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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 4

End of Lecture 3:

We know about rigid $\otimes$-∞-categories from Lurie's talk.
Lecture 4: Derived algebraic geometry
Main problem: To construct a meaningful loop space $\mathcal{L}X$ for X a scheme or algebraic stack.

$$S^1 = B\mathbb{Z} = [*/\mathbb{Z}]$$

is a stack. (here $\mathbb{Z}$ = discrete group scheme)

X is a scheme or an algebraic stack ($[Y/G]$)

We want $\mathcal{L}X := \hom(S^1, X)$. If X is a scheme, then $\mathcal{L}X \cong X$ because $* =$ coarse moduli space of S^1.

If X is a stack, then $\mathcal{L}X$ is the interia stack I_X. A point in I_X is (x, h), $x \in X$ and $h \colon x \xrightarrow{\sim} x$.

E.g.,

$$I_{BG} \cong [G/G]$$

$$S^1 \cong * \coprod_{* \coprod *}^{\mathbb{L}} *$$

$$\hom(S^1, X) \cong X \times_{X \times X} X = \text{self intersection of the diagonal}$$

FIGURE $S^1 \sim$ SQUARE

If $X = SpecA$:

$$A \otimes_{A \otimes_k A} A \ \longrightarrow A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_k A} A$$

The arrow means "replaced by" but the RHS is not a commutative ring anymore. But, it is a simplicial commutative k-algebra. (or E_∞-ring, or commutative dg algebra)

$A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_k A} A \in \infty$-category of simplicial commutaive k-algebras.

When X is affine: we want $\mathcal{X} := Spec(A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_k A} A)$ is a derived scheme.

Definition 1. *A* **derived scheme** *is a topological space Y and a sheaf of simplicial commutative ring $\mathcal{A}$ and local conditions.*

When X is a scheme, these local models $A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_k A} A$ glue to give a derived scheme

$$\mathcal{L}X \to X$$

$$\text{schemes} \ \subset \ \text{derived schemes}$$

Derived schemes form an ∞-category. Derived stacks obtained by taking quotients of derived schemes and there is a full sub2-category of algebraic stacks.

Definition 2. *X is a stack.*

$$\mathcal{L}X := \mathbb{R}\mathrm{hom}(S^1, X)$$

where $\mathbb{R}$ means that hom *is taken in the ∞-category of derived stacks.*

The construction for ch makes for this loop space $\mathcal{L}X$.

X algebraic stack (or scheme)

$$ch \colon \mathcal{D}^{ct}_g(X)^\sim \to End_{\mathcal{D}(\mathcal{L}(X))}(\mathbb{1})^{S^1} \cong L^{S^1}_{qcoh}(\mathcal{L}X)^\sim$$

a map of groupoids.

$$L_{perf}(X)^\sim \to End_{L_{perf}(\mathcal{L}(X)}(\mathbb{1})^{S^1} \cong \mathbb{R}\Gamma(\mathcal{L}X, \mathcal{O})^{S^1}$$

Yoneda gives us $T \in \mathcal{D}^{ct}_g(X) \Leftrightarrow (X \to \mathcal{D}^{ct}_g(-)^\sim$ in the ∞-category of functors $\{$derived schemes$\} \to \infty$-groupoids, a rigid $\otimes\infty$-category.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{D}^{ct}_g(-) \\ 1Bord^{or}/X & \end{tikzcd}$$

$$1Bord^{or}/X \to \mathcal{D}g(-)$$

is a 1-dimensional TFT$/X$ with values in a dg category.

$$\text{moduli of “one” circle}/X \subset End(\emptyset \to X) \to L_{qcoh}(-)^{\sim} \leftarrow [\mathcal{L}X/S^1]$$

and the LHS and RHS are isomorphic.

Yoneda says $ch(T_{\in} L_{qcoh}(\mathcal{L}X)^{\sim^{S^1}})$.

Relations between $\mathcal{L}X$ and deRham theory of X:

Theorem 3. *(Folklore - BZ/Nadler) X is a smooth scheme over k with $chark = 0$.*

1.
$$\pi + 0(\mathbb{R}\Gamma(\mathcal{L}X, \mathcal{O})^{S^1}) \cong H^{ev}_{dR}(X/k) \text{ an iso of rings}$$

2.
$$L^{S^1}_{qcoh}(\mathcal{L}(X) \text{“} \cong \text{”} L(X, \mathcal{R}x)$$

$\mathcal{R}x$ is a sheaf of dg-algebras over X. ($= \oplus_i \mathcal{D}^{\leq i}_X[-2i]$). THE RHS in (2) is the ∞-category of quasi-coherent $\mathcal{R}x$-dg modules.

“Prethorem”: $E \in L_{perf}(X)$, $ch(E) =$ usual chern character in $H^{ev}_{dR}(X)$ (using (1))

$L_{qcoh}(X, \mathcal{R}x)$ is some kind of ∞-category of “filtered ∞-modules”

$$L_{qcoh}(X\mathcal{R}x) \overset{\text{“}u=0\text{”}}{\to} L_{qcoh}(Sym(\mathbb{T}[-2])) \cong L_{qcoh}(\mathcal{L}X)$$

THIS DIAGRAM SHOULD HAVE MORE PIECES

$T \to X$ a dgCat$/X$, then

$ch(T)$ is the “algebraic part” of the Hodge realization of T.

Theorem 4. *$T \to X$ is saturated.*

$$ch(T) \in L_{qcoh}(X, \mathcal{R}x) \to L_{qcoh}(X)$$

A LITTLE HERE ABOUT HODGE FILTRATION

Next steps:

$$chK_0^{(2)}(X) \to L^{S^1}_{perf}(\mathcal{L}X) \to K_0^{S^1}(\mathcal{L}X) \overset{ch}{\to} \pi_0 \left(\mathbb{R}\Gamma(\mathcal{L}\mathcal{L}X, \mathcal{O})^{S^1 \times S^1} \right)$$

Want to understand the map from LHS to RHS.

What about a $\mathbb{Z}$-structure on the Hodge realization?

Candidate: topological K-theory for dgcat.