K-theory and T-duality of topological phases

Intensive seminars on topological insulators and K-theory

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Part 1: Overview of topological phases and mathematical preliminaries on K-theory

Overview of topological phases

The meaning of *phase* depends on the class of physical systems being studied, e.g. thermodynamic, classical/quantum mechanical, symmetry-constrained,...., but there are some general features.
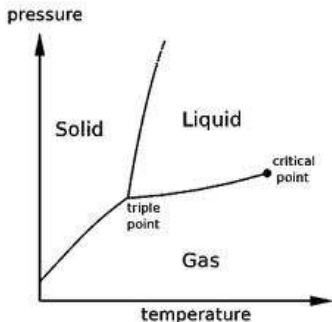
There are typically some parameters specifying a physical *state*, e.g. temperature, pressure, time, ...

The parameter space (*phase space/moduli space*) divides up into various connected pieces (phases) separated by *phase transitions*.

Overview of topological phases

The relevant phase space is often a rich topological space, and the *topological phases* are labelled by *topological invariants* familiar from mathematics.

Examples: winding numbers, homotopy/(co)homology groups, characteristic classes, K-theory, noncommutative generalisations,...

Physicists have been quite creative both in producing models for actual phenomena which realise such invariants, including exotic ones that were not known or studied mathematically!

Overview of topological phases

The role of topological invariants in high-energy physics and string theory is well-known: Gauge theory, anomalies, D-branes, topological field theories. . .

Topological phases in *condensed matter physics* (CMP) have a somewhat different origin. The underlying principles are "universal", and are already being applied in photonics, acoustics, classical mechanical systems, etc.

Most importantly, there are direct experimental realisations of many predicted topological phases!

Typical setup: Linear dynamics (Hilbert spaces, vector bundles, unitary time evolution), Symmetries (representation theory), Spectral gap condition etc.

Bulk-boundary correspondence

An important consequence is the bulk-boundary correspondence —
Topological invariants for bulk physical system may be "invisible"
in a spectral sense (gap condition), but "holographically" detected
as zero modes on a boundary (Index theory).
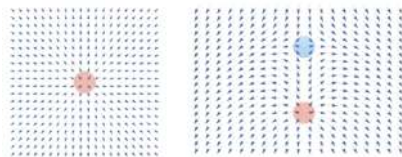
Here are some "cartoons" that physicists have in mind:
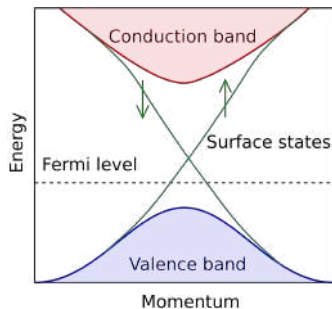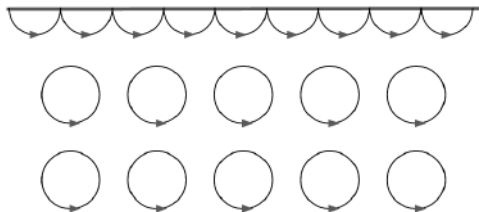
Short history of experimentally found topological phases

At least three Nobel Prizes have been awarded in direct relation to topological phases in CMP: 2016 (Thouless–Kosterlitz–Haldane), 1985 (von Klitzing), 1998 (Laughlin–Störmer–Tsui).

1970s-80s — Kosterlitz–Thouless transitions

1980 — Integer quantum Hall effect
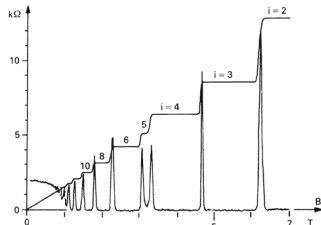
Short history of experimentally found topological phases

1982 — Fractional quantum Hall effect

1986 (maybe 1960) — Aharanov–Bohm effect

2008 — Quantum spin Hall effect (?)
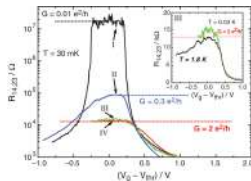
D. Hsieh, et al., Nature 452, 970 (2008)

2009 — $\mathbb{Z}_2$ 3D topological insulator
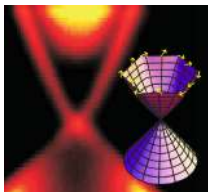
Short history of experimentally found topological phases

2014 — Chern insulator (anomalous quantum Hall effect)
2015 — Weyl semimetals
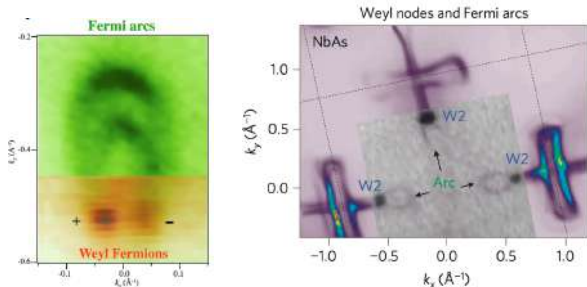

(L) Xu et al, Science **349** 613 (2015); (R) [—] Nature Phys. **11** 748 (2015).

Last ten years — Generalisations and analogues of above, e.g. in photonics

Kitaev's Periodic Table

TABLE 1. Classification of free-fermion phases with all possible combinations of the particle number conservation (Q) and time-reversal symmetry (T). The $\pi_0(C_q)$ and $\pi_0(R_q)$ columns indicate the range of topological invariant. Examples of *topologically nontrivial* phases are shown in parentheses.

q	$\pi_0(C_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		(IQHE)	
1	0			

Above: insulators without time-reversal symmetry (i.e., systems with Q symmetry only) are classified using complex K-theory.

Right: superconductors/superfluids (systems with no symmetry or T-symmetry only) and time-reversal invariant insulators (systems with both T and Q) are classified using real K-theory.

q	$\pi_0(R_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		no symmetry $(p_x+ip_y,$ e.g., SrRu$)$	T only $(^3$He-$B)$
1	$\mathbb{Z}_2$	no symmetry (Majorana chain)	T only $((p_x+ip_y)\uparrow + (p_x-ip_y)\downarrow)$	T and Q (BiSb)
2	$\mathbb{Z}_2$	T only ((TMTSF)$_2$X)	T and Q (HgTe)	
3	0	T and Q		
4	$\mathbb{Z}$			
5	0			
6	0			
7	0			no symmetry

A. Kitaev, arXiv:0901.2686, 2009

Ryu–Schnyder–Furusaki–Ludwig Periodic Table

Cartan	d												
	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

Bott's Periodic Table

Stable homotopy groups

	k (mod 8)							
	0	1	2	3	4	5	6	7
$\pi_k(U(n))$ $k \leq 2n-1$	0	∞	0	∞	0	∞	0	∞
$\pi_k(SO(n))$ $0 < k \leq n-2$	2	2	0	∞	0	0	0	∞
$\pi_k(SO(2n)/U(n))$ $0 < k \leq 2n-2$	2	0	∞	0	0	0	∞	2
$\pi_k(U(2n)/Sp(n))$ $k \leq 4n-1$	0	∞	0	0	0	∞	2	2
$\pi_k(Sp(2n)/Sp(n) \times Sp(n))$ $0 < k \leq 4n+2$	∞	0	0	0	∞	2	2	0
$\pi_k(Sp(n))$ $k \leq 4n+1$	0	0	0	∞	2	2	0	∞
$\pi_k(Sp(n)/U(n))$ $k \leq 2n$	0	0	∞	2	2	0	∞	0
$\pi_k(U(n)/SO(n))$ $k \leq n-1$	0	∞	2	2	0	∞	0	0
$\pi_k(SO(2n)/SO(n) \times SO(n))$ $0 < k \leq n-1$	∞	2	2	0	∞	0	0	0

Reference: R. Bott, Ann. of Math. 70 (1959), 313-337.

Plan of lectures

There are many review articles on the *physics* side, not so much the *mathematical* side (especially why *K*-theory is relevant).

This is an unfortunate gap in the literature, making it hard for mathematicians to appreciate or enter the field seriously.

Actually, the NCG community had already been using K-theory, C^*-algebras, index theory, etc. to study the Quantum Hall effect.

Understanding some basic physical insights lead us to generalise mathematical results in previously ill-motivated directions.

I will proceed by a series of examples, each motivating the introduction of certain mathematical ideas.

Plan of lectures

Su–Schrieffer–Heeger (SSH) model: superalgebra and Toeplitz index theorem

Chern insulator: Bundle invariants and K-theory, spectral flow

3D topological insulator: "Real" K-theory

CT symmetries and tenfold way: Clifford algebras and Bott periodicity

Bulk-boundary correspondence: T-duality and extension theory

Quantum Hall effect: C^*-algebras and noncommutative geometry

Crystallographic and hyperbolic phases: Twisted K-theory, Baum–Connes isomorphism

Weyl semimetals: Differential topology (if there is time)

Mathematical preliminaries: K-theory

Abstractly, topological K-theory is a generalised cohomology functor $X \mapsto K^{-n}(X), n \in \mathbb{Z}$ on locally compact Hausdorff spaces.

Concretely, elements of $K^0(X)$ are usually represented by vector bundles over X.

K-theory is much more useful than just a classification framework (contrary to impression from physics literature).

Functoriality (right-way maps) guarantees coherent comparison between classifications over different spaces, and wrong-way maps (topological index) relate to analysis (spectra of operators) through index theorems.

Computable through Bott periodicity, Baum–Connes maps (related to T-duality), Mayer–Vietoris sequences (locality property).

Mathematical preliminaries: Complex K-theory

For compact Hausdorff X, consider finite-rank complex vector bundles $\mathcal{E}$ over X and their isomorphism classes $[\mathcal{E}]$. These form a commutative monoid $\mathcal{V}(X)$ w.r.t. Whitney (direct) sum $\oplus$. Thus $[\mathcal{E}] + [\mathcal{F}] \equiv [\mathcal{E} \oplus \mathcal{F}] \equiv [\mathcal{F}] + [\mathcal{E}]$ but there are *no inverses*.

$K^0(X)$ is the resulting abelian group obtain by Grothendieck's *universal* completion. Why do this?

Usually the $\oplus$ structure is physical (combining two subsystems together). Whenever we want an *abelian group*-valued topological invariant f, we can always factor through $K^0(X)$ uniquely.

$$\mathcal{V}(X) \xrightarrow{\quad \iota \quad} K^0(X)$$
$$f \searrow \qquad \downarrow \exists!\, g$$
$$G$$

Mathematical preliminaries: Grothendieck construction

Notation: drop $[\cdot]$ (taking isomorphism classes is implied)
Explicitly, we consider *pairs* $\mathcal{V}(X) \times \mathcal{V}(X)$, written as a formal *difference bundle* $\boldsymbol{\mathcal{E}} \equiv \mathcal{E}_+ \ominus \mathcal{E}_-$.

Impose the relation

$$\boldsymbol{\mathcal{E}} \sim \boldsymbol{\mathcal{F}} \Leftrightarrow \exists \mathcal{G} \text{ s.t. } \mathcal{E}_+ + \mathcal{F}_- + \mathcal{G} = \mathcal{F}_+ + \mathcal{E}_- + \mathcal{G}.$$

$K^0(X) := (\mathcal{V}(X) \times \mathcal{V}(X))/_\sim$, and we write K-theory classes as $[\boldsymbol{\mathcal{E}}] \equiv [\mathcal{E}_+ \ominus \mathcal{E}_-]$.

The identity is $[\mathbf{0}] = [0 \ominus 0]$ and the inverse is $-[\boldsymbol{\mathcal{E}}] = [\mathcal{E}_- \ominus \mathcal{E}_+]$.

Clearly $[\mathcal{G} \ominus \mathcal{G}] = \mathbf{0}$ is always trivial, and adding such "trivial differences" to $\boldsymbol{\mathcal{E}}$ does not change its K-theory class.

Mathematical preliminaries: Grothendieck construction

As an example, compute $K^0(\mathrm{pt})$, so $\mathcal{E}$ are vector spaces.

Isomorphism classes are just $\mathbb{C}^m$ labelled by dimension $m \in \mathbb{N}$, thus $\mathcal{V}(\mathrm{pt}) = (\mathbb{N}, +)$.

Cancellation law holds in $\mathbb{N}$, so the Grothendieck completion is pairs $m_+ \ominus m_- \in \mathbb{N} \times \mathbb{N}$ with relation

$$m_+ \ominus m_- \sim n_+ \ominus n_- \Leftrightarrow m_+ + n_- = n_+ + m_-$$

This is just the construction of the integers $\mathbb{Z}$, so $K^0(\mathrm{pt}) = \mathbb{Z}$.

Positive integers $m \leftrightarrow [m \ominus 0] = [m+1 \ominus 1] = [m+2 \ominus 2] = \dots$
Negative integers $-m \leftrightarrow [0 \ominus m]$.

Mathematical preliminaries: Grothendieck construction

The "super", or $\mathbb{Z}_2 = \{\pm\}$ graded, point of view will be useful for physical applications later, where $\pm$ is associated to positive/negative energy sectors (or particle/antiparticle).

Start with *graded* vector bundles, denoted $\boldsymbol{\mathcal{E}} = \mathcal{E}_+ \ominus \mathcal{E}_-$, and take monoid $\boldsymbol{\mathcal{V}}(X)$ of *graded* isomorphism classes (i.e. $E_\pm \cong F_\pm$). Addition means $\boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{F}} = (\mathcal{E}_+ \oplus \mathcal{F}_+) \ominus (\mathcal{E}_- \oplus \mathcal{F}_-)$.

Define the trivial submonoid $\boldsymbol{\mathcal{T}}(X)$ as those $\boldsymbol{\mathcal{T}}$ which admit an odd bundle automorphism $\mathcal{I} = \begin{pmatrix} 0 & \mathcal{I}_\mp \\ \mathcal{I}_\pm & 0 \end{pmatrix}$, $\mathcal{I}^2 = 1$.

E.g. if $\mathcal{E}_+ \cong \mathcal{E}_- \cong \mathcal{G}$, then $\boldsymbol{\mathcal{E}} = \mathcal{G} \ominus \mathcal{G}$ is considered trivial (as before).

Mathematical preliminaries: Grothendieck construction

Declare $\mathcal{E} \sim \mathcal{F}$ if there exists $\mathcal{T}, \mathcal{T}' \in \mathcal{T}(X)$ such that $\mathcal{E} \oplus \mathcal{T} \cong \mathcal{F} \oplus \mathcal{T}'$.

Then define $K^0(X) = \mathcal{V}(X)/_\sim$.

This is an abelian group, in which $-[\mathcal{E}] = [\mathcal{E}^{\mathrm{op}}] := [\mathcal{E}_- \ominus \mathcal{E}_+]$ (the oppositely-graded bundle).

In this "super" picture, we are encoding the freedom to add or remove "particle-antiparticle" pairs $[\mathcal{G} \ominus \mathcal{G}]$, and taking K-theory classes means extracting the invariant that is leftover after such perturbations.

This should remind you of the notion of Fredholm index.

Mathematical preliminaries: C^*-algebra K-theory

It is useful (and essential for some generalisations) to formulate $K^0(X)$ in terms of *operator K*-theory of the corresponding C^*-algebra $C(X)$.

Recall: Bounded operators $\mathcal{L}(\mathscr{H})$ on a (complex) Hilbert space is a Banach $*$-algebra with respect to adjoint and operator norm.

Definition: A C^*-algebra $\mathcal{A}$ is a norm-closed $*$-subalgebra of the bounded operators $\mathcal{L}(\mathscr{H})$ on a Hilbert space.

Commutative example: Continuous functions $C(X)$ with pointwise multiplication and complex conjugation, and supremum norm. Can think of $f \in C(X)$ as multiplication operator M_f on $L^2(X)$ w.r.t some suitable measure.

Noncommutative example: $\mathcal{L}(\mathscr{H})$ itself, matrix algebras $M_n(\mathbb{C}), M_n(\mathcal{A})$

Mathematical preliminaries: Noncommutative topology

Gelfand–Naimark theorem The categories of compact Hausdorff spaces and unital commutative C^*-algebras are (anti)-equivalent.

The morphisms are, respectively, continuous maps and $*$-homomorphisms.

In one direction the correspondence on objects is $X \mapsto C(X)$. A continuous map $f : X \to Y$ corresponds to pullback/precomposition $f^* : C(Y) \to C(X)$.

In the other direction, if $\mathcal{A}$ is a unital commutative C^*-algebra, the character/maximal ideal space/spectrum/state space $\Omega(\mathcal{A})$ of nonzero $*$-homomorphisms $\mathcal{A} \to \mathbb{C}$ is compact in topology of pointwise convergence.

Mathematical preliminaries: Noncommutative topology

For $\mathcal{A} = C(X)$, the spectrum $\Omega(\mathcal{A})$ comprises the evaluation characters $\mathrm{ev}_x, x \in X$. Thus $\Omega(C(X)) \cong X$.

For $X = \Omega(\mathcal{A})$, the algebra $C(X)$ comprises the *Gelfand transforms* $\widehat{a}$ labelled by $a \in \mathcal{A}$. Here, the complex-valued function $\widehat{a}$ has value at $\rho \in \Omega(\mathcal{A})$ equal to $\rho(a)$. Thus $C(\Omega(\mathcal{A})) \cong \mathcal{A}$.

A construction on an actual spaces X has an analogue in terms of the C^*-algebra $\mathcal{A} = C(X)$. Dropping the commutativity requirement formally defines the construction for the "noncommutative space" $\mathcal{A}$.

Mathematical preliminaries: Vector bundles and projections

Example: By the Serre–Swan theorem, a vector bundle $\mathcal{E} \to X$ corresponds to a finitely-generated projective (f.g.p.) module over $C(X)$. This proceeds by taking the space of sections $\Gamma(\mathcal{E})$, which admits pointwise multiplication by $C(X)$.

