Theory and Applications of Crossed Complexes

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by

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Declaration

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not been already accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

Director of Studies Candidate

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Summary

We prove a ‘slightly non-abelian’ version of the classical Eilenberg-Zilber theorem: if K, L are simplicial sets, then there is a strong deformation retraction of the fundamental crossed complex of the cartesian product $K \times L$ onto the tensor product of the fundamental crossed complexes of K and L. This satisfies various side-conditions and associativity/interchange laws, as for the chain complex version. Given simplicial sets $K_0, \ldots, K_r$, we discuss the r-cube of homotopies induced on $\pi(K_0 \times \ldots \times K_r)$ and show these form a *coherent system*.

We introduce a definition of a double crossed complex, and of the associated *total* (or *codiagonal*) crossed complex. We introduce a definition of homotopy colimits of diagrams of crossed complexes. We show that the homotopy colimit of crossed complexes can be expressed as the total complex of a certain ‘twisted’ simplicial crossed complex, analogous to Bousfield and Kan’s definition of simplicial homotopy colimits as the diagonal of a certain bisimplicial set. Using the Eilenberg-Zilber theorem we show that the fundamental crossed complex functor preserves these homotopy colimits up to a strong deformation retraction. This is applied to give a small crossed resolution of a semidirect product of groups.

We consider a simplicial enrichment of the category of crossed complexes, and investigate the coherent homotopy structure up to which a simplicial enrichment may be given to the fundamental crossed complex functor.

We end with a definition of homotopy coherent functors from a small category to the category of crossed complexes, and suggest a definition of homotopy colimits of such functors and of a small crossed resolution of an arbitrary group extension.

Contents

Chapter 0

Introduction

The motivation for this thesis has come from two directions: firstly, from a wish to give a definition of homotopy colimits in a situation where cartesian products are replaced by tensor products, and secondly from an investigation of small resolutions for groups which arise as products, semidirect products or extensions. Both of these rely on the Eilenberg-Zilber theorem, and it has turned out that we have considered the second by translating it into the language of the first.

We have also chosen to carry out this investigation in the context of crossed complexes. This is a category of algebraic objects similar to chain complexes but with some non-abelian information in dimensions one and two. The crossed complex has been around since [42], and following [14] and [19] it may also be thought of as a reduced form of a simplicial groupoid. The extra structure shared by chain and crossed complexes which is not available (currently) for simplicial groups is that of having a "geometrically-motivated" tensor product, and this has been essential for our work.

The Eilenberg-Zilber theorem in its original form [22, 21] gives for simplicial sets K, L a chain homotopy equivalence

$$C_N(K) \otimes C_N(L) \quad \simeq \quad C_N(K \times L)$$

where $C_N(K)$ is the normalised free chain complex on the simplicial set K. This theorem is now part of the general knowledge of algebraic topology, but although it seems clear that it is true for crossed complexes also there has been no explicit proof given. Writing πK for the fundamental crossed complex of a simplicial set K, we have obtained a strong deformation retraction of $\pi(K \times L)$ onto $\pi K \otimes \pi L$ satisying certain side conditions and interchange relations, exactly as in the chain complex case except that in low dimensions the formulæ for the tensor product and the homotopy equivalence contain non-abelian information.

We have also extended some of the basic constructions available for crossed complexes of groupoids, defining a double crossed complex as well as a total crossed complex which behaves nicely with respect to the tensor product. A total crossed complex functor for simplicial crossed complexes has also been defined, and this has been used in defining homotopy colimits.

Since limits and colimits of topological spaces, simplicial sets or chain complexes do not behave well when the spaces, etc., are varied up to homotopy equivalence, it is natural to consider the notions of homotopy limits and homotopy colimits. For example the mapping cylinder, double mapping cone and 'telescope' are all well known examples of homotopy colimit constructions. However the topological space or algebraic structure which represents a particular homotopy colimit will itself only be determined up to homotopy equivalence, and this has led to much interest in setting up formal machinery to provide particular nice models for homotopy limits and colimits for arbitrary diagrams. In this thesis we have given a definition for homotopy colimits in the category of crossed complexes.

In fact the diagrams over which the homotopy colimit is taken need only be functorial up to homotopy rather than on the nose, and there has been a lot of work recently on notions of *lax* or *homotopy coherent* functors and their homotopy limits and colimits. This work has been carried out mainly in the context of simplicially-enriched categories, or sometimes **Cat**- or **Top**-enriched categories. In this thesis we have tried to extend such ideas to monoidal closed categories which satisfy an Eilenberg-Zilber type theorem, although we have not completely achieved this ambition.

The standard crossed resolution [11] $C(G)$ of a group G is defined by applying the fundamental crossed complex functor to the simplicial set given by the nerve of G. This gives a complex of groups whose first homology group is G with all higher homology groups trivial, and which is also *free* in that it has a presentation where the only relations are those defining the boundary maps and quotienting out degenerate simplices.

However $C(G)$ is not the only resolution of G with this freeness property, and there may be other models which are smaller. For instance an application of the Eilenberg-Zilber theorem shows that $C(G) \otimes C(H)$ is a deformation retract of the standard resolution of $G \times H$, and is free by the definition of the tensor product. We have given in this thesis a resolution of a semidirect product of groups which is a deformation retract of the standard resolution and which takes the form of a *twisted* tensor product. Also we have given a candidate for a resolution of an arbitrary extension of groups as a more general twisted tensor product. Both of these arose by considering the data in terms of a homotopy colimit of an appropriate (lax) functor.

0.1 Structure of Thesis

We begin in chapter 1 by considering the notion of a double crossed complex, analogous to the bisimplicial set or to the bichain complex in the abelian situation. Our definition of a double crossed complex is essentially that of a crossed complex of groupoids *internal* to the category of crossed complexes of groupoids (similar to the definition of a double category as a category internal to the category of small categories).

A "total" functor is then defined from the category of double groupoids to the category of crossed modules, and this is extended to a functor

$$\mathbf{Crs}^{(2)} \xrightarrow{\text{Tot}} \mathbf{Crs}$$

from double crossed complexes to crossed complexes. The total crossed complex D of a double crossed complex C is essentially that given by generators $c_{i,j} \in D_n$ for all elements of $C_{i,j}$ with $i + j = n$, subject to certain "geometrical" relations which are similar to those in the Brown-Higgins definition of the tensor product of crossed complexes [12]. In fact our definition is constructed so that given a pair of crossed complexes A, B there is an obvious double crossed complex whose total crossed complex is the tensor product $A \otimes B$.

We also define a total functor from the category of simplicial crossed complexes.

In chapter 2 the definition of homotopy between crossed complex homomorphisms is recalled, in terms of homomorphisms $h : \mathcal{I} \otimes C \to D$ from cylinder objects and of degree one maps $(\phi_n : C_n \to D_{n+1})$, and it is shown that a homotopy from an idempotent endomorphism to the identity can be replaced by a splitting homotopy, which satisfies certain extra 'side-conditions' of the form $h^2 = 0$ and $h\delta h = -h$. In particular deformation retractions can be replaced by strong deformation retractions.

For X a bisimplicial set and ∇X the simplicial set given by the Artin-Mazur diagonal [1], a natural comparison map is given from $\pi\nabla X$ to the total complex of the fundamental double crossed complex of X. This is shown to give the diagonal approximation $a : \pi(K \times L) \to \pi K \otimes \pi L$ in the case $X_{p,q} = K_p \times L_q$. The shuffle map b in the other direction is given, and $b \circ a$ is shown to be the identity map on the tensor product. The associativity relations are also proved for both a and b, as well as an 'interchange' relation.

As an elementary application, it is shown how the diagonal approximation map gives a coalgebra structure on the fundamental crossed complex of a simplicial set, and a multiplication structure on the simplicial nerve of a crossed complex.

We then show that for simplicial sets K, L, there is a natural homotopy

$$\mathcal{I} \otimes \pi(K \times L) \xrightarrow{h} \pi(K \times L)$$

between $a \circ b$ and the identity, and it is proved that h satisfies some interchange relations with respect to a and b.

For simplicial sets K, L, M the deformation retraction h induces two distinct deformation retractions of $\pi(K \times L \times M)$ onto $\pi K \otimes \pi L \otimes \pi M$. However these are themselves homotopy equivalent. In fact there is shown to be a *coherent system* of such homotopies in each dimension; if $K_0, K_1, \dots, K_r$ are simplicial sets, then the homotopy coherence information is recorded by an r-fold homotopy

$$\mathcal{I}^{\otimes r} \otimes \pi(K_0 \times K_1 \times \ldots \times K_r) \longrightarrow \pi(K_0 \times K_1 \times \ldots \times K_r)$$

satisfying certain boundary conditions.

In chapter 3 we examine the usual definition of homotopy colimits in the category of categories [38] and of simplicial sets [4], and consider an alternative definition of the latter which uses the Artin-Mazur diagonal of a bisimplicial set rather than the usual diagonal. Thomason [38] showed that the nerve functor from **Cat** to simplicial sets preserves homotopy colimits up to weak homotopy equivalence. With the alternative definition we prove in theorem 3.2.12 that the nerve functor preserves homotopy colimits up to isomorphism.

