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DRAFT VERSION ONLY

1 Lecture 1: Secondary K-Theory

Secondary K-theory = categorified version of algebraic K-theory

Here categorified means the objects, vector bundles of coherent sheaves, are going to be replaced by a categorified version.

Some "motivations":

- $K^{(2)}(\mathfrak{k})$ = secondary K-theory of the ring $\mathfrak{k}$
- $K^{(2)}(\mathfrak{k})$ encodes interesting "higher" invariants of $X = Spec\mathfrak{k}$

$$H^1(X, \mathcal{G}_m) \cong Pic(X) \to K_0(X)$$

$$H^2_{et}(X, \mathcal{G}_m) \to K_0^{(2)}(X)$$

- $K^{(2)}(\mathfrak{k})$ seems close to a K-group of "motives"/X=spec but in the non-commutative setting (E.g. we will see the deRham realization)

Remark 1. *The categorification process makes $K^{(2)}(\mathfrak{k})$ more "discrete" than $K(\mathfrak{k})$.*

Definition 2. *($K_0^{(2)}(\mathfrak{k})$) $\mathfrak{k}$ commutative ring.*
$C(\mathfrak{k})$ is the category of complexes of $\mathfrak{k}$-modules $dgCat/\mathfrak{k}$ is the category of $C(\mathfrak{k})$-enriched categories

We will use $dgCat/\mathfrak{k}$ to construct a symmetric monoidal category $H_0^{Mor}(dgCat/\mathfrak{k})$ which will be the categorified version of the symmetric monoidal category $\mathfrak{k}$-mod.

Consider $T \in dgCat/\mathfrak{k}$, then $T{-}dgmod$ is (left) dg modules/$T = \underline{Hom}(T, C(\mathfrak{k})$

$E_x \in C(\mathfrak{k})$, $\forall x \in T$

$$T(x, y) \otimes E_x \to E_y, \text{ in } C(\mathfrak{k})$$

$T - dgmod \supset$ quasi-isomorphisms

$$(f \colon E \to E' / \forall x \in T E_x \to E'_x \text{ is a q-iso}$$

Definition 3. *$\mathcal{D}(T) = [quasi - isom]^{-1}(T - dgmod)$*

$\bullet f\colon T \to T'$ in $dgCat/\mathfrak{k}$ is a **Morita equivalence** *of $f^*\colon \mathcal{D}(T') \to \mathcal{D}(T)$ is an equivalence of categories.*

Definition 4.

$$H_0^{Mor}(dgCat/\mathfrak{k}) := [Morita\ equiv]^{-1}(dgCat/\mathfrak{k})$$

Theorem 5. *$T, T' \in dgCat/\mathfrak{k}$ (T has flat $\hom/\mathfrak{k}$)*

$$[T,T']_{H_0^{Mor}} \cong \left\{ E \in \mathcal{D}(T \otimes_{\mathfrak{k}} T'^{op}) | \forall x \in T, E_x \in \mathcal{D}_c(T'^{op}) \right\} / \sim$$

c is compact objects.

Remark 6. *$\mathcal{D}$ is triangulated category, $x \in \mathcal{D}_c$ if*

$$x, \oplus_\alpha y_\alpha] \cong \oplus_\alpha [x, y_\alpha]$$

Remark 7. *$H_0^{Mor}(dgCat/\mathfrak{k})$ is the 1-truncation of a natural 2-Cat ($(2,\infty)$-Cat)*

- *objects: same $dgCat/\mathfrak{k}$*
- $\underline{\hom}(T,T') := \left\{ E \in \mathcal{D}(T \otimes^{\mathbb{L}} T'^{op}) \mid E_x \textit{compact for all } n \right\} \subset \mathcal{D}(T \otimes T'^{op})$