A free module over $C(X)$ has the form $C(X)^n$, which corresponds to the section space of the trivial bundle $X \times \mathbb{C}^n$. A f.g.p. module is a direct summand of $C(X)^n$, corresponding to the section space of some subbundle of $X \times \mathbb{C}^n$.

The projection onto the direct summand is an *idempotent* p in $M_n(C(X))$. Equivalence of vector bundles corresponds to *similarity* of idempotents ($p \sim upu^{-1}$ where u is invertible in $M_n(C(X))$.

Mathematical preliminaries: Vector bundles and projections

The ambient trivial bundle may have arbitrarily large rank n, and we were taking direct sums of bundles anyway, so we might as well consider the corresponding idempotent to live in the direct limit $M_\infty(C(X)) = \cup_{n \in \mathbb{N}} M_n(C(X))$.

This confers a great advantage: Similarity of idempotents becomes equivalent to *homotopy* of idempotents!

By doubling the dimension $n \to 2n$, there is enough room to exhibit a homotopy from $\mathrm{diag}(U, U^{-1})$ to the identity. Let

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \ t \in [0, \frac{\pi}{2}],$$

$$U_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R(t) \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} R(t)^{-1}.$$

Then $P_t = U_t \, \mathrm{diag}(p, 0) U_t^{-1}$ is a homotopy from $\mathrm{diag}(upu^{-1}, 0)$ to $\mathrm{diag}(p, 0)$.

Mathematical preliminaries: C^*-algebra K_0-theory

For C^*-algebras, we can use projections and unitaries in place of idempotents and invertibles. Analogous to giving the bundles a Hermitian metric.

Definition: For a unital C^*-algebra $\mathcal{A}$, let $\mathcal{V}(\mathcal{A})$ be the monoid of homotopy classes of projections in $M_\infty(\mathcal{A})$. Then $K_0(\mathcal{A})$ is the Grothendieck completion of $\mathcal{V}(\mathcal{A})$.

One often constructs vector bundles from a physical system, and wants to classify them "up to homotopy". The above formulation of K-theory exhibits this idea.

Functoriality: if $\pi : \mathcal{A} \to \mathcal{A}'$, then there is a map $\pi_* : K_0(\mathcal{A}) \to K_0(\mathcal{A}')$ by taking $\pi_*[p] = [\pi(p)]$.

Mathematical preliminaries: C^*-algebra K_1-theory

Let $U_n(\mathcal{A})$ denote the group of unitaries in $M_n(\mathcal{A})$. By appending 1 along the diagonal, we can form $U_\infty(\mathcal{A}) = \cup_{n \in \mathbb{N}} U$.

A convenient way to define $K_1(\mathcal{A})$ is as the homotopy classes of unitaries in $U_\infty(\mathcal{A})$.

Example: Let $\mathcal{A} = \mathbb{C}$. The unitary groups $U(n)$ are all connected, so $K_1(\mathbb{C}) = K^{-1}(\mathrm{pt}) = 0$.

Example: Let $\mathcal{A} = C(S^1)$. Homotopy class of $u : S^1 \to U(n)$ is given by the winding number of $\det u : S^1 \to U(1)$. Using Bott periodicity, we can actually show that $K_1(C(S^1)) = K^{-1}(S^1) \cong \mathbb{Z}$ with generator z with winding number 1.

Mathematical preliminaries: Suspensions in K-theory

The suspension of $\mathcal{A}$ is $C_0(\mathbb{R}, \mathcal{A})$, the *non-unital* C^*-algebra of functions $\mathbb{R} \to \mathcal{A}$ which vanish at ∞.

A nonunital C^*-algebra $\mathcal{B}$ is a "noncommutative locally compact space". It can be unitised to $\mathcal{B}^+$ ("one-point compactification"). Take $\mathcal{B} \times \mathbb{C}$ with $(b_1, \lambda)(b_2, \mu) = (b_1 b_2 + \lambda b_2 + \mu b_1, \lambda\mu)$.

There is a morphism $\pi : \mathcal{B}^+ \to \mathbb{C}$ which induces π_* in K-theory. K_0 is extended to non-unital algebras $\mathcal{B}$ by defining

$$K_0(\mathcal{B}) = \ker (\pi_* : K_0(\mathcal{B}^+) \longrightarrow K_0(\mathbb{C})).$$

There is a natural isomorphism between $K_0(C_0(\mathbb{R}, \cdot))$ and $K_1(\cdot)$. There is also a notion of suspension using Clifford algebras, based around ideas of Karoubi, Atiyah–Bott–Shapiro, Kasparov.

Mathematical preliminaries: Suspensions in K-theory

In the topological K-theory of spaces, one similarly suspends $X \to \mathbb{R}^n \times X$ and uses "K-theory with compact supports" to define $K^{-n}(X)$.

Either way, we obtain a sequence of generalised (co-)homology functors $K_n(\cdot)$ or $K^{-n}(\cdot)$, and there are long exact sequences (LES) associated to each $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, i.e.

$$\ldots \xrightarrow{\partial} K_n(\mathcal{A}) \to K_n(\mathcal{B}) \to K_n(\mathcal{C}) \xrightarrow{\partial} K_{n+1}(\mathcal{A}) \to \ldots$$

The connecting maps ∂ are a kind of topological index measuring obstructions to lifting e.g. projections/unitaries from $\mathcal{C}$ to $\mathcal{B}$, and are important for formulating bulk-boundary correspondences.

K-theory has *Bott periodicity*, i.e. $K_2(\cdot) \simeq K_0(\cdot)$. This means that the LES wrap around itself and is cyclic with just six terms.

Part 2: Toeplitz index theory and the SSH model

Quantum mechanics of a particle

We will now study the simplest physical model whose topological invariant is $K^{-1}(S^1) \cong \mathbb{Z}$. Then we will see that the bulk-boundary correspondence in this case is an expression of the Toeplitz index theorem [T (unpublished)].

Some basic principles of quantum mechanics:

- Space of quantum states (the "wavefunctions") is a complex Hilbert space $\mathcal{H}$.

- Dynamics of a closed system is given by unitary time evolution $U_t = e^{-iHt}$. Its generator $H = H^\dagger$ is called the Hamiltonian, and is the observable corresponding to energy.

- If there is a group of symmetries, it is represented unitarily on $\mathcal{H}$ and commutes with H.

Quantum mechanics of a particle

The Hilbert space of a (free, nonrelativistic) electron moving in Euclidean space $\mathbb{R}^d$ is $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$ where $\mathbb{C}^N$ encodes some "internal" degrees of freedom (e.g. spin).

The Euclidean group acts on $L^2(\mathbb{R}^d)$, but this symmetry is usually "broken".

In a crystalline material, the electron moves in a potential V provided by some regular lattice of atoms, so the Hamiltonian $H = H_{\text{free}} + V$ only has a lattice $\mathbb{Z}^d$ of translation symmetry (and maybe some others like reflection, rotation, etc. which we ignore for now).

In the tight-binding approximation, one assumes that it suffices to use a Hilbert subspace of wavefunctions that are localised around the atomic sites. Then one uses $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ instead, and the lattice translations act regularly on this subspace.

SSH model

Consider a 1D model $\mathscr{H} = l^2(\mathbb{Z}) \otimes \mathbb{C}^2 = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ which has sublattice symmetry. This means that there are two translation invariant sublattices, which we call A and B.
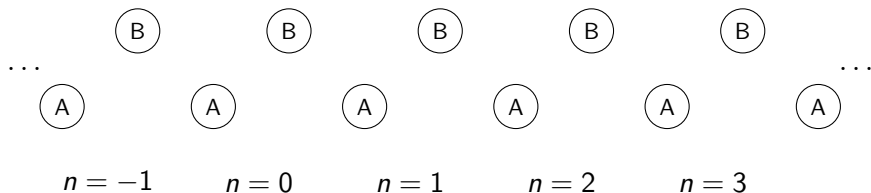

Thus $\mathscr{H} = l^2(\mathbb{Z}_{(A)}) \oplus l^2(\mathbb{Z}_{(B)}) \equiv \mathscr{H}_A \oplus \mathscr{H}_B$ is graded by the sublattice operator $S = 1_{\mathscr{H}_A} \oplus -1_{\mathscr{H}_B}$. (We could also use $-S$.)

SSH model

We wish to study Hamiltonians $H = H^\dagger$ which commute with the translations but anticommute with S. The latter condition, $HS = -SH$, means that S inverts the spectrum of H about 0.

Later we will see that QM quite generally allows this sort of "odd" symmetry besides the usual "even" ones which commute with H.

Thus the operator H always exchanges $A \leftrightarrow B$. For example, the "hopping Hamiltonians" H_{blue} and H_{red} illustrated below translate an A degree of freedom (d.o.f.) to an adjacent B d.o.f.

$$n = -1 \qquad n = 0 \qquad n = 1 \qquad n = 2 \qquad n = 3$$
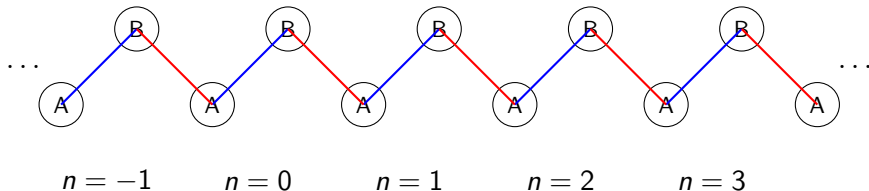
SSH model

Because $\mathbb{Z}$ is an abelian group, we can Fourier transform $l^2(\mathbb{Z})$ to $L^2(S^1)$. For mathematicans, S^1 is the Pontryagin dual group of unitary characters (irreps) of $\mathbb{Z}$. The characters are $\chi_k : n \mapsto e^{ikn}$ labelled by the circle coordinate $k \in [0, 2\pi]/_{0 \sim 2\pi}$.

Physicists call this the Brillouin zone of quasimomenta, or sometimes simply "momentum space".

Thus we have $\mathscr{H} = L^2(S^1) \oplus L^2(S^1)$, with S acting as

$$S(k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and translation by $n \in \mathbb{Z}$ becomes pointwise multiplication by e^{ink}.

SSH model

Since H is odd w.r.t S,

$$H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}$$

We assume that $k \mapsto H(k)$ is continuous, so that $k \mapsto U(k)$ is continuous[1].

We also assume that H is gapped (at 0), i.e. 0 is not in its spectrum. Then $(H(k))^2 > 0$ so we need $U(k)U(k)^* \neq 0$, so that $U(k) \in \mathbb{C}^* \cong \mathsf{GL}(1)$.
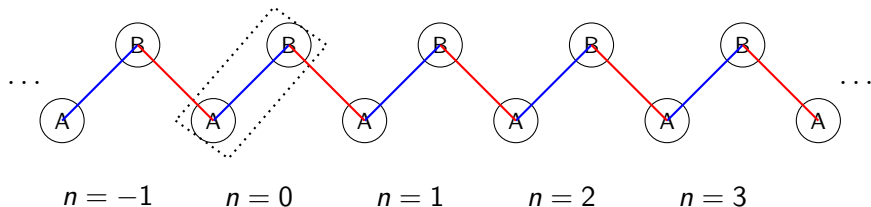
We can homotope H to $\mathrm{sgn}(H)$, which corresponds to replacing $U(k)$ by $U(k)/|U(k)| \in \mathsf{U}(1)$. There is an obvious topological invariant, which is the winding number of $U : S^1 \to \mathsf{U}(1)$.

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[1]This is related to decay of the hopping terms as the hopping range goes to infinity.

SSH model

What is the position space meaning of U and Wind(U)?

The "hopping term" U_{blue} taking A to B rightwards within a unit cell is represented, after Fourier transform, by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, whereas the term U_{red} taking B to A rightwards changes unit cell and is represented by $\begin{pmatrix} 0 & e^{\mathrm{i}k} \\ 0 & 0 \end{pmatrix}$.

SSH model

The general Hamiltonian is a self-adjoint combination of powers of $U_{\mathrm{blue}}, U_{\mathrm{red}}$ which is also required to be gapped and compatible with S, so that after Fourier transform, it has the form

$$H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}^* \tag{1}$$

Consider the "fully-dimerised" Hamiltonian
$H_{\mathrm{blue}} = U_{\mathrm{blue}} + U_{\mathrm{blue}}^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has winding number 0.

Another fully-dimerised Hamiltonian is
$H_{\mathrm{red}} = U_{\mathrm{red}} + U_{\mathrm{red}}^{\dagger} = \begin{pmatrix} 0 & e^{ik} \\ e^{-ik} & 0 \end{pmatrix}$ which has winding number 1.

SSH model

There is an important subtlely in the above analysis.

It is clear that H_{blue} and H_{red} are unitarily equivalent by translating the A sublattice by one unit, or equivalently, choosing a different unit cell convention such that T_{red} is an intra-cell hopping term rather than an inter-cell one:
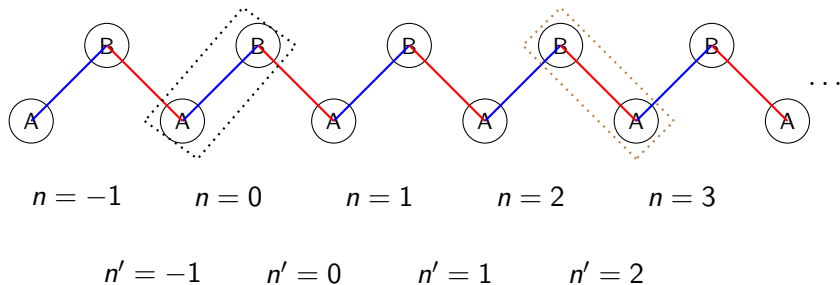
SSH model

Generally, in labelling the atomic sites by $\mathbb{Z}$, we have chosen an origin with respect to which the Fourier transform is taken. A different choice means that we have to conjugate everything in Fourier space by a large gauge transformation e^{imk}.

In the SSH model, we have to choose an origin for each of the A and B sublattices. The two choices shown previously are related by $\begin{pmatrix} e^{ik} & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to shifting the origin of the A lattice by one unit.

Thus the absolute winding number has some inherent ambiguity, although the *change* in winding number does not — this is already familiar from the notion of *polarisation*.

The *difference* between the winding numbers for H_{blue} and H_{red} is a gauge-invariant signature that they cannot be deformed into one another without violating the gapped condition.

SSH model

Nevertheless, when a boundary of the chain is specified, the unit cells are well-defined — this corresponds to a "gauge-fixing".

With respect to this choice, the bulk topological invariant of H (the winding number) is well-defined. Then we may say that H_{blue} is a trivial phase and H_{red} is nontrivial.
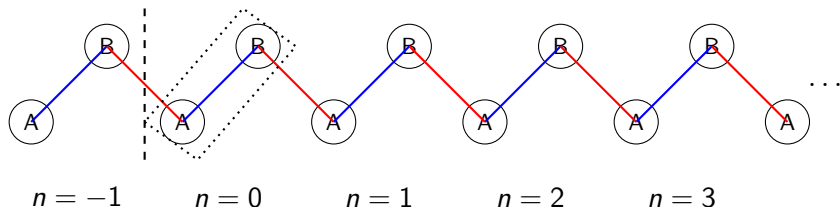
When the boundary is introduced, the red hopping terms is "cut", leaving behind a "dangling zero mode' of A type. This is why it makes sense to say that H_{red} is nontrivial.

SSH model

The general H will acquire some number of "dangling zero modes" of A type and B type when it is truncated to the right half-line.

We can define a boundary invariant for H as the difference in the number of resulting A and B zero modes.

It turns out that bulk invariant (winding number of U) equals boundary invariant, as a consequence of an index theorem.

Strictly speaking we should also define a notion of "addition of phases", and then the equality of invariants is an isomorphism $\mathbb{Z} \to \mathbb{Z}$ of groups. The language for this is naturally in K-theory, but we shall begin with the above simple "2-level" model first.

Toeplitz operators

Recall that the *Hardy space* $\mathcal{H}^2 \equiv \mathcal{H}(S^1)$ is defined to be the Hilbert subspace of $l^2(\mathbb{Z}) \cong L^2(S^1)$ spanned by the basis vectors e_n with $n \geq 0$.

Applying the projection $p : L^2(S^1) \to \mathcal{H}$ is the operation of truncating to non-negative Fourier coefficients, or to the right-half line in our picture.

After applying p, we see that the intra-cell hopping operator U_{blue} remains unitary, whereas the inter-cell U_{red} becomes only an isometry $\widetilde{U}_{\text{red}}$:

$$\widetilde{U}_{\text{red}} \widetilde{U}_{\text{red}}^{\dagger} = 1 - p_{n_A=0}$$

where $p_{n_A=0}$ is the projection onto the A-site at $n = 0$.

Toeplitz operators

In fact, $\widetilde{U}_{\mathrm{red}}$ is a *Toeplitz* operator with *symbol* the invertible function $S^1 \ni k \mapsto U_{\mathrm{red}}(k) = e^{\mathrm{i}k}$.
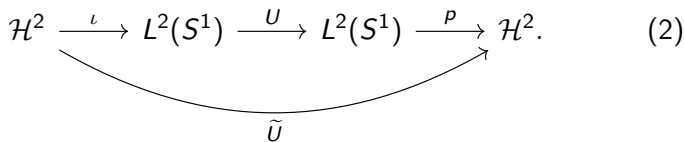
It is furthermore a *Fredholm* operator with index equal to $-\mathrm{Wind}(u) = -1$.

Recall that a bounded operator is *Fredholm* if its kernel and cokernel are finite-dimensional. Its (analytic) *Fredholm index* is the difference of these dimensions.