We then define a notion of homotopy colimits in the monoidal closed category of crossed complexes. We show that our first coend definition of homotopy colimits can be rewritten in terms of the total complex of a particular simplicial crossed complex, as defined in chapter 1. The main result of this thesis is theorem 3.3.11 in which we use the Eilenberg-Zilber theorem of chapter 2 to prove that the fundamental crossed complex functor from simplicial sets to crossed complexes preserves homotopy colimits up to strong deformation retraction.

We also recall that semidirect products of groups are given by the homotopy colimit in **Cat** of the diagram corresponding to the group action. Applying the standard crossed resolution functor to the diagram and then taking the homotopy colimit in **Crs**, we thus obtain a crossed resolution of a semidirect product which is a deformation retract of the standard one. This is expressed in terms of a twisted tensor product of standard resolutions.

In chapter 4 we use the Eilenberg-Zilber theorem to investigate a simplicial-set-enriched structure on the category of crossed complexes. We show that with respect to such a structure the nerve functor from crossed complexes to simplicial sets has a simplicial enrichment, but that the fundamental crossed complex functor only has an enrichment up to a system of higher homotopies given by those of section 2.3.2. We also investigate how the adjunction between the nerve and fundamental crossed complex functor behaves with respect to the simplicial enrichment. We do not present any applications of the results found here, although we expect a tidy treatment of homotopy colimits of lax functors into crossed complexes would rely on the structures presented here. Also this chapter is intended as input for the work by Brown, Golasiński, Porter and the author [7] in which a systematic treatment of equivariant homotopy theory for crossed complexes is being developed.

In the final chapter we give a tentative 'low-tech' definition of homotopy colimits for lax/coherent diagrams of crossed complexes, taking our inspiration from [39] and [16]. The implications for giving a small resolution of an arbitrary group extension are also discussed. We end with some remarks about possible future directions for the development of the work in this thesis.

Chapter 1

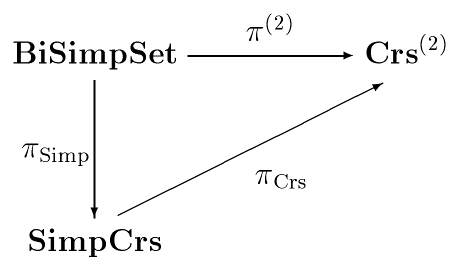
Double Crossed Complexes

1.0 Introduction

In this chapter we introduce double crossed complexes as the "rank 2" generalisation of crossed complexes of groupoids. The fundamental crossed complex functor

$$\mathbf{SimpSet} \xrightarrow{\ \pi\ } \mathbf{Crs}$$

is extended to functors between the categories of bisimplicial sets, simplicial crossed complexes and double crossed complexes:



and the tensor product of crossed complexes is extended to total crossed complex functors on the categories of simplicial crossed complexes and double crossed complexes:

$$\begin{array}{ccc} & & \mathbf{Crs}^{(2)} \\ & & \downarrow{\scriptstyle \mathrm{Tot}} \\ \mathbf{SimpCrs} & \xrightarrow{\ \text{S-Tot}\ } & \mathbf{Crs} \end{array}$$

The structure of the chapter is as follows. In the first section, we recall the definitions of categories, groupoids, crossed modules and crossed complexes. Also the definition of a double category as a category internal to **Cat** is discussed. The notion of a double crossed complex is then introduced, as a crossed complex of groupoids internal to **Crs**.

In the second section, we show how to associate a crossed module to a double groupoid, and extend this to a definition of the total crossed complex associated to a double crossed complex. A construction of a double crossed complex from a pair of crossed complexes is then given such that the associated total complex is their tensor product.

In the third section, we begin by recalling the definitions of simplicial and cosimplicial objects and the fundamental crossed complex functor on simplicial sets. This functor is then extended to the categories of bisimplicial sets and simplicial crossed complexes. We also define the total crossed complex associated with a simplicial crossed complex.

1.1 Definitions

1.1.1 Groupoids and crossed complexes

We begin by recalling some standard definitions.

Definition 1.1.1 A *(small) category* $\mathbf{C}$ consists of

1. an object set $\mathrm{Ob}(\mathbf{C})$,
2. a set of arrows (morphisms) $\mathrm{Arr}(\mathbf{C})$,
3. source and target functions s, t from $\mathrm{Arr}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{C})$,
4. a function $\mathrm{Ob}(\mathbf{C}) \xrightarrow{e} \mathrm{Arr}(\mathbf{C})$ which gives the identity arrow at an object,
5. a partially defined function $\mathrm{Arr}(\mathbf{C}) \times \mathrm{Arr}(\mathbf{C}) \xrightarrow{m} \mathrm{Arr}(\mathbf{C})$ which gives the composite of two arrows.

We will usually write e_x or 1_x for $e(x)$ and $a \circ b$ or $a \cdot b$ for $m(b, a)$. The data satisfy the following axioms:

1. The composite $a \circ b$ of two arrows is defined if and only if $t(a) = s(b)$, and then $s(a \circ b) = s(a)$ and $t(a \circ b) = t(b)$,
2. $s(e_x) = t(e_x) = x$ for all $x \in \mathrm{Ob}(\mathbf{C})$, and $a \circ e_{t(a)} = e_{s(a)} \circ a = a$ for all $a \in \mathrm{Arr}\mathbf{C}$,
3. If either of $a \circ (b \circ c)$ or $(a \circ b) \circ c$ are defined then both are and they are equal.

Definition 1.1.2 A *functor* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ between two categories is given by a pair of functions $\mathrm{Ob}(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{D})$, $\mathrm{Arr}(\mathbf{C}) \longrightarrow \mathrm{Arr}(\mathbf{D})$ which commute with the source, target and identity functions of the two categories and which respect the compositions.

For $\mathbf{C}$ a category and $x, y \in \mathrm{Ob}(\mathbf{C})$, the set of arrows a such that $s(a) = x$ and $t(a) = y$ is written $\mathbf{C}(x, y)$ and termed a *hom-set*. If $\mathbf{C}(x, y)$ is empty whenever x, y are distinct (that is, if $s = t$), then $\mathbf{C}$ is termed *totally disconnected*.

A *groupoid* is a category in which every morphism is an isomorphism, that is, for any arrow a there exists a (necessarily unique) arrow a^{-1} such that $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$. A *monoid* is a category whose object set is a singleton, and a *group* is a monoid which is a groupoid.

Definition 1.1.3 Suppose $\mathbf{C}$, $\mathbf{D}$ are two groupoids over the same object set and $\mathbf{C}$ is totally disconnected. Then an *action* of $\mathbf{D}$ on $\mathbf{C}$ is given by a partially defined function

$$\begin{array}{ccc} \mathrm{Arr}(\mathbf{D}) \times \mathrm{Arr}(\mathbf{C}) & \xrightarrow{\alpha} & \mathrm{Arr}(\mathbf{C}) \\ (d, c) & \longmapsto & c^d \end{array}$$

which satisfies:

1. c^d is defined if and only if $t(c) = s(d)$, and then $t(c^d) = t(d)$,
2. $(c_1 \circ c_2)^{d_1} = c_1^{d_1} \circ c_2^{d_1}$ and $(e_x)^{d_1} = e_y$,
3. $c_1^{d_1 \circ d_2} = (c_1^{d_1})^{d_2}$ and ${c_1}^{e_x} = c_1$,

for all $c_1, c_2 \in \mathbf{C}(x, x)$, $d_1 \in \mathbf{D}(x, y)$, $d_2 \in \mathbf{D}(y, z)$.

For example if $\mathbf{C}'$ is the largest totally disconnected subcategory of a groupoid $\mathbf{C}$ then $\mathbf{C}$ acts on $\mathbf{C}'$ by $a^c = c^{-1} \circ a \circ c$. Note that definition 1.1.3 makes sense when $\mathbf{C}, \mathbf{D}$ are categories rather than groupoids. However we will not need this extra generality.

Suppose $\mathbf{D}$ is a groupoid with object set O and $\mathbf{C}$ is a totally disconnected groupoid over O equipped with a $\mathbf{D}$-action. If each group $\mathbf{C}(x, x)$ is abelian then $\mathbf{C}$ will be termed a $\mathbf{D}$-*module*, and if $\mathbf{C}, \mathbf{C}'$ are $\mathbf{D}$-modules then a functor $\mathbf{C} \longrightarrow \mathbf{C}'$ defines a homomorphism of $\mathbf{D}$-modules iff it is the identity on the object set and respects the actions of $\mathbf{D}$. The category of $\mathbf{D}$-modules and their homomorphisms will be written $\mathbf{Mod}_{\mathbf{D}}$.

Definition 1.1.4 A *crossed module of groupoids* consists of a pair of groupoids $\mathbf{C}$,$\mathbf{D}$ over a common object set, with $\mathbf{C}$ totally disconnected, together with an action of $\mathbf{D}$ on $\mathbf{C}$ and a functor $\mathbf{C} \xrightarrow{\delta} \mathbf{D}$ which is the identity on the object set and satisfies

1. $\delta(c^d) = d^{-1} \circ \delta c \circ d$,
2. $c^{\delta c'} = c'^{-1} \circ c \circ c'$

for $c, c' \in \mathbf{C}(x, x)$, $d \in \mathbf{D}(x, y)$.

A *crossed module of groups* is a crossed module of groupoids as above in which $\mathbf{C}$, $\mathbf{D}$ are groups.

