The reduction of five dimensional Chern–Simons theories

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The effects of gauge symmetries on Lagrangian gauge theories which include the Chern–Simons term are studied herein. It is found that the Chern–Simons term appended to a five dimensional Yang–Mills theory makes nontrivial contributions to the classical field equations in four and two dimensions depending on the gauge group, but makes no contribution to the field equations in three dimensions, regardless of the gauge group. In addition to the reduction of Yang–Mills–Chern–Simons, the pure Chern-Simons theory is reduced in five dimensions—a topological field theory—to two and three dimensions. The potential physical observability of solutions to the resulting field equations is examined by determining if a nontrivial spontaneous symmetry breaking is permitted.

I. INTRODUCTION

The Chern–Simons term was discovered by mathematicians attempting to define a further level of conformal invariant—the secondary characteristic—in fiber bundles.[1] The first results in physics generated by use of the Chern–Simons term came ten years later in gauge theories: it was shown that the quantum theory of a Yang–Mills–Chern–Simons Lagrangian in $(2+1)$ dimensions leads to a quantized coupling constant and a mass for the gauge field,[2,3] and, soon thereafter it was observed that the Chern–Simons term is closely related to "consistent" chiral anomalies in any even dimensional quantum Yang–Mills theory.[4] Since then, there have been found classical static, solitary-wavelike solutions within both $(2+1)$ Maxwell–Higgs–Chern–Simons[5,6] and $(2+1)$ Higgs–Chern–Simons Lagrangian theories.[7–9] Also, it has been shown that the nonlinear, planar Schrödinger equation with coupling to non-Abelian Chern–Simons gauge fields possesses static, zero-energy solutions that satisfy self-duality equations.[10,11] Responding to the incentive to examine higher (odd) dimensional theories, the existence of chiral fermions interacting with gauge fields in four dimensions from a quantum Yang–Mills–Chern–Simons theory in $(4+1)$ dimensions has been recently demonstrated.[12] Insofar as the pure Chern–Simons theory is concerned, Witten has shown that by choosing a nonstandard Lie algebra inner product on a $(2+1)$ Chern–Simons ISO(2,1) gauge theory, the theory is equivalent to a $(2+1)$ gravity theory.[13] Pure Chern–Simons Lagrangians are examples of topological field theories and mathematicians now hope these will give new topological invariants of knots and three-manifolds.[14,15]

In this article we shall study the effects of gauge symmetry to obtain a dimensional reduction of two five dimensional, non-Abelian gauge theories containing the Chern–Simons term: the Yang–Mills–Chern–Simons theory and the pure Chern–Simons theory. In the next section we define the Yang–Mills and Yang–Mills–Chern–Simons Lagrangians on a principal fiber bundle over an oriented pseudo-Riemannian five manifold E, and give the field equations. From the invariance of the connection (the vector potential on the bundle) under a symmetry we derive bundle symmetry equations. Unfortunately, coset space dimensional reduction (CSDR) cannot be applied directly to this Lagrangian, because the Chern–Simons term is not gauge invariant.[16] This problem is avoided if our reduction takes place on the field equations, instead of the Lagrangian. In Sec. III we dimensionally reduce the nongauge invariant five

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dimensional Yang–Mills–Chern–Simons theory. We present the procedure and show it to be locally equivalent to CSDR for the gauge invariant Yang–Mills theory (cf. Ref. 16). We give a geometric interpretation to the Prasad–Sommerfield (PS) limit of the Yang–Mills–Higgs equations. When the reduction is applied to the Yang–Mills–Chern–Simons Lagrangian, we show that the Chern–Simons term contributes nontrivially to the reduced Yang–Mills field equations in four and two dimensions, but cancels to make no contribution in three dimensions. In Sec. IV we display the field equations reduced from a pure Chern–Simons theory in five dimensions in the "BPS limit." We examine the two and three dimensional field equations, and discuss their potential for nontrivial electromagnetic fields arising from spontaneous symmetry breaking.

II. THE BUNDLE SYMMETRY EQUATIONS

Let $\pi:P\rightarrow E$ be a principal fiber bundle with a compact, semisimple structure group G, over a $(4+1)$ dimensional manifold E, which we take to be extended space–time. We assume a G-invariant metric on E, and a connection ω on P. In this section, we shall first define the Yang–Mills and Yang–Mills–Chern–Simons functionals and give their field equations. Next, we derive a general equation for the Lie derivative of a connection ω in the direction of an infinitesimal automorphism generated by a group of automorphisms of P. When, in particular, the automorphisms are gauge symmetries, the bundle symmetry equations result. The bundle symmetry equations are used in the next section to set curvature components in the field equations equal to expressions involving particle fields.