The product of two Fredholm operators is again Fredholm, and the index of a product is the sum of the indices. Furthermore, the index is invariant under a compact perturbation.

Toeplitz operators

The *Toeplitz operator* $\widetilde{U}$ with continuous symbol $U \in C(S^1)$ is the compression of the multiplication operator U on $L^2(S^1)$ to the Hardy space $\mathcal{H}^2$,

$$\mathcal{H}^2 \xrightarrow{\iota} L^2(S^1) \xrightarrow{U} L^2(S^1) \xrightarrow{p} \mathcal{H}^2. \tag{2}$$

$$\widetilde{U}$$

The operator $\widetilde{U}$ is Fredholm iff its symbol U is invertible everywhere (i.e. U is an invertible element of $C(S^1)$).

Toeplitz/Göhberg–Krein index theorem: (analytic) Fredholm index of a Toeplitz operator is minus the topological winding number of its symbol.

Index theorem and bulk-boundary correspondence

The boundary invariant for H_{red} defined earlier can be thought of as $\ker \widetilde{U}_{\mathrm{red}}^{\dagger} - \ker \widetilde{U}_{\mathrm{red}}$, which is just (-1) times the Fredholm index of the operator $\widetilde{U}_{\mathrm{red}}$.

This integer must equal the bulk invariant $\mathrm{Wind}\, U_{\mathrm{red}}$ by the index theorem.

Thus the SSH model provides a physical interpretation of this index theorem. Alternatively, the index theorem explains why there is a bulk-edge correspondence of *topologically meaningful* integers.

Addition of phases

Higher winding numbers may be obtained by "dimerising" across more unit cells, which causes more bonds to be intercepted by the boundary, leaving more zero modes unpaired.

A more natural alternative is to consider direct sums of the basic SSH model, so that there are $2N$ sites per unit cell. Then a S-symmetric Hamiltonian has the form

$$H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^\dagger & 0 \end{pmatrix}, \quad U(k) \in \mathrm{GL}(N, \mathbb{C}). \qquad (3)$$

which still has a bulk topological invariant given by the ordinary winding number of $\det(U)$.

Allowing N to be finite but arbitrarily large, we can take direct sums $H(k) \oplus H'(k)$ as the operation of "adding physical systems" or phases. Physically, the arbitrarily large N is available because $L^2(\mathbb{R}) \cong l^2(\mathbb{Z}) \otimes L^2(\mathbb{R}/\mathbb{Z})$ and the second factor is "internal".
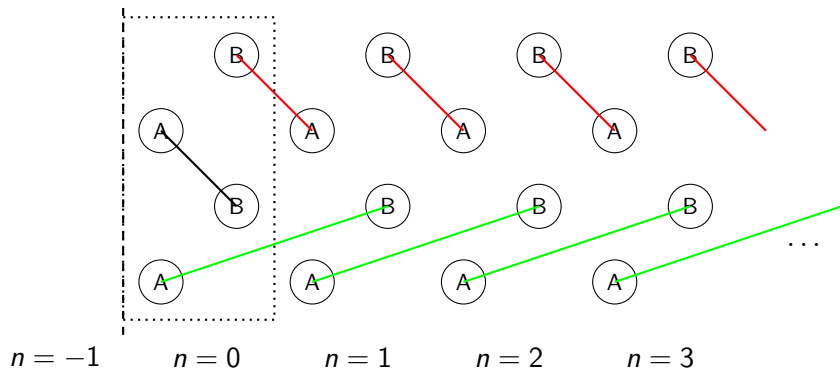
Addition of phases

Consider H_{green} with winding number -1. The truncated $\widetilde{H}_{\text{green}}$ acquires a dangling B zero mode.
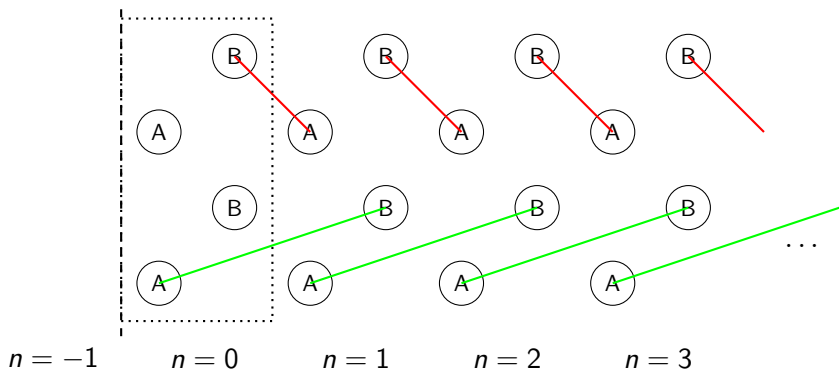
Addition of phases

The bulk story says that $H_{\mathrm{red}} \oplus H_{\mathrm{green}}$ has total winding number zero $\Rightarrow$ "trivial".

Addition of phases

The boundary story is: A and B boundary zero modes at position $n = 0$ can be paired and gapped out by turning on a boundary term which is compatible (anticommutes) with $S|_{n=0}$,

$$H_{\mathrm{bdry}} = \begin{pmatrix} 0 & a \\ \overline{a} & 0 \end{pmatrix},$$

which has spectrum $\pm|a|$. Thus the two zero modes in combination are *not* topologically protected, in accordance with the cancelling winding numbers.

K-theoretic topological phases

We need to be able to (1) consider Hamiltonians with N finite but arbitrarily large, and (2) extract a topological invariant for $U \oplus U'$ which is additive with respect to $\oplus$, in accordance with the addition of boundary zero modes.

This is precisely what K-theory allows us to do. In more detail, allowing for direct sums and noting that invertible matrices can be retracted to unitary ones, the S-symmetric Hamiltonians are classified by the homotopy classes of maps from S^1 into the infinite unitary group $\mathscr{U} = \cup_{N \in \mathbb{N}} U(N)$.

This is $K_1(S^1)$ which can be shown to be $\mathbb{Z}$ by Bott periodicity. The generator may be represented as follows:

For each N, the winding number of a map $S^1 \to \mathrm{U}(N)$ can be defined as the ordinary winding number of its determinant, and there is a map with winding number 1. $N = 1$ was the case we'd analysed.

K-theoretic phases

The winding number is a homomorphism with respect to matrix multiplication, in that Wind(UV)=Wind(U)+Wind(V).

Recall the homotopy

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \ t \in [0, \frac{\pi}{2}],$$

$$U_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R(t) \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} R(t)^{-1}.$$

which takes $u \oplus v$ to $\mathrm{diag}(uv, 1)$. Then we see that the winding number is also additive w.r.t. direct sums (of K-theoretic phases).

Toeplitz index in K-theory

On the analytic side, we can extend the discussion of Toeplitz operators to that on $(\mathcal{H}^2)^{\oplus N}$ for any finite N.

A continuous function $N \times N$ matrix-valued function U on S^1 defines a multiplication operator by U on $(L^2(S^1))^{\oplus N}$, whose truncation to $(\mathcal{H}^2)^{\oplus N}$ is the Toeplitz operator $\widetilde{U}$ with matrix-valued symbol U.

As before, $\widetilde{U}$ is Fredholm iff U is invertible, and the index of $\widetilde{U}$ is equal to $-\mathrm{Wind}(U)$. The index is unchanged if $\widetilde{U}$ is modified into $\widetilde{U} \oplus 1_{N'}$ acting on N' extra copies of $\mathcal{H}^2$.

Thus we can consider the map $[U] \mapsto \mathrm{Index}(\widetilde{U})$ as a homomorphism $K^{-1}(S^1) \to \mathbb{Z}$ (which is an isomorphism).

Theorem: Let $U \in C(S^1, \mathrm{GL}(N, \mathbb{C}))$ represent a class $[U]$ in $K^{-1}(S^1)$. The analytic index map $[U] \mapsto \mathrm{Index}(\widetilde{U})$ is equal to the topological index $-\mathrm{Wind}(U)$.

Part 3: Chern insulator, $\mathbb{Z}_2$-topological insulator, and "Real" K-theory

Topological phases classified by bundle invariants

Previously, we saw an example of a class of physical systems (Hamiltonians compatible with $\mathbb{Z}$ translations and sublattice symmetry S), for which the gapped phases have a homotopy classification by $K^{-1}(S^1) \cong \mathbb{Z}$.

A geometric way to think about $K^{-1}(S^1)$ is as a classification of "large gauge transformations" of a trivial bundle over S^1.

Each sublattice corresponds to sections of a trivial line bundle over S^1, and the gapped Hamiltonian effectively implements an isomorphism between the two bundles.

There are bundle isomorphisms which are not connected to the identity map.

Let us now see how non-trivial bundles can appear in similar models, but with different symmetries.

Topological phases classified by bundle invariants

Recall the typical tight-binding Hilbert space $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^m$. The character (momentum) space of $\mathbb{Z}^d$ is $\mathbb{T}^d$, called the *Brillouin torus* in physics.

Fourier transform[2] diagonalises $\mathbb{Z}^d$ invariant operators on $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^m$ into multiplication operators on $L^2(\mathbb{T}^d \times \mathbb{C}^m) \cong L^2(\mathbb{T}^d) \otimes \mathbb{C}^m$.

The Hamiltonian becomes a fibered family of *Bloch Hamiltonians* $k \mapsto H(k), k \in \mathbb{T}^d$, each with m eigenvalues.
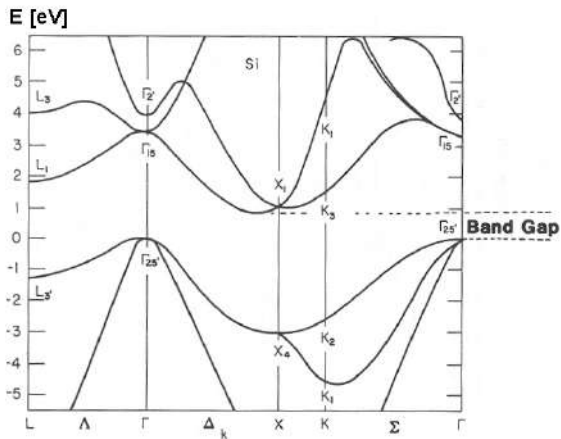
For a *band insulator*, the spectra of $H(k)$ assemble into continuous bands over $\mathbb{T}^d$, with a gap at a particular energy called the *Fermi energy* (which we set to zero)

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[2]The full-blown theory is Bloch–Floquet theory, which will be discussed in another talk.

Topological phases classified by bundle invariants

Topological phases classified by bundle invariants

Electrons obey the Pauli exclusion principle, so they fill up the low energy states up until a certain Fermi energy E_F.

In order for electrons to act as charge carriers, there needs to be vacant electronic states to move into. Then the electrons can increase their energy (accelerate) when an electric field is applied.

If E_F lies in a band gap, an electron must traverse this energy barrier to access the next available state for conduction, thus the *insulating* property.

Topological phases classified by bundle invariants

The bundle $\mathcal{E} = \mathbb{T}^d \times \mathbb{C}^m$ is graded by the gapped Hamiltonian into a positive energy sub-eigenbundle and a negative energy sub-eigenbundle $\mathcal{E}_F$. These are respectively called the conduction and valence bundles.

The negative energy valence bundle can be topologically non-trivial when $d \geq 2$!

This has important consequences: the composite bands are trivial and come from a tight-binding Hilbert space, but $\mathcal{E}_F$ by itself may not! In other words, there is a topological obstruction to localisability of wavefunctions.

This prompts us to seek a classification of possible $\mathcal{E}_F$ (in an appropriate category).

Topological phases classified by bundle invariants

The simplest class systems that exhibits this kind of topological phase is in $d = 2$. No additional symmetry besides $\mathbb{Z}^2$ translations is present.

It is known that there is a non-trivial line bundle over $\mathbb{T}^2$, which is just the pullback of the Hopf (or tautological line) bundle over S^2 under a degree-1 map $\mathbb{T}^2 \to S^2$ (e.g. the collapse map to the 2-cell).

It is fairly easy to construct a Hamiltonian whose $\mathcal{E}_F$ is such a non-trivial complex line bundle. This is sometimes called the Chern insulator phase because the topological invariant is the first Chern class.

Chern topological insulator

We seek a classification complex vector bundles $\mathcal{E}_F$ over $\mathbb{T}^2$. These are classified by the (first) Chern class $c_1(\mathcal{E}_F) = c_1(\det \mathcal{E}_F)$.

Thus we look just at line bundles, or principal $U(1)$ bundles. They are classified by
$[\mathbb{T}^2, BU(1)] = [\mathbb{T}^2, \mathbb{CP}^\infty] = [\mathbb{T}^2, K(\mathbb{Z}, 2)] = H^2(\mathbb{T}^2; \mathbb{Z})$.

This is a fancy way of saying that there is a universal tautological line bundle $\mathcal{E}_{\mathrm{taut}} \to \mathbb{CP}^\infty$, and any line bundle over X is obtained by pulling back $\mathcal{E}_{\mathrm{taut}}$ under a classifying map $X \to \mathbb{CP}^\infty$.

The isomorphism class of the pullback bundle only depends on the homotopy class of the classifying map. $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ so
$[\mathbb{T}^2, BU(1)] = H^2(\mathbb{T}^2; \mathbb{Z})$.

Chern topological insulator

Because $\mathbb{T}^2$ has low dimension, it suffices to look at classifying maps $\mathbb{T}^2 \to \mathbb{CP}^1 \subset \mathbb{CP}^\infty$.

It is known that $\mathbb{CP}^1$ is homeomorphic to S^2. The physicist's way to see this is through the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

They span the traceless 2×2 Hermitian matrices, and have the (Clifford algebra) property $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$.

Then the spectrum of the linear combination $\boldsymbol{h} \cdot \boldsymbol{\sigma}$ is easily computed: the square is a scalar $|\boldsymbol{h}|^2$, so the eigenvalues are $\pm|\boldsymbol{h}|$.

Each unit vector $\widehat{\boldsymbol{h}} \in S^2 \subset \mathbb{R}^3$ defines a point in $\mathbb{CP}^1$ by taking the (-1) eigenspace of $\widehat{\boldsymbol{h}} \cdot \boldsymbol{\sigma}$. This may be shown to be a bijection.

Chern topological insulator

A family of 2×2 Bloch Hamiltonians $H(k)$ over $\mathbb{T}^2$ can be written as $H(k) = \boldsymbol{h}(k) \cdot \boldsymbol{\sigma}$, where $\boldsymbol{h} : \mathbb{T}^2 \to \mathbb{R}^3$ is some 3-vector field.

The gapped condition is simply that $\boldsymbol{h}$ is nowhere vanishing, so the *unit* vector field $\widehat{\boldsymbol{h}} : \mathbb{T}^2 \to S^2$ may be defined.

The valence line bundle $\mathcal{E}_F$ is the collection of (-1) subspaces for $\widehat{\boldsymbol{h}}(k) \cdot \boldsymbol{\sigma}$, $k \in \mathbb{T}^2$.

By the earlier identification of $S^2 \cong \mathbb{CP}^1$, $\mathcal{E}_F$ is nothing but the pullback of the tautological bundle over $\mathbb{CP}^1$ under the classifying map $\widehat{\boldsymbol{h}}$.

The Chern class of $\mathcal{E}_F$ in $H^2(\mathbb{T}^2, \mathbb{Z})$ is the homotopy class of $\widehat{\boldsymbol{h}} : \mathbb{T}^2 \to S^2$ (i.e. the Brouwer degree, or the number to times S^2 is "wrapped" around by $\widehat{\boldsymbol{h}}$).

Chern topological insulator

Choose any nowhere vanishing $\boldsymbol{h}$ such that $\widehat{\boldsymbol{h}}$ has nonzero degree n. Then the Hamiltonian $H(k) = \boldsymbol{h}(k) \cdot \boldsymbol{\sigma}$ has a nontrivial valence bundle with Chern class n, i.e. a *Chern insulator* phase.

The topological invariant which distinguishes these phases is $c_1(\mathbb{T}^2, \mathbb{Z}) \in H^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$.

We can be quite explicit and use the following vector field:

$$\mathbf{h}(k) = (\cos k_1, \cos k_2, m + a\cos(k_1 + k_2) + b(\sin k_1 + \sin k_2))$$

where m, a, b are real parameters. The Chern class of $\mathcal{E}_F$ may be computed to be

$$c_1(\mathcal{E}_F) = \mathrm{sgn}(-m - a) + \frac{1}{2}[\mathrm{sgn}(m - a + 2b) + \mathrm{sgn}(m - a - 2b)].$$

Chern topological insulator phase diagram

Schematically a partial phase diagram would look like:
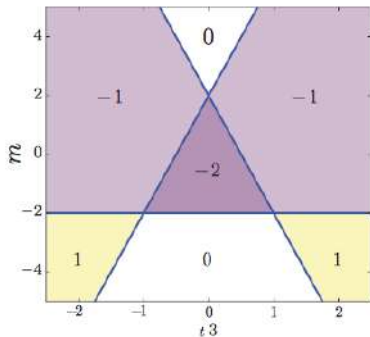
Sticlet et al, PRB 85 (2012)

We should, however, be careful. Such models introduce extra non-physical constraints. They can only realise some of the possible phases, and may also artificially introduce some extra ones.

Chern topological insulator via curvature methods

Usually in the physics literature, the Chern insulator is discussed in the context of "Berry connection" and "Berry curvature".

When $\mathcal{E}_F$ is a line bundle, the (gauge-dependent) Berry connection is usually defined (locally) as

$$A = \frac{1}{i}\langle\psi_-|\mathrm{d}\psi_-\rangle$$

where $|\phi_i\rangle$ is a local section of $\mathcal{E}_F$. Alternatively, there is a canonical Grassmann connection on the tautological bundle over $\mathbb{CP}^1$ which is pulled back to $\mathcal{E}_F$.