$T, T' \in dgCatmathfrakk$
$T \otimes_{\mathfrak{k}} T' \in dgCat/\mathfrak{k}$
$Ob(T \otimes_{\mathfrak{k}} T')((x,x'),(y,y')) = T(x,y) \otimes_{\mathfrak{k}} T'(x',y')$
This $\otimes$ make dgCat/$\mathfrak{k}$ into a closed $\otimes$-category
It can be derived: $T, T; \mapsto T \otimes_{\mathfrak{k}}^{\mathbb{L}} T'$
$\left(\otimes_{\mathfrak{k}}^{\mathbb{L}}, H_0^{Mor}(dgCat/\mathfrak{k})\right)$ is a closed symmetric monoidal category.
Localization
T a dgCat/$\mathfrak{k}$
subset of morphisms $= S \subset [T]$ =homotopy category of T
objects same as T
$[T](x,y) := H^0(T(x,y))$

Definition 8. *A* **localization** *of T along S is*

$$T \xrightarrow{l} L_S T$$

in $H_0^{Mor}(dgCat/\mathfrak{k})$ such that $\forall T'$,

$$[L_s T, T'] \overset{l^*}{\hookrightarrow} [T, T']$$

and the image of $\mathcal{C}^* = \{f\colon T \to T' | (*)\}$.

$$\begin{tikzcd} {[t]} \arrow[r] \arrow[d] & {[T']} \\ S^{-1}[t] \arrow[ur] & \end{tikzcd}$$

Proposition 9. *Localizations always exist.*

Examples:

- $C_{perf}(\mathfrak{k}) \in dgCat/\mathfrak{k}$ (perfect complexes)$\supset$ quasi-isomorphisms
  $L_{perf}(\mathfrak{k} := L_{quasi-isom} C_{perf(\mathfrak{k}) \in H_0^{Mor}}(dgCat/\mathfrak{k})$
- T a dgCat/$\mathfrak{k}$
  $T - dgmod_{perf} \subset T - dgmod$
  $\mathcal{D}_c(T) \subset \mathcal{D}(T)$
- X scheme/$\mathfrak{k}$
  quasi-isomorphisms$\subset$ $Perf(X)$ = dgCat of perfect complexes of $\mathcal{O}_X$-modules on X

$L_{perf}(X) := L_{quasi-isom} Perf(X)$

$$[L_{perf}(X)] \cong \mathcal{D}_{perf}(X)$$

$$L_{perf}(T) \cong \mathcal{D}_c(T)$$

$$L_{perf}(X)(E, E') \cong \mathbb{R}\underline{\text{hom}}(E, E')$$

$H_0^{Mor}(dgCat/\mathfrak{k})$	$\mathfrak{k} - mod$
$\otimes_{\mathfrak{k}}^{\mathbb{L}}$	$\otimes_{\mathfrak{k}}$
$\otimes - closed$	$\otimes - closed$
$saturateddg - categoies :=$	$Proj\ ft(\mathfrak{k}) =$
$dualizableobjectsinH_0^{Mor}(dgCat/\mathfrak{k})$	$\{dualizableobjectsin\mathfrak{k} - Mod\}$
$T \hookrightarrow T' \to T/T'$	$shortexactsequences$
$thehookis(dg)fullyfaithfuldgfunctor$	

$$K_0^{(2)}(\mathfrak{k}) = \frac{\mathbb{Z}\left[saturateddgCat\right]}{T' = T + T/T'}$$

Examples

- X smooth proper over $\mathfrak{k}$, then $L_{perf}(X)$ is saturated.

$$L_{perf}(X) \in L_0^{(2)}(\mathfrak{k})$$

- more generally: $\mathfrak{k} = \mathbb{C}$

$$K_0(Varover\mathfrak{k}) \xrightarrow{\exists} K_0^{(2)}(\mathfrak{k})$$

where LHS os $\mathbb{Z}\left[\text{Varieties over } \mathfrak{k}\right]/\{X = Y + U, Y \hookrightarrow \text{ closed } U = X - Y\}$??? The existence is a theorem of Bondal-Lunts-Larson.