We denote the set of all real valued k-forms on E by $\Lambda^k(E)$. Then, given a G-invariant metric on E we can define a Hodge star operator on E, $*: \Lambda^k(E)\rightarrow\Lambda^{5-k}(E)$. By pulling back the metric on E with the projection π, we can similarly define a star operator $\bar{*}$, which acts on $\bar{\Lambda}^k(P,\mathscr{G})$—the Lie algebra, $\mathscr{G}$, valued k-forms on P which vanish on vertical vectors (determined by the connection), $\bar{*}:\bar{\Lambda}^k(P,\mathscr{G})\rightarrow\bar{\Lambda}^{5-k}(P,\mathscr{G})$. This operator is used to define the Lagrangian densities. The Yang–Mills action density is

$$\mathscr{L}_{\mathrm{YM}}=-\tfrac{1}{2}\,\mathrm{tr}\,\Omega\wedge\bar{*}\Omega.$$

In $\mathscr{L}_{\mathrm{YM}}$, the star operator $\bar{*}$ acts on the bundle curvature $\Omega\in\bar{\Lambda}^2(P,\mathscr{G})$, where $\Omega=d\omega+\omega\wedge\omega$. The Chern–Simons term, $\mathrm{CS}(\omega)$, on P is

$$\mathrm{CS}(\omega)=3\,\mathrm{tr}\int_0^1\omega\wedge\Omega_s\wedge\Omega_s\,ds,$$

where $\Omega_s\equiv s\Omega+(s^2-s)\omega\wedge\omega$. We define the Yang–Mills–Chern–Simons action density by

$$\mathscr{L}_{\mathrm{YMCS}}=\mathscr{L}_{\mathrm{YM}}+\frac{\alpha}{3}\,\mathrm{CS}(\omega).$$

α is the coupling constant. The action $\int_E\mathscr{L}_{\mathrm{YMCS}}$ is gauge invariant for gauge transformations, $g:E\rightarrow G$, capable of being continuously deformed to the identity. Thus, if the homotopy group is nontrivial the Yang–Mills–Chern–Simons action can be gauge dependent. This fact has been used to argue the quantization of the coupling constant.[2,3]

In forming the variation of $\mathscr{L}_{\mathrm{YM}}$, we find that the field equations for the Yang–Mills action density are

$$\bar{*}D\bar{*}\Omega=0. \tag{2.1}$$

D is the exterior covariant derivative using the connection ω. These are the Yang–Mills equations in coordinate free notation.[17] The variation of the Chern–Simons term is[18]

$$\frac{d}{dt}(\mathrm{CS}(\omega_t))|_{t=0}=3\ \mathrm{tr}(\tau\wedge\Omega\wedge\Omega)+\text{exact form},$$

where $\omega_t\equiv\omega+t\tau$, for $\tau\in\bar{\Lambda}^1(P,\mathscr{G})$. For $G=\mathrm{SU}(2)$, there is no contribution from the Chern–Simons term to the field equations. Therefore assuming that the gauge group is other than $\mathrm{SU}(2)$ and observing that τ is arbitrary, the field equations for the Yang–Mills–Chern–Simons action density over a five dimensional extended space–time with gauge group G are found to be

$$\bar{*}\bar{D}\bar{*}\Omega=\frac{\alpha}{2}\ \bar{*}(\Omega\wedge\Omega). \tag{2.2}$$

Before we examine gauge symmetries, we recall that a bundle automorphism is a diffeomorphism, $f:P\to P$, which commutes with the G action. Consider the following theorem about bundle automorphisms:

Theorem 2.1: We are given a principal fiber bundle (PFB) $\pi:P\to E$ with structure group G, connection ω, and a group of automorphisms K, which act on P on the left. Denote $\mathscr{K}$ and $\mathscr{G}$ as the Lie algebras of K and G, respectively. Then for each $k\in\mathscr{K}$, k generates a vector field on P, $\tilde{k}$, and satisfies

$$\pounds_{\tilde{k}}\omega=D\phi+\Omega(\tilde{k},\cdot).$$

Ω is the curvature determined by ω. ϕ is defined by $\omega(\tilde{k})$ and is equivariant.