The Berry curvature $\mathcal{F} = dA$ of this connection is gauge-invariant, and integrates over $\mathbb{T}^2$ to give an *integral* Chern number. This is because $\mathcal{F}$ is an *integral* class in $H^2(\mathbb{T}^2, \mathbb{R})$ (de Rham cohomology).

Chern topological insulator via curvature methods

In terms of the vector field $\boldsymbol{h}$, one can show that

$$c(\mathcal{E}_\mathcal{F}) = \frac{1}{4\pi} \int_{\mathbb{T}^2} \mathrm{d}k \, \widehat{\mathbf{h}}(k) \cdot \frac{\partial \widehat{\mathbf{h}}(k)}{\partial k_1} \wedge \frac{\partial \widehat{\mathbf{h}}(k)}{\partial k_2}$$

where the integrand is the Berry curvature.

In NCG language, this is the Chern character

$$\frac{\mathrm{i}}{2} \mathrm{Tr}(P \mathrm{d}P \wedge \mathrm{d}P)$$

for the projection $p_F \in M_2(C(\mathbb{T}^2))$,

$$p_F(k) = \frac{1}{2}(1 - \hat{\boldsymbol{h}}(k) \cdot \boldsymbol{\sigma})$$

K-theoretic topological insulator

Since we know some K-theory, e.g. that it provides a classification of complex vector bundles over a space, we could also consider $K^0(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. As before, K-theory allows the natural "addition" of phases.

Under the Chern character map from K-theory to (rational) cohomology, the two factors of $\mathbb{Z}$ correspond to the rank (number of bands) and the first Chern class.

The rank invariant is often ignored (so one is looking at *reduced* K-theory), but actually it has physical meaning as the density of valence states. Furthermore, we will see in a later talk that T-duality exchanges rank $\leftrightarrow$ Chern in this case!

K-theoretic topological insulator

In anticipation of a later talk, let us note that $C(\mathbb{T}^2)$ is the (reduced) group C^*-algebra of the underlying symmetry group $\mathbb{Z}^2$.

The group C^*-algebra is generated by the image of the regular representation of $\mathbb{Z}^2$ (in this case, by multiplication operators $e^{\mathrm{i} n \cdot k}$ after Fourier transform).

In computing $K^0(\mathbb{T}^2) = K_0(C(\mathbb{T}^2))$, we are equivalently computing $K_0(C^*(\mathbb{Z}^2))$ as the group of topological phases of $\mathbb{Z}^2$-symmetric gapped Hamiltonians. We will replace $\mathbb{Z}^2$ by more general *nonabelian* groups later.

K-theoretic topological insulator

In a sense, the Berry curvature way of understanding the Chern insulator is not necessary, and it is not actually easy to measure directly (as far as I know).

It is, however, useful in analogy to electromagnetism (as a $U(1)$ gauge theory), and there are analogues of Dirac monopoles and strings in condensed matter systems.

The latter appear in Weyl semimetals, discovered in 2015, and they are now a big field of study.

In the setting of *torsion* topological phases that *K*-theory methods come to the forefront, because these are not typically detected (mathematically) by curvature methods. Remarkable, these torsion phases actually have been found (physically)!

Time-reversal and quaternions

In quantum theory, symmetries may be represented (projectively) by *antiunitary* operators (Wigner's theorem). The basic example is fermionic time-reversal Θ, with $\Theta^2 = -1$.

Example: On $\mathbb{C}^2$, take $\Theta = \mathcal{K} \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where $\mathcal{K}$ is complex conjugation. This is a *quaternionic structure* identifying $\mathbb{C}^2 \cong \mathbb{H}$.

Example: On a complex vector bundle $\mathcal{E} \to T$, a quaternionic structure is an antilinear bundle map squaring to -1, which reduces the structure group from $\mathrm{U}(2n)$ to $\mathrm{Sp}(n)$. Fiberwise, this identifies $\mathbb{C}^{2n}$ with $\mathbb{H}^n$.

Notice that the complex dimension must be even — this is "Kramers degeneracy". So ψ and $\Theta\psi$ are orthogonal (as state vectors or sections) and called "Kramers partners".

"Quaternionic" bundles

Symplectic characteristic classes, connections, . . . , are known, e.g. $\mathrm{Sp}(1) = \mathrm{SU}(2)$. A higher-dimensional $d \geq 4$ base manifold is needed to host non-trivial symplectic bundles, which is why they are not really relevant in the 3D world of CMP.

Nevertheless, recall that our base manifold $\mathbb{T}^d$ is a *momentum space*, not position space.

Time-reversal also implements momentum reversal $\theta : k \mapsto -k$, so that the Brillouin torus $\mathbb{T}^d$ is naturally a space with involution (a Real space in the sense of Atiyah).

This means that time-reversal as a bundle map Θ is required to be an antiunitary lift of θ satisfying $\Theta^2 = -1$.

"Quaternionic" bundles

The appearance of the involution θ is familiar from the fact that the Fourier transform of a real-valued function $f : n \mapsto f(n)$ (i.e. invariant under complex conjugation) is a complex valued function $\widehat{f} : S^1 \to \mathbb{C}$ satisfying the modified reality condition

$$\overline{\widehat{f}(k)} = \widehat{f}(-k) \equiv \widehat{f}(\theta(k)).$$

From the representation theory (harmonic analysis) point of view, the complex character χ_k is mapped to the conjugate character $\overline{\chi_k} = \chi_{\theta(k)}$.

Time-reversal Θ in position space (say acting on $l^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2n}$) is complex conjugation κ composed with some unitary matrix U. Accordingly, its Fourier transform effects $k \mapsto \theta(k)$ followed by multiplication by U.

"Quaternionic" bundles

For the Brillouin torus, the involution θ has fixed point set F with 2^d points. The fibre over each fixed point (and hence the whole bundle) is even dimensional (Kramers degenerate) over $\mathbb{C}$.

Think of each circle $\mathbb{T}$ as unit complex numbers e^{ik} with θ acting by conjugation, so $k = 0, \pi$ are fixed. (Notice that these are the two real-valued characters of $\mathbb{Z}$.) The Brillouin torus is the d-fold product of this Real circle.
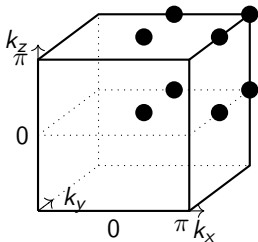
"Quaternionic bundles"

Quite generally, for involutive base spaces (X, θ), complex (hermitian) bundles $\mathcal{E}$, with a "Quaternionic" lift Θ of θ as above, form a category of "Quaternionic" ("Q") bundles.

This category has not really been studied much mathematically. Let us consider the basic example $X = \mathbb{T}^2$. The involution θ is orientation preserving.

Naïvely, Θ implements an antilinear isomorphism between $\mathcal{E}$ and $\theta^*\mathcal{E}$, so

$$c_1(\mathcal{E}) = c_1(\overline{\theta^*\mathcal{E}}) = -c_1(\theta^*\mathcal{E}) = -c_1(\mathcal{E}) = 0.$$

Thus $\mathcal{E}$ trivialises as a complex bundle. But physicists Fu–Kane–Mele constructed a $\mathbb{Z}_2$-topological invariant in 2006!

2D FKM invariant

It will suffice to take $\mathcal{E}$ to be a trivial complex rank-2 bundle.

Although there are non-vanishing global sections $u_1 \perp u_2$, it turns out that the "generalised Kramers pairing" condition $\Theta u_1(k) = u_2(\theta(k))$ cannot be globally implemented in general.

One way to measure the obstruction is to form the 2×2 "sewing matrix" function

$$\omega_{ab}(k) = \langle u_a(\theta(k)) | \Theta u_b(k) \rangle.$$

At each fixed point $k \in F$, $\omega(k)$ is an antisymmetric matrix, and we can assign the sign $\pm = \frac{\sqrt{\det(\omega)}}{\mathsf{Pf}(\omega)}(k)$ to that point.

The product-of-signs (POS) over F is defined to be the 2D FKM number $\nu \in \mathbb{Z}_2$ (One checks that this product is gauge-invariant).

3D FKM invariant
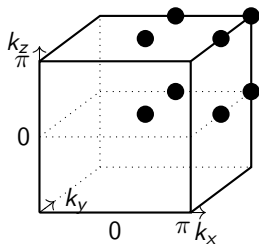
For $\mathbb{T}^3$, there are six choices of 2D tori which are stable under θ. These have $k_{x/y/z} = 0$ or π. So there are six 2D FKM invariants $\nu_{i,0}, \nu_{i,\pi}, i = x, y, z$.

However, not all six are independent!

Define the *strong* FKM number ν_0 to be the POS at all eight fixed points. There are constraints

$$\nu_{i,0} = \nu_0 \nu_{i,\pi}, \qquad i = x, y, z.$$

So only four independent $\mathbb{Z}_2$ numbers classify 3D topological insulators with time-reversal symmetry.

Systematic classification of "Quaternionic bundles"

The FKM constructions are ad-hoc, and it's not clear how to generalise them or what their functorial properties are.

A recent mathematical work showed that there is a *characteristic class* $\kappa : \mathrm{Vect}_Q^{2m}(X, \theta) \to H^2_{\mathbb{Z}_2}(X, F; \mathbb{Z}(1))$ which is even a bijection in the above cases (any many other low-dimensional cases).

This means that tools from algebraic topology like Poincaré duality can be used to formulate a topological bulk-boundary map. This in fact correctly predicts the appearance of the famous Dirac cone surface state [Gomi+Sato+T].
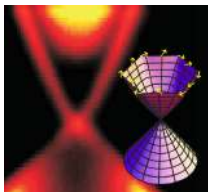
K-theoretic $\mathbb{Z}_2$ topological insulator

More directly, there is a stable classification of "Quaternionic" bundles using Dupont's KQ-theory, which is the quaterionic version of Atiyah's KR-theory for "Real" vector bundles.

Here, $KQ^0(X, \theta)$ is the Grothendieck group of isomorphism classes of "Quaternionic" bundles over (X, θ).

Using a stable splitting of the torus, it is not hard to show that $\widetilde{KQ}^0(\mathbb{T}^2, \theta) = \mathbb{Z}_2$ and $\widetilde{KQ}^0(\mathbb{T}^3, \theta) = \mathbb{Z}_2^4$, as realised by FKM's constructions.

Kitaev in 2009 calculated this by using the Baum–Connes isomorphism and Poincaré duality to convert to a KO-theory computation. This is in fact a type of T-duality!

Real noncommutative topology

In the complex world, we saw that the K-theory of the group C^*-algebra of the group of symmetries ($\mathbb{Z}^d$) gave us the classification of $\mathbb{Z}^d$-symmetric topological phases.

This philosophy of using C^*-algebraic language will allow generalisation to nonabelian groups, and we want to do the same in the real ($\mathbb{R}$) setting.

In the real world, there is again a correspondence between l.c.h. $\mathbb{Z}_2$-spaces (X, θ) and commutative C^*-algebras over $\mathbb{R}$, due to Arens–Kaplansky.

Specifically, the latter are all of the form

$$C_0(X, \theta) := \{f \in C_0(X) \,:\, f(\theta(x)) = \overline{f(x)} \ \forall x \in X\}.$$

Real noncommutative topology

Roughly speaking, (X, θ) is the "Real spectrum" of the real C^*-algebra.

An illustrative example is the *real* group C^*-algebra of $\mathbb{Z}^d$, which is $C^*_{\mathbb{R}}(\mathbb{Z}^d) \cong C(\mathbb{T}^d, \theta)$.

Note that $C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C} \cong C^*(\mathbb{Z}^d)$, and $C^*_{\mathbb{R}}(\mathbb{Z}^d)$ is the real subalgebra under complex conjugation.

The spectrum of $C^*_{\mathbb{R}}(\mathbb{Z}^d)$ is defined to be that of the complexification (which is $\mathbb{T}^d$). The Real structure θ on $\mathbb{T}^d$ is induced from complex conjugation (thus it is conjugation of the characters in $\mathbb{T}^d$).

The fixed points are the characters that map into $\mathbb{R}$.

Real noncommutative topology

When there is time-reversal symmetry T, the symmetry group is abstractly $\mathbb{Z}^d \times \{1, T\}$.

However, there are two additional pieces of data. (1) T needs to act as an antilinear operator Θ, and (2) $\Theta^2 = \pm 1$ so that T is only *projectively* represented as an involution.

Then the correct notion of group C^*-algebra should be the *twisted crossed product* $\mathbb{C} \rtimes_{\phi, \sigma} G$. Here, ϕ denotes an action of G on $\mathbb{C}$ (e.g. T acts by conjugation), and σ is a 2-cocycle which encodes projective data.

The ordinary $C^*(G)$-algebra is the special case where ϕ, σ are trivial.

Real noncommutative topology

So for $G = \mathbb{Z}^d \rtimes \{1, t\}$ with $\Theta^2 = -1$, we have

$$C^*(G, \phi, \sigma) = \mathbb{C} \rtimes_{\phi, \sigma} (\mathbb{Z}^d \rtimes \{1, T\}) = C^*(\mathbb{Z}^d) \rtimes_{\phi, \sigma} \{1, T\}$$
$$\cong (C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C}) \rtimes_{1 \otimes \phi, 1 \otimes \sigma} \{1, T\} \cong C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}.$$

The f.g.p modules for $C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}$ are precisely sections of "Quaternionic bundles" over $(\mathbb{T}^d, \theta)$.

So instead of computing $KQ^0(\mathbb{T}^d, \theta)$ for the time-reversal invariant topological insulators, we can compute

$$KO_0(C^*_{\mathbb{R}}(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{H}) \cong KO_4(C^*_{\mathbb{R}}(\mathbb{Z}^d)).$$

$\mathbb{Z}_2$ topological insulator

By the Baum–Connes isomorphism in real K-theory,

$$KO_4(C^*_{\mathbb{R}}(\mathbb{Z}^d)) \cong KO_4(B\mathbb{Z}^d)$$

where the classifying space $B\mathbb{Z}^d$ is $\mathbb{R}^d/\mathbb{Z}^d$ which is a (different!!) torus.

By Poincaré duality, this is $KO^{d-4}(\mathbb{T}^d)$. Then classical algebraic topology techniques can be used.

Note that we can think of $\mathbb{R}^d/\mathbb{Z}^d$ as the unit cell (fundamental domain) in position space, so we have done a kind of topological Fourier transform!

Part 4: Wigner's theorem, generalised symmetries and Bott-Periodic Table of topological phases

Symmetries in quantum mechanics

In the examples from the previous talks, topological phases were defined with respect to a group of symmetries G.

Symmetry compatibility constrained the set of possible gapped Hamiltonians, such that there can be disconnected regions (topological phases) in "Hamiltonian space" labelled by a K-theory group.

Some of the symmetry group elements are unusual. They can be (1) antilinear, (2) odd/even, (3) represented projectively.

Mathematicians have done a lot with unitary representation theory. But one message from the physicists is that working over $\mathbb{C}$ is sometimes too convenient and misses interesting phenomena.

Symmetries in quantum mechanics

Pure states in quantum mechanics (QM) are elements of the projective Hilbert space $\mathbb{P}\mathscr{H}$, usually represented by a normalised vector $|\psi\rangle$.

QM symmetries are a bit unusual in that they only need to preserve transition probabilities between any pair of states, i.e. the symmetric function

$$p : ([\psi_1], [\psi_2]) \mapsto |\langle\psi_1|\psi_2\rangle|^2 \in [0, 1].$$

I have used Dirac's bra-ket notation for the inner product $\langle\cdot|\cdot\rangle$.

A classical theorem of Wigner[3] says that any automorphism of $(\mathbb{P}\mathscr{H}, p)$ is implemented by a unitary or antiunitary operator on $\mathscr{H}$.

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[3]see [D.S. Freed, Geom. Top. Monogr. 18 83–89 (2012)] for a modern geometric proof.

Symmetries in quantum mechanics

Recall that an antiunitary operator on $\mathscr{H}$ is a complex-antilinear bijection U such that $\langle U\psi_1 | U\psi_2 \rangle = \langle \psi_2 | \psi_1 \rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathscr{H}$.

Modifying the (anti)unitary implementing operator U by an overall phase does not change the automorphism of $(\mathbb{P}\mathscr{H}, p)$.

Thus the target group of "QM automorphisms" is the *projective unitary-antiunitary* (PUA) group of $\mathcal{H}$, and it is "PUA-representation theory" which is needed.

In contrast, ordinary unitary representation theory (e.g. of locally compact second countable topological groups) is the study of homomorphisms into unitary operators $\mathcal{U}(\mathscr{H})$.

Symmetries in quantum mechanics

Example: Complex conjugation κ is antiunitary.

Example: *Fermionic* time-reversal T, which squares to -1 instead of $+1$, e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \kappa$. This is a *quaternionic structure*.

Recall that $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong \mathrm{Spin}(3)$ is a double cover of $\mathrm{SO}(3)$, which as groups/manifolds is

$$\{\pm 1\} \cong \mathbb{Z}_2 \hookrightarrow S^3 \cong \mathrm{SU}(2) \twoheadrightarrow \mathbb{RP}^3 \cong \mathrm{SO}(3).$$

The failure of $\mathrm{SO}(3)$ to lift into $\mathrm{SU}(2)$ means that the latter acting on $\mathbb{C}^2$ gives a *projective representation* of $\mathrm{SO}(3)$.

Symmetries in quantum mechanics

Notation: For each $g \in G$, we write g for its representative operator on $\mathscr{H}$.

There is a homomorphism $\phi : G \to \{\pm 1\}$, which specifies whether g is unitary ($\phi(g) = +1$) or antiunitary ($\phi(g) = -1$).