Definition 1.1.5 A *crossed complex of groupoids* C is given by

1. a crossed module of groupoids $C_2 \xrightarrow{\delta_2} C_1$ with object set C_0,

2. for each $i \geq 3$, a C_1-module C_i and a functor $C_i \xrightarrow{\delta_i} C_{i-1}$ which is the identity on the object set and respects the C_1-actions.

These data satisfy the following conditions for $i \geq 3$:

1. $\delta_i \circ \delta_{i-1}$ is zero, that is, maps $c_i \in C_i$ to $e_{t(c_i)} \in C_{i-2}$,

2. the image of δ_2 acts trivially on C_i.

A *crossed complex of groups* is a crossed complex of groupoids in which C_0 is a singleton, and hence each C_i, $i \geq 1$, is a group.

A crossed complex of groupoids is often written diagrammatically as follows

$$\cdots\cdots\xrightarrow{\delta_5} C_4 \xrightarrow{\delta_4} C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

The category of crossed complexes of groupoids and their homomorphisms will be denoted by **Crs**.

1.1.2 Internal categories and double groupoids

If **C**, **D** are two small categories, then their product $\mathbf{C} \times \mathbf{D}$ is that category with object set $\mathrm{Ob}(\mathbf{C}) \times \mathrm{Ob}(\mathbf{D})$ and set of arrows $\mathrm{Arr}(\mathbf{C}) \times \mathrm{Arr}(\mathbf{D})$ and the structure maps defined componentwise. The internal hom object $[\mathbf{C}, \mathbf{D}]$ is the category whose objects are all functors from **C** to **D** and whose arrows are the natural transformations between them. The category **Cat** of all small categories is complete, cocomplete and cartesian closed, as is the full subcategory **Gpd** of groupoids. In particular the completeness means that internal categories in **Cat** may be considered.

Definition 1.1.6 A category $\mathcal{C}$ internal to a category **D** is given by objects and morphisms

$$\mathbf{Arr}(\mathcal{C}) \underset{e}{\overset{s,t}{\rightleftarrows}} \mathbf{Ob}(\mathcal{C}) \qquad \mathbf{Arr}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathbf{Arr}(\mathcal{C}) \xrightarrow{m} \mathbf{Arr}(\mathcal{C})$$

where $\mathbf{Arr}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathbf{Arr}(\mathcal{C})$ is the pullback in **D** of (s, t). These data are required to satisfy

1. $e \circ s = 1$ and $e \circ t = 1$, the identity morphism at $\mathbf{Ob}(\mathcal{C})$ in **D**,

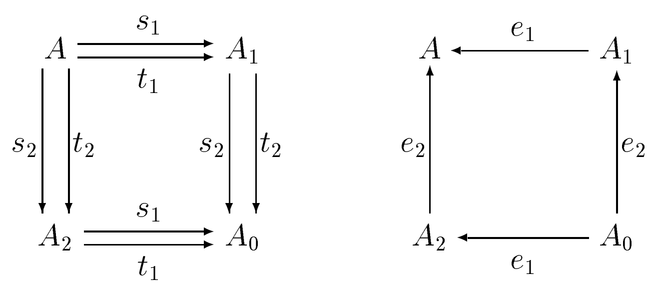
2. $m \circ s = \pi_2 \circ s$ and $m \circ t = \pi_1 \circ t$, where π_1, π_2 are the projection maps from the pullback to $\mathbf{Arr}(\mathcal{C})$,

3. $l \circ m = 1$ and $r \circ m = 1$, where l, r are the maps to the pullback from $\mathbf{Arr}(\mathcal{C})$ induced by $(1, s \circ e), (t \circ e, 1)$ respectively,

4. $(1, m) \circ m = (m, 1) \circ m$.

Thus a category internal to the category of sets is just a small category as in definition 1.1.1. A category internal to the category of small categories is termed a double category and may be defined more explicitly as follows:

Definition 1.1.7 A *double category* $\mathcal{A}$ is given by a set A of squares, sets A_1, A_2 of horizontal and vertical arrows, and a set A_0 of vertices, and functions s_i, t_i, e_i for $i = 1, 2$ as shown in the diagrams below:



together with partially defined horizontal compositions $\circ_1 \colon A \times A \to A$, $\circ_1 \colon A_2 \times A_2 \to A_2$, and vertical compositions $\circ_2 \colon A \times A \to A$, $\circ_2 \colon A_1 \times A_1 \to A_1$, such that the following axioms are satisfied:

1. The horizontal data $(A, A_1, s_1, t_1, e_1, \circ_1)$ and $(A_2, A_0, s_1, t_1, e_1, \circ_1)$ define category structures.

2. The vertical data $(A, A_2, s_2, t_2, e_2, \circ_2)$ and $(A_1, A_0, s_2, t_2, e_2, \circ_2)$ define category structures.

3. The horizontal structure maps $s_1, t_1, e_1, \circ_1$ are functorial with respect to the vertical category structures (and hence *vice-versa*). That is

   (a) $s_i s_j = s_j s_i$, $t_i t_j = t_j t_i$ and $s_i t_j = t_j s_i$ for $\{i, j\} = \{1, 2\}$.

   (b) $s_i(a \circ_j b) = s_i a \circ_j s_i b$ for $\{i, j\} = \{1, 2\}$.

   (c) $t_i(a \circ_j b) = t_i a \circ_j t_i b$ for $\{i, j\} = \{1, 2\}$.

   (d) $e_i(a \circ_j b) = e_i a \circ_j e_i b$ for $\{i, j\} = \{1, 2\}$.

   (e) $e_1 e_2 = e_2 e_1$.

   (f) The horizontal and vertical compositions satisfy an *interchange law* — if the expressions $(a \circ_1 b) \circ_2 (c \circ_1 d)$ and $(a \circ_2 c) \circ_1 (b \circ_2 d)$ are both defined, then they are equal.

A *double groupoid* is a double category in which all the category structures are groupoids. Note that taking inverses in one direction is automatically functorial in the other. In the case that all the category structures are monoids, or groups, we have the following well-known proposition.

Proposition 1.1.8 *Double monoids are abelian monoids.*

Proof: Suppose $\mathcal{A} = (A, \{*_1\}, \{*_2\}, \{*_0\})$ is a double monoid, and $g, h \in A$. Then $e_1 e_2 = e_2 e_1$ gives $e_1 *_1 = e_2 *_2 = *$ say, and so

$$\begin{array}{rcccl} g \circ_1 h & = & (g \circ_2 *) \circ_1 (* \circ_2 h) & = & (g \circ_1 *) \circ_2 (* \circ_1 h) = \; g \circ_2 h \\ g \circ_1 h & = & (* \circ_2 g) \circ_1 (h \circ_2 *) & = & (* \circ_1 h) \circ_2 (g \circ_1 *) = \; h \circ_2 g \end{array}$$

Thus $\circ_1 = \circ_2$ and the multiplication is commutative. □

1.1.3 Double crossed complexes

The category **Crs** of crossed complexes of groupoids is also complete, cocomplete and cartesian closed (see [26] for details of this last construction). In this section we introduce a notion of a double crossed complex of groupoids by considering crossed complexes of groupoids internal to the category **Crs**.

Definition 1.1.9 A *double crossed complex of groupoids* consists of

1. A collection of sets $C_{i,j}$ for $i, j \geq 0$,
2. source, target and identity maps

$$C_{i,j} \underset{e_1}{\overset{s_1, t_1}{\rightleftarrows}} C_{0,j} \qquad\qquad C_{j,i} \underset{e_2}{\overset{s_2, t_2}{\rightleftarrows}} C_{j,0}$$

   for $i \geq 1$, $j \geq 0$, with $s_1 = t_1$ and $s_2 = t_2$ for $i \geq 2$,

3. partially defined compositions and actions

$$C_{i,j} \times C_{i,j} \xrightarrow{\circ_1} C_{i,j} \qquad\qquad C_{1,j} \times C_{k,j} \xrightarrow{\alpha_1} C_{k,j}$$

$$C_{j,i} \times C_{j,i} \xrightarrow{\circ_2} C_{j,i} \qquad\qquad C_{j,1} \times C_{j,k} \xrightarrow{\alpha_2} C_{j,k}$$

   for $i \geq 1$, $j \geq 0$, $k \geq 2$,

4. horizontal and vertical boundary maps

$$C_{i,j} \xrightarrow{\delta_i^{\mathrm{h}}} C_{i-1,j} \qquad\qquad C_{j,i} \xrightarrow{\delta_i^{\mathrm{v}}} C_{j,i-1}$$

   for $i \geq 2$, $j \geq 0$.

