T2-Duality: Lifting T-Duality to M-theory via adjusted Lie 3-groups

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Based on joint work with Gianni Gagliardo and Christian Sämann

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  - Prequantized: a gerbe with connection $(\mathcal{L}, B) \to P$ of curvature H.
  - This defines, along with g and ϕ, a 2d σ-model with action $S^{g,B,\phi}[X]$, $X \in Map(\Sigma_2, P)$.

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  - This defines, along with g and ϕ, a 2d σ-model with action $S^{g,B,\phi}[X]$, $X \in Map(\Sigma_2, P)$.
- **Ramond-Ramond fields** :
  - Field Strength: $F^\bullet \in \Omega^{even/odd}(P)$ with $dF^j + H \wedge F^{j-2} = 0$.
  - Prequantized: a cocycle $RR \in K^\bullet_{diff,(\mathcal{L},B)}(P)$, which is even for IIA and odd for IIB.

MOTIVATION: LIFTING T-DUALITY AND S-DUALITY

T-Duality is a relation between $(P, g, \phi, \mathcal{L}, B)$ and $(\hat{P}, \hat{g}, \hat{\phi}, \widehat{\mathcal{L}}, \hat{B})$ which:

- May be defined when P, $\hat{P}$ are a $\mathbb{T}^n$-bundle and a $(\mathbb{T}^n)^*$-bundle over some manifold M.
- Implies the path integral of $S^{g,B,\phi}[X]$ over $Map(\Sigma_2, P)$ equals that of $S^{\hat{g},\hat{B},\hat{\phi}}[\hat{X}]$ over $Map(\Sigma_2, \hat{P})$.
- Implies there is an iso $K^{\bullet}_{diff,(\mathcal{L},B)}(P) \to K^{\bullet+1}_{diff,(\widehat{\mathcal{L}},\hat{B})}(\hat{P})$, $RR \mapsto \widehat{RR}$.

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This means:

- IIA/B with background $(P, g, \phi, \mathcal{L}, B, RR)$ equivalent to IIB/A with background $(\hat{P}, \hat{g}, \hat{\phi}, \widehat{\mathcal{L}}, \hat{B}, \widehat{RR})$.
- Well-defined QFTs without well-defined P can be obtained by gluing along T-Dualities (**T-folds**).

MOTIVATION: LIFTING T-DUALITY AND S-DUALITY

Brief and Incomplete History of T-Duality:

1. [Bergshoeff et al., 1995; Buscher, 1987, 1988; Hassan, 2000] **Buscher rules** describe $(g, B, \phi, RR) \mapsto (\hat{g}, \hat{B}, \hat{\phi}, \widehat{RR})$ when $P = M^{10-n} \times \mathbb{T}^n$ and $\mathcal{L} = P \times BU(1)$.
2. [Bouwknegt et al., 2004; Bunke & Schick, 2005] **Topological T-Duality** describes $(P, \mathcal{L}) \mapsto (\hat{P}, \widehat{\mathcal{L}})$.
3. [Baraglia & Hekmati, 2015; García-Fernández, 2019] **Generalized Geometry** describes $(P, g, \phi, E, B, RR) \mapsto (\hat{P}, \hat{g}, \hat{\phi}, \widehat{E}, \hat{B}, \widehat{RR})$, for E Courant algebroid of inf. symmetries of $\mathcal{L}$.
4. [Kim & Sämann, 2022; Nikolaus & Waldorf, 2020; Waldorf, 2024] **Geometric T-Duality** describes $(P, g, \phi, \mathcal{L}, B) \mapsto (\hat{P}, \hat{g}, \hat{\phi}, \widehat{\mathcal{L}}, \hat{B})$.

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4. [Kim & Sämann, 2022; Nikolaus & Waldorf, 2020; Waldorf, 2024] **Geometric T-Duality** describes $(P, g, \phi, \mathcal{L}, B) \mapsto (\hat{P}, \hat{g}, \hat{\phi}, \widehat{\mathcal{L}}, \hat{B})$.

About the latter approach:

- It uses an adjusted Lie 2-group $T\mathbb{D}_n^{F_2}$ to describe the cocycle data necessary to construct a T-Dual.
- It shows how to use such cocycle data to construct a T-Dual.
- It allows for a definition of a T-fold as a $T\mathbb{D}_n^{F_2} \rtimes Aut(T\mathbb{D}_n^{F_2})$-bundle.
- Iso $K^{\bullet}_{diff,(\mathcal{L},B)}(P) \to K^{\bullet+1}_{diff,(\widehat{\mathcal{L}},\hat{B})}(\hat{P})$, $RR \mapsto \widehat{RR}$ proven abstractly, but without explicit cocycle data.
- Thus, no 'General topology Buscher rules for RR fields' or cocycle description of RR fields in T-folds.

MOTIVATION: LIFTING T-DUALITY AND S-DUALITY

S-Duality is a symmetry of the space of IIB backgrounds $(P, g, \phi, \mathcal{L}, B, RR)$ mixing ϕ, $(\mathcal{L}, B)$, RR without changing the full path integral [Schwarz, 1995]. This means:

- A model for IIB backgrounds treating ϕ, $(\mathcal{L}, B)$, RR as a single entity is desirable for S-Duality.
- Such a model is necessary for defining **S-folds** ; i.e., QFTs obtained by gluing along S-Dualities.

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The preferred method in the literature to approach this is lifting to **M-theory** [Hull & Townsend, 1995; Kriz & Sati, 2005; Witten, 1995]:

- An M-theory background on a $11d$ spacetime of the form $M^9 \times \mathbb{T}_{IIA} \times \mathbb{T}_M$ determines a IIA background on $M^9 \times \mathbb{T}_{IIA}$ and then T-dualizing along $\mathbb{T}_{IIA}$ a IIB background on $M^9 \times \mathbb{T}_{IIB}$.
- Under this procedure, the geometric action of $SL(2, \mathbb{Z})$ on $\mathbb{T}^2$ covers S-Duality of the IIB background.

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This raises the following question:

*Can **topology-changing T-Dualities** be lifted to **M-theory** in a compatible way with **S-Duality**?*
i.e.
Is there a span of higher groups which, given an M-theory background over a $\mathbb{T}^n$-fibration, describes the construction of associated IIA and IIB backgrounds in a $SL(n, \mathbb{Z})$-equivariant way?

Such construction would lead to cocycle data for T-folds and S-folds with RR fields.
Our work constitutes a first step towards this goal.

MOTIVATION: LIFTING T-DUALITY AND S-DUALITY

An M-theory background consists of an $11d$ spacetime manifold P with metric g and an M-brane. Whatever this is, it should determine $G^4 \in \Omega^4(P, \mathbb{R})$ and $G^7 \in \Omega^7(P, \mathbb{R})$ with

$$dG^4 = 0, \quad dG^7 + G^4 \wedge G^4 = 0. \tag{1}$$

- A model for the M-brane yielding such (G^4, G^7), admitting a dimensional reduction to $(\mathcal{L}, B)$, $[RR] \in K^0_{diff,(\mathcal{L},B)}$, minimal in some sense, is a cocycle in **J-twisted cohomotopy theory** [Banerjee et al., 2026; Fiorenza et al., 2020b, 2021a, 2021b].