We wish to study *dynamical symmetries*. Recall that time evolution in QM is given by a strongly-continuous 1-parameter group of unitaries $\mathbb{R} \ni t \mapsto U_t = e^{-iHt}$. The self-adjoint generator H, given by Stone's theorem, is called the *Hamiltonian*.

Usually, if g is a dynamical symmetry, then $g U_t g^{-1} = U_t$.

Symmetries in quantum mechanics

We can also consider *time-reversing* symmetries: $gU_tg^{-1} = U_{-t}$.

Thus there is another homomorphism $\tau : G \to \{\pm 1\}$ which encodes the time-arrow preserving/reversing property,

$$gU_tg^{-1} = U_{\tau(g)t}.$$

Since $U_t = e^{-iHt}$, we can rewrite this as

$$g(iH)g^{-1} = \tau(g)(iH).$$

Removing the i factor, we get

$$g(H)g^{-1} = \phi \cdot \tau(g)H =: c(g)H$$

where the product homomorphism c gives G a grading, and encodes (anti-)commutativity of g with the Hamiltonian H.

Symmetries in quantum mechanics

Often, c is assumed to be trivial so symmetries $\Leftrightarrow$ commute with H, but this is not generally the case..

The letter c is meant to suggest "charge-conjugation", or "particle-hole" symmetry: notice that if $c(g) = -1$, then g reflects the spectrum of H about 0.

By definition, $\phi \cdot c \cdot \tau \equiv 1$, and any two of these three homomorphisms are independent.

Thus it is not enough to say that G is the symmetry group. We must also specify the data of ϕ, c.

One last ingredient is needed to account for phase ambiguities in g.

Symmetries in quantum mechanics

Given (G, ϕ), a PUA-rep on $\mathscr{H}$ is map[4] $\theta : g \mapsto \rho_g$ such that ρ_g is (anti)unitary according to $\phi(g) = \pm 1$, and such that $\rho_x \rho_y = \sigma(x, y) \rho_{xy}$ for some function $\sigma : G \times G \to \mathrm{U}(1)$. Then

$$\sigma(x, y)\sigma(xy, z) = \sigma(y, z)^{\phi(x)}\sigma(x, yz).$$

follows from associativity. This is the (generalised) 2-cocycle condition $\delta\sigma = 1$ in the sense of group cohomology.

Modifying $\theta_x \mapsto \lambda_x \theta_x$, where $\lambda : x \mapsto \lambda_x \in \mathrm{U}(1)$, corresponds to multiplying σ by a 2-coboundary $\delta\lambda : (x, y) \mapsto \lambda_x \lambda_y^{\phi(x)}/\lambda_{xy}$. Such phase modifications do not matter physically, so only the cocycle class $[\sigma]$ is invariant.

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[4]For infinite topological groups, we would technically need ρ and the subsequent 2-cocycle σ to be a Borel map. For $\sigma \equiv 1$, θ is a homomorphism which is automatically continuous.

Symmetries in quantum mechanics

What about the condition $gH = c(g)Hg$?

For H which are gapped (at 0), $\Gamma := \mathrm{sgn}(H)$ gives a $\mathbb{Z}_2$-grading of $\mathscr{H}$ and there is a homotopy from H to Γ through gapped self-adjoint operators (at least for bounded H, otherwise a truncation is assumed).

Given (G, c, ϕ, σ), define an sPUA-rep to be a PUA-rep on a super-Hilbert space $(\mathscr{H}, \Gamma)$, such that G *graded* commutes with Γ.

If we agree not to distinguish homotopic gapped H, then the Γ in a sPUA-rep for (G, c, ϕ, σ) represents a "topological class/phase" of symmetry-compatible gapped Hamiltonians.

There could be many ways state the precise equivalence relation defining "phase", but each at least contains the previous homotopy equivalence. E.g. allowing homotopies within direct sums will lead to a K-theory classification.

CT symmetries

The basic illustrative example is the "CT-group" $\{1, C\} \times \{1, T\}$, so-called because C, T are respectively the **C**harge-conjugation and **T**ime-reversal symmetries.

The diagonal element $CT = TC$ is denoted S, for **S**ublattice.

By convention, τ, c are defined to be

$$\tau(C) = +1, \; c(C) = -1, \quad \tau(T) = -1, \; c(T) = +1. \quad (4)$$

Note that $\phi(C) = -1 = \phi(T)$, so the representatives C, T are antiunitary, whereas the diagonal element S is represented by a unitary S.

CT symmetries

In general, only some subgroup of $A \subset G$ is present for a given physical system.

Even though $C^2 = 1 = T^2$ and $CT = TC$, the operators C, T only need to be involutions up to a phase, and they also commute only up to a phase.

These phase ambiguities are encoded in a 2-cocycle σ. The symmetry data is (A, σ), with $\phi, c : A \to \{\pm 1\}$ implicitly.

Q: What are all the possibilities for C, T ("CT-symmetry classes"), i.e. the possible (A, σ)?

CT symmetries

Proposition: There are exactly ten classes, corresponding to the $8 + 2$ Morita classes of real $+$ complex (graded) Clifford algebras / super-Brauer group over $\mathbb{R}$ and $\mathbb{C}$ / ten superdivision algebras over $\mathbb{R}$. They are labelled by the squares of C and T (where present).

Sketch of proof: Note that $\mathsf{T}^2 = \lambda$ for some $\lambda \in \mathsf{U}(1)$, so $\lambda \mathsf{T} = \mathsf{T}^3 = \mathsf{T}\lambda = \bar{\lambda}\mathsf{T}$, and $\lambda \in \{\pm 1\}$. Thus $\mathsf{T}^2 = \pm 1$, and similarly $\mathsf{C}^2 = \pm 1$. Note that this sign is invariant under $\mathsf{T} \mapsto \mu\mathsf{T}$. Next, we use the phase freedom in defining $\mathsf{C}, \mathsf{T}, \mathsf{S}$ to "standardize" them; specifically, we can arrange for $\mathsf{TC} = \mathsf{CT}$ and $\mathsf{S}^2 = +1$. Therefore we just need to assign, for each of the five possible subgroups $A \subset G$:

$$\{1\}, \{1, S\}, \{1, T\}, \{1, C\}, \{1, C, T, S\},$$

a ± 1 sign to C, T (where present).

Tenfold way

Symmetry	C^2	T^2	Clifford algebra	Graded Morita class
T		$+1$	$Cl_{1,2}$	$Cl_{0,0}$
C, T	-1	$+1$	$Cl_{2,2}$	$Cl_{1,0}$
C	-1		$Cl_{2,1}$	$Cl_{2,0}$
C, T	-1	-1	$Cl_{3,1}$	$Cl_{3,0}$
T		-1	$Cl_{3,0}$	$Cl_{4,0}$
C, T	$+1$	-1	$Cl_{0,4}$	$Cl_{5,0}$
C	$+1$		$Cl_{0,3}$	$Cl_{6,0}$
C, T	$+1$	$+1$	$Cl_{1,3}$	$Cl_{7,0}$
N/A	N/A		$\mathbb{C}l_1$	$\mathbb{C}l_0$
S	$S^2 = +1$		$\mathbb{C}l_2$	$\mathbb{C}l_1$

Tenfold way

We can identify A with the image of (ϕ, c). More generally, A arises as a *quotient* of the full symmetry group G by the kernel G_0 of (ϕ, c), i.e.

$$1 \to G_0 \to G \xrightarrow{\phi, c} A \to 1,$$

and there is a 2-cocycle $\tilde{\sigma}$ on $G \times G$ which projects onto σ on $A \times A$.

G_0 is the "nice" subgroup which may be studied by (projective) unitary representation theory.

We had studied $G = \mathbb{Z}^d \times \{1, S\}$ and $G = \mathbb{Z}^d \times \{1, T\}$ previously.

Generally, A need not map back into G! Obstruction is again cohomological and leads to twists in the classification scheme.

Clifford algebras

In connection with geometry, it is usual to define the (real) *Clifford algebra* for a vector space V with a quadratic form Q to be the free (tensor) algebra (over $\mathbb{R}$) subject to $v^2 = Q(v), v \in V$.

This is a "quantization" of the exterior algebra ($Q \equiv 0$), i.e. the same underlying vector space but modified multiplication law.

Complexification $(\cdot) \otimes_{\mathbb{R}} \mathbb{C}$ yields the complex Clifford algebra.

Complex Clifford algebras

We will proceed more concretely (corresponding to taking a standard form for Q, using Sylvester's law of inertia).

Define the *complex* Clifford algebra $\mathbb{C}l_n$ to be the complex unital algebra generated by anticommuting elements $f_i, i = 1, \ldots, n$ that square to $+1$.

For example, $\mathbb{C}l_0 \cong \mathbb{C}$, $\mathbb{C}l_1 \cong \mathbb{C}[\frac{1+f_1}{2}] \oplus \mathbb{C}[\frac{1-f_1}{2}]$, and $\mathbb{C}l_2 \cong M_2(\mathbb{C})$ with $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Real Clifford algebras

The *real* Clifford algebra $Cl_{r,s}$ is generated (over $\mathbb{R}$) by anticommuting elements $e_i, f_j, i = 1, \ldots, r, j = 1, \ldots, s$ such that $e_i^2 = -1, f_j^2 = +1$.

For example, $Cl_{0,0} \cong \mathbb{R}$, $Cl_{1,0} \cong \mathbb{C}$, $Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$ and $Cl_{1,1} \cong M_2(\mathbb{R})$ with $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

We also have $Cl_{2,0} \cong \mathbb{H}$ and $Cl_{0,2} \cong M_2(\mathbb{R})$. One can also show that $Cl_{0,8} \cong M_{16}(\mathbb{R}) \cong Cl_{8,0}$.

When we complexify, ie_i squares to $+1$, so $Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}l_{r+s}$ as complex algebras. E.g. the complexifications of $Cl_{1,0}$ and $Cl_{0,1}$ are both $\mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}l_1$, and the complexifications of $Cl_{2,0}, Cl_{1,1}, Cl_{0,2}$ are all $M_2(\mathbb{C}) \cong \mathbb{C}l_2$.

Clifford algebras

It turns out that there are (algebraic) "Bott periodicity" identities

$$Cl_{r+1,s+1} \cong Cl_{r,s} \otimes_{\mathbb{R}} Cl_{1,1}$$

$$Cl_{r+8,0} \cong Cl_{r,0} \otimes_{\mathbb{R}} Cl_{8,0}$$

$$Cl_{0,s+8} \cong Cl_{0,s} \otimes_{\mathbb{R}} Cl_{0,8}$$

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

Since $Cl_{1,1}, Cl_{8,0}, Cl_{0,8}, \mathbb{C}l_2$ are each matrix algebras, the Morita class (representation theory) of $\mathbb{C}l_n$ only depends on n (mod 2) while that of $Cl_{r,s}$ only depends on $r - s$ (mod 8).

In total, there are 8+2=10 Morita classes of real/complex Clifford algebras, and each is a matrix algebra over $\mathbb{R}/\mathbb{C}/\mathbb{H}$ or a direct sum of two matrix algebras (of the same dimension over the same (skew)-field).

Graded Clifford algebras

It is mathematically convenient and physically essential to regard the Clifford algebras as $\mathbb{Z}_2$-graded real/complex C^*-algebras.

Define e_i, f_j to be odd, e_i to be skew-adjoint, f_j to be self-adjoint, and taking the (unique) C^*-norm e.g. from their matrix realisations.

Then we can define $\widehat{\mathscr{M}_{r,s}}$ and $\widehat{\mathscr{M}_n^{\mathbb{C}}}$ to be the free abelian groups generated by the distinct irreducible unitary (*orthogonal* in the real case) super-reps of $Cl_{r,s}$ and $\mathbb{C}l_n$ respectively.

Graded Clifford algebras

We can regard the grading operator in a super-rep as representing an extra Clifford generator f_{s+1} in the *ungraded* sense, so the ungraded versions $\mathscr{M}_{r,s}$ and $\mathscr{M}_n^{\mathbb{C}}$ are related to the graded ones via $\widehat{\mathscr{M}}_{r,s} \cong \mathscr{M}_{r,s+1}$ and $\widehat{\mathscr{M}}_n^{\mathbb{C}} \cong \mathscr{M}_{n+1}^{\mathbb{C}}$.

Furthermore, the $\mathbb{Z}_2$-graded tensor product gives an isomorphism $Cl_{r_1,s_1} \hat{\otimes} Cl_{r_2,s_2} \cong Cl_{r_1+r_2,s_1+s_2}$, and the $\mathbb{Z}_2$-graded tensor product $W \hat{\otimes} V$ of modules $W \in \widehat{\mathscr{M}}_{r_1,s_2}, V \in \widehat{\mathscr{M}}_{r_2,s_2}$ is canonically a $Cl_{r_1+r_2,s_1+s_2}$-module.

Thus there are natural pairings $\widehat{\mathscr{M}}_{r_1,s_1} \otimes_{\mathbb{Z}} \widehat{\mathscr{M}}_{r_2,s_2} \to \widehat{\mathscr{M}}_{r_1+r_2,s_1+s_2}$ which give $\oplus_{r,s \geq 0} \widehat{\mathscr{M}}_{r,s}$ the structure of a bigraded *ring*

Similarly, we get a graded ring $\oplus_{n \geq 0} \widehat{\mathscr{M}}_n^{\mathbb{C}}$ in the complex case.

Tenfold way and Clifford algebras

It is now an exercise to get a one-to-one correspondence between the ten C, T possibilities and the Clifford algebras.

For example, if $A = \{1, C, T, S\}$ and $C^2 = -1, T^2 = +1$, then $\{C, iC, iCT\}$ are odd Clifford generators. From their squares, we see that we have a graded representation of $Cl_{2,1}$, in which $e_1 = C, e_2 = iC, f_1 = iCT$.

In other words, an element of $\widehat{\mathcal{M}}_{r,s}$ or $\widehat{\mathcal{M}}_n^{\mathbb{C}}$ is nothing but a sPUA-rep for the corresponding (A, σ).

Remark: The two $A = \{1, T\}$ cases are a bit tricky: T gives the graded Hilbert space $\mathscr{H}$ a real or quaternionic structure. Take $\{i, T, iT\Gamma\}$ as generators of the *ungraded* Clifford algebra $Cl_{1,2}$ when $T^2 = +1$ and $Cl_{3,0}$ when $T^2 = -1$. The sPUA-reps are classified by $\mathcal{M}_{1,2} \cong \widehat{\mathcal{M}}_{1,1} \cong \widehat{\mathcal{M}}_{0,0}$ or $\mathcal{M}_{3,0} \cong \mathcal{M}_{4,1} \cong \widehat{\mathcal{M}}_{4,0}$.

Tenfold way

Symmetry	C^2	T^2	Clifford algebra	Graded Morita class
T		$+1$	$Cl_{1,2}$	$Cl_{0,0}$
C, T	-1	$+1$	$Cl_{2,2}$	$Cl_{1,0}$
C	-1		$Cl_{2,1}$	$Cl_{2,0}$
C, T	-1	-1	$Cl_{3,1}$	$Cl_{3,0}$
T		-1	$Cl_{3,0}$	$Cl_{4,0}$
C, T	$+1$	-1	$Cl_{0,4}$	$Cl_{5,0}$
C	$+1$		$Cl_{0,3}$	$Cl_{6,0}$
C, T	$+1$	$+1$	$Cl_{1,3}$	$Cl_{7,0}$
N/A		N/A	$\mathbb{C}l_1$	$\mathbb{C}l_0$
S		$S^2 = +1$	$\mathbb{C}l_2$	$\mathbb{C}l_1$

A version of the Tenfold way.

Atiyah–Bott–Shapiro construction

An interesting construction of Atiyah–Bott–Shapiro expresses the K-theory rings of a point in terms of Clifford modules:

$$K^n(\star) \cong \widehat{\mathscr{M}}_n^{\mathbb{C}}/\iota^*\widehat{\mathscr{M}}_{n+1}^{\mathbb{C}}, \qquad KO^{r-s}(\star) \cong \widehat{\mathscr{M}}_{r,s}/\iota^*\widehat{\mathscr{M}}_{r,s+1}$$

This quotienting operation has a physical interpretation: a $\mathbb{C}l_n$-module (sPUA-rep for some CT-class) which admits a $\mathbb{C}l_{n+1}$-module structure should really be considered sPUA-reps for some *other* CT-class.

In this sense, the K-theory groups of a point classify the "topological" phases for 0-dimensional systems. The "noncommutative topology" comes from the CT-symmetries.

Karoubi K-theory

In "super" language, there is yet another formulation due originally to Karoubi.

The idea is to consider a graded vector bundle (or f.g.p. module) $\mathcal{E}$ as an *ungraded one* together with a grading operator Γ (which squares to the identity bundle endomorphism).

If Γ can be homotoped to $-\Gamma$, this means that particles and antiparticle sectors can be exchanged in a continuous manner. Their distinction is not intrinsic so their difference should be considered "topologically trivial".

For example, if $\boldsymbol{\mathcal{E}} \equiv (\mathcal{E}, \Gamma)$ admits a trivialising odd operator $\mathcal{I}$ in the earlier sense, then $(\cos t)\Gamma + (\sin t)\mathcal{I},\ t \in [0, \pi]$ shows that Γ and $-\Gamma$ are homotopic (through gradings).

Karoubi K-theory

More precisely, consider triples $(\mathcal{E}, \Gamma_1, \Gamma_2)$ where $\mathcal{E}$ is a vector bundle and Γ_i are gradings. This represents the ordered difference between $(\mathcal{E}, \Gamma_1)$ and $(\mathcal{E}, \Gamma_2)$.

If $\Gamma_1 \sim_{\text{homotopic}} \Gamma$ (through gradings), declare $(\mathcal{E}, \Gamma_1, \Gamma_2)$ to be trivial.