These data are such that

1. for each $j \geq 0$ the horizontal structure $((C_{i,j})_{i\geq 0}, s_1, t_1, e_1, \circ_1, \alpha_1, (\delta_i^{\mathrm{h}})_{i\geq 2})$ defines a crossed complex,

2. for each $i \geq 0$ the vertical structure $((C_{i,j})_{j\geq 0}, s_2, t_2, e_2, \circ_2, \alpha_2, (\delta_j^{\mathrm{v}})_{j\geq 2})$ defines a crossed complex,

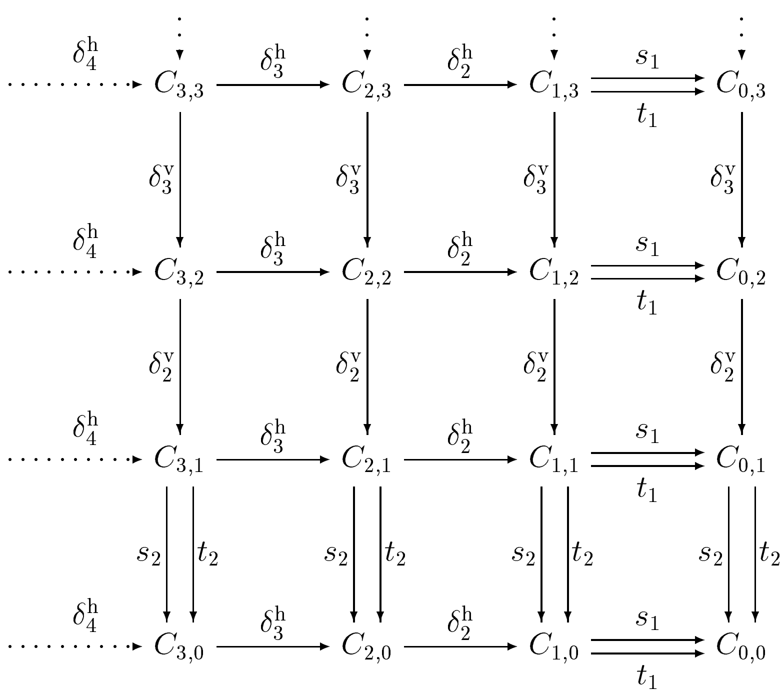
3. the horizontal structure maps commute with the vertical structure maps. That is:

   (a) the functions $s_1, t_1, e_1, \delta^{\mathrm{h}}$ define crossed complex morphisms between the vertical crossed complexes, as do $s_2, t_2, e_2, \delta^{\mathrm{v}}$ between the horizontal ones,

   (b) for each $i, j \geq 1$ the structure $(C_{i,j}, C_{0,j}, C_{i,0}, C_{0,0}, (s_k, t_k, e_k, \circ_k)_{k=1,2})$ defines a double groupoid,

   (c) the horizontal and vertical actions satisfy an interchange law — if the expressions $\alpha_2(\alpha_1(r, q), \alpha_1(p, a))$ and $\alpha_1(\alpha_2(r, p), \alpha_2(q, a))$ are both defined, then they are equal.

A double crossed complex of groupoids may be represented diagrammatically as follows



The category of double crossed complexes of groupoids and their homomorphisms will be written $\mathbf{Crs}^{(2)}$.

A *reduced* double crossed complex consists of a double crossed complex as defined above such that the set $C_{0,0}$ is a singleton. Note that this is not the same as a crossed complex of groups internal to the category of crossed complexes of groups, in which $C_{i,0}$ and $C_{0,i}$ are singletons for all $i \geq 0$ and hence $C_{i,j}$ is an abelian group for all $i, j \geq 1$.

Our intention is to show that the double crossed complex plays a rôle similar to that of the bichain complex in the abelian situation, or to that of the bisimplicial set. Note that taking the diagonal of a double crossed complex does not define a crossed complex as we might have liked. In the next section, however, we will see that there is an appropriate notion of a *codiagonal* or *total* crossed complex of a double crossed complex.

1.2 Some Algebraic Constructions

1.2.1 The total module of a double groupoid

If $\mathbf{C}$, $\mathbf{D}$ are categories over a common object set O, then the *free product* of $\mathbf{C}$ and $\mathbf{D}$, written $\mathbf{C} *_O \mathbf{D}$, is the coproduct of $\mathbf{C}$ and $\mathbf{D}$ in the category $\mathbf{Cat}_O$ of categories over O and functors which are the identity on objects. Alternatively, writing $\mathbf{O}$ for the subcategory of $\mathbf{C}$ and $\mathbf{D}$ with object set O and no non-identity arrows, the free product may be defined as the following pushout in $\mathbf{Cat}$

$$\begin{array}{ccc} \mathbf{O} & \hookrightarrow & \mathbf{C} \\ \downarrow & & \vdots \\ \mathbf{D} & \cdots\rightarrow & \mathbf{C} *_O \mathbf{D} \end{array}$$

Definition 1.2.1 Suppose that $\mathcal{A} = (A, A_1, A_2, A_0)$ is a double groupoid. Then define the *total crossed module* of $\mathcal{A}$ to be the crossed module $\mathbf{C} \xrightarrow{\delta} \mathbf{D}$ where $\mathbf{D}$ is the groupoid $A_1 *_{A_0} A_2$. The crossed $\mathbf{D}$-module $\mathbf{C}$ has generators a corresponding to the squares in A with source and target functions both given by $t_1 t_2$, identities given by $e_1 e_2$ and the boundary map given by

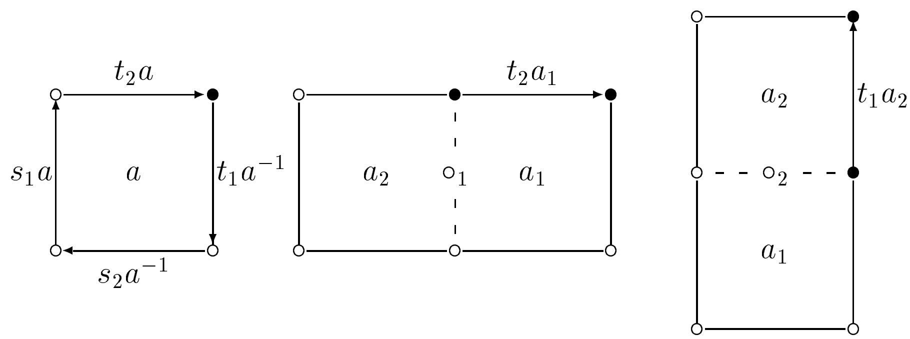
$$a \stackrel{\delta}{\longmapsto} t_1 a^{-1} \circ s_2 a^{-1} \circ s_1 a \circ t_2 a,$$

which are subject to the relations

$$\begin{array}{rcll} a_1 \circ a_2^{t_2 a_1} & = & a_2 \circ_1 a_1 & \text{if } s_1 a_1 = t_1 a_2, \\ a_1^{t_1 a_2} \circ a_2 & = & a_1 \circ_2 a_2 & \text{if } t_2 a_1 = s_2 a_2. \end{array}$$

for $a_1, a_2 \in A$.

The base points, boundary maps and composition relations for $\mathbf{C}$ may be seen geometrically from the following diagrams:



We will see below that this definition of the total crossed module generalises a construction of Brown and Higgins which associates a crossed module to a pair of groupoids.

1.2.2 The total complex of a double complex

Suppose $\mathbf{D}_1$, $\mathbf{D}_2$ are groupoids over a common object set O. Then a functor $\mathbf{D}_1 \xrightarrow{f} \mathbf{D}_2$ which is the identity on O induces a functor $\mathbf{Mod}_{\mathbf{D}_2} \xrightarrow{f^*} \mathbf{Mod}_{\mathbf{D}_1}$. If $\mathbf{C}$ is a $\mathbf{D}_2$-module then the module $f^*(\mathbf{C})$ has the same underlying groupoid as $\mathbf{C}$ and $\mathbf{D}_1$ acts on this by $(d, c) \longmapsto c^{f(d)}$.

The left adjoint f_* to the functor f^* defines the *induced module* construction. If $\mathbf{C}$ is a $\mathbf{D}_1$-module then the induced module $f_*(\mathbf{C})$ may be defined as follows. Let $\mathbf{E}$ be the totally disconnected category over O generated by arrows $(c, d) \in \mathbf{E}(y, y)$ for all $c \in \mathbf{C}(x, x), d \in \mathbf{D}_2(x, y)$, subject to the relations

1. $(c_1, d) \circ (c_2, d) = (c_1 \circ c_2, d)$,
2. $(e_x, d) = e_y$,
3. $(c, f(d_1) \circ d_2) = (c^{d_1}, d_2)$,
4. $(c, d) \circ (c', d') = (c', d') \circ (c, d)$

where $c, c_1, c_2 \in \mathbf{C}(x, x)$, $c' \in \mathbf{C}(w, w)$, $d, d_1 \in \mathbf{D}(x, y)$, $d_2 \in \mathbf{D}(y, z)$, $d' \in \mathbf{D}(w, y)$. Then $\mathbf{D}_2$ acts on $\mathbf{E}$ by $(c, d)^{d_2} = (c, d \circ d_2)$, and this defines $f_*(\mathbf{C})$.

If $\mathbf{C} \xrightarrow{\delta} \mathbf{D}_1$ is a crossed module and f is as above, then an induced crossed $\mathbf{D}_2$-module $f_*\mathbf{C}$ may also be defined [8]. Let $\mathbf{E}$ be the category-with-$\mathbf{D}_2$-action given by the same presentation as in the previous paragraph except that the commutativity relation (4) is replaced by

4′. $(c', d')^{-1} \circ (c, d) \circ (c', d') = (c, d \circ d'^{-1} \circ f\delta(c') \circ d')$

Then the induced crossed module $f_*\mathbf{C}$ is

$$\begin{array}{rcl}\mathbf{E} & \xrightarrow{\quad\delta\quad} & \mathbf{D}_2 \\ (c,d) & \longmapsto & d^{-1}\circ f\delta(c)\circ d\end{array}$$

In particular, if $\mathbf{D}$ is the free product $\mathbf{D}_1 *_O \mathbf{D}_2$ and $\mathbf{C}$ a (crossed) module over $\mathbf{D}_1$, say, then we write $\mathbf{C}^*$ for the induced (crossed) module over $\mathbf{D}$.