Proof: Let $\tilde{k}$ denote the vector field generated by $k\in\mathscr{K}$

$$\tilde{k}(p)\equiv\frac{d}{ds}[(\exp sk)p]|_{s=0}$$

and define a Lie algebra valued function on P, $\phi:P\to\mathscr{G}$, by $\phi(p)\equiv\omega_p(\tilde{k})\in\mathscr{G}$. The definition of the Lie derivative therefore gives the following expression, for $X_p\in T_pP$:

$$\pounds_{\tilde{k}}\omega(X_p)=d(\omega(\tilde{k}))(X_p)+d\omega(\tilde{k}(p),X_p)=d\phi(X_p)+d\omega(\tilde{k}(p),X_p).$$

In the adjoint representation

$$\omega\wedge\phi(X_p)=[\omega(X_p),\phi(p)],$$

$$d\omega(\tilde{k},X)(p)=\tilde{k}\omega(X)(p)-X\omega(\tilde{k})(p)-\omega([\tilde{k},X])(p),$$

$$\tfrac{1}{2}[\omega,\omega](\tilde{k},X)(p)=-[\omega(X_p),\omega(\tilde{k}_p)].$$

We may add zero in the form of $\omega\wedge\phi(X_p)+\frac{1}{2}[\omega,\omega](\tilde{k},X)(p)=0$, to get

$$\pounds_{\tilde{k}}\omega(X_p)=d\phi(X_p)+\omega\wedge\phi(X_p)+d\omega(\tilde{k}(p),X_p)+\tfrac{1}{2}[\omega,\omega](\tilde{k},X)(p)=(D\phi+\Omega(\tilde{k},\cdot))(X_p).$$

That ϕ is equivariant is seen from the following line:

$$\phi(pg)=\omega(\tilde{k}(pg))=\omega(R_{g*}\tilde{k}(p))=\mathrm{ad}(g^{-1})\omega(\tilde{k}_p)=\mathrm{ad}(g^{-1})\phi(p),$$

for $g\in G$ and R the right action by G. ■

A generalized gauge symmetry is a bundle automorphism which leaves the connection invariant—that is, preserves the horizontal and vertical tangent spaces. Gauge transformations are a special case of this. It is easy to show that when $k \in \mathscr{K}$ generates a symmetry, then $\pounds_{\tilde{k}}\omega = 0$,[19,20] and is equivalent to Forgács and Manton's formula for a gauge symmetry.[16] Therefore from Theorem 2.1, when k is a symmetry of the connection

$$D\phi = -\Omega(\tilde{k}, \cdot). \tag{2.3}$$

These are the bundle symmetry equations. We assume that $\tilde{k}$ is a Killing vector if a metric is given.

III. THE REDUCTION

In this section we dimensionally reduce the Yang–Mills and Yang–Mills–Chern–Simons (YMCS) theories on $\mathbf{R}^5$ equipped with a Minkowski metric. The nongauge invariance of the Chern–Simons term creates difficulties for coset space dimensional reduction. These difficulties are avoided by weakening the global requirements of coset space dimensional reduction to local requirements, and applying the local symmetries directly to the field equations rather than the Lagrangian. We assume that a generalized gauge symmetry group acting locally on the bundle P defines a local foliation of P by the infinitesimal action of a basis for the Lie algebra of the symmetry group, as defined in Theorem 2.1. We define local coordinates for the field equations by mapping infinitesimal actions to as many coordinate frame axes as possible. The bundle symmetry equations are then used. To remove any dependence on the order of the reductions, consistency conditions are employed. The reduced equations are defined on the local foliation. The algebraic considerations that break the gauge group are unchanged (see Ref. 16).

A. Yang–Mills reduction

We verify the reduction procedure in this section for the Yang–Mills theory in five dimensions, for which the results are well known. In addition, by using a geometric formulation we are able to express the Prasad–Sommerfield (PS) limit in a geometric form. This would allow a similar "limit" to be taken in the YMCS and pure Chern–Simons theories.

We choose to express the field equations by providing a smooth local coordinate basis for the bundle $P, \pi: P \to E$, where the local symmetry vector field $\tilde{k}_1$ is a member of the basis: $\{\tilde{X}_\mu, \tilde{X}_a, \tilde{k}_1\}$ for $\mu = 1,...,4$ and $a = 1,...,\dim(\mathscr{G})$. The basis vectors are chosen so that $\{\tilde{X}_a\}$ are all vertical, while $\{\tilde{X}_\mu\}$ are all horizontal. The Yang–Mills equations, ${}^*D^*\Omega = 0$, are evaluated on the basis to give ${}^*D^*\Omega(\tilde{X}_\mu) = 0$, and ${}^*D^*\Omega(\tilde{k}_1) = 0$. The Yang–Mills equations vanish identically on the vertical vectors, $\{\tilde{X}_a\}$. In coordinates, $({}^*D^*\Omega)^\alpha = D_\beta\Omega^{\alpha\beta} = 0$, where α, $\beta = 1,...,5$. The Einstein summation convention is enforced. Let the index $\alpha = 5$ correspond to the $\tilde{k}_1$ axis. From the bundle symmetry equations, $\Omega^{5\beta} = -D^\beta\phi$, we find that