Next lecture: few more examples
Ultimate goal: Does there exist is Chern character

$$K_0^{(2)}(\mathfrak{k}) \to ?$$

$$K^{(2)}(X) \to ?$$

$K^{(2)}(X) =$ spectrum of scheme (stack) X.

2 Lecture 2

Reminders (and compliments):

$\mathfrak{k}$ commutative ring.

$$(H_0^{Mor}(dgCat/\mathfrak{k}), \otimes_{\mathfrak{k}}^{\mathbb{L}}) \subset (H_0^{Ct}(dgCat/\mathfrak{k}), \otimes_{\mathfrak{k}}^{\mathbb{L}})$$

(dualizable = saturated dgCat)

same objects

$$[T, T'] = \left\{E \in \mathcal{D}(T \otimes T'^{op}), E_x \text{ is compact}\right\} \quad [T, T'] = \mathcal{D}(T \otimes T'^{op})/\text{iso}$$

(rigid $\otimes$-cat)

Proposition 10. *$T \in H_0^{Mor}(dgCat/\mathfrak{k})$ is saturated $\Leftrightarrow T \sim B$.*

B is a dg algebra/$\mathfrak{k}$ such that

- B is compact $\in \mathcal{D}(\mathfrak{k})$ ("proper")
- B is compact $\in \mathcal{D}(B \otimes_{\mathfrak{k}} B^{op})$ ("smoothness")

$\alpha \in H^2_{ct}(X\mathbb{G}_m), X = Spec\mathfrak{k}$ then
A_α is Azumaya $\mathfrak{k}$-algebra, smooth and proper, then
$L_{perf}(A_\alpha)$ is saturated, which implies
$L_{perf}(A_\alpha) \in K_0^{(2)}(\mathfrak{k})$, then
$H^2(X, \mathbb{G}_m) \to K_0^{(2)}(\mathfrak{k})$

More generally: A is a $\mathfrak{k}$-algebra

- projective of finite type over $\mathfrak{k}$
- $\forall L$ field, $\forall k \to L$

$$A \otimes_{\mathfrak{k}} L \text{has finite homological dimension}$$

Then $L_{perf}(A)$ is saturated. (e.g., A = quiver algebra)

Basic:

- $\mathfrak{k} \to K_0^{(2)}(\mathfrak{k})$ is a functor in $\mathfrak{k}$

$$\mathfrak{k} \to \mathfrak{k}' \quad - \otimes_{\mathfrak{k}} \mathfrak{k}' \colon H_0^{Mor}(dgCat/\mathfrak{k}) \to H_0^{Mor}(dgCat/\mathfrak{k})$$

- $K_0^{(2)}(\mathfrak{k}) = \pi_0(K^{(2)}(\mathfrak{k}))$
  where $K^{(2)}(\mathfrak{k})$ is a commutative ring spectrum.

Theorem 11.

$$\mathfrak{k} = colim_{filtered}\mathfrak{k}_\alpha, \quad K^{(2)}(\mathfrak{k}) = colim K^{(2)}(\mathfrak{k}_\alpha)$$

We want:

- extend $K^{(2)}$ to schemes and algebraic stacks
- $ch \colon K^{(2)}(X) \to ?$
  X is a scheme stack

The above needs ∞-cagoy theory. Also uses cobordism hypothesis in dimension 1.
$ch :=$ "deRham realization of "$N.C.Motives$"$/X$

2.1 Really begin lecture 2: Segal categories

For ideas of what higher categories are see Jacob's talk.

$(\infty, 1)$-categories: "$(1, \infty)$-category" $=$ ∞-categories such that n-morphisms are invertible for all $n \geq 1$ $=$ ∞-categories whose <u>hom</u> are ∞-groupoids.

$$|\cdot| \colon \infty - groupoids \leftrightarrow \text{homotopy types} \colon \Pi_\infty = SSet$$

"Naive definition": A $(1, \infty)$-categoy is a $SSet$-enriched category $=$ $\mathcal{S}$-categories:

$\mathcal{S}$-categories are badly behaed with respect to the category of functors.