We can now introduce a *total complex* functor on the category of double crossed complexes of groupoids.

$$\mathbf{Crs}^{(2)} \xrightarrow{\text{Tot}} \mathbf{Crs}$$

Suppose C is a double crossed complex. Then the associated total complex is the crossed complex $\text{Tot}(C)$ defined as follows

- The set $\text{Tot}(C)_0 = O$ is given by $C_{0,0}$.
- The groupoid $\text{Tot}(C)_1 = P$ is given by the free product of $C_{1,0}$ and $C_{0,1}$ over O.
- The crossed module $\delta_2\colon \text{Tot}(C)_2 \to P$ is given by the coproduct of the induced crossed P-modules $C^*_{2,0}$ and $C^*_{0,2}$ and the total crossed P-module associated to the double groupoid $(C_{1,1}, C_{0,1}, C_{1,0}, C_{0,0})$ as discussed in section 1.2.1.
- For $m \geq 3$, the abelian P-module $\text{Tot}(C)_m$ is defined as the coproduct of abelian P-modules $M_0, M_1, \ldots, M_m$. Each M_i is in turn defined from a P-module N_i by imposing the relation $a^{\delta_2 b} = a$ for all $a \in N_i$, $b \in \text{Tot}(C)_2$ such that $t(a) = t(b)$. The P-modules N_0, N_m are given by the induced modules $C^*_{m,0}, C^*_{0,m}$ respectively. For $1 \leq i \leq m-1$ we give the abelian P-module N_i in terms of generators and relations. Generators a of N_i correspond to elements of $C_{m-i,i}$, with source and target functions given by t_1t_2 and identities by e_1e_2. In the case $i = 1$ these are subject to the relations

$$\begin{array}{rcll} a^{t_2 b} & = & \alpha_1(b,a) & \text{if } t_1 a = s_1 b, \\ a_1 \circ a_2 & = & a_1 \circ_1 a_2 & \text{if } t_1 a_1 = t_1 a_2, \\ a_1^{t_1 a_2} \circ a_2 & = & a_1 \circ_2 a_2 & \text{if } t_2 a_1 = s_2 a_2 \end{array}$$

for $a, a_1, a_2 \in C_{m-1,1}$, $b \in C_{1,1}$, and in the case $i = m-1$ to the relations

$$\begin{array}{rcll} a^{t_1 b} & = & \alpha_2(b,a) & \text{if } t_2 a = s_2 b, \\ a_1 \circ a_2^{t_2 a_1} & = & a_2 \circ_1 a_1 & \text{if } s_1 a_1 = t_1 a_2, \\ a_1 \circ a_2 & = & a_1 \circ_2 a_2 & \text{if } t_2 a_1 = t_2 a_2 \end{array}$$

for $a, a_1, a_2 \in C_{1,m-1}$, $b \in C_{1,1}$. For $2 \leq i \leq m-2$ the relations are

$$\begin{array}{rcll} a^{t_2 b_1} & = & \alpha_1(b_1,a) & \text{if } t_1 a = s_1 b_1, \\ a^{t_1 b_2} & = & \alpha_2(b_2,a) & \text{if } t_2 a = s_2 b_2, \\ a_1 \circ a_2 & = & a_1 \circ_1 a_2 & \text{if } t_1 a_1 = t_1 a_2, \\ a_1 \circ a_2 & = & a_1 \circ_2 a_2 & \text{if } t_2 a_1 = t_2 a_2 \end{array}$$

where $a, a_1, a_2 \in C_{m-i,i}$, $b_1 \in C_{1,i}$, $b_2 \in C_{m-i,1}$. The boundary map δ_m is the module homomorphism induced by the functions $N_i \longrightarrow \mathrm{Tot}(C)_{m-1}$ given on generators by

$$a \longmapsto \begin{cases} \delta^{\mathrm{h}}_m a & \text{for } i = 0 \\ \delta^{\mathrm{v}}_m a & \text{for } i = m \\ \delta^{\mathrm{h}}_{m-1} a \;\circ\; \left((t_2 a)^{-1} \circ (s_2 a)^{t_1 a}\right)^{(-1)^{m-1}} & \text{for } i = 1 \\ \left((t_1 a)^{-1} \circ (s_1 a)^{t_2 a}\right) \;\circ\; (\delta^{\mathrm{v}}_{m-1} a)^{-1} & \text{for } i = m-1 \\ \delta^{\mathrm{h}}_{m-i} a \;\circ\; (\delta^{\mathrm{v}}_i a)^{(-1)^{m-i}} & \text{for } 2 \le i \le m-2 \end{cases}$$

Collecting the various formulæ together we can give the following definition of Tot in terms of generators and relations.

Proposition 1.2.2 *Suppose C is a double crossed complex of groupoids. Then $Tot(C)$ is the crossed complex of groupoids given by generators $c_{i,j} \in Tot(C)_n$ for all $c_{i,j} \in C_{i,j}$ with $n = p + q$, satisfying the following relations*

1. $$\begin{aligned} sc_{1,0} &= s_1 c_{1,0} \\ sc_{0,1} &= s_2 c_{0,1} \\ tc_{i,0} &= t_1 c_{i,0} && \textit{for } i \ge 1 \\ tc_{0,j} &= t_2 c_{0_j} && \textit{for } j \ge 1 \\ tc_{i,j} &= t_1 t_2 c_{i,j} && \textit{for } i, j \ge 1 \end{aligned}$$

2. $$\begin{aligned} \delta_2 c_{1,1} &= (t_1 c_{1,1})^{-1} \circ (s_2 c_{1,1})^{-1} \circ s_1 c_{1,1} \circ t_2 c_{1,1} \\ \delta_i c_{i,0} &= \delta^{\mathrm{h}}_i c_{i,0} && \textit{for } i \ge 2 \\ \delta_j c_{0,j} &= \delta^{\mathrm{v}}_j c_{0,j} && \textit{for } j \ge 2 \\ \delta_{i+1} c_{i,1} &= \delta^{\mathrm{h}}_i c_{i,1} \;\circ\; \left((t_2 c_{i,1})^{-1} \circ (s_2 c_{i,1})^{t_1 c_{i,1}}\right)^{(-1)^i} && \textit{for } i \ge 2 \\ \delta_{j+1} c_{1,j} &= \left((t_1 c_{1,j})^{-1} \circ (s_1 c_{1,j})^{t_2 c_{1,j}}\right) \;\circ\; (\delta^{\mathrm{v}}_j c_{1,j})^{-1} && \textit{for } j \ge 2 \\ \delta_{i+j} c_{i,j} &= \delta^{\mathrm{h}}_i c_{i,j} \circ (\delta^{\mathrm{v}}_j c_{i,j})^{(-1)^i} && \textit{for } i, j \ge 2 \end{aligned}$$

3. $$\begin{aligned} \alpha_1(c_{1,j}, c_{i,j}) &= c_{i,j}^{t_2 c_{1,j}} && \textit{for } i \ge 2 \\ \alpha_2(c_{i,1}, c_{i,j}) &= c_{i,j}^{t_1 c_{i,1}} && \textit{for } j \ge 2 \end{aligned}$$

4. $$\begin{aligned} c_{1,j} \circ_1 c'_{1,j} &= c'_{1,j} \circ c_{1,j}^{t_2 c'_{1,j}} && \textit{for } j \ge 1 \\ c_{i,j} \circ_1 c'_{i,j} &= c_{i,j} \circ c'_{i,j} && \textit{for } j = 0 \textit{ or } i \ge 2 \\ c_{i,1} \circ_2 c'_{i,1} &= c_{i,1}^{t_1 c'_{i,1}} \circ c'_{i,1} && \textit{for } i \ge 1 \\ c_{i,j} \circ_2 c'_{i,j} &= c_{i,j} \circ c'_{i,j} && \textit{for } i = 0 \textit{ or } j \ge 2 \end{aligned}$$

1.2.3 Tensor products and double complexes

In this section we will consider a functor

$$\mathbf{Crs} \times \mathbf{Crs} \xrightarrow{\otimes^{(2)}} \mathbf{Crs}^{(2)}$$

whose composite with the functor Tot defined above gives the tensor product of crossed complexes as defined in [12].

Definition 1.2.3 Suppose C, D are crossed complexes. Then the double crossed complex $C \otimes^{(2)} D$ is defined as follows

- Each set $(C \otimes^{(2)} D)_{i,j}$ is given by the cartesian product $C_i \times D_j$. Elements (c, d) will be written $c \otimes d$.
- The horizontal crossed complex structures are defined by the crossed complex structure on C and the vertical structures by that on D. That is

$$
\begin{array}{rclrcl}
s_1(c \otimes d) &=& s(c) \otimes d & s_2(c \otimes d) &=& c \otimes s(d) \\
t_1(c \otimes d) &=& t(c) \otimes d & t_2(c \otimes d) &=& c \otimes t(d) \\
e_1(c \otimes d) &=& e(c) \otimes d & e_2(c \otimes d) &=& c \otimes e(d) \\
(c \otimes d) \circ_1 (c' \otimes d) &=& (c \circ c') \otimes d & (c \otimes d) \circ_2 (c \otimes d') &=& c \otimes (d \circ d') \\
\alpha_1(c_1 \otimes d, c \otimes d) &=& c^{c_1} \otimes d & \alpha_2(c \otimes d_1, c \otimes d) &=& c \otimes d^{d_1} \\
\delta_i^{\mathrm{h}}(c \otimes d) &=& \delta_i(c) \otimes d & \delta_j^{\mathrm{v}}(c \otimes d) &=& c \otimes \delta_j(d)
\end{array}
$$

  where defined.