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- We use a simpler model for the M-brane: a 2-gerbe with connection $(\mathcal{L}, C) \to P^{11}$ of curvature G^4.
- Accordingly, our RR fields are also prequantized in something simpler than $K^\bullet_{diff,(\mathcal{L},B)}(P^{10})$.
- This is sufficient to define **T_2-Duality**, which lifts T-Duality and incorporates S-Duality.
- We expect U-Dualities to have an underlying T_2-Duality.

Motivation: Lifting T-Duality and S-Duality

Our main results can be summarized as follows.

Theorem [Gagliardo et al., 2026]

There is a span of Lie 3-groups $T_2\mathbb{B}_n^{l,F_2} \leftarrow T_2\mathbb{D}_n^{F_2} \rightarrow T_2\mathbb{B}_n^{r,F_2}$ such that:

1. A $T_2\mathbb{B}_n^{l,F_2}$-bundle over M determines a $\mathbb{T}^n$-bundle $P \rightarrow M$ and a 2-gerbe $\mathcal{L} \rightarrow P$.
2. A $T_2\mathbb{B}_n^{r,F_2}$-bundle determines a complicated fibration $\widehat{\mathcal{P}} \rightarrow M$ and a 2-gerbe $\widehat{\mathcal{L}} \rightarrow \widehat{\mathcal{P}}$.
3. It admits as a subspan $T\mathbb{B}_{n-1}^{F_2} \leftarrow T\mathbb{D}_{n-1}^{F_2} \rightarrow T\mathbb{B}_{n-1}^{F_2}$, which controls T-Duality of KR fields along $\mathbb{T}^{n-1}$.
4. It is equivariant under natural actions of $GL(n, \mathbb{Z})$, inducing S-Duality of IIB backgrounds.

The whole construction is enhanced with adjusted connections, prequantizing all the IIA and IIB KR and RR field strengths that can be constructed from a $G^4 \in \Omega^4_{cl}(P^{11}, \mathbb{R})$.

A review of T-Duality

Definition

For T an abelian Lie group, a ***T*-gerbe with connection** $(\mathcal{L}, B)$ over the manifold M is:

1. $\lambda_{ijk} : U_{ijk} \to T$ with $\lambda_{ijk}\lambda_{ikl} = \lambda_{ijl}\lambda_{jkl}$,
2. $\Lambda_{ij} \in \Omega^1(U_{ij}, \mathfrak{t})$ with $\Lambda_{ij} - \Lambda_{ik} + \Lambda_{jk} = \lambda_{ijk}^* \theta^T$,
3. $B_i \in \Omega^2(U_i, \mathfrak{t})$ with $B_i - B_j = d\Lambda_{ij}$,

for $\{U_i\}_{i \in I}$ an open cover of M. Its **curvature** is $H \in \Omega^3_{cl}(M, \mathfrak{t})$ defined by $H := dB_i$.

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- We treat $\mathcal{L}$ as having a well-defined total space, where $B \in \Omega^2(\mathcal{L}, \mathfrak{t})$ is well-defined and satisfies $dB = H$. This can be formalized, for example, with simplicial manifolds.
- When $T = \mathbb{T}^1$, then $[H] \in H^3(M, \mathbb{Z})$ and we just say *gerbe* (instead of $\mathbb{T}^1$-gerbe).
- Coboundaries are called **isomorphisms**, higher coboundaries are called **2-isomorphisms**, etc.
- Taking higher cocycles, one defines ***p*-gerbes with connection**

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Consider tuples $(P, A, \mathcal{L}, B)$ where

- $(P, A) \to M$ is a $\mathbb{T}^n$-bundle with connection of curvature $F_0 \in \Omega^2_{cl}(M, \mathbb{R}^n)$, and
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Then

1. Suppose $H = H_0 + A \wedge \hat{F}_0$ for some $H_0 \in \Omega^3(M, \mathbb{R})$, $\hat{F}_0 \in \Omega^2(M, (\mathbb{R}^n)^*)$. Then
$$d\hat{F}_0 = 0, \quad dH_0 + F_0 \wedge \hat{F}_0 = 0. \tag{2}$$
2. Assume $\hat{F}_0$ is the curvature of a $(\mathbb{T}^n)^*$-bundle with connection $(\widehat{P}, \hat{A}) \to M$.
3. Assume $\hat{H} := H_0 + F_0 \wedge \hat{A} \in \Omega^3_{cl}(\widehat{P}, \mathbb{R})$ is the curvature of a gerbe with connection $(\widehat{\mathcal{L}}, \hat{B}) \to \widehat{P}$.
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Definition [Belov et al., 2007; Bunke & Schick, 2005; Kahle & Valentino, 2014]

$(\hat{P}, \hat{A}, \widehat{\mathcal{L}}, \hat{B})$ is **T-dual** of $(P, A, \mathcal{L}, B)$ if over $P \times_M \hat{P}$ there exists an iso $\tau : \mathcal{L} \to \widehat{\mathcal{L}}$ with $\tau^*\hat{B} - B = A \wedge \hat{A}$.

- Topologically, a T-dual for $(P, \mathcal{L})$ exists iff $[\mathcal{L}] \in F_2H^3(P, \mathbb{Z})$ (Serre filtration of $P \to M$).
- T-duality implies an equivalence of 2d σ-models with targets P, $\widehat{P}$ [Belov et al., 2007] and an iso $K^\bullet_{diff,(\mathcal{L},B)}(P) \to K^{\bullet+1}_{diff,(\mathcal{L},B)}(\hat{P})$ [Kahle & Valentino, 2014].

A REVIEW OF T-DUALITY

A **Lie crossed module** is $\mathcal{G} = (G, H, t, \triangleright)$ where G, H are Lie groups, $t : H \to G$ is a hom. and $\triangleright : G \times H \to H$ is an action by automorphisms satisfying

$$t(g \triangleright h) = gt(h)g^{-1}, \qquad t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}. \tag{3}$$

A $\mathcal{G}$-**bundle** over a manifold M is described over a cover $\{U_i\}_{i\in I}$ by $g_{ij} : U_{ij} \to G$, $h_{ijk} : U_{ijk} \to H$ with

$$t(h_{ijk})g_{ij}g_{jk} = g_{ik}, \qquad h_{ikl}h_{ijk} = h_{ijl} \cdot g_{ij} \triangleright h_{jkl}. \tag{4}$$

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An **adjustment** [Rist et al., 2026] on $\mathcal{G}$ is a map $\kappa : G \times \mathfrak{g} \to \mathfrak{h}$, linear in $\mathfrak{g}$, and satisfying

$$\kappa(g_2 g_1, v) = g_2 \triangleright \kappa(g_1, v) + \kappa(g_2, g_1 v g_1^{-1} - t\kappa(g_1, v)), \qquad \kappa(t(h), v) = h \cdot V \triangleright h^{-1}. \tag{5}$$