$$D_\mu\Omega^{\alpha\mu} + D_5(\Omega^{\alpha 5}) = \begin{cases} D_\mu\Omega^{\nu\mu} + D_5(D^\nu\phi) = 0, & \text{if} \quad \alpha = \nu, \\ -D_\mu D^\mu\phi + D_5(D^5\phi) = 0, & \text{if} \quad \alpha = 5, \end{cases} \tag{3.1}$$

when μ, $\nu = 1,...,4$. In the first equation

$$-D_5(D^\nu\phi) = -D_5(\Omega^{\nu 5}) = -d(\Omega^{\nu 5})(\tilde{k}_1) - [\omega, \Omega^{\nu 5}](\tilde{k}_1) = -[\phi, \Omega^{\nu 5}] = [D^\nu\phi, \phi].$$

The expression $d(\Omega^{\nu 5})(\tilde{k}_1)$ vanishes. To see this we note that $\pounds_{\tilde{k}_1}\omega = 0$ implies that $\pounds_{\tilde{k}_1}\Omega = 0$, because $\Omega = d\omega + \omega \wedge \omega$, and, the Lie derivative both commute with the exterior derivative and has the Leibnizian property on wedge products. From the definition of the Lie derivative $(\pounds_{\tilde{k}_1}\Omega)(\tilde{X}_\nu, \tilde{k}_1) = 0$, $d(\Omega^{\nu 5})(\tilde{k}_1)$ is seen to vanish. For the second Eq. in (3.1), the curvature $\Omega^{\alpha\beta}$ is antisymmetric in the indices and so implies that $-D^5\phi = \Omega^{55} = 0$. We conclude that the equations after reduction are

$$D_\mu\Omega^{\nu\mu} = [D^\nu\phi, \phi], \quad D_\mu D^\mu\phi = 0.$$

When ϕ is taken to be the Higgs field, these equations are precisely the Yang–Mills–Higgs equations in the PS limit on a four-manifold. They may be returned to coordinate-free notation

$$\text{(a)}\quad \bar{*}D\bar{*}F=[D\phi,\phi]\equiv -J,$$

$$\text{(b)}\quad \bar{*}D\bar{*}(D\phi)=0.$$

The curvature F is that of the foliation. To avoid complications in the notation, permit us to denote the Hodge star operator on the foliation again by $*$. The current J is conserved.

We reduce to three-space by assuming another symmetry. The infinitesimal symmetry we denote by $\tilde{k}_2$. The Frobenius condition

$$[\tilde{k}_1,\tilde{k}_2]=\lambda_1\tilde{k}_1+\lambda_2\tilde{k}_2 \tag{3.2}$$

for λ_1,λ_2 both real functions on P, is required for a local foliation of P (involutive distribution). The reduction here is very similar to that above with $\mu=4$ corresponding to the $\tilde{k}_2$ axis. Let ϕ be denoted by ϕ_1, and a new field $\phi_2\equiv\omega(\tilde{k}_2)$. From (a), we get

$$\begin{aligned} D_jF^{ij}&=[D^i\phi_1,\phi_1]+[D^i\phi_2,\phi_2],\\ D_jD^j\phi_2&=[D\phi_1(\tilde{k}_2),\phi_1]. \end{aligned} \tag{3.3}$$

And (b) can be written as

$$D_iD^i\phi_1=-D_4D^4\phi_1. \tag{3.4}$$

Theorem 3.1: $D_4D^4\phi_1=D(\omega([\tilde{k}_2,\tilde{k}_1]))(\tilde{k}_2)+[\phi_2,D\phi_1(\tilde{k}_2)]$.

