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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 3

End of Lecture 2:
We talked abojut ∞-categories using the formalism of Segal categories. Here we will present some $|infty$-categories that we will present later. These have shown up in David's lectures.

$$(C, W) \to L_w C \in \infty - category$$

where C is a category.

- $L_{perf}(\mathfrak{k}) \in dgCat$
- $(Perf(\mathfrak{k}), q - isom) \to LPerf(\mathfrak{k}) \in \infty - category$
- $(dgCat/\backslash\mathfrak{k}, Moritaeq) \to \mathcal{D}_g^{Mor}(\mathfrak{k}) = \infty -$ categorey of dg Category
- $[LPerf(\mathfrak{k})] \cong \mathcal{D} + perf(\mathfrak{k})]$
- $[\mathcal{D}_g^{Mor}(\mathfrak{k})] \cong H_0^{Mor}(dgCat/\backslash\mathfrak{k})]$
- $E, F \in LPerf(\mathfrak{k})$

$$\pi_i(LPerf(\mathfrak{k})(E, F), \circ) \cong Ext^{-i}_{\mathcal{D}(\mathfrak{k})}(E, F)$$

$$T, T' \in \mathcal{D}_g^{Mor}(\mathfrak{k})$$

- $\pi_1(\mathcal{D}_g^{Mor}(T, T'), E) \cong aut(E), \quad E \in \mathcal{D}(T, T')$

- $\pi_i(\mathcal{D}_g^{Mor}(\mathfrak{k})(T,T'),E) \cong Ext_{\mathcal{D}}^{1-i}(T \otimes T')$
  imples

  $\pi_i(\mathcal{D}_g^{Mor}(\mathfrak{k})(T,T)) \cong HH^{1-i}(T) =$ Hochschild cohomology $=: Ext^{1-i}_{|mathcalD(T \otimes T)}(T,T)$

Adjunctions and limits: $f\colon A \to B$ in ∞-category.

Definition 1. *f has a right adjoint if there exists:*

$$g\colon B \to RA \hookleftarrow A \textit{fibrant model}$$

and $u \in \underline{\hom}(A,RA)(i,gf)$ such that $\forall a \in A, \forall b \in B$

$$RA(i(a),g(b))$$

$$B(f(a),b) \xrightarrow[g]{} RA(gf(a),g(b)) \xrightarrow{w-equiv} RA(i(a),g(b))$$

This gives a notion of limit and colimit in a given $|infty$-category.

$A \in \infty$-cat, I ∞-category

$$A \xrightarrow{c_*} \mathbb{R}\underline{\hom}(I,A)$$

has right adjoint as limit and left adjoint as colimit.

M is a (nice) model category.
Then L_wM has all limits and colimits and they "are" the holim and hocolim defined in model category theory.

$$\begin{tikzcd} L_wM & \mathbb{R}\underline{\hom}(I,L_wM) \arrow[l, "lim"'] \arrow[d, "\sim"] \\ & L_w(M^I) \arrow[ul, "holim"] \end{tikzcd}$$

Lecture 3 begins now: Want to talk about $\}^{Mor}(X)$,$\mathcal{D}_g^{ct}(X)$,$\mathcal{D}_g^{sat}(X)$, when X is now a scheme, an algebraic stack, or any ∞-stack.

$$\mathcal{D}_g^{Mor}(X) := \lim_{Spec\mathfrak{k} \to X} \mathcal{D}_g^{Mor}(\mathfrak{k}) \in \infty - \underline{cat}$$

This is equal to $\Gamma(X, \underline{\mathcal{D}}_g^{Mor}$

Remark 2.

$$LPerf(X) := \lim_{spec\mathfrak{k} \to X} LPerf(\mathfrak{k}), \infty\textit{-category model for}[Lperf(X)] \cong \mathcal{D}_{Perf}(X)$$

$$\underline{End}_{\mathcal{D}_g^{ct}(X)}(1) \cong LQCoh(X)\text{“} = Q(X)\text{”}$$

The Chern Character

The idea of the construction of ch:
Let $T \in \mathcal{D}_g^{ct}(X)$, $T =$ *"sheaf of dgcat/X.*

What is $ch(T) =?$

We consider th loops space

$$\text{"}\underline{\text{hom}}(S^1, X)\text{"} = \mathcal{L}X \xrightarrow{\pi} X$$

$\pi^*(T) \in \mathcal{D}_g^{ct}(\mathcal{L}X)$
$\pi^*(T)$ *coms equipped with a natural auto-equivalence* $m \colon \pi^*(T) \xrightarrow{\sim} \pi^*(T)$

$m =$ *monodromy along the loop*

$\mathcal{L}X \times S^1 \xrightarrow{ev} X$
then self homotopy/equivalence of |pi and self equivalence onπ^*.
$\mathcal{D}_g^{ct}(Y)$ *is a rigid9*∞-$\otimes$-*category.*

Then $Tr(\mathcal{M}) \in \mathcal{D}_g^{ct}(\mathcal{L}X)(1,1) \cong LQCoh(\mathcal{L}X) =$ "$Q(\mathcal{L}X)$"
key result:
$Tr(\mathcal{M})$ *is* S^1-*invariant*

This follows from Lurie's theorem on $\widetilde{1Bord}$

Remark 3.
- *We can replace* $\mathcal{D}_g^{ct}(-)$ *by any functor "Schemes"* $\xrightarrow{\mathcal{A}} \otimes - \infty$-*category rigid.*

$$\mathcal{A}(X) \xrightarrow{ch} End_{\mathcal{A}(\mathcal{L}X)}(1)^{S^1}$$

  If $\mathcal{A} = LPerf$, *then* "$\cong$" "$HC^-(X)$ *and ch s usual Chern character.*

- $$ch(T) \in \mathcal{D}qCoh^{S^1}(\mathcal{L}X) \overset{\text{"}\sim\text{"}}{\to} \mathcal{D}(\mathcal{D}_X - mod)$$

  The LHS is deRham realization of T as a noncommutative variety over X.

$\otimes\infty$-category i a$\otimes - \infty$-category is a "monoid object in ∞-category."
commutative monoid in Sets = Γ-object in Set
Γ = category of pointed finite sets

$$M \colon \Gamma \to \text{Set}$$

where $M_0 = *$ and $M_n \to M_1^n$ an iso.

A $\otimes - \infty$-category is

$$A \colon \Gamma \xrightarrow{\sim} \infty - category$$

such that

$$A_0 \xrightarrow{\sim} * \text{ and } A_n \xrightarrow{\sim} A_1 \times \cdots \times A_1$$

These are equivalences of ∞-categories.

$\otimes - \infty-$category from an ∞-category: $\infty - cat^{\otimes}$.
Example:
If A is a $\times - \infty$-category, then $[A_1]$ is endowed with a natural $\otimes$-structures

Remark:
$\otimes - \infty$-category can be obtained by localization of $\otimes$-category with A_0 a set of maps

E.g., $dgCat/\mathfrak{k}$, $\otimes_{\mathfrak{k}}$, Morita