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FOLIATIONS AND GROUPS OF DIFFEOMORPHISMS

BY WILLIAM THURSTON

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John Mather has described a close relation between framed codimension-one Haefliger structures (these form a class of singular foliations), and the group of compactly supported diffeomorphisms of $\boldsymbol{R}^1$, with discrete topology [11], [12], [14]. In this announcement I will describe generalizations of his ideas to higher codimension Haefliger structures and groups of diffeomorphisms of arbitrary manifolds. See Haefliger [7] for a development of Haefliger structures and their classifying spaces.

I would like to thank Boyd Anderson, André Haefliger and John Mather for long discussions and proddings about this material, and many others for helpful conversations and relevant information.

Let $\mathrm{Diff}^r(M^p)$ denote the group of C^r diffeomorphisms of M^p, a closed manifold. Let $\mathrm{Diff}_0^r(M^p)$ denote the connected component of the identity.

THEOREM 1. $\mathrm{Diff}_0^\infty(M^p)$ *is a simple group.*

The proof makes use of both the theorem of Epstein [4] that the commutator subgroup of $\mathrm{Diff}_0(M^p)$ is simple, and of the result of M. Herman [9] which gives the case M^p is a p-torus.

THEOREM 2. $B\bar{\Gamma}_p^\infty$ *is* $(p+1)$*-connected, where* $B\bar{\Gamma}_p^\infty$ *is the classifying space for framed, codimension* p, C^∞, *Haefliger structures.*

The more usual notation is $F\Gamma_p^\infty = B\bar{\Gamma}_p^\infty$. Haefliger proved [6] that $B\bar{\Gamma}_p^r$ is p-connected for $1 \leqq r \leqq \infty$; Mather proved that $B\bar{\Gamma}_1^\infty$ is 2-connected.

Theorem 2 means that two C^∞ foliations of a manifold coming from nonsingular vector fields are homotopic as Haefliger structures if and only if the normal bundles are isomorphic.

Theorems 1 and 2 are proven by showing they are related; cf. Theorem 4 for a statement of a relationship.

COROLLARY. $P_1^{[p/2]}$ *is nontrivial in* $H^*(B\Gamma_p^\infty; \boldsymbol{R})$ *where* P_1 *is the first real Pontrjagin class of the normal bundle to the canonical Haefliger structure.*

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Thus, Bott's vanishing theorem [1], which says real Pontrjagin classes in $B\Gamma_p^r$ $(r\geqq 2)$ vanish above dimension $2p$, gives a sharp bound on dimensions.

This corollary in the case $p=2$ follows easily from Theorem 2.

For higher codimensions, product foliations then yield examples.

THEOREM 3. *$B\bar{\Gamma}_p^0$ is contractible.*

Again, Mather proved this when the codimension is one.

This means topological Haefliger structures are completely determined up to homotopy by their normal micro-bundles.

Theorem 3 implies that Bott's vanishing theorem is quite false in the topological case—any normal micro-bundle is the normal micro-bundle for a topological foliation. In fact, if the micro-bundle is differentiable, it even admits a Haefliger structure Lipschitz close to being differentiable.

A little background and notation is necessary before the statement of the more general relationships. Let G be a topological group. Let G_δ be G with discrete topology. Then the map $G_\delta \to G$ is a continuous map which has a homotopy-theoretic fiber $\bar{G}$. $\bar{G}$ is also a topological group: the explicit construction for $\bar{G}$ is the space of paths α in G ending at the identity $e=\alpha(1)$, with discrete topology on $\alpha(0)$. Then multiplication is pointwise. There are maps, now,

$$\bar{G} \to G_\delta \to G \to B\bar{G} \to BG_\delta \to BG,$$

and any two consecutive arrows define a fibration.

BG is the classifying space for G-bundles. BG_δ classifies flat G-bundles: for instance, $B\,\mathrm{Diff}^\infty(M^n)_\delta$ has an associated M-bundle, with discrete structure group: i.e., a C^∞ foliation transverse to the fibers of the bundle. Thus, $B\,\mathrm{Diff}^\infty(M^p)_\delta$ classifies "foliated M^p-bundles". Finally, $B\bar{G}$ classifies G-bundles with a flat structure, together with a global trivialization defined (up to homotopy); e.g. $B\,\overline{\mathrm{Diff}}^\infty(M^p)$ classifies "foliated M^p-products".