An **adjusted connection** on a $\mathcal{G}$-bundle is $A_i \in \Omega^1(U_i, \mathfrak{g})$, $\Lambda_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$, $B_i \in \Omega^2(U_i, \mathfrak{h})$ with

$$\Lambda_{ik} = \Lambda_{jk} + g_{jk}^{-1} \triangleright \Lambda_{ij} - g_{ik}^{-1} \triangleright (h_{ijk} \nabla_i h_{ijk}^{-1}), \qquad A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} - t\Lambda_{ij}, \tag{6}$$

$$B_j = g_{ij}^{-1} \triangleright B_i + d\Lambda_{ij} + A_j \triangleright \Lambda_{ij} + \frac{1}{2}[\Lambda_{ij}, \Lambda_{ij}] - \kappa(g_{ij}^{-1}, F). \tag{7}$$

Its **curvature** is

$$F := dA_i + \frac{1}{2}[A_i, A_i] - tB_i, \qquad H_i := dB_i + A_i \triangleright B_i - \kappa(A_i, F_i) \tag{8}$$

satisfying (for $\nabla_i^\kappa := d + A_i \triangleright - t\kappa(A_i, \ )$)

$$\nabla_i^\kappa F_i = tH_i, \quad \nabla_i^\kappa H_i + \kappa(F_i, F_i) = 0. \tag{9}$$

A REVIEW OF T-DUALITY

Let $V_0 := \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ and $V_1 = \mathbb{R}$. Consider the canonical pairing

$$\langle \cdot, \cdot \rangle : V_0 \otimes V_0 \to V_1, \quad \langle v_1 + A_1, v_2 + A_2 \rangle = \iota_{v_1} A_2. \tag{10}$$

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Theorem [Ganter, 2018; Kim & Sämann, 2022; Nikolaus & Waldorf, 2020; Waldorf, 2024]

The data $(V_0, V_1, \langle \cdot, \cdot \rangle)$ determines a span of adjusted Lie 2-groups

$$T\mathbb{B}_n^{F_2} \leftarrow T\mathbb{D}_n^{F_2} \rightarrow T\mathbb{B}_n^{F_2} \tag{11}$$

with the following properties.

1. $T\mathbb{D}_n^{F_2}$-bundles with connection determine $F_0 \in \Omega^2(M, \mathbb{R}^n)$, $\hat{F}_0 \in \Omega^2(M, (\mathbb{R}^n)^*)$, $H_0 \in \Omega^3(M, \mathbb{R})$ with

$$dF_0 = 0, \quad d\hat{F}_0 = 0, \quad dH_0 + \langle F_0 \wedge \hat{F}_0 \rangle = 0. \tag{12}$$

2. $T\mathbb{B}_n^{F_2}$-bundles are equivalent to tuples $(P, A, \mathcal{L}, B)$ admitting a T-Dual.
3. If $(P, A, \mathcal{L}, B)$ and $(\hat{P}, \hat{A}, \widehat{\mathcal{L}}, \hat{B})$ are related by the span, then they are T-dual. In this case, the curvatures of the connections are:

$$A \to F_0, \quad B \to H_0 + \langle A \wedge \hat{F}_0 \rangle, \tag{13}$$

$$\hat{A} \to \hat{F}_0, \quad \hat{B} \to H_0 + \langle F_0 \wedge \hat{A} \rangle. \tag{14}$$

A REVIEW OF T-DUALITY

- Given $(P, A, \mathcal{L}, B)$, we construct a T-dual as follows.
  1. Write it in terms of cocycle data for a $T\mathbb{B}_n^{F_2}$-bundle.
  2. Find cocycle data for a $T\mathbb{D}_n^{F_2}$-bundle lifting the preceding one through the left homomorphism.
  3. Project to new cocycle data for a $T\mathbb{B}_n^{F_2}$-bundle with the right homomorphism.
  4. Reinterpret the result as $(\hat{P}, \hat{A}, \widehat{\mathcal{L}}, \hat{B})$,

- Different lifts of $(P, A, \mathcal{L}, B)$ to a $T\mathbb{D}_n^{F_2}$-bundle are related by the action of $Aut(T\mathbb{D}_n^{F_2})$ on $T\mathbb{D}_n^{F_2}$.

- This means that a T-fold can be defined as a $T\mathbb{D}_n^{F_2} \rtimes Aut(T\mathbb{D}_n^{F_2})$-bundle.

Higher Forms of T-Duality. Introducing T2-Duality.

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Then

1. Suppose $G = G_0 - A \wedge H_0$ for some $G_0 \in \Omega^4(M, \mathbb{R})$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$. Then

$$dH_0 = 0, \quad dG_0 - F_0 \wedge H_0 = 0. \tag{15}$$

2. Assume $H_0 \in \Omega^3_{cl}(M, (\mathbb{R}^n)^*)$ is the curvature of a $(\mathbb{T}^n)^*$-gerbe with connection $(\widehat{\mathcal{P}}, \hat{B}) \to M$.
3. Assume $\hat{G} := G_0 - F_0 \wedge \hat{B} \in \Omega^4_{cl}(\widehat{\mathcal{P}}, \mathbb{R})$ is the curvature of a 2-gerbe with connection $(\widehat{\mathcal{L}}, \hat{C}) \to \widehat{\mathcal{P}}$.
4. Note $\hat{G} - G = d(A \wedge \hat{B})$ over $P \times_M \widehat{\mathcal{P}}$.

Higher Forms of T-Duality. Introducing T2-Duality.

- Let $(P, A) \to M$ be a $\mathbb{T}^n$-bundle with connection of curvature $F_0 \in \Omega^2_{cl}(M, \mathbb{R}^n)$.
- Let $(\mathcal{L}, C) \to P$ be a 2-gerbe with connection of curvature $G \in \Omega^4_{cl}(P, \mathbb{R})$.

Then

1. Suppose $G = G_0 - A \wedge H_0$ for some $G_0 \in \Omega^4(M, \mathbb{R})$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$. Then
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3. Assume $\hat{G} := G_0 - F_0 \wedge \hat{B} \in \Omega^4_{cl}(\widehat{\mathcal{P}}, \mathbb{R})$ is the curvature of a 2-gerbe with connection $(\widehat{\mathcal{L}}, \hat{C}) \to \widehat{\mathcal{P}}$.
4. Note $\hat{G} - G = d(A \wedge \hat{B})$ over $P \times_M \widehat{\mathcal{P}}$.

Definition [Gagliardo et al., 2026]

We say $(\widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ is a **T_2-dual of type F_3** of $(P, A, \mathcal{L}, C)$ if the above is satisfied and there is an iso $\tau : \mathcal{L} \to \widehat{\mathcal{L}}$ over $P \times_M \widehat{\mathcal{P}}$ with $\tau^* \hat{C} - C = A \wedge \hat{B}$.

One can easily prove:

- Topologically, a T_2-dual of type F_3 for $(P, \mathcal{L})$ exists iff $[\mathcal{L}] \in F_3 H^4(P, \mathbb{Z})$ (Serre filtration of $P \to M$).
- T_2-duality of type F_3 implies an equivalence of 3d σ-models with targets P, $\widehat{\mathcal{P}}$.

Higher Forms of T-Duality. Introducing T2-Duality.