Proof: Note that for the moment we do not make use of the fact that we have a local foliation. From the definition of the exterior covariant derivative

$$D\phi_1(\tilde{k}_2)=d\phi_1(\tilde{k}_2)+[\omega(\tilde{k}_2),\phi_1]=d\phi_1(\tilde{k}_2)+[\phi_2,\phi_1].$$

Since $\tilde{k}_2$ is a symmetry we use the definition of the Lie derivative to give $(\pounds_{\tilde{k}_2}\omega)(\tilde{k}_1)=\tilde{k}_2(\phi_1)-\omega([\tilde{k}_2,\tilde{k}_1])=0$. Therefore, $d\phi_1(\tilde{k}_2)=\omega([\tilde{k}_2,\tilde{k}_1])$. We substitute this into our expression for $D\phi_1(\tilde{k}_2)$ to give

$$D\phi_1(\tilde{k}_2)=\omega([\tilde{k}_2,\tilde{k}_1])+[\phi_2,\phi_1]. \tag{3.5}$$

With $D\phi_2(\tilde{k}_2)=-F^{44}=0$ we conclude that

$$D(D\phi_1(\tilde{k}_2))(\tilde{k}_2)=D(\omega([\tilde{k}_2,\tilde{k}_1]))(\tilde{k}_2)+[\phi_2,D\phi_1(\tilde{k}_2)].\qquad\blacksquare$$

Using Theorem 3.1, Eq. (3.4) can be rewritten, so that the field equations become

$$\bar{*}D\bar{*}(D\phi_1)=D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_2)+[D\phi_1(\tilde{k}_2),\phi_2],$$

$$\bar{*}D\bar{*}(D\phi_2)=[D\phi_1(\tilde{k}_2),\phi_1].$$

Alternatively, we could have reversed the order of the two reductions giving

$$\bar{*}D\bar{*}(D\phi_1)=-[D\phi_1(\tilde{k}_2),\phi_2],$$

$$\bar{*}D\bar{*}(D\phi_2)=-D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_1)-[D\phi_1(\tilde{k}_2),\phi_1].$$

The correct field equations should not depend on the order in which the reductions were performed, thereby leading us to the compatibility equations

$$[D\phi_1(\tilde{k}_2),\phi_1]=-\tfrac{1}{2}D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_1),$$
$$[D\phi_1(\tilde{k}_2),\phi_2]=-\tfrac{1}{2}D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_2). \tag{3.6}$$

We now take into account that $\tilde{k}_1$ and $\tilde{k}_2$ define a local foliation of P. Using Eqs. (3.2), (3.5), and the compatibility equations, we find that the reduced field equations are

$$\bar{*}D\bar{*}H=[D\phi_1,\phi_1]+[D\phi_2,\phi_2],$$
$$\bar{*}D\bar{*}(D\phi_1)=-\lambda_1[\phi_1,\phi_2]-[[\phi_2,\phi_1],\phi_2], \tag{3.7}$$
$$\bar{*}D\bar{*}(D\phi_2)=\lambda_2[\phi_1,\phi_2]+[[\phi_2,\phi_1],\phi_1]$$

on the three dimensional local foliation M_3. If either $\tilde{k}_1$ or $\tilde{k}_2$ is an infinitesimal time symmetry so that M_3 is Riemannian, and viewing ϕ_1 and ϕ_2 as Higgs fields, Eqs. (3.7) are a general form of the Yang–Mills–Higgs field equations on M_3.

While Eqs. (3.7) are too difficult to solve, a certain class of solitary wavelike solutions—Bogomol'nyi–Prasad–Sommerfeld (BPS) magnetic monopoles—are known when the self-couplings of the Higgs fields are removed by the Prasad–Sommerfield (PS) limit. The geometric language used in this article can attach a geometric meaning to the PS limit in the Yang–Mills–Higgs theory. In the PS limit the right hand side of the second and third Yang–Mills–Higgs equations in (3.7) vanish. Equations (3.2), (3.5), (3.6), and the Yang–Mills–Higgs equations (3.7) inform us that the PS limit is equivalent to

$$D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_i)=0, \tag{3.8}$$

where $i=1,2$. This says that $\omega([\tilde{k}_1,\tilde{k}_2])$ is covariantly constant in the directions of symmetry. Commuting infinitesimal symmetries automatically satisfy this condition. If the symmetries do not commute but $\omega([\tilde{k}_1,\tilde{k}_2])=0$, then the commutator is horizontal with respect to ω. This implies, using the local foliation (3.2), that ϕ_1 and ϕ_2 are parallel or antiparallel in the Lie algebra. Finally, if $\omega([\tilde{k}_1,\tilde{k}_2])\neq 0$ then Eq. (3.2) and PS equations (3.8) imply that the Higgs fields are covariantly constant, $D\phi_1=D\phi_2=0$, on each local leaf in the foliation.

B. Yang–Mills–Chern–Simons reduction

The YMCS equations with gauge group G were found to be

$$\bar{*}D\bar{*}\Omega=\frac{\alpha}{2}\ \bar{*}(\Omega\wedge\Omega). \tag{2.2}$$

The dimensional reduction of the left hand side was computed in the previous section, thus our attention is drawn to the right hand side. We evaluate it on the basis $\{\tilde{X}_\mu,\tilde{X}_a,\tilde{k}_1\}$ to give

(a) $\bar{*}(\Omega\wedge\Omega)(\tilde{X}_\mu)$,

(b) $\bar{*}(\Omega\wedge\Omega)(\tilde{k}_1)$.