$\oplus$ gives a monoid of triples, and quotienting by the trivial submonoid of triples gives $\mathbf{K}(X) = \mathbf{K}(C(X))$.

$$[\mathcal{E}, \Gamma_1, \Gamma_2] = -[\mathcal{E}, \Gamma_2, \Gamma_1] \qquad \text{(indeed a group)}$$
$$[\mathcal{E}, \Gamma_1, \Gamma_2] + [\mathcal{E}, \Gamma_2, \Gamma_3] = [\mathcal{E}, \Gamma_1, \Gamma_3] \qquad \text{(path independence)}$$
$$[\mathcal{E}, \Gamma_1, \Gamma_2] + [\mathcal{E}, \Gamma_1', \Gamma_2'] \text{ if } \Gamma_i \sim_h \Gamma_i' \qquad \text{(homotopy independence)}$$

Karoubi K_0-theory and projections

Karoubi's $\mathbf{K}(\cdot)$ is another model for $K_0(\cdot)$.

From the first formulation to Karoubi's formulation:
A virtual (or graded) bundle $\boldsymbol{\mathcal{E}} = \mathcal{E}_+ \ominus \mathcal{E}_-$ representing $[\boldsymbol{\mathcal{E}}]$ has associated underlying ungraded bundle $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ with grading operator $\Gamma = \operatorname{diag}(1_{\mathcal{E}_+}, -1_{\mathcal{E}_-})$.
The associated Karoubi K-theory class is
$[\mathcal{E}_+, -1, 1] + [\mathcal{E}_-, 1, -1] = [\mathcal{E}, -\Gamma, \Gamma]$.

From a Karoubi class $[\mathcal{E}, \Gamma_1, \Gamma_2]$, take the formal difference of the two negatively graded subbundles, $[\mathcal{E}_{-,1} \ominus \mathcal{E}_{-,2}]$.

Karoubi K-theory

Karoubi's formulation also works for graded C^*-algebras $\mathcal{A}$, not just $C(X)$. A triple $[\mathcal{E}, \Gamma_1, \Gamma_2]$ comprises an *ungraded* f.g.p. $\mathcal{A}$-module $\mathcal{E}$, and operators (module maps) Γ, $\Gamma^2 = 1$ that turn $\mathcal{E}$ into a *graded* $\mathcal{A}$- module.

Example: take $\mathbb{C}l_1$ as a graded C^*-algebra. An ungraded module is $\mathbb{C} \oplus \mathbb{C}$ where the Clifford generator acts as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A grading operator for this module looks like $\begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}$ for $u \in \mathbb{C}^*$, so a triple looks like

$$[\mathbb{C} \oplus \mathbb{C}, \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix}].$$

Since any $u \sim_h v$, the triples are trivial, and $\mathbf{K}(\mathbb{C}l_1) = 0$.

Karoubi K-theory and unitaries

In fact, $K_1(\mathcal{A}) \cong \mathbf{K}(\mathcal{A}\widehat{\otimes}\mathbb{C}l_1)$ ("Clifford suspension").

In Karoubi's formulation, $K^{-1}(S^1) \cong \mathbf{K}(C(S^1)\widehat{\otimes}\mathbb{C}l_1)$ is generated by

$$\left[S^1 \times (\mathbb{C} \oplus \mathbb{C}), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{-ik} \\ e^{ik} & 0 \end{pmatrix} \right]$$

where the $\mathbb{C}l_1$ generator acts as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (this was called a sublattice operator S). So the S compatible gradings (Hamiltonians) are homotopically classified by $K^{-1}(S^1)$.

Generally, a Karoubi triple representing a class in $K^{-1}(X)$ has the form

$$\left[\mathcal{E} \oplus \mathcal{E}, \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix} \right]$$

where u, v are regarded as unitaries in some $M_n(C(S^1))$. This corresponds to $[u^{-1}v]$ in the unitary formulation.

Karoubi K-theory and relative phases

This last formulation of K-theory is a "relative" one. It measures an obstruction to deforming one grading operator (gapped Hamiltonian) into another while respecting the module action (the symmetries).

Generally, a base grading operator Γ_1 needs to be fixed first in order to assign an "absolute" K-theory class to Γ_2.

In the SSH model example where $\mathcal{A} = C(S^1)\widehat{\otimes}\mathbb{C}l_1$, there is no canonical basepoint.

Dimension shifts

The 0-dimensional classification can be related to the $d \geq 1$ cases by "K-theoretic dimension shift".

Sketch: $\mathbb{Z}^d$ or $\mathbb{R}^d$ symmetries introduce a momentum space $\check{\mathbb{T}}^d$ or $\check{\mathbb{R}}^d$ (we use the $\check{(\cdot)}$ notation to avoid ambiguity). So we might want to compute $K_0(C(\check{\mathbb{T}}^d))$ or $K_0(C_0(\check{\mathbb{R}}^d))$ for the topological phases. There is a contribution from the d-fold suspension of a point.

Extra CT-symmetries are accounted for by graded tensor product with a Clifford algebra, which implements suspension.

Both suspensions shift the K-theory degree, so that the $d \geq 1$ classification is just a shift of the $d = 0$ one.

Periodic Table

n	C^2	T^2	$KO^{d-n}(\star)$ or $K^{d-n}(\star)$			
			$d=0$	$d=1$	$d=2$	$d=3$
0		$+1$	$\mathbb{Z}$	0	0	0
1	$+1$	$+1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
2	$+1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
3	$+1$	-1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
4		-1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
5	-1	-1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
6	-1		0	0	$\mathbb{Z}$	0
7	-1	$+1$	0	0	0	$\mathbb{Z}$
0	N/A		$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	$S^2 = +1$		0	$\mathbb{Z}$	0	$\mathbb{Z}$

Only the "strong" phases, corresponding to pullback under $\mathbb{T}^d \to S^d$ are accounted for in this Table.

Part 5: T-duality and the bulk-boundary correspondence

T-duality

Loosely speaking, T-duality is an equivalence of two seemingly different physical theories when some circles or tori are present in the model. It originated in string theory, but the mathematical aspect is quite general and applicable elsewhere.

The basic example comes from considering a closed string propagating on a circle with radius R (think of this as one of the compactified spatial dimensions).

There are two basic quantised quantities: (1) winding number, and (2) momentum. Their contributions to the energy are proportional to R and $\frac{1}{R}$ respectively.

So if we pass to a dual circle with radius $\frac{1}{R}$, the "same" theory is obtained with winding number and momentum exchanged.

Origin of tori in solid state physics

More generally, a circle bundle has a bunch of topological invariants, and there exists canonically a "T-dual" circle bundle with the same invariants but permuted (Bouwknegt–Evslin–Mathai).

For torus bundles, the T-dual is a "noncommutative torus bundle"!

In solid state physics, the duality is between momentum space (e.g. Brillouin torus) and position space (e.g. unit cell or fundamental domain).

This is most useful when the momentum "space" is noncommutative, which happens when the symmetry group is non-abelian.

Origin of tori in solid state physics

A lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ defines two related tori: the *unit cell* $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and the *Pontryagin dual* of characters $\check{\mathbb{T}}^d = \mathsf{Hom}(\mathbb{Z}^d, \mathsf{U}(1))$.

$\mathbb{T}$ and $\check{\mathbb{T}}$ are related through two self-dual exact sequences of groups.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T} \longrightarrow 0 \quad \text{position space}$$

$$\updownarrow$$

$$0 \longrightarrow \check{\mathbb{Z}} \longrightarrow \check{\mathbb{R}} \longrightarrow \check{\mathbb{T}} \longrightarrow 0 \quad \text{momentum space}$$

$$\boxed{\begin{array}{l} \text{self-dual under} \\ \mathrm{Hom}(\,\cdot\,, \mathrm{U}(1)) \end{array}}$$

Notation: In these slides, $\check{(\cdot)}$ generally denotes a momentum space quantity to distinguish from a position space quantity. We also suppress the dimension and write $\mathbb{Z}, \mathbb{R}, \mathbb{T}$, unless it becomes necessary to specify d.

Origin of tori in solid state physics

In solid state physics, $p \in \check{\mathbb{R}}$ is the momentum variable, and $\check{\mathbb{Z}} \equiv \mathrm{Hom}(\mathbb{T}, \mathrm{U}(1)) \subset \check{\mathbb{R}}$ is the *reciprocal lattice* of momenta whose value is trivial on the real space lattice $\mathbb{Z}$.

The momentum space quotient $\check{\mathbb{R}}/\check{\mathbb{Z}}$ is the *Brillouin torus* of *quasimomenta*, and coincides with the Pontryagin dual $\check{\mathbb{T}}$.

In position space, the unit cell $\mathbb{R}/\mathbb{Z}$ is also a torus $\mathbb{T}$, and mathematically it is a classifying space $B\mathbb{Z}$ (the quotient of contractible $\mathbb{R}$ by a free $\mathbb{Z}$ action).

The Brillouin torus is of fundamental importance because $\mathbb{Z}$-translation invariant (Hamiltonian) operators become "diagonalised" over $\check{\mathbb{T}}$ under a Fourier transform.

Bloch–Floquet transform

Recall that electron motion in a crystalline material is described by a Hamiltonian $H = H^\dagger$ acting on $L^2(\mathbb{R})$, e,g, $H = -\nabla^2 + V$, with V invariant under $\mathbb{Z} \subset \mathbb{R}$.

Bloch–Floquet transform decomposes H into $H = \int_{k \in \check{\mathbb{T}}}^{\oplus} H_k$. where each *Bloch Hamiltonian* acts on the Bloch waves that are k-quasiperiodic,

$$f_k(x + 1) = e^{ik} f_k(x), \qquad \text{(Bloch wave condition)}$$

Intuitively, one solves Schrödinger's equation on the unit cell $\mathbb{T}$ for each quasi-periodic boundary condition labelled by $k \in \check{\mathbb{T}}$, then integrates over all such conditions.

Bloch–Floquet transform

H_k typically has discrete spectrum $\{E_i(k)\}_{i\in\mathbb{N}}$, and as k varies over **B**, we obtain bands of spectra.

Equivalently, we are transforming $L^2(\mathbb{R})$ into the section space $L^2(\breve{\mathbb{T}}, \mathcal{E})$ of a certain *Hilbert bundle* $\mathcal{E} \to \breve{\mathbb{T}}$. The fibre $\mathcal{E}_k$ is the infinite-dimensional Hilbert space of k-quasi-periodic Bloch waves, or equivalently, the L^2-sections of a line bundle $\mathcal{L}_k \to \mathbb{T}$.

A $f \in L^2(\mathbb{R})$ has a Bloch decomposition

$$f(x) = \int_{k\in\breve{\mathbb{T}}} \breve{f}_k(x)\,\mathrm{dk}$$

whose k-component $\breve{f}_k$ is obtained from f by a *Bloch sum*

$$\breve{f}_k(x) = \sum_{m\in\mathbb{Z}} e^{-2\pi\mathrm{i}km} f(x + m).$$

Bloch–Floquet transform

In $f(x) = \int_{k \in \breve{\mathbb{T}}} \breve{f}_k(x)\, \mathrm{d}k$, we note that $\breve{f}_k(x)$ is a function of

$$(x, k) \in \mathbb{R} \times \breve{\mathbb{T}}$$

which is equivariant under $\mathbb{Z}$-translations according to

$$f(x + m, k) = e^{\mathrm{i}km} f(x, k)$$

Thus $\breve{f}$ can equally be regarded as a section of the *Poincaré line bundle* $\mathcal{P} \to \mathbb{T} \times \breve{\mathbb{T}}$,

$$\mathcal{P} = \mathbb{R} \times \breve{\mathbb{T}} \times \mathbb{C}/_{\sim_{\mathbb{Z}}}, \qquad m \cdot (x, k; z) = (x + m, k; e^{\mathrm{i}km} z),$$

putting $\mathbb{T}$ and $\breve{\mathbb{T}}$ on a more equal footing. Later, we will see that $\mathcal{P}$ implements T-duality.

Geometric intuition behind bulk-boundary correspondence

All topological insulators are "spectrally identical" in the bulk, so they are hard to probe directly.

At an interface between two systems with different bulk invariants, the insulating gap should close for "continuous interpolation" of the invariants.

Thus one expects gapless (metallic/conducting) boundary modes, which can be measured as a signature of the *change* in bulk invariants across the interface.

Furthermore, such boundary modes inherit *topological protection* from the bulk.

Geometric intuition behind bulk-boundary correspondence

Example: In the semiclassical picture of the Quantum Hall Effect, electrons in a 2D sample subject to a uniform perpendicular magnetic field move in cyclotron orbits.
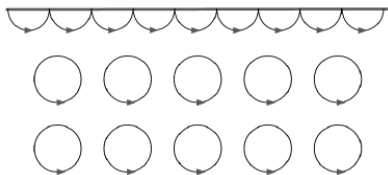
Its (quantized) angular momentum turns into a (quantized) linear momentum when intercepted by the boundary, giving rise to (quantized) unidirectional conductance along the boundary.

We stress that in position space, the bulk-boundary correspondence should be a kind of "geometric restriction".

Momentum space model for bulk-boundary correspondence

Suppose there is a boundary C^*-algebra of observables $\mathcal{A}$, and we also have an extra transverse $\mathbb{Z}$-symmetry. Then the bulk algebra $\check{\mathcal{B}}$ would be $\mathcal{A} \otimes C^*(\mathbb{Z})$.

More generally, $\mathbb{Z}$ could act by automorphisms α of $\mathcal{A}$. Then $\check{\mathcal{B}} = \mathcal{A} \rtimes_\alpha \mathbb{Z}$.

There is a canonical "Toeplitz extension",

$$0 \longrightarrow \boxed{\begin{array}{c} \text{Boundary} \\ \text{algebra} \\ \mathcal{A} \otimes \mathcal{K} \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{Bulk-with-} \\ \text{boundary} \\ \text{algebra} \\ \mathcal{T}(\mathcal{A}, \alpha) \end{array}} \xrightarrow{\mathrm{ev}_\infty} \boxed{\begin{array}{c} \text{Bulk} \\ \text{algebra } \check{\mathcal{B}} \end{array}} \longrightarrow 0,$$

where $\mathcal{T}(\mathcal{A}, \alpha)$ is the bulk-with-boundary algebra in which the transverse translations are turned into unilateral shifts.

Momentum space model for bulk-boundary correspondence

Special case: in the SSH model where the boundary is just a point, we have $\mathcal{A} = \mathbb{C}$ and the algebra $\mathcal{T}$ containing the Toeplitz operators fits into the extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C^*(\mathbb{Z}) = C(\breve{\mathbb{T}}) \longrightarrow 0.$$

A deep result of Pimsner–Voiculescu says that the K-theories of $\mathcal{T}(\mathcal{A}, \alpha)$ and $\mathcal{A}$ are the same, so that the six-term long exact sequence for the Toeplitz extension becomes

$$
\begin{array}{ccc}
K_0(\mathcal{A}) \xrightarrow{\ 1-\alpha_* \ } K_0(\mathcal{A}) \xrightarrow{\ j_* \ } K_0(\breve{\mathcal{B}}) \\
\uparrow{\scriptstyle \partial} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle \partial} \\
K_1(\breve{\mathcal{B}}) \xleftarrow[\ j_* \ ]{} K_1(\mathcal{A}) \xleftarrow[\ 1-\alpha_* \ ]{} K_1(\mathcal{A})
\end{array}
$$

Momentum space model for bulk-boundary correspondence

In the SSH example, $\partial : K_1(C(\breve{\mathbb{T}})) \to K_0(\mathcal{K}) \cong K_0(\mathbb{C})$ is the topological index ($=$ winding number). It is an isomorphism in this case.

In the Chern insulator example, $\partial : K_0(C(\breve{\mathbb{T}}^2)) \to K_1(\breve{\mathbb{T}})$ takes the Chern class to the winding invariant. Physically, this means that the Chern insulator gives rise to a "winding" mode parallel to the boundary. This may be interpreted as spectral flow/"charge punping".

(a)
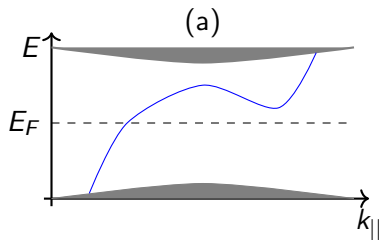
Generally, ∂ is interpreted as the *topological bulk-boundary map*,

Position space bulk-boundary correspondence?

This framework was first used by Kellendonk–Richter–Schulz-Baldes to prove equality of Hall and edge conductance in the Quantum Hall Effect.

As it stands, connecting homomorphims ∂ are somewhat abstract, and we would like a geometrical understanding which is consistent with the heuristics.

It turns out that T-duality is precisely the tool to make this connection. The slogan is that T-duality implements a "geometric Fourier transform" which converts ∂ into a restriction-to-boundary map in position space.

Commutative T-duality

The basic commutative T-duality is an isomorphism between the K-theories of $\mathbb{T}^d$ and $\check{\mathbb{T}}^d$ with a degree shift of d.