More generally, for $p, q \geq 1$,

- Let $(P, A) \to M$ be a $(p-1)$-$\mathbb{T}^n$-gerbe with connection of curvature $F_0^{p+1} \in \Omega^{p+1}(M, \mathbb{R}^n)$.
- Let $(\mathcal{L}, B) \to P$ be a $(p+q-1)$-gerbe with connection of curvature $F^{p+q+1} \in \Omega_{cl}^{p+q+1}(P, \mathbb{R})$.

Higher Forms of T-Duality. Introducing T2-Duality.

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- Let $(\mathcal{L}, B) \to P$ be a $(p+q-1)$-gerbe with connection of curvature $F^{p+q+1} \in \Omega_{cl}^{p+q+1}(P, \mathbb{R})$.

Definition

A ***T_q^p*-dual of type *F_{p+q}*** for $(P, A, \mathcal{L}, B)$ is a tuple $(\hat{P}, \hat{A}, \widehat{\mathcal{L}}, \hat{B})$ with

1. $(\hat{P}, \hat{A}) \to M$ is a $(q-1)$-$(\mathbb{T}^n)^*$-gerbe with connection,
2. $(\widehat{\mathcal{L}}, \hat{B}) \to \hat{P}$ is a $(p+q-1)$-gerbe with connection,
3. There is an iso $\tau : \widehat{\mathcal{L}} \to \mathcal{L}$ over $P \times_M \hat{P}$ with $\tau^* \hat{B} - B = A \wedge \hat{A}$.

This implies that the path integrals of certain $(p+q)$-dimensional σ-models with targets P, $\hat{P}$ coincide.

Higher Forms of T-Duality. Introducing T2-Duality.

More generally, for $p, q \geq 1$,

- Let $(P, A) \to M$ be a $(p-1)$-$\mathbb{T}^n$-gerbe with connection of curvature $F_0^{p+1} \in \Omega^{p+1}(M, \mathbb{R}^n)$.
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Definition

A **T_q^p-dual of type F_{p+q}** for $(P, A, \mathcal{L}, B)$ is a tuple $(\hat{P}, \hat{A}, \widehat{\mathcal{L}}, \hat{B})$ with

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This implies that the path integrals of certain $(p+q)$-dimensional σ-models with targets P, $\hat{P}$ coincide.

Some special cases:

- $p = 1$, $q = 1$ is standard **T-Duality** .
- $p = 2$, $q = 2$ is **electric-magnetic duality** or **S-Duality** .
- $p = q \in 2\mathbb{Z}$ is studied in [Fiorenza et al., 2018, 2020a] under the name **higher T-Duality** .
- $p = q \in \mathbb{Z}$ is studied in [Chatzistavrakidis et al., 2021] under the name **higher T-Duality** .
- $p = 1$, $q = 2$ and generalizations are studied in [Gagliardo et al., 2026] under the name **T_2-Duality** .

Higher Forms of T-Duality. Introducing T2-Duality.

- Let $(P, A) \to M$ be a $\mathbb{T}^n$-bundle with connection of curvature $F_0 \in \Omega^2_{cl}(M, \mathbb{R}^n)$.
- Let $(\mathcal{L}, C) \to P$ be a 2-gerbe with connection of curvature $G \in \Omega^4_{cl}(P, \mathbb{R})$.

Higher Forms of T-Duality. Introducing T2-Duality.

- Let $(P, A) \to M$ be a $\mathbb{T}^n$-bundle with connection of curvature $F_0 \in \Omega^2_{cl}(M, \mathbb{R}^n)$.
- Let $(\mathcal{L}, C) \to P$ be a 2-gerbe with connection of curvature $G \in \Omega^4_{cl}(P, \mathbb{R})$.

Then

1. Suppose $G = G_0 - A \wedge H_0 - \frac{1}{2} A \wedge A \wedge \hat{F}_0$ for some $G_0 \in \Omega^4(M, \mathbb{R})$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$, $\hat{F}_0 \in \Omega^2(M, \Lambda^2(\mathbb{R}^n)^*)$. Then
$$d\hat{F}_0 = 0, \quad dH_0 + F_0 \wedge \hat{F}_0 = 0, \quad dG_0 - F_0 \wedge H_0 = 0. \tag{16}$$
2. Assume $\hat{F}_0 \in \Omega^2_{cl}(M, \Lambda^2(\mathbb{R}^n)^*)$ is the curvature of a $\Lambda^2(\mathbb{T}^n)^*$-bundle $\hat{P} \to M$ with connection $\hat{A}$.
3. Assume $\hat{H} := H_0 + F_0 \wedge \hat{A} \in \Omega^3_{cl}(\hat{P}, (\mathbb{R}^n)^*)$ is the curvature of a $(\mathbb{T}^n)^*$-gerbe $(\widehat{\mathcal{P}}, \hat{B}) \to P$.
4. Assume $\hat{G} := G_0 - F_0 \wedge \hat{B} \in \Omega^4_{cl}(\widehat{\mathcal{P}}, \mathbb{R})$ is the curvature of a 2-gerbe with connection $(\widehat{\mathcal{L}}, \hat{C}) \to \widehat{\mathcal{P}}$.
5. Note $\hat{G} - G = d(A \wedge \hat{B} - \frac{1}{2} A \wedge A \wedge \hat{A})$ over $P \times_M \widehat{\mathcal{P}}$.

HIGHER FORMS OF T-DUALITY. INTRODUCING T2-DUALITY.

- Let $(P, A) \to M$ be a $\mathbb{T}^n$-bundle with connection of curvature $F_0 \in \Omega^2_{cl}(M, \mathbb{R}^n)$.
- Let $(\mathcal{L}, C) \to P$ be a 2-gerbe with connection of curvature $G \in \Omega^4_{cl}(P, \mathbb{R})$.

Then

1. Suppose $G = G_0 - A \wedge H_0 - \frac{1}{2} A \wedge A \wedge \hat{F}_0$ for some $G_0 \in \Omega^4(M, \mathbb{R})$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$, $\hat{F}_0 \in \Omega^2(M, \Lambda^2(\mathbb{R}^n)^*)$. Then

$$d\hat{F}_0 = 0, \quad dH_0 + F_0 \wedge \hat{F}_0 = 0, \quad dG_0 - F_0 \wedge H_0 = 0. \tag{16}$$

2. Assume $\hat{F}_0 \in \Omega^2_{cl}(M, \Lambda^2(\mathbb{R}^n)^*)$ is the curvature of a $\Lambda^2(\mathbb{T}^n)^*$-bundle $\hat{P} \to M$ with connection $\hat{A}$.
3. Assume $\hat{H} := H_0 + F_0 \wedge \hat{A} \in \Omega^3_{cl}(\hat{P}, (\mathbb{R}^n)^*)$ is the curvature of a $(\mathbb{T}^n)^*$-gerbe $(\widehat{\mathcal{P}}, \hat{B}) \to P$.
4. Assume $\hat{G} := G_0 - F_0 \wedge \hat{B} \in \Omega^4_{cl}(\widehat{\mathcal{P}}, \mathbb{R})$ is the curvature of a 2-gerbe with connection $(\widehat{\mathcal{L}}, \hat{C}) \to \widehat{\mathcal{P}}$.
5. Note $\hat{G} - G = d(A \wedge \hat{B} - \frac{1}{2} A \wedge A \wedge \hat{A})$ over $P \times_M \widehat{\mathcal{P}}$.

