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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

May 20, 2009

DRAFT VERSION ONLY

1 Lecture 2

Reminders (and compliments):

$\mathfrak{k}$ commutative ring.

$$(H_0^{Mor}(dgCat/\mathfrak{k}), \otimes_{\mathfrak{k}}^{\mathbb{L}}) \subset (H_0^{Ct}(dgCat/\mathfrak{k}), \otimes_{\mathfrak{k}}^{\mathbb{L}})$$

(dualizable = saturated dgCat)

same objects

$$[T, T'] = \left\{ E \in \mathcal{D}(T \otimes T'^{op}), E_x \text{ is compact} \right\} \quad [T, T'] = \mathcal{D}(T \otimes T'^{op})/\text{iso}$$

(rigid $\otimes$-cat)

Proposition 1. *$T \in H_0^{Mor}(dgCat/\mathfrak{k})$ is saturated $\Leftrightarrow T \sim B$.*

B is a dg algebra/$\mathfrak{k}$ such that

- B is compact $\in \mathcal{D}(\mathfrak{k})$ ("proper")
- B is compact $\in \mathcal{D}(B \otimes_{\mathfrak{k}} B^{op})$ ("smoothness")

$\alpha \in H^2_{ct}(X\mathbb{G}_m), X = Spec\mathfrak{k}$ then
A_α is Azumaya $\mathfrak{k}$-algebra, smooth and proper, then
$L_{perf}(A_\alpha)$ is saturated, which implies
$L_{perf}(A_\alpha) \in K_0^{(2)}(\mathfrak{k})$, then
$H^2(X, \mathbb{G}_m) \to K_0^{(2)}(\mathfrak{k})$

More generally: A is a $\mathfrak{k}$-algebra

- projective of finite type over $\mathfrak{k}$
- $\forall L$ field, $\forall k \to L$

$$A \otimes_{\mathfrak{k}} L \text{has finite homological dimension}$$

Then $L_{perf}(A)$ is saturated. (e.g., A = quiver algebra)

Basic:

- $\mathfrak{k} \to K_0^{(2)}(\mathfrak{k})$ is a functor in $\mathfrak{k}$

$$\mathfrak{k} \to \mathfrak{k}' \quad - \otimes_{\mathfrak{k}} \mathfrak{k}' \colon H_0^{Mor}(dgCat/\mathfrak{k}) \to H_0^{Mor}(dgCat/\mathfrak{k})$$

- $K_0^{(2)}(\mathfrak{k}) = \pi_0(K^{(2)}(\mathfrak{k}))$
  where $K^{(2)}(\mathfrak{k})$ is a commutative ring spectrum.

Theorem 2.

$$\mathfrak{k} = colim_{filtered} \mathfrak{k}_\alpha, \quad K^{(2)}(\mathfrak{k}) = colim K^{(2)}(\mathfrak{k}_\alpha)$$

We want:

- extend $K^{(2)}$ to schemes and algebraic stacks
- $ch \colon K^{(2)}(X) \to ?$
  X is a scheme stack

The above needs ∞-cagoy theory. Also uses cobordism hypothesis in dimension 1.
$ch :=$ "deRham realization of "$N.C.Motives$"$/X$

1.1 Really begin lecture 2: Segal categories

For ideas of what higher categories are see Jacob's talk.

$(\infty, 1)$-categories: "$(1, \infty)$-category" = ∞-categories such that n-morphisms are invertible for all $n \geq 1$ = ∞-categories whose hom are ∞-groupoids.

$$|\cdot| \colon \infty - groupoids \leftrightarrow \text{homotopy types} \colon \Pi_\infty = SSet$$

"Naive definition": A $(1, \infty)$-categoy is a $SSet$-enriched category = $\mathcal{S}$-categories:

$\mathcal{S}$-categories are badly behaed with respect to the category of functors.

