_1$. We have $\langle m \rangle = \mathbb{Z} \leq \Gamma_0$.

We need the following crucial Proposition to complete the argument.

Proposition 6.19. Consider a subgroup $\Gamma_0 \leq \Gamma_1$. Then there are commutative diagrams

$$\begin{tikzcd}
\mathcal{N}\Gamma_0 \arrow[rr] \arrow[dr, "\mathrm{tr}_{\Gamma_0}"'] & & \mathcal{N}\Gamma_1 \arrow[dl, "\mathrm{tr}_{\Gamma_1}"] \\
& \mathbb{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
L^0(\mathcal{N}\Gamma_0) \arrow[rr] \arrow[dr, "\sigma^{(2)}_{\Gamma_0}"'] & & L^0(\mathcal{N}\Gamma_1) \arrow[dl, "\sigma^{(2)}_{\Gamma_1}"] \\
& \mathbb{R} &
\end{tikzcd}$$

Remark 6.20. The above proposition is fundamental to our calculations. We constructed our 4-manifold in such a way that the matrix of the intersection form over $\mathbb{Z}\Gamma_1$ contains as its entries only linear combinations of powers of a single non-trivial group element $m \in \Gamma' \leq \Gamma_1$. Since our groups Γ are torsion-free, this gives an inclusion of groups

$$\mathbb{Z} \cong \langle m \rangle \leq \Gamma_1$$

to which Proposition 6.19 can be applied. Thus the L^2-signature for Γ can be calculated for this particular Hermitian form as an integral over the circle of certain twisted signatures.

Proof of Proposition 6.19. A homomorphism $\mathcal{N}\Gamma_0 \to \mathcal{N}\Gamma_1$ is given by completing

$$a \otimes \mathrm{Id} \in \mathrm{End}(l^2\Gamma_0 \otimes_{\mathbb{C}\Gamma_0} \mathbb{C}\Gamma_1)$$

to a bounded operator on $l^2\Gamma_2$ for any $a \in \mathcal{N}\Gamma_1$. Since

$$\langle (e_0 \otimes e_1)(a \otimes \mathrm{Id}), e_0 \otimes e_1 \rangle = \langle (e_0)a, e_0 \rangle \cdot \langle e_1, e_1 \rangle = \langle (e_0)a, e_0 \rangle$$

it follows that the first diagram commutes. For details see Theorem 3.3 of Lück's book on L^2-homology. This reference also contains the statement that tensoring an $\mathcal{N}\Gamma_0$-module with $\mathcal{N}\Gamma_1$ is a flat functor. To see that the second diagram above commutes just observe that diagonalising a hermitian matrix over $\mathcal{N}\Gamma_0$ and then tensoring up the ± 1-eigenspaces to $\mathcal{N}\Gamma_1$ diagonalises the induced matrix over $\mathcal{N}\Gamma_1$. Thus the commutativity of the first diagram proves the claim. $\square$

By Proposition 6.19, since $\Gamma' \leq \Gamma_1$, we get a commuting diagram:

$$\begin{tikzcd}
L^0(\mathcal{N}\Gamma') \arrow[r] \arrow[dr, "\widetilde{\sigma}^{(2)}_{\Gamma'}"'] & L^0(\mathcal{N}\Gamma_1) \arrow[d, "\widetilde{\sigma}^{(2)}_{\Gamma_1}"] \\
 & \mathbb{R}
\end{tikzcd}$$

so just using the using the subgroup Γ' gives us the correct L^2-signature. We now *claim* that:

$$H_2(W; \mathcal{N}\Gamma') \cong H_2(W_0(k); \mathcal{N}(\langle m \rangle) \otimes_{\mathbb{Z}[m^{\pm 1}]} \mathcal{N}\Gamma')$$

The intersection form on the latter group has the desired signature $\widetilde{\sigma}^{(2)}$ for the proof of the lemma.

To see the claim, note that $\Gamma' \leq \pi_1(W)/\pi_1(W)^{(2)}$. Compute the second homology geometrically; the 2-handles from $W_0(k)$ give a basis in the covering space with deck group Γ'. The linking of the b curve with the m 1-handle means that its attaching circle is not null-homotopic and so it doesn't give a generator of $H_2(W; \mathcal{N}\Gamma)$. We can then compute intersection numbers using $\langle m \rangle \cong \mathbb{Z}$, which then gives us the numbers with $m \in \Gamma'$, hence the tensoring up as a Hermitian module. Now Proposition 6.19 applies again; the map $L^0(\mathbb{N}\mathbb{Z}) \to L^0(\mathcal{N}\Gamma')$ is exactly tensoring up. So ultimately:

$$\widetilde{\sigma}^{(2)}_{\Gamma_1}(K, \rho_K) = \widetilde{\sigma}^{(2)}_{\Gamma_1}(W) = \widetilde{\sigma}^{(2)}_{\Gamma'}(W) = \widetilde{\sigma}^{(2)}_{\mathbb{Z}}(W_0(k)) = \widetilde{\sigma}^{(2)}_{\mathbb{Z}}(k)$$

Note that about surjectivity of $\pi_1 \xrightarrow{\rho} \Gamma$ was not essential in our choices: using the image of ρ is enough. $\square$

Remark 6.21. Note on Proposition 6.19: it is rather remarkable that a signature on a smaller group extends to a bigger one. This is not usually true for invariants of L-groups. It comes because the trace function extends from $\mathcal{N}\Gamma_1 \to \mathbb{C}$ to $\mathcal{N}\Gamma_2 \to \mathbb{C}$.