Definition

We say $(\hat{P}, \hat{A}, \widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ is a **T_2-dual of type F_2** of $(P, A, \mathcal{L}, C)$ if the above is satisfied and there is an iso $\tau : \mathcal{L} \to \widehat{\mathcal{L}}$ over $P \times_M \widehat{\mathcal{P}}$ with $\tau^* \hat{C} - C = A \wedge \hat{B} - \frac{1}{2} A \wedge A \wedge \hat{A}$.

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

CONSTRUCTING T2-DUALS ON NON-TRIVIAL TOPOLOGY VIA LIE 3-GROUPS

A **Lie 2-crossed module** is $\mathcal{G} = (G, H, L, t, \triangleright, \{\cdot,\cdot\})$ with G, H, L Lie groups, $L \xrightarrow{t} H \xrightarrow{t} G$ homs with $t^2 = 1$, $\triangleright : G \times L \to L$, $\triangleright : G \times H \to H$ actions by automorphisms and $\{\cdot,\cdot\} : H \times H \to L$ satisfying

$$t(g \triangleright l) = g \triangleright t(l), \quad t(g \triangleright h) = gt(h)g^{-1}, \quad t\{h_1, h_2\}t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}, \tag{17}$$

as well as some other axioms for $\{\cdot,\cdot\}$.
A $\mathcal{G}$-**bundle** over M is described over $\{U_i\}_{i\in I}$ by $g_{ij} : U_{ij} \to G$, $h_{ijk} : U_{ijk} \to H$, $l_{ijkl} : U_{ijkl} \to L$ with

$$t(h_{ijk})g_{ij}g_{jk} = g_{ik}, \qquad h_{ikl}h_{ijk}t(l_{ijkl}) = h_{ijl} \cdot g_{ij} \triangleright h_{jkl}, \tag{18}$$

$$l_{ijkl} \cdot (g_{ij} \triangleright h_{jkl}^{-1}) \triangleright l_{ijlm} \cdot g_{ij} \triangleright l_{jklm} = h_{ijk}^{-1} \triangleright l_{iklm}\{h_{ijk}, g_{ik} \triangleright h_{klm}\} \cdot (g_{ij} \triangleright (g_{jk} \triangleright h_{klm}))^{-1} \triangleright l_{ijkm}. \tag{19}$$

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A $\mathcal{G}$-**bundle** over M is described over $\{U_i\}_{i\in I}$ by $g_{ij} : U_{ij} \to G$, $h_{ijk} : U_{ijk} \to H$, $l_{ijkl} : U_{ijkl} \to L$ with

$$t(h_{ijk})g_{ij}g_{jk} = g_{ik}, \qquad h_{ikl}h_{ijk}t(l_{ijkl}) = h_{ijl} \cdot g_{ij} \triangleright h_{jkl}, \tag{18}$$

$$l_{ijkl} \cdot (g_{ij} \triangleright h_{jkl}^{-1}) \triangleright l_{ijlm} \cdot g_{ij} \triangleright l_{jklm} = h_{ijk}^{-1} \triangleright l_{iklm}\{h_{ijk}, g_{ik} \triangleright h_{klm}\} \cdot (g_{ij} \triangleright (g_{jk} \triangleright h_{klm}))^{-1} \triangleright l_{ijkm}. \tag{19}$$

A **Lie almost 2-crossed module** [Gagliardo et al., 2026] is $\mathcal{G} = (G, H, L, t, \triangleright, \{\cdot,\cdot\}, \alpha, \beta)$ with G, H, L Lie groups, $L \xrightarrow{t} H \xrightarrow{t} G$ homs with $t^2 = 1$, $\triangleright : G \times L \to L$ action by automorphisms and $\triangleright : G \times H \to H$, $\alpha : G \times G \times H \to L$, $\beta : G \times H \times H \to L$, $\{\cdot,\cdot\} : H \times H \to L$ satisfying

$$t(g \triangleright l) = g \triangleright t(l), \quad t(g \triangleright h) = gt(h)g^{-1}, \quad t\{h_1, h_2\}t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}, \tag{20}$$

$$(g_1 g_2) \triangleright h = g_1 \triangleright (g_2 \triangleright h)t\alpha(g_1, g_2, h), \quad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)t\beta(g, h_1, h_2), \tag{21}$$

as well as some other axioms for $\{\cdot,\cdot\}$, α, β.
A $\mathcal{G}$-**bundle** over M is described over $\{U_i\}_{i\in I}$ by $g_{ij} : U_{ij} \to G$, $h_{ijk} : U_{ijk} \to H$, $l_{ijkl} : U_{ijkl} \to L$ with (18) and a deformed version of (19) including β and α.

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

- In [Gagliardo et al., 2025] we define **adjustments** and **adj. connections** for Lie 2-crossed modules.
- However, for T_2-Duality of type $F2$ we need to work with Lie **almost** 2-crossed modules.
- We give a consistent notion of adj. con. **ONLY** for the Lie almost 2-crossed modules of T_2-Duality.
- **Future work** : include this in a general formalism of adjustments for Lie almost 2-crossed modules.

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

Let $V_0 = \mathbb{R}^n$, $V_1 = (\mathbb{R}^n)^*$ and $V_2 = \mathbb{R}$. Consider the pairing

$$\langle \cdot, \cdot \rangle_1 : V_0 \otimes V_1 \to V_2, \quad \langle v, A \rangle_1 = \iota_v A. \tag{22}$$

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

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$$\langle \cdot, \cdot \rangle_1 : V_0 \otimes V_1 \to V_2, \quad \langle v, A \rangle_1 = \iota_v A. \tag{22}$$

Theorem 1 ([Gagliardo et al., 2026])

The data $(V_0, V_1, \langle \cdot, \cdot \rangle_1)$ determines a span of adjusted Lie 2-crossed modules

$$T_2\mathbb{B}_n^{l,F_3} \leftarrow T_2\mathbb{D}_n^{F_3} \rightarrow T_2\mathbb{B}_n^{r,F_3} \tag{23}$$

with the following properties.