Definition 3. *A* **Segal category** *(or* ∞-**category***) is*

$$A \colon \Delta^{op} \to SSet$$

such that

- A_0 *is discrete (i.e., a set)*
- $A_n \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ *is a weak equivalence*

with $[1] \to [n]$, $0 \to i$, $n \to i+1$.

Remark 4. *A is an $\mathcal{S}$-category.*

$$A\colon \Delta^{op} \to SSet$$
$$n \mapsto \coprod_{(a_0,\ldots,a_n)} A(a_0,a_1) \times \ldots \times A(a_{n-1},a_n)$$

Here $\mathcal{S}-Cat \overset{full}{\hookrightarrow}$ Segal Category $\overset{full}{\subset}$ Fun($\Delta^{op}, SSet$).

A Segal category A comes from an $\mathcal{S}$-category iff $A_n \to A_1 \times_{A_0} \ldots \times_{A_0} A_1$ are isomorphisms.

Definition 5. *A is a ∞-category, $[A]$ = category with same objects as A and $[A](x,y) := \Pi_0 A(x,y)$, where $[A] = 1$-truncation.*

$f\colon A \to B$ a morphism between ∞-categories is:

- fully-faithful $A(x,y) \to B(fx,fy)$weak equivalence
- essentially surjective if $[A] \to [B]$ is so.

Theorem 6. *(Main theorem)(Hirschowitz-Simpson/Pellisier/Bergner) There exist a model categry structure on $\{F\colon \Delta^{op} \to SSet \mid F_0 \, discrete\}$ such that*

- *weak equivalences $\cap$\{Segal categories\} = fully faithful and essentially surjective*
- *cofibrations = morphisms*
- *fibrant objects = Segal category plus Reedy fibration condition*
- *It is a $\otimes$-monoidal category fo the direct product. (most important property)*

$$\mathcal{S} - Cat \hookrightarrow PrSeCat$$

This is a Quillen equivalence.

So

$$H_0(\mathcal{S}-Cat) \cong H_0(\infty - cat)$$

Consequence:
A, B ∞-categories

$$\mathbb{R}\mathfrak{h}\underline{\mathrm{hom}}(A,B) := \underline{\mathrm{hom}}(A, RB)$$

where RB is a fibrant model for B.

$\mathbb{R}\underline{\mathrm{hom}}(A,B)$, an ∞-category, “is” the ∞-category of weak ∞-functors $A \to B$.

$SeCat^{fib}$+$\underline{\mathrm{hom}}$ $\Rightarrow$ Cat enriched over ∞-categories = an example of a $(2,\infty)$-category.

Localization:
A is an ∞-category
$S \subset [A]$
$A \to L_S A$ of ∞-categories such that

$$
\begin{tikzcd}
\mathbb{R}\underline{\mathrm{hom}}(L_S A, B) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{hom}}(A, B) \arrow[d] \\
\mathrm{hom}(S^{-1}[A], [B]) \arrow[r] & \mathrm{hom}([A], [B])
\end{tikzcd}
$$

Remark 7. *$L_S A$ always exists.*

We can start with A, a category (monoidal-∞).

E.g., $A = M$ a model category $S = W$, then $L_W M$.

Theorem 8. *M is nice enough and simplicial, then $L_W M \cong \underline{M}^{cofib}$*

$$L_W M(x, y) \cong Map_M(x, y)$$

Under same assumptions as above, $\forall A$, a category

$$
\begin{tikzcd}
\mathbb{R}\underline{\mathrm{hom}}(A, L_W M) \arrow[r] \arrow[dr] & L_W(M^A) \arrow[d] \\
 & (\underline{(M^A)}^{fib,cofib}
\end{tikzcd}
$$

where all arrows are isomorphisms.
Next lecture:

- limits and colimits of ∞-categories
- ∞-categories of dg-categories
- $\otimes - \infty$-cat + rigity

Lecture 4:

- Derived algebraic geometry
- construct ch