Expression (a) becomes

$$\frac{1}{4!}(2\epsilon_{5\mu\nu\rho\zeta}D_{[\nu}\phi F_{\rho\zeta]}+2\epsilon_{5\mu\nu\rho\zeta}F_{[\nu\rho}D_{\zeta]}\phi=\frac{1}{2}\bar{*}(D\phi\wedge F+F\wedge D\phi).$$

Similarly, (b) yields

$$\frac{1}{4!}\epsilon_{5\mu\nu\rho\zeta}F_{[\mu\nu}F_{\rho\zeta]}=\bar{*}(F\wedge F).$$

We conclude that the YMCS field equations after a reduction with a $\tilde{k}_1$ symmetry are

$$\begin{aligned}\bar{*}\bar{D}\bar{*}F&=[D\phi,\phi]+\frac{\alpha}{4}(D\phi\wedge F+F\wedge D\phi)\equiv -J,\\ \bar{*}\bar{D}\bar{*}(D\phi)&=\frac{\alpha}{2}\bar{*}(F\wedge F)\end{aligned}\tag{3.9}$$

on a four-manifold. Although the action density is not gauge invariant, J is still a conserved current.

For a further reduction by an infinitesimal $\tilde{k}_2$ symmetry, we again use the notation $\phi_1=\omega(\tilde{k}_1)$ and $\phi_2=\omega(\tilde{k}_2)$. Performing the reduction

$$\bar{*}\bar{D}\bar{*}H=[D\phi_1,\phi_1]+[D\phi_2,\phi_2]-\frac{\alpha}{6}\bar{*}(2D\phi_1\wedge D\phi_2-2D\phi_2\wedge D\phi_1+D\phi_1(\tilde{k}_2)H+HD\phi_1(\tilde{k}_2)),$$

$$\bar{*}\bar{D}\bar{*}(D\phi_1)=D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_2)+[D\phi_1(\tilde{k}_2),\phi_2]-\frac{\alpha}{4}\bar{*}(D\phi_2\wedge H+H\wedge D\phi_2),$$

$$\bar{*}\bar{D}\bar{*}(D\phi_2)=[D\phi_1(\tilde{k}_2),\phi_1]+\frac{\alpha}{4}\bar{*}(D\phi_1\wedge H+H\wedge D\phi_1)$$

and performed in the reversed order

$$\bar{*}\bar{D}\bar{*}H=[D\phi_1,\phi_1]+[D\phi_2,\phi_2]+\frac{\alpha}{6}\bar{*}(2D\phi_1\wedge D\phi_2-2D\phi_2\wedge D\phi_1-D\phi_2(\tilde{k}_1)H-HD\phi_2(\tilde{k}_1)),$$

$$\bar{*}\bar{D}\bar{*}(D\phi_1)=-[D\phi_1(\tilde{k}_2),\phi_2]-\frac{\alpha}{4}\bar{*}(D\phi_1\wedge H+H\wedge D\phi_1),$$

$$\bar{*}\bar{D}\bar{*}(D\phi_2)=-D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_1)-[D\phi_1(\tilde{k}_2),\phi_1]+\frac{\alpha}{4}\bar{*}(D\phi_2\wedge H+H\wedge D\phi_2).$$

The compatibility equations are found to be

$$2D\phi_1\wedge D\phi_2-2D\phi_2\wedge D\phi_1=-D\phi_1(\tilde{k}_2)H-HD\phi_1(\tilde{k}_2),$$

$$2[D\phi_1(\tilde{k}_2),\phi_2]-\frac{\alpha}{2}\bar{*}(D\phi_2\wedge H+H\wedge D\phi_2)=-D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_2),$$

$$2[D\phi_1(\tilde{k}_2),\phi_1]+\frac{\alpha}{2}\bar{*}(D\phi_1\wedge H+H\wedge D\phi_1)=-D(\omega([\tilde{k}_1,\tilde{k}_2]))(\tilde{k}_1).$$

Subject to the compatibility equations and $D\phi_1(\tilde{k}_2) = -D\phi_2(\tilde{k}_1)$, the reduced field equations on the three-manifold are found to reproduce the Yang–Mills–Higgs equations of (3.7). Therefore the addition of the Chern–Simons term to the Yang–Mills functional in five dimensions does not alter the reduced field equations in three dimensions. The cancellation of reduced Chern–Simons terms appears to continue to higher ($n>5$) odd dimensional Yang–Mills–Chern–Simons theories on $\mathbf{R}^n$ reduced by two infinitesimal symmetries.