Proposition 1.2.4 *The above definitions for the structure maps of $C \otimes^{(2)} D$ are consistent with the double crossed complex axioms.*

Proof: Clear. As an illustration, note that $t_1(c \otimes d) = s_1(c' \otimes d')$ implies $d = d'$ as well as $tc = sc'$, so we are indeed able to define the horizontal compositions by those of C. We are actually using the fact that the coproduct of crossed complexes of groupoids (but *not* of crossed complexes of groups) is given by disjoint union, and so the copower can be defined by a cartesian product. $\Box$

We now define the tensor product $C \otimes D$ of two crossed complexes C, D to be the total complex of $C \otimes^{(2)} D$. More explicitly, we have the following presentation.

Proposition 1.2.5 *Given crossed complexes of groupoids C, D, the tensor product $C \otimes D$ is the crossed complex of groupoids given by generators $c_i \otimes d_j \in (C \otimes D)_{i+j}$ for all $c_i \in C_i$, $d_j \in D_j$, satisfying the following relations*

1.
$$
\begin{array}{rcll}
s(c_1 \otimes d_0) &=& sc_1 \otimes d_0 & \\
s(c_0 \otimes d_1) &=& c_0 \otimes sd_1 & \\
t(c_i \otimes d_0) &=& tc_i \otimes d_0 & \text{for } i \geq 1 \\
t(c_0 \otimes d_j) &=& c_0 \otimes td_j & \text{for } j \geq 1 \\
t(c_i \otimes d_j) &=& tc_i \otimes td_j & \text{for } i, j \geq 1
\end{array}
$$

2.
$$
\begin{array}{rcll}
\delta_2(c_1 \otimes d_1) &=& (tc_1 \otimes d_1)^{-1} \circ (c_1 \otimes sd_1)^{-1} \circ sc_1 \otimes d_1 \circ c_1 \otimes td_1 & \\
\delta_i(c_i \otimes d_0) &=& \delta_i c_i \otimes d_0 & \text{for } i \geq 2 \\
\delta_j(c_0 \otimes d_j) &=& c_0 \otimes \delta_j d_j & \text{for } j \geq 2 \\
\delta_{i+1}(c_i \otimes d_1) &=& \delta_i c_i \otimes d_1 \circ \left((c_i \otimes td_1)^{-1} \circ (c_i \otimes sd_1)^{tc_i \otimes d_1}\right)^{(-1)^i} & \text{for } i \geq 2 \\
\delta_{j+1}(c_1 \otimes d_j) &=& \left((tc_1 \otimes d_j)^{-1} \circ (sc_1 \otimes d_j)^{c_1 \otimes td_j}\right) \circ (c_1 \otimes \delta_j d_j)^{-1} & \text{for } j \geq 2 \\
\delta_{i+j}(c_i \otimes d_j) &=& \delta_i c_i \otimes d_j \circ (c_i \otimes \delta_j d_j)^{(-1)^i} & \text{for } i, j \geq 2
\end{array}
$$

3. $$\begin{array}{rcll} c_i^{c_1} \otimes d_j & = & (c_i \otimes d_j)^{c_1 \otimes t d_j} & \text{for } i \geq 2 \\ c_i \otimes d_j^{d_1} & = & (c_i \otimes d_j)^{t c_i \otimes d_1} & \text{for } j \geq 2 \end{array}$$

4. $$\begin{array}{rcll} c_i \otimes (d_1 \circ d_1') & = & (c_i \otimes d_1)^{t c_i \otimes d_1'} \circ c_i \otimes d_1' & \text{for } i \geq 1 \\ c_i \otimes (d_j \circ d_j') & = & c_i \otimes d_j \circ c_i \otimes d_j' & \text{for } i = 0 \text{ or } j \geq 2 \\ (c_1 \circ c_1') \otimes d_j & = & c_1' \otimes d_j \circ (c_1 \otimes d_j)^{c_1' \otimes t d_j} & \text{for } j \geq 1 \\ (c_i \circ c_i') \otimes d_j & = & c_i \otimes d_j \circ c_i' \otimes d_j & \text{for } j = 0 \text{ or } i \geq 2 \end{array}$$

Proof: Follows directly by substitution of the definitions of 1.2.3 into the formulæ of proposition 1.2.2. □

It should be noted that our definition of Tot in the previous section was guided by the principle that the definitions of the tensor product in **Crs** given here and in [12] should agree.

Remark 1.2.6 If $\mathbf{G}$, $\mathbf{H}$ are groupoids, then we may form a double groupoid from them by considering

$$\begin{array}{ccc} \mathrm{Arr}(\mathbf{G}) \times \mathrm{Arr}(\mathbf{H}) & \rightrightarrows & \mathrm{Ob}(\mathbf{G}) \times \mathrm{Arr}(\mathbf{H}) \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathrm{Arr}(\mathbf{G}) \times \mathrm{Ob}(\mathbf{H}) & \rightrightarrows & \mathrm{Ob}(\mathbf{G}) \times \mathrm{Ob}(\mathbf{H}) \end{array}$$

with the horizontal structure maps induced from $\mathbf{G}$ and the vertical ones from $\mathbf{H}$. It is clear in this case that the associated total crossed module, as defined in section 1.2.1, is precisely that encountered previously by Brown and Higgins in [12] as the tensor product of $\mathbf{G}$, $\mathbf{H}$ regarded as crossed complexes which are trivial above dimension one.

1.3 Functors from Simplicial Categories

1.3.1 Simplicial sets

Let Δ be the category with objects the ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$ and arrows the order preserving functions between them. Recall that the arrows are in fact generated by the injections $d(i) : [n-1] \to [n]$ $(0 \leq i \leq n)$ which miss out the ith element and the surjections $s(i) : [n+1] \to [n]$ $(0 \leq i \leq n)$ which repeat the ith element.

A simplicial object in a category $\mathbf{C}$ is a functor $C_\bullet$ from Δ^{op} to $\mathbf{C}$. Equivalently, by considering the images under $C_\bullet$ of $[n]$, $d(i)$, and $s(i)$, a simplicial object may be given by a family of objects (C_n) of $\mathbf{C}$ together with arrows $d_i : C_n \to C_{n-1}$ (face maps) and

$s_i : C_n \to C_{n+1}$ (degeneracy maps) in $\mathbf{C}$ which satisfy the usual simplicial relations:

$$\begin{array}{rcll} d_i d_j & = & d_{j-1} d_i & \text{for } i < j \\ d_i s_j & = & \left\{ \begin{array}{ll} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for } i > j \end{array} \right. & \\ s_i s_j & = & s_{j+1} s_i & \text{for } i \leq j \end{array}$$

We will write **SimpC** for the category $[\Delta^{\text{op}}, \mathbf{C}]$ of simplicial objects in $\mathbf{C}$.

Similarly a cosimplicial object $C^\bullet : \Delta \to \mathbf{C}$ may be given by a family of objects (C^n) and coface and codegeneracy arrows d^i, s^i satisfying the dual relations.

In particular, we will consider the category of simplicial objects in **Set**, the category of sets, together with the *fundamental crossed complex* functor

$$\mathbf{SimpSet} \xrightarrow{\ \pi\ } \mathbf{Crs}$$

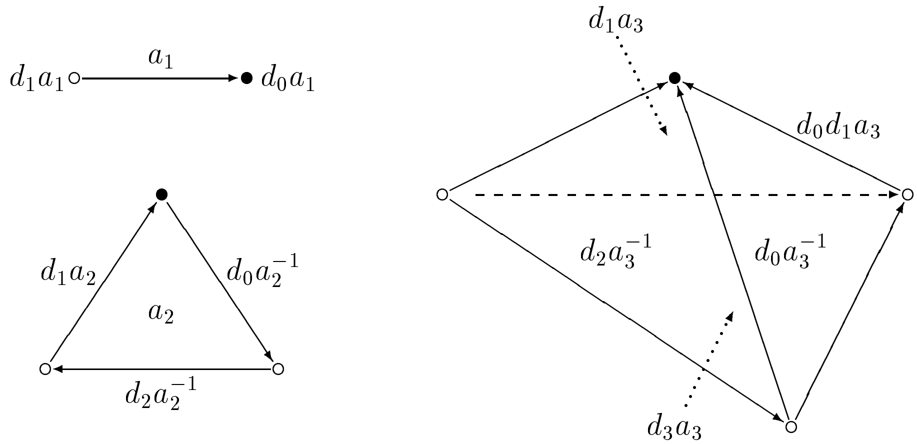
from simplicial sets to crossed complexes which is defined as follows:

Definition 1.3.1 *For $K_\bullet$ a simplicial set, $\pi(K_\bullet)$ is the crossed complex C generated by $[a_n] \in C_n$ for all n-simplices $a_n \in K_n$, such that the following relations hold:*

$$\begin{array}{rcl} [s_0 a_0] & = & e_{[a_0]} \ \textit{in } \pi(K_\bullet)_1 \\ [s_i a_n] & = & e_{t[a_n]} \ \textit{in } \pi(K_\bullet)_{n+1} \quad \textit{for } n \geq 1 \\ s[a_1] & = & [d_1 a_1] \\ t[a_n] & = & [d_0^n a_n] \quad \textit{for } n \geq 1 \\ \delta_2[a_2] & = & [d_0 a_2]^{-1} \circ [d_2 a_2]^{-1} \circ [d_1 a_2] \\ \delta_3[a_3] & = & [d_1 a_3] \circ [d_2 a_3]^{-1} \circ [d_0 a_3]^{-1} \circ [d_3 a_3]^{[d_0 d_1 a_3]} \\ \delta_n[a_n] & = & \displaystyle\prod_{i=0}^{n-1} [d_i a_n]^{(-1)^{i+1}} \ \circ \ \left([d_n a_n]^{[d_0 d_1 \ldots d_{n-2} a_n]} \right)^{(-1)^{n+1}} \quad \textit{for } n \geq 4 \end{array}$$

We will often omit the brackets around the generators.