Let $\mathrm{Diff}_K(\boldsymbol{R}^p)$ be the group of diffeomorphisms of $\boldsymbol{R}^p$ with compact support. Then again, $B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\times \boldsymbol{R}^p$ has a foliation of codimension p transverse to the $\boldsymbol{R}^p$-factors. Thus, there is a classifying map

$$B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p) \times \boldsymbol{R}^p \to B\bar{\Gamma}_p^r.$$

(The image is in $B\bar{\Gamma}_p^r$ since there is a natural trivialization of the normal bundle to the foliation.)

The foliation agrees with the trivial, product foliation in a neighborhood of ∞ in the $\boldsymbol{R}^p$ factors. Thus, one obtains a map of the p-fold suspension of $B\,\overline{\mathrm{Diff}}_K^r \boldsymbol{R}^p$,

$$S^p(B\,\overline{\mathrm{Diff}}_K^r \boldsymbol{R}^p) \to B\bar{\Gamma}_p^r.$$

This defines an adjoint map $B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}_p^r)$ to the p-fold loop space of $B\bar{\Gamma}_p^r$.

THEOREM 4. *The map* $B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p)\to\Omega^p(B\bar{\Gamma}_p^r)$ *induces an isomorphism on homology.*

This theorem is due to Mather in the case $p=1$.

The map is certainly not a homotopy equivalence since $\pi_1(B\,\overline{\mathrm{Diff}}_K^r\boldsymbol{R}^p)$ is highly nonabelian while $\pi_1(\Omega^p B\bar{\Gamma}_p^r)=\pi_{p+1}(B\bar{\Gamma}_p^r)$ is abelian.

Similarly, there is a map $B\,\overline{\mathrm{Diff}}^r(M^p)\times M^p\to B\Gamma_p^r$ which is a lifting of the classifying map for the tangent bundle of M^p, so there is a commutative diagram

$$\begin{array}{ccc} & & B\Gamma_p^r \\ & \nearrow & \downarrow \\ B\,\overline{\mathrm{Diff}}^r(M^p)\times M^p & \to & BO_p \end{array}$$

Let X be the space of liftings of the classifying map for $T(M^p)$ in BO_p to $B\Gamma_p^r$. Then we have a map $B\,\overline{\mathrm{Diff}}^r(M^p)\to X$.

THEOREM 5. *The map*

$$B\,\overline{\mathrm{Diff}}^r(M^p)\to X$$

induces an isomorphism on homology.

Again, this is not a homotopy equivalence since $\pi_1(X)$ is abelian.

For the case $r=0$, we assume M^p is a differentiable manifold.

COROLLARY. (a) $B\,\overline{\mathrm{Homeo}}(M^p)$ *is acyclic, where* $\mathrm{Homeo}(M^p)=\mathrm{Diff}^0(M^p)$ *is the group of homeomorphisms of* M^p.

(b) *The map* $B\,\mathrm{Homeo}(M^p)_\delta\to B\,\mathrm{Homeo}(M^p)$ *induces an isomorphism on homology.*

This corollary is implied by Theorems 3 and 5. Cf. Mather [**13**], who showed $B\,\mathrm{Homeo}_K(\boldsymbol{R}^p)_\delta$ is acyclic.

COROLLARY. *The following groups are isomorphic, where k is the first positive integer such that one of them is nontrivial:*

(i) $H_k(B\,\overline{\mathrm{Diff}}^r(M^p);Z)$,

(ii) $H_k(B\,\overline{\mathrm{Diff}}_K^r(\boldsymbol{R}^p);Z)$,

(iii) $H_{k+p}(B\Gamma_p^r;Z)$.

CONJECTURE. *This first k is* $p+1$, *for* $r=\infty$.

Mather's theorem [**11**] shows this for $p=1$. Bott and Haefliger showed that all differentiable characteristic classes (in some sense) vanish below this dimension, $H_{2p+1}(B\bar{\Gamma}_p^r; Z)$ [2], [3].

In [16] I sketched examples showing there is a surjective homomorphism

$$H_3(B\bar{\Gamma}_1^\infty; Z) \twoheadrightarrow \mathbf{R},$$

using the Godbillon-Vey invariant gv [5]. Recently I have extended this to arbitrary codimension, so there is a surjective homomorphism

$$H_{2p+1}(B\bar{\Gamma}_p^\infty; Z) \twoheadrightarrow \mathbf{R}.$$

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Department of Mathematics, University of California, Berkeley, California 94720

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139