1. *$T_2\mathbb{D}_n^{F_3}$-bundles with connection determine $F_0 \in \Omega^2(M, \mathbb{R}^n)$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$, $G_0 \in \Omega^4(M, \mathbb{R})$ with*

$$dF_0 = 0, \quad dH_0 = 0, \quad dG_0 - \langle F_0 \wedge H_0 \rangle_1 = 0 \tag{24}$$

2. *$T_2\mathbb{B}_n^{l,F_3}$-bundles with connection are equivalent to $(P, A, \mathcal{L}, C)$ admitting a T_2-dual of type F3.*
3. *$T_2\mathbb{B}_n^{r,F_3}$-bundles with connection are equivalent to $(\widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ arising as a T_2-dual of type F3.*
4. *If $(P, A, \mathcal{L}, C)$ and $(\widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ are related by the span, then they are T_2-dual. In this case, the curvatures of the connections are:*

$$A \to F_0, \quad C \to G_0 - \langle A \wedge H_0 \rangle_1, \tag{25}$$

$$\hat{B} \to H_0, \quad \hat{C} \to G_0 - \langle F_0 \wedge \hat{B} \rangle_1. \tag{26}$$

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

Let $V_0 = \mathbb{R}^n \oplus \Lambda^2(\mathbb{R}^n)^*$, $V_1 = (\mathbb{R}^n)^*$ and $V_2 = \mathbb{R}$. Consider the canonical pairings

$$\langle\cdot,\cdot\rangle_0 : V_0 \otimes V_0 \to V_1, \quad \langle v_1 + B_1, v_2 + B_2\rangle_0 = \iota_{v_1} B_2, \qquad \langle\cdot,\cdot\rangle_1 : V_0 \otimes V_1 \to V_2, \quad \langle v + B, A\rangle_1 = \iota_v A. \tag{27}$$

Constructing T2-Duals on Non-Trivial Topology via Lie 3-Groups

Let $V_0 = \mathbb{R}^n \oplus \Lambda^2(\mathbb{R}^n)^*$, $V_1 = (\mathbb{R}^n)^*$ and $V_2 = \mathbb{R}$. Consider the canonical pairings

$$\langle\cdot,\cdot\rangle_0 : V_0 \otimes V_0 \to V_1, \quad \langle v_1 + B_1, v_2 + B_2\rangle_0 = \iota_{v_1} B_2, \qquad \langle\cdot,\cdot\rangle_1 : V_0 \otimes V_1 \to V_2, \quad \langle v + B, A\rangle_1 = \iota_v A. \tag{27}$$

Theorem 2 ([Gagliardo et al., 2026])

The data $(V_0, V_1, V_2, \langle\cdot,\cdot\rangle_0, \langle\cdot,\cdot\rangle_1)$ determines a span of Lie almost 2-crossed modules

$$T_2\mathbb{B}_n^{l,F_2} \leftarrow T_2\mathbb{D}_n^{F_2} \to T_2\mathbb{B}_n^{r,F_2}. \tag{28}$$

1. *$T_2\mathbb{D}_n^{F_2}$-bundles with connection determine $F_0 \in \Omega^2(M, \mathbb{R}^n)$, $\hat{F}_0 \in \Omega^2(M, \Lambda^2(\mathbb{R}^n)^*)$, $H_0 \in \Omega^3(M, (\mathbb{R}^n)^*)$, $G_0 \in \Omega^4(M, \mathbb{R})$ with*

$$dF_0 = 0, \quad d\hat{F}_0 = 0, \quad dH_0 + \langle F_0 \wedge \hat{F}_0\rangle_0 = 0, \quad dG_0 - \langle F_0 \wedge H_0\rangle_1 = 0. \tag{29}$$

2. *$T_2\mathbb{B}_n^{l,F_2}$-bundles with connection are equivalent to $(P, A, \mathcal{L}, C)$ admitting a T_2-dual of type F2.*
3. *$T_2\mathbb{B}_n^{r,F_2}$-bundles with connection are equivalent to $(\hat{P}, \hat{A}, \widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ arising as a T_2-dual of type F2.*
4. *If $(P, A, \mathcal{L}, C)$ and $(\hat{P}, \hat{A}, \widehat{\mathcal{P}}, \hat{B}, \widehat{\mathcal{L}}, \hat{C})$ are related by the span, then they are T_2-dual. In this case, there is also a $(\mathbb{T}^n)^*$-gerbe with connection $(\mathcal{P}, B) \to P$ and the curvatures of the connections are:*

$$A \to F_0, \quad B \to H_0 + \langle A \wedge \hat{F}_0\rangle_0, \quad C \to G_0 - \langle A \wedge H_0\rangle_1 - \frac{1}{2}\langle A \wedge \langle A \wedge \hat{F}_0\rangle_0\rangle_1, \tag{30}$$

$$\hat{A} \to \hat{F}_0, \quad \hat{B} \to H_0 + \langle F_0 \wedge \hat{A}\rangle, \quad \hat{C} \to G_0 - \langle F \wedge \hat{B}\rangle_1. \tag{31}$$

CONSTRUCTING T2-DUALS ON NON-TRIVIAL TOPOLOGY VIA LIE 3-GROUPS

A $T_2\mathbb{D}_n^{F_2}$-bundle with connection is described over $\{U_i\}_{i\in I}$ by

$$\begin{aligned}
(v^0_{ij},\hat v^0_{ij}) : U_{ij} \to \mathbb{R}^n \times \Lambda^2(\mathbb{R}^n)^*, &\qquad (A_i,\hat A_i)\in\Omega^1(U_i,\mathbb{R}^n)\times\Omega^1(U_i,\Lambda^2(\mathbb{R}^n)^*),\\
(\lambda^0_{ijk},\hat\lambda^0_{ijk},v^1_{ijk}) : U_{ijk}\to\mathbb{Z}^n\times\Lambda^2(\mathbb{Z}^n)^*\times(\mathbb{R}^n)^*, &\quad \Lambda_{ij}\in\Omega^1(U_{ij},(\mathbb{R}^n)^*),\quad B_i\in\Omega^2(U_i,(\mathbb{R}^n)^*),\\
(\lambda^1_{ijkl},[v^2_{ijkl}]) : U_{ijkl}\to(\mathbb{Z}^n)^*\times\mathbb{R}/\mathbb{Z} &\qquad \Xi_{ijk}\in\Omega^1(U_{ijk},\mathbb{R}),\quad \Sigma_{ij}\in\Omega^2(U_{ij},\mathbb{R}),\quad C_i\in\Omega^3(U_i,\mathbb{R})
\end{aligned}\tag{32}$$

satisfying

$$\begin{aligned}
v^0_{ij}-v^0_{ik}+v^0_{jk} &= -\lambda^0_{ijk},\\
\hat v^0_{ij}-\hat v^0_{ik}+\hat v^0_{jk} &= -\hat\lambda^0_{ijk},\\
v^1_{ijk}-v^1_{ijl}+v^1_{ikl}-v^1_{jkl} &= \langle v^0_{ij},\hat\lambda^0_{jkl}\rangle_0-\lambda^1_{ijkl},\\
[v^2_{ijkl}-v^2_{ijkm}+v^2_{ijlm}-v^2_{iklm}+v^2_{jklm}] &= [-\langle v^0_{ij},\lambda^1_{jklm}\rangle_1+\langle\lambda^0_{ijk},\langle v^0_{ik},\hat\lambda^0_{klm}\rangle_0\rangle^{low}_1+\langle v^0_{ij},\langle v^0_{jk},\hat\lambda^0_{klm}\rangle_0\rangle^{low}_1],\\
A_j-A_i &= dv^0_{ij},\\
\hat A_j-\hat A_i &= d\hat v^0_{ij},\\
\Lambda_{ij}-\Lambda_{ik}+\Lambda_{jk} &= dv^1_{ijk}+\langle dv^0_{ij},\hat v^0_{jk}\rangle_0,\\
B_j-B_i &= d\Lambda_{ij}+\langle A_i\wedge d\hat v^0_{ij}\rangle_0,\\
\Xi_{ijk}-\Xi_{ijl}+\Xi_{ikl}-\Xi_{jkl} &= dv^2_{ijkl}-\langle dv^0_{ij},v^1_{jkl}\rangle_1+\langle v^0_{ij},\langle dv^0_{ij},\hat\lambda^0_{jkl}\rangle_0\rangle^{low}_1,\\
\Sigma_{ij}-\Sigma_{ik}+\Sigma_{jk} &= d\Xi_{ijk}-\langle dv^0_{ij}\wedge\Lambda_{jk}\rangle_1+\langle dv^0_{ij}\wedge\langle dv^0_{ij},\hat v^0_{jk}\rangle_0\rangle^{low}_1,\\
C_j-C_i &= d\Sigma_{ij}+\langle dv^0_{ij}\wedge B_j\rangle_1+\langle A_i\wedge\langle A_i\wedge d\hat v^0_{ij}\rangle_0\rangle^{low}_1.
\end{aligned}\tag{33}$$