We should warn the reader that the von Neumann algebra $\mathcal{N}\Gamma$ is not functorial in Γ. This has to do with the specific choice of the Hilbert-space $l^2\Gamma$. Proposition 6.19 gives the best possible functoriality which is valid for all groups. If Γ is amenable, then the equality of the reduced and maximal C^*-algebras (which are functorial!) implies that the projection $\Gamma \to \{1\}$ induces a homomorphism of C^*-algebras $\varepsilon : C^*\Gamma \to \mathbb{C}$. For example, if $\Gamma = \mathbb{Z}$ then this is given by evaluating a continuous function at $1 \in S^1$. This clearly does *not* extend to $\mathcal{N}\mathbb{Z} = L^\infty(S^1)$.

Proofs

Proposition 6.22. Let M be a finitely generated abelian group. Then,

$$\operatorname{Ext}^1(M,\mathbb{Z}) \cong \operatorname{Hom}(\operatorname{Tor}(M),\mathbb{Q}/\mathbb{Z}).$$

Proof. We have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \to 0$$

Since $\mathbb{Q}$ is injective, $\operatorname{Ext}^1(M,\mathbb{Q}) = 0$ and we get the sequence

$$0 \to \operatorname{Hom}(M,\mathbb{Z}) \xrightarrow{\mu^*} \operatorname{Hom}(M,\mathbb{Q}) \xrightarrow{\nu^*} \operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z}) \to \operatorname{Ext}^1(M,\mathbb{Z}) \to 0$$

Now, by the structure theorem for f.g. abelian groups, we can write $M = \mathbb{Z}^n \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_q}$. Then

$$\operatorname{Hom}(M,\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}^n,\mathbb{Z}) \oplus \operatorname{Hom}(\mathbb{Z}_{n_1},\mathbb{Z}) \oplus \cdots \oplus \operatorname{Hom}(\mathbb{Z}_{n_q},\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}^n,\mathbb{Z})$$

since $\operatorname{Hom}(A,\mathbb{Z}) = 0$ for any torsion module A. Similarly, $\operatorname{Hom}(M,\mathbb{Q}) = \operatorname{Hom}(\mathbb{Z}^n,\mathbb{Q})$.

This gives us

$$\operatorname{Ext}^1(M,\mathbb{Z}) \cong \operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z})/\operatorname{im}\nu^* = \operatorname{Hom}(M,\mathbb{Q}/\mathbb{Z})/\operatorname{Hom}(\mathbb{Z}^n,\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(\operatorname{Tor}(M),\mathbb{Q}/\mathbb{Z}).$$

□

Proposition 6.23. There is an isomorphism $\Phi\colon \operatorname{Tor} H^n(X) \to \operatorname{Hom}(\operatorname{Tor} H_{n-1}(X),\mathbb{Q}/\mathbb{Z})$.

Proof. For a function $f \in \operatorname{Tor} H^n(X)$, we let $\Phi(f)$ be the map $x \mapsto \frac{f(y)}{s}$, where $x \in \operatorname{Tor} H_{n-1}(X)$, and $sx = dy$ for some $s \neq 0 \in \mathbb{Z}$ and $y \in S_n(X)$. Here we use the terminology that the homology groups are calculated from the chain complex

$$\cdots \to S_{n+1}(X) \xrightarrow{d} S_n(X) \xrightarrow{d} S_{n-1}(X) \to \ldots.$$

Our map is clearly a homomorphism. To prove surjectivity, take $\phi \in \mathrm{Hom}(\mathrm{Tor}\, H_{n-1}(X), \mathbb{Q}/\mathbb{Z})$. We want to find $f \in \mathrm{Tor}\, H^n(X)$ such that $\phi(x) = \Phi(f)(x)\ \forall x$. So let $x \in \mathrm{Tor}\, H_{n-1}(X)$ with $sx = dy$ for some $s \in \mathbb{Z} \setminus \{0\}$ and $y \in S_n(X)$. Put $f = \phi \circ d$. Then $f(y) = \phi(d(y)) = \phi(sx) = s\phi(x)$, and so $\phi(x) = \frac{f(y)}{s} = \Phi(f)(x)$ as required.

It remains to prove injectivity. Suppose $\Phi(f)$ is the zero map for some $f \in \mathrm{Tor}\, H^n(X)$. This means that $\Phi(f)(x) = \frac{f(y)}{s} \equiv 0$ for all $x \in \mathrm{Tor}\, H_{n-1}(X)$, which means that $\frac{f(y)}{s} \in \mathbb{Z}$, so s divides $f(y)$.

Since $f \in \mathrm{Tor}\, H^n(X)$ we can find $m \in \mathbb{Z}$ and $g \in S^{n-1}(X)$ such that $mf = g \circ d$ (i.e. $mf \in \mathrm{im}\, d^*$). Now

$$mf(y) = g(d(y)) = g(sx) = sg(x)$$

so that $\frac{f(y)}{s} = \frac{g(x)}{m} \in \mathbb{Z}$ for all x. But if m divides $g(x)$ for all x, then we may write $g = mg'$ for some $g' \in S^{n-1}(X)$. Then $mf = mg' \circ d \Rightarrow f = g' \circ d \in \mathrm{im}\, d^*$, so f is the zero element in $\mathrm{Tor}\, H^n(X)$. $\square$