Definition 12. *A* **Segal category** *(or* ∞-**category***) is*

$$A\colon \Delta^{op} \to SSet$$

such that

- A_0 *is discrete (i.e., a set)*
- $A_n \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ *is a weak equivalence*

with $[1] \to [n]$, $0 \to i$, $n \to i+1$.

Remark 13. *A is an* $\mathcal{S}$*-category.*

$$A\colon \Delta^{op} \to SSet$$

$$n \mapsto \coprod_{(a_0,\ldots,a_n)} A(a_0,a_1) \times \ldots \times A(a_{n-1},a_n)$$

Here $\mathcal{S} - Cat \overset{full}{\hookrightarrow}$ *Segal Category* $\overset{full}{\subset}$ *Fun*$(\Delta^{op}, SSet)$.

A Segal category A comes from an $\mathcal{S}$*-category iff* $A_n \to A_1 \times_{A_0} \ldots \times_{A_0} A_1$ *are isomorphisms.*

Definition 14. *A is a* ∞*-category,* $[A]$ = *category with same objects as A and* $[A](x,y) := \Pi_0 A(x,y)$, *where* $[A] = 1$*-truncation.*

$f\colon A \to B$ a morphism between ∞-categories is:

- fully-faithful $A(x,y) \to B(fx,fy)$weak equivalence
- essentially surjective if $[A] \to [B]$ is so.

Theorem 15. *(Main theorem)(Hirschowitz-Simpson/Pellisier/Bergner) There exist a model categry structure on* $\{F\colon \Delta^{op} \to SSet \mid F_0\,discrete\}$ *such that*

- *weak equivalences* $\cap\{$*Segal categories*$\}$ = *fully faithful and essentially surjective*
- *cofibrations = morphisms*
- *fibrant objects = Segal category plus Reedy fibration condition*
- *It is a* $\otimes$*-monoidal category fo the direct product. (most important property)*

$$\mathcal{S} - Cat \hookrightarrow PrSeCat$$

This is a Quillen equivalence.

So

$$H_0(\mathcal{S}-Cat) \cong H_0(\infty-cat)$$

Consequence:
A, B ∞-categories

$$\mathbb{R}\mathfrak{h}\underline{\mathrm{hom}}(A,B) := \underline{\mathrm{hom}}(A, RB)$$

where RB is a fibrant model for B.

$\mathbb{R}\underline{\mathrm{hom}}(A,B)$, an ∞-category, "is" the ∞-category of weak ∞-functors $A \to B$.

$SeCat^{fib}$+$\underline{\mathrm{hom}}$ $\Rightarrow$ Cat enriched over ∞-categories = an example of a $(2,\infty)$-category.

Localization:
A is an ∞-category
$S \subset [A]$
$A \to L_S A$ of ∞-categories such that

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{hom}}(L_S A, B) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{hom}}(A,B) \arrow[d] \\
\mathrm{hom}(S^{-1}[A],[B]) \arrow[r] & \mathrm{hom}([A],[B])
\end{tikzcd}$$

Remark 16. *$L_S A$ always exists.*

We can start with A, a category (monoidal-∞).

E.g., $A = M$ a model category $S = W$, then $L_W M$.

Theorem 17. *M is nice enough and simplicial, then $L_W M \cong \underline{M}^{cofib}$*

$$L_W M(x,y) \cong Map_M(x,y)$$

Under same assumptions as above, $\forall A$, a category

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{hom}}(A, L_W M) \arrow[r] \arrow[dr] & L_W(M^A) \arrow[d] \\
& \underline{((M^A)}^{fib,cofib}
\end{tikzcd}$$

where all arrows are isomorphisms.
Next lecture:

- limits and colimits of ∞-categories
- ∞-categories of dg-categories
- $\otimes - \infty$-cat + rigity

Lecture 4:

- Derived algebraic geometry
- construct ch