In this case, T-duality is the *Fourier–Mukai transform*, implemented the Poincaré line bundle, and is summarised by the following diagram,

$$
\begin{array}{ccc}
& \mathcal{P} & \\
& \downarrow & \\
& \mathbb{T}^d \times \check{\mathbb{T}}^d & \\
{}^{p}\swarrow & & \searrow{}^{\check{p}} \\
\mathbb{T}^d & & \check{\mathbb{T}}^d.
\end{array}
$$

The first Chern class of $\mathcal{P}$ can be represented by the 2-form $\sum_{i=1}^{d} \mathrm{d}x_i \wedge \mathrm{d}k_i$. Then $T : K^0(\mathbb{T}^d) \xrightarrow{\sim} K^{-d}(\check{\mathbb{T}}^d)$ is

$$
T : [\mathcal{E}] \mapsto [\check{\mathcal{E}}] = [\check{p}_*(p^*(\mathcal{E}) \otimes \mathcal{P})].
$$

Commutative T-duality

In terms of Chern character,

$$\mathrm{ch}(\breve{\mathcal{E}}) = \int_{\mathbb{T}^d} \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{P})$$

This is already interesting in $d = 2$. Then $K^0(\mathbb{T}^2) \cong H^{\mathrm{even}}(\mathbb{T}^2)$ is given by the rank r and first Chern class c, i.e.

$$\mathrm{ch}(\mathcal{E}) = r(\mathcal{E}) + c(\mathcal{E})\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

The T-dual $\breve{\mathcal{E}}$ has

$$\mathrm{ch}(\breve{\mathcal{E}}) = \int_{\mathbb{T}^2} \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{P}) = c(\mathcal{E}) + r(\mathcal{E})\mathrm{d}k_1 \wedge \mathrm{d}k_2$$

$$= r(\breve{\mathcal{E}}) + c(\breve{\mathcal{E}})\mathrm{d}k_1 \wedge \mathrm{d}k_2.$$

Thus rank and first Chern number are interchanged!

Real commutative T-duality

The bundle $\mathcal{P}$ has a natural antilinear lift of the involution $\mathrm{id} \times \theta$ on $\mathbb{T} \times \check{\mathbb{T}}$, so that $\mathcal{P}$ is a Real bundle.

There are real versions of the Fourier–Mukai transform,

$$KO^{-n+d}(\mathbb{T}^d) \cong KR^{-n}(\check{\mathbb{T}}^d)$$

$$KSp^{-n+d}(\mathbb{T}^d) \cong KQ^{-n}(\check{\mathbb{T}}^d)$$

Under these isomorphisms, the $\mathbb{Z}_2$ FKM invariants become Stiefel–Whitney classes in a certain sense.

T-duality as Fourier transform

We can think of T-duality as a kind of Fourier transform at the level of topological invariants.

Recall that the ordinary Fourier transform takes $f : \mathbb{Z} \to \mathbb{C}$ to $\check{f} : \check{\mathbb{T}} \to \mathbb{C}$, implemented by the kernel $P(m, k) = e^{2\pi i k m}$.

$\check{f}$ is obtained by pulling back $f = f(m)$ to a function $f = f(m, k)$ on $\mathbb{Z} \times \check{\mathbb{T}}$, multiplying by kernel $P(m, k)$, then integrating out $\mathbb{Z}$.

Let $\iota : \mathbb{Z}^{d-1} \to \mathbb{Z}^d$ be $(m_1, \ldots, m_{d-1}) \mapsto (m_1, \ldots, m_{d-1}, 0)$, and $\partial : \check{f} \mapsto \partial \check{f}$ be integration (push-forward) along d-th circle in $\check{\mathbb{T}}^d$. Only Fourier components with $m_d = 0$ survive, so

$$
\begin{array}{ccc}
f & \xrightarrow[\text{FT}]{\sim} & \check{f} \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \partial} \\
\iota^* f & \xrightarrow[\text{FT}]{\sim} & \partial \check{f}
\end{array}
$$

commutes, with ι^* the restriction map to $m_d = 0$.

T-duality as Fourier transform

This observation is consistent with the geometric (or physical) intuition of the bulk-boundary correspondence as being a restriction-to-boundary map,
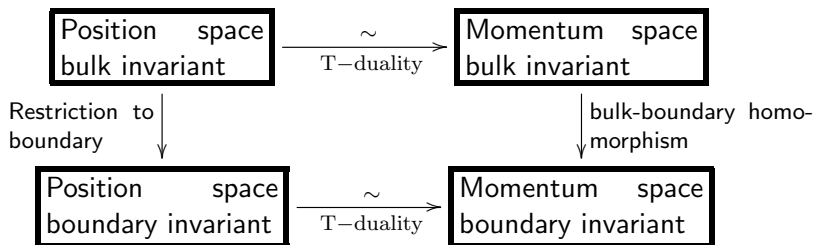
Mathematically the RHS is formulated generally using crossed product C^*-algebras, while the LHS is "geometric".

Twisted crossed products

Definition: A C^*-dynamical system $(\mathcal{A}, G, \alpha)$ has a locally compact group G acting on a C^*-algebra $\mathcal{A}$ by automorphisms α_g.

The crossed product $\mathcal{A} \rtimes_\alpha G$ is obtained by taking the compactly supported functions $G \to \mathcal{A}$ with α-convolution product, and completing in a certain C^*-norm.

The action can also be twisted by a 2-cocycle σ, in which case the crossed product is twisted, $\mathcal{A} \rtimes_{\alpha,\sigma} G$.

We had an example where $G = 1$, $\mathcal{T}$ acted on $\mathbb{C}$ by complex conjugation, and there was a 2-cocycle σ which encoded $\mathsf{T}^2 = -1$.

Twisted crossed products

The rotation algebra $C(S^1) \rtimes_{R_\theta} \mathbb{Z}$ where R_θ is rotation by angle θ, is also known as the noncommutative torus A_θ.

It can also be constructed as a *twisted crossed product* $\mathbb{C} \rtimes_{\mathrm{id},\sigma} \mathbb{Z}^2$, which is the algebra of magnetic translations. Two translations $T_{\boldsymbol{m}}, T_{\boldsymbol{n}}$ differ from $T_{\boldsymbol{m+n}}$ by a phase $\sigma(\boldsymbol{m}, \boldsymbol{n}) = e^{\mathrm{i}\theta n_1 m_2}$ where θ is the magnetic field strength.

Let $\mathcal{A} = C(\Omega)$ where Ω is a compact disorder space. The algebra $C(\Omega) \rtimes_{\alpha,\sigma} \mathbb{Z}^d$ was used by Bellissard et al to model disorder in the quantum Hall effect.

Induced algebra

When $\mathbb{Z}$ acts on $\mathcal{A}$ by α, there is another natural associated algebra.

$\mathbb{Z}$ acts freely on $\mathbb{R}$ on the right. The *induced algebra* $\mathcal{B} = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A}, \alpha)$ consists of continuous functions $f : \mathbb{R} \to \mathcal{A}$ satisfying $\mathbb{Z}$-equivariance

$$f(x + 1) = \alpha^{-1}(f(x)), \qquad x \in \mathbb{R}.$$

There is a left action τ^{α} of $\mathbb{R}$ on $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A}, \alpha)$ by translation

$$\tau_y^{\alpha} f(x) \coloneqq f(x - y).$$

We can think of the induced algebra as a bundle of algebras over $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with fibre $\mathcal{A}$, i.e. a *mapping torus* for α.

Two long exact seqences

Let the bulk and boundary algebras be related by $\check{\mathcal{B}} = \mathcal{A} \rtimes_\alpha \mathbb{Z}$ as before.

The *momentum space* Toeplitz extension

$$0 \to \mathcal{A} \to \mathcal{T}(\mathcal{A}, \alpha) \to \check{\mathcal{B}} \to 0$$

gave the long exact sequence

$$
\begin{array}{ccccc}
K_0(\mathcal{A}) & \overset{1-\alpha_*}{\longrightarrow} & K_0(\mathcal{A}) & \longrightarrow & K_0(\check{\mathcal{B}}) \\
{\scriptstyle \partial} \uparrow & & & & \downarrow {\scriptstyle \partial} \\
K_1(\check{\mathcal{B}}) & \longleftarrow & K_1(\mathcal{A}) & \overset{1-\alpha_*}{\longleftarrow} & K_1(\mathcal{A}),
\end{array}
$$

Two long exact sequences

There is also a *geometric* sequence for $\mathcal{B} = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A}, \alpha)$,

$$0 \to S\mathcal{A} \to \mathcal{B} \xrightarrow{\mathrm{ev_0}} \mathcal{A} \to 0,$$

where $S\mathcal{A}$ is the suspension $C_0((0,1), \mathcal{A})$ and $\mathrm{ev_0}$ is evaluation at 0. This SES gives, using the suspension isomorphism $K_0(S(\cdot)) \cong K_1(\cdot)$, the long exact sequence

$$
\begin{array}{ccccc}
K_0(\mathcal{A}) & \longrightarrow & K_1(\mathcal{B}) & \xrightarrow{\iota^*} & K_1(\mathcal{A}) \\
{\scriptstyle 1 - \alpha_*} \uparrow & & & & \downarrow {\scriptstyle 1 - \alpha_*} \\
K_0(\mathcal{A}) & \xleftarrow{\iota^*} & K_0(\mathcal{B}) & \longleftarrow & K_1(\mathcal{A}),
\end{array}
\tag{6}
$$

where $\iota^* \equiv (\mathrm{ev_0})_*$ is restriction to fibre at 0 (the boundary).

Noncommutative T-duality

Take $\mathcal{A} = \mathbb{C}$ and $\alpha = \mathrm{id}$. Commutative T-duality
$K_\bullet(C(\mathbb{T})) \xrightarrow{\sim} K_{\bullet+1}(C(\check{\mathbb{T}}))$ is a special case of

$$T \equiv T_\alpha^\bullet : K_\bullet(\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha)) \xrightarrow{\sim} K_{\bullet+1}(\mathcal{A} \rtimes_\alpha \mathbb{Z})$$

$T_\alpha^\bullet$ is defined by Connes' Thom isomorphism

$$K_\bullet(\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha)) \xrightarrow{\sim} K_{\bullet+1}(\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha) \rtimes_{\tau^\alpha} \mathbb{R})$$

followed by the isomorphism from Green's imprimitivity theorem

$$\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha) \rtimes_{\tau^\alpha} \mathbb{R} \cong \mathcal{A} \rtimes_\alpha \mathbb{Z} \otimes \mathcal{K}$$

which induces the K-theory isomorphism

$$K_{\bullet+1}(\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha) \rtimes_{\tau^\alpha} \mathbb{R}) \cong K_{\bullet+1}(\mathcal{A} \rtimes_\alpha \mathbb{Z}).$$

T-duality characterisation

[HMT] In fact, we can abstractly characterise T as the *unique* transformation of the functors $K_\bullet(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\,\cdot\,))$ and $K_{\bullet+1}((\,\cdot\,) \rtimes \mathbb{Z})$, from the category of $\mathbb{Z}$-C^*-algebras to abelian groups, which

- normalizes to standard T-duality for $\bullet = 0$, $\mathcal{A} = \mathbb{C}$, $\alpha = \mathrm{id}$;
- is natural in the appropriate sense;
- is compatible with suspensions.

The same constructions can be carried out in real K-theory, with a modification of the normalisation condition.

Physically, the functors $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\,\cdot\,)$ and $(\,\cdot\,) \rtimes \mathbb{Z}$ introduce a position and momentum space circle respectively. Then T-duality exchanges them K-theoretically, with a degree shift.

Paschke's map

Paschke[5] had previously constructed a concrete map $\gamma_\alpha^\bullet$ which intertwines the two LES,

$$\ldots \xrightarrow{1-\alpha_*} K_{\bullet+1}(\mathcal{A}) \quad \tau \downarrow \gamma_\alpha^\bullet \quad K_\bullet(\mathcal{A}) \xrightarrow{1-\alpha_*} \ldots$$

with top vertex $K_{\bullet+1}(\mathcal{A} \rtimes_\alpha \mathbb{Z})$, arrow ∂, bottom vertex $K_\bullet(\mathrm{Ind}_\mathbb{Z}^\mathbb{R}(\mathcal{A}, \alpha))$, and map ι^*.

One can show that his $\gamma_\alpha^\bullet$ satisfies the characterisation of (and is thus equal to) T.

[5]On the mapping torus of an automorphism. Proc. Amer. Math. Soc. 88 481–485 (1983)

Paschke's map

$$\gamma_\alpha^0 : K_0(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A}, \alpha)) \to K_1(\mathcal{A} \rtimes_\alpha \mathbb{Z})$$

is constructed as follows. For $p = \{p_t\}_{t \in [0,1]} \in \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(\mathcal{A}, \alpha)$, a path of unitaries $t \mapsto w_t \mathcal{A}$ is found such that

$$p_t = \mathrm{Ad}(w_t)(p_0), \qquad t \in [0,1].$$

In particular, $p_1 = \mathrm{Ad}(w_1)(p_0)$. Then

$$\gamma_\alpha^0[p] := [L^* w_1 p_0 + 1 - p_0],$$

where $L \in (\mathcal{A} \rtimes_\alpha \mathbb{Z})$ is the unitary implementing α (i.e. $\alpha(a) = LaL^*, a \in \mathcal{A}$).

The $\bullet = 1$ case is defined by compatibility with suspensions.

T-duality simplifies bulk-boundary correspondence

To summarize: T-duality is like a topological Fourier transform, switching between momentum space picture and geometric real space picture, and exchanges the bulk-boundary homomorphism ∂ with geometrical restriction ι^*.

$$
\begin{array}{ccc}
K_\bullet(\mathcal{B}) & \xrightarrow{\underset{T}{\sim}} & K_{\bullet+1}(\check{\mathcal{B}}) \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \partial} \\
K_\bullet(\mathcal{A}) & \xrightarrow[=]{\sim} & K_\bullet(\mathcal{A})
\end{array} \ ,
$$

The abstract characterisation of $T \equiv T_\alpha^\bullet = \gamma_\alpha^\bullet$ allows generalisation to actions by $\mathbb{Z}^d$.

For even more complicated groups, the Connes–Thom isomorphism generalises to Baum–Connes+Poincaré duality. The RHS is very hard to describe, but we can use the geometric LHS to do the description, and then T-dualise.

Part 6: Applications to lattices with defects, and hyperbolic and crystallographic topological phases

Main messages of this talk

1. Geometry is also important: crystalline phases, FQHE, bulk-boundary relations, defects, Berry phases...

2. Non-abelian symmetries can be Fourier transformed in **N**on-**C**ommutative **G**eometry

3. Although NC-spaces are hard to picture, T-duality provides a complementary picture, e.g. of BEC in position-space as the "obvious" restriction-to-boundary map.

4. Applications: Screw dislocations, hyperbolic geometry (fractional BEC, hyperbolic Kane–Mele invariant), and crystallographic space groups (torsion phases).

Screw dislocations

The integer Heisenberg group $\mathrm{Heis}^{\mathbb{Z}}$

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad a, b, c, \in \mathbb{Z},$$

is a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.

Imagine this as two "horizontal" translations a, b which fail to commute up to some "vertical" translation c — this describes the symmetries of a uniform distribution of screw dislocations of a standard Euclidean lattice.
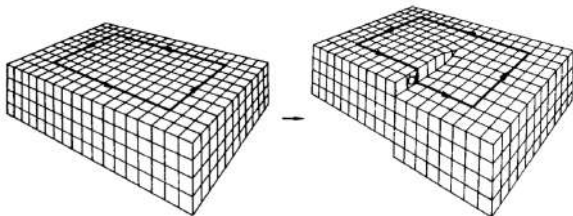
Screw dislocations

$\mathrm{Heis}^{\mathbb{Z}}$ is a discrete subgroup of the real Heisenberg group $\mathrm{Heis}^{\mathbb{R}}$ (which is topologically $\mathbb{R}^3$)

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad a, b, c \in \mathbb{R},$$

The classifying space $B\mathrm{Heis}^{\mathbb{Z}}$ is the quotient $\mathrm{Heis}^{\mathbb{R}}/\mathrm{Heis}^{\mathbb{Z}}$, which is a manifold called Nil (the name comes from "nilpotent geometry", a departure from Euclidean geometry)

Physically, Nil replaces $\mathbb{T}^3$ as the unit cell.

Because $\mathrm{Heis}^{\mathbb{Z}}$ is nonabelian, "momentum space" is not $\breve{\mathbb{T}}^3$ but $C^*(\mathrm{Heis}^{\mathbb{Z}})$, which is an interesting noncommutative "space".

Screw dislocations

Roughly speaking, $C^*(\mathrm{Heis}^{\mathbb{Z}})$ is a family of noncommutative 2-tori $\mathcal{A}_\theta$ over $\breve{\mathbb{T}}$, where over $\theta \in \breve{\mathbb{T}}$ the rotation parameter is θ itself.

Nil is a principal circle bundle over $\mathbb{T}^2$, with Chern class the generator of $H^2(\mathbb{T}^2, \mathbb{Z})$. It is sometimes called a "twisted torus" by physicists because it is also a (non-principal) torus bundle over a circle $\mathbb{T}$, with monodromy given by the $\mathrm{SL}(2, \mathbb{Z})$ matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

There is isomorphism between the K-theories of $\mathrm{Nil} = B\mathrm{Heis}^{\mathbb{Z}}$ and $C^*(\mathrm{Heis}^{\mathbb{Z}})$, using the Baum–Connes isomorphism or crossed product by $\mathrm{Heis}^{\mathbb{R}}$, which is a nonabelian version of T-duality.

Screw dislocations

We can introduce a 2D boundary as a vertical plane along a, c, containing the screw axes. So the boundary has symmetries

$$\mathbb{Z}^2 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b, c \in \mathbb{Z} \right\} \subset \mathrm{Heis}^{\mathbb{Z}}.$$

with b generating the transverse translations. Abstractly,

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathrm{Heis}^{\mathbb{Z}} = \mathbb{Z}^2 \rtimes \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1$$

with the semidirect product defined by the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ via the $\mathrm{SL}(2, \mathbb{Z})$ matrix mentioned above.

Then the boundary algebra is $\mathcal{A} = C^*(\mathbb{Z}^2) \cong C(\breve{\mathbb{T}}^2)$ and the bulk algebra is $\breve{\mathcal{B}} = C^*(\mathrm{Heis}^{\mathbb{Z}}) = C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}$.