Interpretation within M-theory

Interpretation within M-theory

Let $P^{11} = M^9 \times \mathbb{T}^1_{IIA} \times \mathbb{T}^1_M$, let $A = A^{IIA} + A^M$ be a connection on $P^{11} \to M^9$ with curvature $F_0^{IIA} + F_0^M$ and let $G \in \Omega^4_{cl}(P^{11}, \mathbb{R})$ be the field strength of an M2-brane.

1. Assume $G = G_0 - A^{IIA} \wedge H_0^{IIA} - A^M \wedge H_0^M - A^{IIA} \wedge A^M \wedge \hat{F}_0$. Then

$$d\hat{F}_0 = 0, \quad dH_0^M + F_0^{IIA} \wedge \hat{F}_0 = 0, \quad dH_0^{IIA} + F_0^M \wedge \hat{F}_0 = 0, \quad dG_0 - F_0^{IIA} \wedge H_0^{IIA} - F_0^M \wedge H_0^M = 0. \tag{34}$$

2. Then on $M^9 \times \mathbb{T}^1_{IIA}$ we have $H := H_0^M + A^{IIA} \wedge \hat{F}_0$, $F^2 := F_0^M$ and $F^4 := G_0 - A^{IIA} \wedge H_0^{IIA}$ with

$$dH = 0, \quad dF^2 = 0, \quad dF^4 + H \wedge F^2 = 0. \tag{35}$$

3. Prequantize $\hat{F}_0$ to $(M^9 \times \mathbb{T}^{IIB}, \hat{A})$. Then here we have $\hat{H} := H_0^M + F_0^{IIA} \wedge \hat{A}$, $\hat{F}^3 := H_0^{IIA} + F_0^M \wedge \hat{A}$ with

$$d\hat{H} = 0, \quad d\hat{F}^3 = 0. \tag{36}$$

4. To complete (F^2, F^4), $(\hat{F}^3)$ to RR field strengths, we need $G^7 \in \Omega^7(P^{11}, \mathbb{R})$ with $dG^7 + G \wedge G = 0$.
5. Without G^7, we can't prequantize $(H, F^2, F^4, \hat{H}, \hat{F}^3)$ in twisted K-theory. But we do something else.

Interpretation within M-theory

- Given $H \in \Omega^3(P^{10}, \mathbb{R})$ $F^2 \in \Omega^2(P^{10}, \mathbb{R})$, $F^4 \in \Omega^3(P^{10}, \mathbb{R})$ with

$$dH = 0, \quad dF^2 = 0, \quad dF^4 + H \wedge F^2 = 0. \tag{37}$$

Then:

1. Prequantize H as curvature of $\mathbb{T}^1$-gerbe with connection $(\mathcal{L}, B) \to P^{10}$. Call it **KR field** .
2. Prequantize F^2 as curvature of $\mathbb{T}^1$-bundle with connection $(P^{11}, A^M) \to P^{10}$. Call it **D0-brane** .
3. Then $F^4 - H \wedge A \in \Omega^4_{cl}(P^{11}, \mathbb{R})$, so prequantize it as curvature of 2-gerbe with connection $(\mathcal{L}, C) \to P^{11}$. Call it **D2-brane** .

Interpretation within M-theory

- Given $H \in \Omega^3(P^{10}, \mathbb{R})$ $F^2 \in \Omega^2(P^{10}, \mathbb{R})$, $F^4 \in \Omega^3(P^{10}, \mathbb{R})$ with

$$dH = 0, \quad dF^2 = 0, \quad dF^4 + H \wedge F^2 = 0. \tag{37}$$

  Then:
  1. Prequantize H as curvature of $\mathbb{T}^1$-gerbe with connection $(\mathcal{L}, B) \to P^{10}$. Call it **KR field** .
  2. Prequantize F^2 as curvature of $\mathbb{T}^1$-bundle with connection $(P^{11}, A^M) \to P^{10}$. Call it **D0-brane** .
  3. Then $F^4 - H \wedge A \in \Omega^4_{cl}(P^{11}, \mathbb{R})$, so prequantize it as curvature of 2-gerbe with connection $(\mathcal{L}, C) \to P^{11}$. Call it **D2-brane** .

- Given $\hat{H} \in \Omega^3(\hat{P}^{10}, \mathbb{R})$, $\hat{F}^3 \in \Omega^3(\hat{P}^{10}, \mathbb{R})$, $\hat{F}^5 \in \Omega^5(\hat{P}^{10}, \mathbb{R})$ with

$$d\hat{H} = 0, \quad d\hat{F}^3 = 0, \quad d\hat{F}^5 + \hat{H} \wedge \hat{F}^3 = 0. \tag{38}$$

  Then:
  1. Prequantize $\hat{H}$ as curvature of $\mathbb{T}^1$-gerbe with connection $(\widehat{\mathcal{P}}^{KR}, \hat{B}^{KR}) \to \hat{P}^{10}$. Call it **KR field** .
  2. Prequantize $\hat{F}^3$ as curvature of $\mathbb{T}^1$-gerbe with connection $(\widehat{\mathcal{P}}^{D1}, \hat{B}^{D1}) \to \hat{P}^{10}$. Call it **D1-brane** .
  3. Then $F^5 + \frac{1}{2}\hat{B}^{KR} \wedge \hat{F}^3 + \frac{1}{2}\hat{B}^{D1} \wedge \hat{H} \in \Omega^5_{cl}(\widehat{\mathcal{P}}, \mathbb{R})$, so prequantize it as curvature of 3-gerbe with connection $(\mathfrak{L}, D) \to \widehat{\mathcal{P}}$. Call it **D3-brane** .