The first two relations say that degenerate simplices in each K_n may be ignored. The other relations are boundary relations and are often known as the homotopy addition theorem [41, IV.6]. They may be seen geometrically as follows



Note that other equivalent presentations of the functor may be given by choosing alternative basepoints or signs for the generators. The presentation given here is that which leads to the tidiest formulæ later.

1.3.2 Simplicial crossed complexes

We now consider the category **SimpCrs** of simplicial objects in the category of crossed complexes of groupoids. To fix the notation we shall consider the crossed complex structures as being 'horizontal' and the simplicial structures as being 'vertical', as in the following definition.

Definition 1.3.2 A *simplicial crossed complex (of groupoids) C* is given by

1. A collection of sets $C_{i,j}$ for $i, j \geq 0$,
2. source, target and identity maps

$$C_{i,j} \underset{e}{\overset{s,t}{\rightleftarrows}} C_{0,j}$$

   for $i \geq 1$, $j \geq 0$, with $s = t$ for $i \geq 2$,

3. partially defined compositions and actions

$$C_{i,j} \times C_{i,j} \xrightarrow{\circ} C_{i,j} \qquad C_{1,j} \times C_{k,j} \xrightarrow{\alpha} C_{k,j}$$

   for $i \geq 1$, $j \geq 0$, $k \geq 2$,

4. (horizontal) boundary maps

$$C_{i,j} \xrightarrow{\delta_i} C_{i-1,j}$$

   for $i \geq 2$, $j \geq 0$,

5. (vertical) face maps and degeneracy maps

$$C_{i,j+1} \underset{s_q}{\overset{d_p}{\rightleftarrows}} C_{i,j}$$

for $i, j \geq 0$, $0 \leq p \leq j+1$, $0 \leq q \leq j$.

These data are such that

1. for each $j \geq 0$ the horizontal structure $((C_{i,j})_{i\geq 0}, s, t, e, \circ, \alpha, (\delta_i)_{i\geq 2})$ defines a crossed complex of groupoids,

2. for each $i \geq 0$ the vertical structure $((C_{i,j})_{j\geq 0}, (d_p), (s_q))$ defines a simplicial set,

3. the face and degeneracy maps define homomorphisms between the horizontal crossed complex structures.

Note that the (horizontal) source maps s should not be confused with the (vertical) degeneracy maps s_q.

The formulæ of definition 1.3.1 may also be used to also define a functor

$$\mathbf{SimpCrs} \xrightarrow{\pi_{\mathrm{Crs}}} \mathbf{Crs}^{(2)}$$

from the category of simplicial crossed complexes to the category of double crossed complexes, simply by taking the definition of π internal to the category $\mathbf{Crs}$. If C is a simplicial crossed complex, then $\pi_{\mathrm{Crs}}(C)$ has vertical crossed complexes structures given by applying π to the simplicial sets $((C_{i,j})_{j\geq 0}, (d_p), (s_q))$ for each $i \geq 0$, and horizontal crossed complex structures those induced from the crossed complexes $((C_{i,j})_{i\geq 0}, s, t, e, \circ, \alpha, (\delta_i)_{i\geq 2})$ for each $j \geq 0$.

A bisimplicial object $C_{\bullet,\bullet}$ in a category $\mathbf{C}$ is a simplicial object in $\mathbf{SimpC}$, or alternatively a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{C}$. We will write $C_{m,n}$ for the image of $([m],[n])$ under $C_{\bullet,\bullet}$, and define the horizontal and vertical face and degeneracy maps d^{h}_i, s^{h}_i, d^{v}_i, s^{v}_i by the images of $(d(i), 1)$, $(s(i), 1)$, $(1, d(i))$, $(1, s(i))$ respectively. The category $[\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{C}]$ of all such bisimplicial objects will be denoted $\mathbf{BiSimpC}$.

Note we can define a functor

$$\mathbf{BiSimpSet} \xrightarrow{\pi_{\mathrm{Simp}}} \mathbf{SimpCrs}$$

from bisimplicial sets to simplicial crossed complexes by taking definition 1.3.1 internal to the category of simplicial sets. Furthermore the composite functor $\pi_{\mathrm{Simp}} \circ \pi_{\mathrm{Crs}}$ gives the *fundamental double crossed complex* of a bisimplicial set

$$\mathbf{BiSimpSet} \xrightarrow{\pi^{(2)}} \mathbf{Crs}^{(2)}$$

If K, L are simplicial sets then we can form a bisimplicial set which in dimension (i, j) has the set $K_i \times L_j$, with the horizontal face and degeneracy maps coming from K and the vertical ones from L. This gives a functor

$$\mathbf{SimpSet} \times \mathbf{SimpSet} \xrightarrow{\times^{(2)}} \mathbf{BiSimpSet}$$

Note that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{SimpSet} \times \mathbf{SimpSet} & \xrightarrow{\times^{(2)}} & \mathbf{BiSimpSet} \\
\Big\downarrow {\scriptstyle \pi \times \pi} & & \Big\downarrow {\scriptstyle \pi^{(2)}} \\
\mathbf{Crs} \times \mathbf{Crs} & \xrightarrow{\otimes^{(2)}} & \mathbf{Crs}^{(2)}
\end{array}$$

1.3.3 The total complex of a simplicial crossed complex

Suppose C is a simplicial crossed complex as in definition 1.3.2. We have seen above how to define a double crossed complex $\pi_{\mathrm{Crs}}(C)$ from C. We can therefore make the following definition:

Definition 1.3.3 The *(simplicial) total functor* from simplicial crossed complexes to crossed complexes is the composite of the functor π_{Crs} and the total crossed complex functor defined in section 1.2.2.

$$\begin{array}{ccc}
\mathbf{SimpCrs} & \overset{\text{S-Tot}}{\dashrightarrow} & \mathbf{Crs} \\
& {\scriptstyle \pi_{\mathrm{Crs}}} \searrow \quad \nearrow {\scriptstyle \mathrm{Tot}} & \\
& \mathbf{Crs}^{(2)} &
\end{array}$$

This construction will play an important part in the definition of homotopy colimits of crossed complexes in chapter 3.

We have immediately

Proposition 1.3.4 *The following diagram commutes*

$$\begin{array}{ccc}
\mathbf{BiSimpSet} & \xrightarrow{\pi^{(2)}} & \mathbf{Crs}^{(2)} \\
\Big\downarrow {\scriptstyle \pi_{\mathrm{Simp}}} & & \Big\downarrow {\scriptstyle \mathrm{Tot}} \\
\mathbf{SimpCrs} & \xrightarrow{\text{S-Tot}} & \mathbf{Crs}
\end{array}$$

Proof: Since S-Tot $= \pi_{\mathrm{Crs}} \circ$ Tot and $\pi^{(2)} = \pi_{\mathrm{Simp}} \circ \pi_{\mathrm{Crs}}$, the result follows by associativity of functor composition. (Diagrammatically: putting in the diagonal arrow $\nearrow \pi_{\mathrm{Crs}}$ gives two commutative triangles). $\square$

We can also present the total complex of a simplicial crossed complex in terms of generators and relations.