Screw dislocations

As before, there is a "momentum space" extension algebra $\mathcal{T}(\mathcal{A}, \alpha)$ inducing a topological bulk-boundary map

$$\partial : K_0(\breve{\mathcal{B}}) = K_0(C^*(\mathrm{Heis}^{\mathbb{Z}})) \to K_1(\mathcal{A}) = K_1(C(\breve{\mathbb{T}}^2)).$$

Unlike the SSH and Chern insulator examples, ∂ is not surjective in this case. The image is only generated by windings parallel to the screw dislocation!
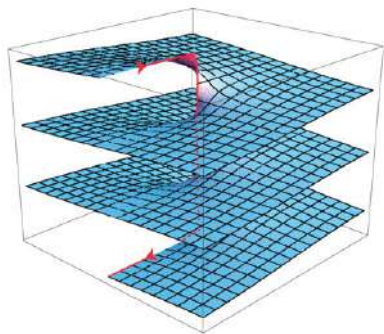
This was predicted by physical arguments in [Ran, Zhang, Vishwanath, Nature Phys. 5 298–303 (2009)]

Screw dislocations and H-flux

As a circle bundle $\mathrm{Nil} \to \mathbb{T}^2$, there is also a T-dual in the sense of BEM, which is a 3-torus $\check{\mathbb{T}}^1 \times \mathbb{T}^2$ with "H-flux" the generator vol of $H^3(\check{\mathbb{T}}^1 \times \mathbb{T}^2)$.

This as a "partial Fourier transform" along the screw axis.

This means that there is an isomorphism between $K^{-1}(\mathrm{Nil})$ and twisted K-theory $K^0(\check{\mathbb{T}}^1 \times \mathbb{T}^2, \mathrm{vol})$. The latter is also the K-theory of the continuous-trace C^*-algebra $CT(\check{\mathbb{T}}^1 \times \mathbb{T}^2, \mathrm{vol})$.

$$
\begin{array}{ccc}
K^0(\check{\mathbb{T}} \times \mathbb{T}^2, \mathrm{vol}) & \xrightarrow{\ \underset{T}{\sim}\ } & K_0(C^*(\mathrm{Heis}^{\mathbb{Z}})) \\[2em]
{\scriptstyle T_{\mathrm{circle}}}\searrow{\scriptstyle\sim} & & {\scriptstyle\sim}\nearrow{\scriptstyle T_{BC}} \\[1em]
& K^1(\mathrm{Nil}) &
\end{array}
$$

Noncommutative Bloch theory

Translations in $\mathbb{Z}^d$ generate a convolution algebra $C^*(\mathbb{Z}^d)$. The Fourier transform is $C(\check{\mathbb{T}}^d)$ with pointwise multiplication.

For general G non-abelian, (the reduced) $C^*(G) \sim$ convolution algebra of translations on $L^2(G)$, and plays the role of "momentum space". Projective symmetries also possible using $C^*(G, \sigma)$.

There is a notion of NC-Bloch theory (Grüber, Mathai–Marcolli) and NC-bundles over $C^*(G)$. Spectral projections of G-invariant Hamiltonian are elements of some matrix algebra over $C^*(G)$, leading us to compute the K-theory of $C^*(G)$.

Put in another way, noncommutative "valence bundles" are classified by $K_0(C^*(G))$. They are Hilbert C^*-modules over $C^*(G)$, or roughly, Hilbert spaces parametrised by $\widehat{G}$.

Momentum space and unit cell duality

A deep result of Baum–Connes says that $C^*(G)$ and (proper) classifying space $\underline{B}G$ are dual for a large class of groups (we will only look at discrete G)

Baum–Connes defined an assembly map
$\mu : K_\bullet(\underline{B}G) \cong K_\bullet(C^*(G))$, which is an isomorphism in many cases.

Here, $K_\bullet(\underline{B}G)$ is short-form for the G-equivariant K-homology of the proper classifying space $\underline{E}G$. The latter is a contractible space on which G acts freely up to finite isotropy. We do not need a precise definition of K-homology, just that it is often computable in terms of ordinary homology.

We have seen some examples: $\mathbb{Z}^d$ acts freely on $\mathbb{R}^d$, and $\mathrm{Heis}^{\mathbb{Z}}$ on $\mathrm{Heis}^{\mathbb{R}}$, so the quotients $\mathbb{T}^d = B\mathbb{Z}^d$ and $\mathrm{Nil} = B\mathrm{Heis}^{\mathbb{Z}}$.

Momentum space and unit cell duality

In fact, $\mathbb{R}^d$ is an $\underline{E}G$ for any crystallographic space group!

Definition: a space group G is a discrete cocompact subgroup of the Euclidean group $\mathbb{R}^d \rtimes \mathrm{O}(d)$. As abstract groups, G has the form

$$1 \longrightarrow \mathbb{Z}^d \longrightarrow G \longrightarrow F \longrightarrow 1$$

Here $\mathbb{Z}^d \subset \mathbb{R}^d$ is the maximal abelian normal subgroup of lattice translations (acting freely), while $F \subset \mathrm{O}(d)$ is the finite *point group* (e.g. reflections, rotations which may have fixed points)

F acts on the normal subgroup $\mathbb{Z}^d$ and thus it has a dual action on the Brillouin torus $\breve{\mathbb{T}}^d$.

Momentum space and unit cell duality

Only some space groups are semidirect products $\mathbb{Z}^d \rtimes_\alpha F$. Most of them are *nonsymmorphic*, meaning that G is a *non-split* extension of F by $\mathbb{Z}^d$.

A group 2-cocycle $\nu : F \times F \to \mathbb{Z}^d$ measures nonsymmorphicity (obstruction to semidirect product).

In the split case, $C^*(G) = C^*(\mathbb{Z}^d) \rtimes_\alpha F = C(\check{\mathbb{T}}^d) \rtimes_{\check{\alpha}} F$. Valence bundles are now F-equivariant bundles over $\check{\mathbb{T}}^d$, so we compute $K_0^F(\check{\mathbb{T}}^d)$ for the G-symmetric topological phases.

Momentum space and unit cell duality

In the nonsymmorphic case, ν induces an *equivariant twist* $\check{\nu}$ so that valence bundles are $\check{\nu}$-twisted F-equivariant bundles, classified by twisted equivariant K-theory $K_F^{0+\check{\nu}}(\check{\mathbb{T}}^d)$ (as suggested by Freed–Moore).

$K_F^{0+\check{\nu}}(\check{\mathbb{T}}^d)$ is hard (even to define)! Using Baum–Connes, we can "compute in position space" instead.
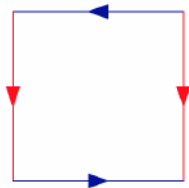
The unit cell is $\mathbb{R}^d/G \cong \mathbb{T}^d/F \equiv \underline{B}G$ (an orbifold).

We compute $K_0(\underline{B}G)$ instead. This can be done by approximating (i.e. spectral sequece) with (Bredon equivariant) homology.

Application: Nonsymmorphic crystalline phases
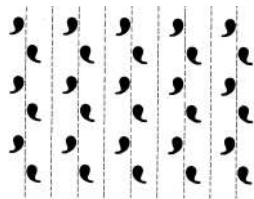
Wallpaper group pg is non-symmorphic: $F = \mathbb{Z}_2$ reflection lifts to *glide reflection* of infinite order.

$$1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(\times 1, \times 2)} \text{pg} = \mathbb{Z} \rtimes \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 1$$

In the semidirect product, the second $\mathbb{Z}$ acts on the first $\mathbb{Z}$ by $x \mapsto -x$.
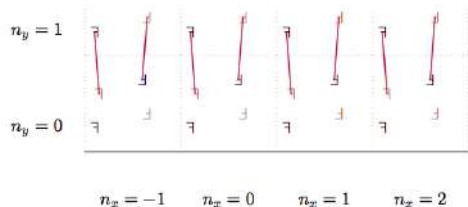
Unit cell is a Klein bottle
$$\mathbf{K} = \mathbb{R}^2/\text{pg} = F\backslash(\mathbb{R}^2/\mathbb{Z}^2) = \mathbb{Z}_2^{\text{free}}\backslash\mathbb{T}^2.$$

Application: Klein topological phase

$$K_1(C^*(\mathrm{pg})) \stackrel{T}{\cong} K_1(B\mathrm{pg}) = K_1(\mathbf{K}) \cong H_1(\mathbf{K}) = \mathbb{Z} \oplus \mathbb{Z}_2$$

.

As far as I know, this $\mathbb{Z}_2$-phase has not been found by other means (Berry curvature methods cannot detect torsion).

In ongoing work, K. Gomi and I show that there is an exotic 1D $\mathbb{Z}_2$ boundary mode, which is the index of the $\mathbb{Z}_2$ phase above.
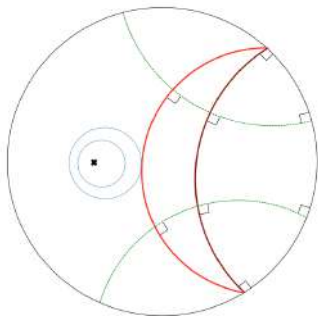

We formulate an index map between twisted K-theory groups, for a "twisted family of Toeplitz operators".

Toward fractions: hyperbolic plane geometry

Hyperbolic plane $\mathbb{H}$ can be indentified with interior of a Euclidean disc but with different metric.

Geodesics in $\mathbb{H}$ (green) are arcs of Euclidean circles intersecting disc boundary at right angles. **PSL**$(2, \mathbb{R})$ acts transitively and isometrically on $\mathbb{H}$ by Möbius transformations, and can be categorised into hyperbolic/parabolic/elliptic transformations. A hyperbolic element effects "translations" and each orbit lies on a *hypercycle* (red), homeomorphic to $\mathbb{R}$. An elliptic element effects "rotations".

Space groups and unit cells in hyperbolic plane
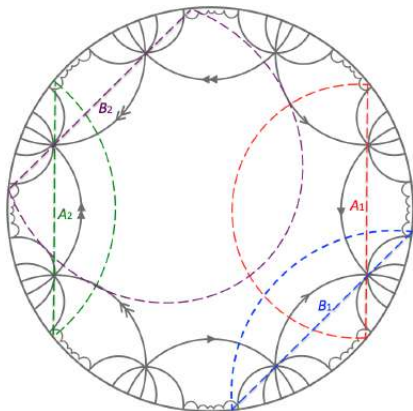
Cayley graph of Γ_2 in $\mathbb{H}$, illustrating that $[A_1, B_1][A_2, B_2] = 1$.

Analogue of lattice $\mathbb{Z}^2$ is a cocompact discrete torsion-free $\Gamma_g \subset \textbf{PSL}(2, \mathbb{R})$, with canonical generators $A_1, B_1, \ldots, A_g, B_g$.

Unit cell is genus g Riemann surface Σ_g.

Analogue of space group is Fuchsian group $\Gamma_{g,\nu}$

$$1 \to \Gamma_{g'} \to \Gamma_{g,\nu} \to F \to 1.$$

ν lists the order of elliptic rotations C_j, $C_j^{\nu_j} = 1$. Unit cell $\Sigma_{g,\nu} = \Sigma_{g'}/F$ is a hyperbolic orbifold.

Euclidean plane quantum Hall effect

In 2D Euclidean space, $\mathbf{A} = -\theta\, y \mathrm{d}x$ is potential for magnetic field $\mathbf{B} = \theta\, \mathrm{d}x \wedge \mathrm{d}y = d\mathbf{A}$. Quantum Hall Hamiltonian is $H_{\mathbf{A},V} = \frac{1}{2}(d + i\mathbf{A})^*(d + i\mathbf{A}) + V$ for a $\mathbb{Z}^2$-invariant potential.

Then $H_{\mathbf{A},V}$ commutes with *magnetic translations* $T_{\mathbf{n}}^{\sigma}$. The latter furnish a projective rep. of $\mathbb{Z}^2$ with 2-cocycle $\sigma(\mathbf{n}, \mathbf{n}') = e^{-2\pi i \theta n_1' n_2}$.

Spectral projections of $H_{\mathbf{A},V}$ lie in $C^*(\mathbb{Z}^2, \overline{\sigma}) \otimes \mathcal{K}(L^2(\mathbb{T}^2))$. Pairs with *conductance* cyclic cocycle τ_{Kubo} (NC-integration) to give *integer* values for the Hall conductance as a NC Chern number.

Suffices here to mention that the NC analogue of the Chern insulator is the Rieffel projection.

Quantum Hall effect in hyperbolic space

Everything works for $\mathbb{H}$: $\mathbf{PSL}(2, \mathbb{R})$ acts transitively and isometrically on $\mathbb{H}$ by Möbius transformations. Invariant $\mathbf{B}$ field is $\theta \, dx \wedge dy / y^2$ in half-plane model of $\mathbb{H}$.

$\mathbb{Z}^2$ is replaced by $\Gamma_{g,\boldsymbol{\nu}} \subset \mathbf{PSL}(2, \mathbb{R})$, and spectral projections for $\Gamma_{g,\boldsymbol{\nu}}$-invariant Hamiltonian lie in $C^*(\Gamma_{g,\boldsymbol{\nu}}, \overline{\sigma}) \otimes \mathcal{K}$.

These projections can be classified by K-theory,

$$K_\bullet(C^*(\Gamma_{g,\boldsymbol{\nu}}, \overline{\sigma})) \cong \begin{cases} \mathbb{Z}^{2 + \sum_{j=1}^r (\nu_j - 1)} & \bullet = 0, \\ \mathbb{Z}^{2g} & \bullet = 1. \end{cases}$$

whose computation uses $K_{\mathrm{orb}}^\bullet(\Sigma_{g,\boldsymbol{\nu}})$ [Farsi'92] and Baum–Connes isomorphism. This is "Riemann surface T-duality".
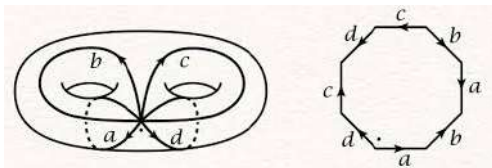
Fractional indices

Main difference: range of $\tau_{\mathrm{Kubo},\mathbb{H}}$ is $\phi\mathbb{Z} \subset \mathbb{R}$, where
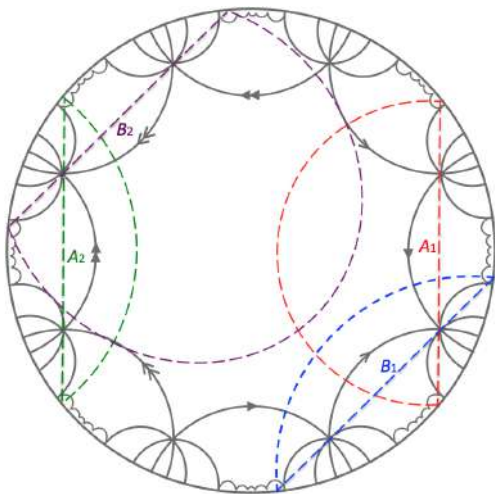
$$\phi = 2(g-1) + \sum_{j=1}^{r}(1 - \frac{1}{\nu_j}) \in \mathbb{Q}$$

is the orbifold Euler characteristic of $\Sigma_{g,\nu}$ [M+M'01].

Another interpretation: Effective geometry due to interactions among electrons (model for FQHE). $\phi \sim$ effective charge

Geometry of bulk and boundary

A hyperbolic $X \in \Gamma$ effects translation along a *hypercycle* $\mathcal{O}_X$ which serves as an effective boundary that partitions $\mathbb{H}$ into "bulk" and "vacuum". Not all complementary directions are transverse!

Fractional bulk-boundary correspondence

It was easy to label points in $\mathbb{Z}^d$ and write down an explicit tight-binding model to truncate.

Very hard to enumerate the lattice points in the hyperbolic half-plane!

Geometrically it is easy to describe: a "boundary" is the hypercycle generated by some choice of hyperbolic symmetry group element.

Topologically, this boundary is also $\mathbb{R}$, but it is embedded in hyperbolic $\mathbb{R}^2$ differently from in Euclidean space. This is reflected in the "geometric factor" ϕ. The boundary unit cell is again $\mathbb{R}/\mathbb{Z} = \mathbb{T}$

Given this geometric bulk-boundary relation, we can formulate the bulk-boundary correspondence (in momentum space) abstractly using T-duality.

Fractional bulk-boundary correspondence

T-duality converts the geometric restriction-to-bounday map

$$\iota^* : K^\bullet_{\mathrm{orb}}(\Sigma_{g,\nu}) \to K^\bullet(\mathbb{T})$$

into some homomorphism between bulk and boundary algebras,

$$\partial : K_\bullet(C^*(\Gamma_{g,\nu}, \overline{\sigma})) \to K_{\bullet+1}(C(\breve{\mathbb{T}})).$$

Deep theorem of Rosenberg–Schochet (Universal Coefficients) says that ∂ is the index map for a *unique* class of extensions of $C^*(\Gamma_{g,\nu}, \overline{\sigma})$ by $C(\mathbf{B}')$.

Desired "Topelitz" algebra is in this class, and its index map ∂ gives the bulk-boundary map. There is also a dual boundary conductance 1-cocycle so that boundary and bulk fractional conductance are equal.

Bulk-boundary map for general geometries

• Take a sensible codimension-1 boundary with enough translational and/or point symmetries, so that the boundary unit cell is a subspace of the bulk unit cell.

• All bulk and boundary unit cell invariants have a T-dual momentum space counterpart.

• Desired index homomorphism ∂ in momentum space $\leftrightarrow$ "Obvious" restriction homomorphism at unit cell level.

• Implicit construction of "Toeplitz" algebra of truncated hopping terms, with momentum space bulk-boundary map ∂.

Part 7: Remarks and discussion