Interpretation within M-theory

Proposition [Gagliardo et al., 2026]

A T_2-Duality correspondence of Type F_2 with $n = 2$ over M determines, and is always determined, by a left background of the form

$$
\begin{array}{l}
M \times B(\mathbb{T}_{IIA})^* \xrightarrow{H_0^{IIA} + A^M \wedge \hat{F}_0} M \times \mathbb{T}_M \\
M \times \mathbb{T}_M \times \mathbb{T}_{IIA} \times B^2U(1) \longrightarrow M \times \mathbb{T}_M, \quad M \times \mathbb{T}_M \times \mathbb{T}_{IIA} \times B^2U(1) \longrightarrow M \times \mathbb{T}_{IIA} \quad \left(G_0 - A^M \wedge H_0^M - A^{IIA} \wedge H_0^{IIA} - A^M \wedge A^{IIA} \wedge \hat{F}_0\right) \\
M \times B(\mathbb{T}_M)^* \xrightarrow{H_0^M + A^{IIA} \wedge \hat{F}_0} M \times \mathbb{T}_{IIA} \\
M \times \mathbb{T}_M \xrightarrow{F_0^M} M \xleftarrow{F_0^{IIA}} M \times \mathbb{T}_{IIA}
\end{array}
$$

.

and a right background of the form

$$
\begin{array}{l}
M \times \mathbb{T}_{IIB} \times B(\mathbb{T}_M^*) \times B(\mathbb{T}_{IIA})^* \times B^2U(1) \longrightarrow M \times \mathbb{T}_{IIB} \times B(\mathbb{T}_{IIA})^*, \quad \longrightarrow M \times \mathbb{T}_{IIB} \times B(\mathbb{T}_M)^* \quad \left(G_0 - F_0^M \wedge \hat{B}^M - F_0^{IIA} \wedge \hat{B}^{IIA}\right) \\
M \times \mathbb{T}_{IIB} \times B(\mathbb{T}_{IIA})^* \xrightarrow{H_0^{IIA} + F_0^M \wedge \hat{A}^{IIB}} M \times \mathbb{T}_{IIB} \xleftarrow{H_0^M + F_0^{IIA} \wedge \hat{A}^{IIB}} M \times \mathbb{T}_{IIB} \times B(\mathbb{T}_M)^* \\
M \times \mathbb{T}_{IIB} \xrightarrow{\hat{F}_0} M
\end{array}
$$

- These curvatures agree with the $(H, F_2, F_4, \hat{H}, \hat{F}_3)$ that we wanted to prequantize.
- Although we cannot construct a $D3$-brane in IIB, the 2-gerbe in the right background seems to be a shadow of it. We also see it as an $M2$, defined over the space $M \times \mathbb{T}_{IIB} \times B(\mathbb{T}^2)^*$.

The fact that T_2-Duality has a dimensional reduction to **T-Duality** of KR fields is expressed as follows.

Theorem 3 ([Gagliardo et al., 2026])

There is a commutative diagram

$$\begin{array}{ccccc} T\mathbb{B}^{F_2}_{n-1} & \longleftarrow & T\mathbb{D}^{F_2}_{n-1} & \longrightarrow & T\mathbb{B}^{F_2}_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ T_2\mathbb{B}^{F_2}_{n} & \longleftarrow & T_2\mathbb{D}^{F_2}_{n} & \longrightarrow & T_2\mathbb{B}^{F_2}_{n} \end{array}, \tag{39}$$

where the vertical arrows are injective in an appropriate sense.

Interpretation within M-theory

The fact that T_2-Duality has a dimensional reduction to **T-Duality** of KR fields is expressed as follows.

Theorem 3 ([Gagliardo et al., 2026])

There is a commutative diagram

$$\begin{array}{ccccc} T\mathbb{B}^{F_2}_{n-1} & \longleftarrow & T\mathbb{D}^{F_2}_{n-1} & \longrightarrow & T\mathbb{B}^{F_2}_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ T_2\mathbb{B}^{F_2}_{n} & \longleftarrow & T_2\mathbb{D}^{F_2}_{n} & \longrightarrow & T_2\mathbb{B}^{F_2}_{n} \end{array}, \tag{39}$$

where the vertical arrows are injective in an appropriate sense.

The fact that T_2-Duality lifts **S-Duality** is expressed as follows.

Theorem 4 ([Gagliardo et al., 2026])

The natural actions of $GL(n, \mathbb{Z})$ on $\mathbb{R}^n$, $(\mathbb{R}^n)^*$ and $\Lambda^2(\mathbb{R}^n)^*$ determine a span

$$T_2\mathbb{B}^{F_2}_{n} \rtimes GL(n, \mathbb{Z}) \longleftarrow T_2\mathbb{D}^{F_2}_{n} \rtimes GL(n, \mathbb{Z}) \longrightarrow T_2\mathbb{B}^{F_2}_{n} \rtimes GL(n, \mathbb{Z}). \tag{40}$$

Conclusions

- T_2**-Duality** is a relation between a field C coupling to 3d worldhseets over a $\mathbb{T}^n$-bundle and a field $\hat{C}$ coupling to 3d worldsheets over a $\Lambda^2(\mathbb{T}^n)^* \ltimes B(\mathbb{T}^n)^*$-bundle.

- The fields C and $\hat{C}$ prequantize the IIA and IIB field content that can be obtained from a closed 4-form in M-theory.

- With this interpretation, T_2-Duality is a **lift of T-Duality** along $\mathbb{T}^{n-1}$ that incorporates **S-Duality** as $GL(n,\mathbb{Z})$-equivariance.

- **Conjecture**: U-Duality in general topology relates an M-brane over a $\mathbb{T}^n$-bundle with an M-brane over a complicated fibration, and it has an underlying T_2-Duality.

- If this is true, then T_2**-folds** (defined as $T_2\mathbb{D}_n^{F_2} \rtimes Aut(T_2\mathbb{D}_n^{F_2})$-bundles) underlie **$U$-folds** .

THANKS!

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Interpretation within M-theory

An Example

- $M = S^2$.
- $n = 2$.
- $(P_{c_1,c_2}, A) \to M$ is a $\mathbb{T}^2$-bundle with connection with Chern class $(c_1, c_2) \in \mathbb{Z}^2 = H^2(S^2, \mathbb{Z})$.
- $(\mathcal{L}_{c_3}, C) \to P_{c_1,c_2}$ is a 2-gerbe with connection with Dixmier-Douady class $c_3 \in \mathbb{Z} = H^4(P_{c_1,c_2}, \mathbb{Z})$.
- There are no T_2-Duals $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}})$ of type F_3.
- There are T_2-Duals $(\hat{P}_{c_3}, \widehat{\mathcal{P}}_{-c_2,c_1}, \widehat{\mathcal{L}})$ of type F_2. They necessarily satisfy:
  - The Chern class of $\hat{P}_{c_3} \to S^2$ is $c_3 \in \mathbb{Z} \cong H^2(S^2, \Lambda^2(\mathbb{Z}^2)^*)$.
  - The Dixmier-Douady class of $\widehat{\mathcal{P}}_{-c_2,c_1} \to \hat{P}_{c_3}$ is $(-c_2, c_1) \in \mathbb{Z}^2 \cong H^3(\hat{P}_{c_3}, (\mathbb{Z}^2)^*)$.
- We give explicit cocycle data for all these bundles/gerbes and their connections.