C. Further reduction to two dimensions

In general we dimensionally reduce with three infinitesimal symmetries, $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, which define a local foliation $[\tilde{k}_1, \tilde{k}_2] = \alpha^i \tilde{k}_i$, $[\tilde{k}_2, \tilde{k}_3] = \beta^i \tilde{k}_i$, $[\tilde{k}_1, \tilde{k}_3] = \delta^i \tilde{k}_i$, where α^i, β^i, and δ^i are real-valued functions and $i=1,2,3$. We shall, however, assume that the infinitesimal symmetries commute, so that $\alpha_i = \beta_i = \delta_i = 0$. This gives a PS geometry (the geometrization of the PS limit). The reduced Yang–Mills field equations from five dimensions are

$$\begin{aligned}
\bar{*}D\bar{*}H &= [D\phi_1, \phi_1] + [D\phi_2, \phi_2] + [D\phi_3, \phi_3], \\
\bar{*}D\bar{*}(D\phi_1) &= [[\phi_1, \phi_2], \phi_2] + [[\phi_1, \phi_3], \phi_3], \\
\bar{*}D\bar{*}(D\phi_2) &= [[\phi_2, \phi_1], \phi_1] + [[\phi_2, \phi_3], \phi_3], \\
\bar{*}D\bar{*}(D\phi_3) &= [[\phi_3, \phi_1], \phi_1] + [[\phi_3, \phi_2], \phi_2].
\end{aligned} \tag{3.10}$$

Similarly, for Yang–Mills–Chern–Simons in five dimensions they reduce to

$$\begin{aligned}
\bar{*}D\bar{*}G &= \sum_{i=1}^{3} [D\phi_i, \phi_i] - \frac{\alpha}{6} \bar{*}(\{D\phi_1, [\phi_3, \phi_2]\} + \{D\phi_2, [\phi_1, \phi_3]\} + \{D\phi_3, [\phi_2, \phi_1]\}), \\
\bar{*}D\bar{*}(D\phi_1) &= [[\phi_1, \phi_2], \phi_2] + [[\phi_1, \phi_3], \phi_3] + \tfrac{2}{3}\alpha \bar{*}(D\phi_2 \wedge D\phi_3), \\
\bar{*}D\bar{*}(D\phi_2) &= [[\phi_2, \phi_1], \phi_1] + [[\phi_2, \phi_3], \phi_3] + \tfrac{2}{3}\alpha \bar{*}(D\phi_3 \wedge D\phi_1), \\
\bar{*}D\bar{*}(D\phi_3) &= [[\phi_3, \phi_1], \phi_1] + [[\phi_3, \phi_2], \phi_2] + \tfrac{2}{3}\alpha \bar{*}(D\phi_1 \wedge D\phi_2)
\end{aligned} \tag{3.11}$$

in two dimensions. Equations (3.10) and (3.11) are clearly different.

IV. THE REDUCTION OF PURE CHERN–SIMONS

The main result in Sec. III was that the addition of the Chern–Simons term to a Yang–Mills theory in five dimensions is removed in the dimensional reduction to three dimensions. However, in regions where the Yang–Mills term is dominated by the Chern–Simons term, so that the theory is effectively a pure Chern–Simons theory, the compatibility equations give rather different results. We begin this section by stating results from the reduction of five dimensional non-Abelian Chern–Simons to lower dimensions. We have withheld the details of the computation because of its similarity with Secs. III A and III B. Next we examine the potential observability of solutions to the field equations.

We shall assume, as we did in Sec. III C, that all the infinitesimal symmetries commute. As there is no longer a metric, we can remove the now empty requirement that the infinitesimal symmetries be Killing vectors. In the reduction to three dimensions, $D\phi_1 = D\phi_2$ is seen to be a compatibility equation. We define $\Phi_+ \equiv \phi_1 + \phi_2$, $\Phi_- \equiv \phi_1 - \phi_2$, and $\Phi = (\Phi_+ + \Phi_-)/2$. Then, the reduced equations to three dimensions can be written as

$$D\Phi \wedge D\Phi = 0, \quad \{D\Phi, H\} = 0, \quad \{[\Phi_+, \Phi_-], H\} = 0. \tag{4.1}$$

While those in two dimensions are found to be

$$[D\phi_1, D\phi_2] = 0, \quad \{[\phi_1, \phi_2], D\phi_3\} = 0, \quad \{[\phi_1, \phi_2], G\} = 0 \tag{4.2}$$

which are cyclic in the Higgs fields ϕ_1, ϕ_2, and ϕ_3.

The reduced Chern–Simons equations in both three and two dimensions permit observable electromagnetic fields in regions where the gauge group is broken down to electromagnetism using "spontaneous symmetry breaking." To see this, we recall that the equations which induce spontaneous symmetry breaking are

$$D\Phi = 0 \tag{4.3}$$

for some Higgs field Φ. In regions where $D\Phi = 0$ (the Higgs vacuum), we assume for simplicity in the expressions below that $|\Phi| = 1$. Equations (4.3) have a general solution for the gauge potential A_i^c when Φ is given[21,22]

$$A_i^c = \epsilon_{abc}\Phi^a \partial_i \Phi^b + \Phi^c A_i, \quad H_{ij} = \Phi F_{ij},$$
$$F_{ij} = \epsilon_{abc}\Phi^a \partial_i \Phi^b \partial_j \Phi^c + \partial_i A_j - \partial_j A_i, \tag{4.4}$$

with $i,j = 1,2,3$. A_i is an arbitrary real-valued function on $\mathbf{R}^3$. Now we let the Higgs field in Eq. (4.3) denote Φ in Eqs. (4.1). In regions where $D\Phi = 0$, F_{ij} in Eq. (4.4) can be shown to satisfy Maxwell's source-free equations. Moreover, with H of the form given by Eq. (4.4) and $D\Phi = 0$, the reduced Chern–Simons equations (4.1) are satisfied with F_{ij} generally nontrivial. Therefore the pure Chern–Simons theory permits nontrivial electromagnetic fields. The third equation in Eq. (4.4) defines a homotopy invariant, $k = \int_\Sigma \epsilon_{ijk} F^{jk}\, dS^i$, where Σ is a closed surface in $\mathbf{R}^3$. The invariant, k, measures the flux in regions where $D\Phi = 0$. Because F_{ij} satisfies Maxwell's source-free equations, we can identify F_{ij} with either the spatial part of the electromagnetic tensor, or, the spatial part of its dual. That is, k can represent either a magnetic or an electric charge, depending on the interpretation given to F_{ij}, and, in addition, the charge is nonzero only if Σ surrounds a region in which $D\Phi \neq 0$. This mimics the 't Hooft–Polyakov magnetic monopole in Yang–Mills–Higgs theory.

One might take for granted that nonzero electromagnetic fields exist in the Higgs vacuum. However it is for this reason that the dimensional reduction of the Euclidean self-duality equations on $\mathbf{R}^4$ to both $\mathbf{R}^3$ (the Bogomol'nyi equations for BPS magnetic monopoles) and $\mathbf{R}^2$ (vortex equations) with gauge group SO(3) or SU(2) are *not* of direct physical interest because quite simply the electromagnetic fields of the solutions vanish under spontaneous symmetry breaking, and therefore they are not conventionally observable. For example, the BPS magnetic monopole solutions to $B_i = D_i\Phi$ with gauge group SO(3) or SU(2) have vanishing magnetic field in regions where $D\Phi = 0$. As we saw above, BPS magnetic monopoles do not introduce a source term into conventional Maxwell's equations in regions of space where spontaneous symmetry breaking has occurred, unlike the Dirac magnetic monopole. In order to detect BPS SO(3) or SU(2) magnetic monopoles non-Abelian field detectors are needed. The reduced self-duality equations on $\mathbf{R}^2$ are given by[11,23]

$$D^{(0,1)}\Phi = 0, \quad H_{12} = [\phi_1, \phi_2], \tag{4.5}$$

where $D^{(0,1)}=D_1+iD_2$ and $\Phi=(\phi_1-i\phi_2)$. These equations can be derived using the reduction in Sec. III with $[k_1, k_2]=0$. With the gauge group SO(3) or SU(2), spontaneous symmetry breaking can be used as it was above to regain electromagnetism. We find that in regions where $D\Phi=0$, H is of the form given by Eq. (4.4) in two dimensions and F_{12} must vanish because the Lie algebra components in the second equation in (4.5) cannot possibly be in the same direction. Therefore, there are no field configurations which can be conventionally detected. Another class of topological field theories is considered in Ref. 24 that contain analogs to the self-duality and Bogomol'nyi equations. The Bogomol'nyi equations in Ref. 24 permit non-vanishing Abelian fields to exist in the Higgs vacuum. In a slightly modified form the reduced self-duality equations are of mathematical interest.[11]

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