Proposition 1.3.5 *Suppose C is a simplicial crossed complex of groupoids. Then* S-Tot(C) *is the crossed complex of groupoids given by generators $[c_{i,j}] \in$ S-Tot$(C)_n$ for all $c_{i,j} \in C_{i,j}$ with $n = i + j$, satisfying the following relations*

1. $$\begin{aligned} [s_0 c_{0,0}] &= e_{[c_{0,0}]} \text{ in S-Tot}(C)_1 \\ [s_k c_{i,j}] &= e_{t[c_{i,j}]} \text{ in S-Tot}(C)_{i+j+1} \quad \text{for } i+j \geq 1,\ 0 \leq k \leq j \end{aligned}$$

2. $$\begin{aligned} s[c_{1,0}] &= [s c_{1,0}] \\ s[c_{0,1}] &= [d_1 c_{0,1}] \\ t[c_{0,j}] &= [d_0^j c_{0,j}] \quad \text{for } j \geq 1 \\ t[c_{i,j}] &= [t d_0^j c_{i,j}] \quad \text{for } i \geq 1,\ j \geq 0 \end{aligned}$$

3. $$\begin{aligned} \delta_i[c_{i,0}] &= [\delta_i c_{i,0}] && \text{for } i \geq 2 \\ \delta_2[c_{0,2}] &= [d_0 c_{0,2}]^{-1} \circ [d_2 c_{0,2}]^{-1} \circ [d_1 c_{0,2}] \\ \delta_3[c_{0,3}] &= [d_1 c_{0,3}] \circ [d_2 c_{0,3}]^{-1} \circ [d_0 c_{0,3}]^{-1} \circ [d_3 c_{0,3}]^{[d_0^2 c_{0,3}]} \\ \delta_j[c_{0,j}] &= \prod_{k=0}^{j-1} [d_k c_{0,j}]^{(-1)^{k+1}} \circ \left([d_j c_{0,j}]^{[d_0^{j-1} c_{0,j}]} \right)^{(-1)^{j+1}} && \text{for } j \geq 4 \\ \delta_2[c_{1,1}] &= [t c_{1,1}]^{-1} \circ [d_1 c_{1,1}]^{-1} \circ [s c_{1,1}] \circ [d_0 c_{1,1}] \\ \delta_{i+1}[c_{i,1}] &= [\delta_i c_{i,1}] \circ \left([d_0 c_{i,1}]^{-1} \circ [d_1 c_{i,1}]^{[t c_{i,1}]} \right)^{(-1)^i} && \text{for } i \geq 2 \\ \delta_3[c_{1,2}] & \\ &= [d_0 c_{1,2}] \circ [s c_{1,2}]^{[d_0 d_1 c_{1,2}]} \circ [d_1 c_{1,2}]^{-1} \circ [t c_{1,2}]^{-1} \circ [d_2 c_{1,2}]^{[t d_0 c_{1,2}]} \\ \delta_{j+1}[c_{1,j}] &= \left([t c_{1,j}]^{-1} \circ [s c_{1,j}]^{[d_0^j c_{1,j}]} \right) \\ &\circ \prod_{k=0}^{j-1} [d_k c_{1,j}]^{(-1)^k} \circ \left([d_j c_{1,j}]^{[t d_0^{j-1} c_{1,j}]} \right)^{(-1)^j} && \text{for } j \geq 3 \\ \delta_{i+j}[c_{i,j}] &= [\delta_i c_{i,j}] \\ &\circ \prod_{k=0}^{j-1} [d_k c_{i,j}]^{(-1)^{i+k+1}} \circ \left([d_j c_{i,j}]^{[t d_0^{j-1} c_{i,j}]} \right)^{(-1)^{i+j+1}} && \text{for } i,j \geq 2 \end{aligned}$$

4. $$[\alpha(c_{1,j}, c_{i,j})] = [c_{i,j}]^{[d_0^j c_{1,j}]} \quad \text{for } i \geq 2$$

5. $$\begin{aligned} [c_{1,j} \circ c'_{1,j}] &= [c'_{1,j}] \circ [c_{1,j}]^{[d_0^j c'_{1,j}]} && \text{for } j \geq 1 \\ [c_{i,j} \circ c'_{i,j}] &= [c_{i,j}] \circ [c'_{i,j}] && \text{for } j = 0 \text{ or } i \geq 2 \end{aligned}$$

Proof: Fairly routine. The least straight-forward boundary relation is that for $\delta_3[c_{1,2}]$ in Tot $\pi_{\mathrm{Crs}}(C)$. In $\pi_{\mathrm{Crs}}(C)$ we have

$$\delta_2^{\mathrm{v}}[c_{1,2}] = [d_0 c_{1,2}]^{-1} \circ_2 [d_2 c_{1,2}]^{-1} \circ_2 [d_1 c_{1,2}]$$

where the inverses are with respect to $\circ_2$. Using the relation $c_{1,1} \circ_2 c'_{1,1} = c_{1,1}^{t_1 c'_{1,1}} \circ c'_{1,1}$ from proposition 1.2.2 we see that $\circ_2$-inverse of $c_{1,1}$ is given by taking $\left(c_{1,1}{}^{(tc_{1,1})^{-1}}\right)^{-1}$ in the total complex, and the above boundary relation becomes

$$\delta_2^{\mathrm{v}}[c_{1,2}] = \left([d_0 c_{1,2}]^{[td_0 c_{1,2}]^{-1} \circ [td_2 c_{1,2}]^{-1} \circ [td_1 c_{1,2}]}\right)^{-1} \circ \left([d_2 c_{1,2}]^{[td_2 c_{1,2}]^{-1} \circ [td_1 c_{1,2}]}\right)^{-1} \circ [d_1 c_{1,2}]$$

Note that this is just

$$\delta_2^{\mathrm{v}}[c_{1,2}] = \left([d_0 c_{1,2}]^{\delta_2 [tc_{1,2}]}\right)^{-1} \circ \left([d_2 c_{1,2}]^{[td_0 c_{1,2}] \circ \delta_2 [tc_{1,2}]}\right)^{-1} \circ [d_1 c_{1,2}]$$

Substituting this into the relation

$$\delta_3 c_{1,2} = \left((t_1 c_{1,2})^{-1} \circ (s_1 c_{1,2})^{t_2 c_{1,2}}\right) \circ (\delta_2^{\mathrm{v}} c_{1,2})^{-1}$$

from proposition 1.2.2, and recalling the crossed complex axiom $a_2^{-1} b_2 a_2 = b_2^{\delta_2 a_2}$, we get the required result. □

Chapter 2

The Eilenberg-Zilber Theorem

2.0 Introduction

In this chapter we prove the Eilenberg-Zilber theorem for crossed complexes: given simplicial sets K, L, there are natural homomorphisms

$$\pi K \otimes \pi L \underset{a}{\overset{b}{\rightleftarrows}} \pi(K \times L)$$

such that $b \circ a$ is the identity, and a homotopy

$$\mathcal{I} \otimes \pi(K \times L) \xrightarrow{\ h\ } \pi(K \times L)$$

between $a \circ b$ and the identity. Associativity and interchange relations for a, b and h are also proved.

We also show that any homotopy between an idempotent crossed complex endomorphism and the identity may be replaced by a homotopy which satisfies certain side-conditions, and in particular if h is the deformation retraction of the Eilenberg-Zilber theorem we may assume that the corresponding degree one map $\phi : x \mapsto h(\iota \otimes x)$ satisfies

$$\phi^2(x) = e, \qquad \phi(b(x)) = e, \qquad a(\phi(x)) = e, \qquad \phi\delta\phi(x) = (\phi(x))^{-1}$$

The Eilenberg-Zilber theorem is also shown to extend to give r-fold homotopies

$$\mathcal{I}^{\otimes r} \otimes \pi(K_0 \times \ldots \times K_r) \longrightarrow \pi(K_0 \times \ldots \times K_r)$$

satisfying certain boundary relations.

The structure of the chapter is as follows. In the first section, we begin with a review of the definitions of homotopy in **Crs**. This is essentially an exposition of material dating back to [42]. A *splitting homotopy* is then defined, and it is proved that any homotopy between an idempotent endomorphism and the identity may be replaced by a splitting homotopy. This result for chain complexes may be found in [30].

In the second section, we define the diagonal approximation map a and the shuffle homomorphism b. We prove that b is a one-sided inverse to a, and that a and b are associative and satisfy an 'interchange' relation. Some connection is shown between the Artin-Mazur diagonal of a bisimplicial set and the diagonal approximation map a, and between a and a construction by Brown and Gilbert of a simplicial group from a braided regular crossed module.

In the third section the homotopy h between $a \circ b$ and the identity is defined, using simplicial operators for the high-dimensional work as in the chain complex situation. We also show that h satisfies four interchange relations with respect to a and b, two of which in the chain complex case were shown by Shih [33]. We then use these relations to show that the higher homotopies on $\pi(K_0 \times \ldots \times K_r)$ induced by h form a *coherent system.*

2.1 Homotopy Theory of Crossed Complexes

2.1.1 Homotopy of morphisms

Let $\mathcal{I}$ be the groupoid

$$0 \underset{\iota^{-1}}{\overset{\iota}{\rightleftarrows}} 1$$

with object set $O = \{0, 1\}$ and non-identity arrows $\iota: 0 \to 1$ and its inverse $\iota^{-1}: 1 \to 0$. We will often regard $\mathcal{I}$ as a crossed complex which in dimensions ≥ 2 has only the trivial groupoid over O. Given any crossed complex C note that there are natural monomorphisms

$$C \underset{i_1}{\overset{i_0}{\rightrightarrows}} \mathcal{I} \otimes C$$

defined on generators by $i_\alpha: c \mapsto \alpha \otimes c$ for $\alpha = 0$ or 1.

Definition 2.1.1 *Suppose C, D are crossed complexes, and $f, g: C \to D$ are homomorphisms between them. A* homotopy *h from f to g, written $h: f \simeq g$, is given by a crossed* *complex homomorphism* $h: \mathcal{I} \otimes C \to D$ *such that the following diagram commutes*

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{i_0} & & \searrow^{f} & \\ \mathcal{I} \otimes C & & \xrightarrow{\quad h \quad} & & D \\ & \nwarrow_{i_1} & & \nearrow_{g} & \\ & & C & & \end{array}$$

The following proposition is standard.

Proposition 2.1.2 *The relation of homotopy given by* $\simeq$ *is an equivalence relation.*

Proof: For reflexivity, we note that i_0 and i_1 have a common one-sided inverse e given by the homomorphism

$$\mathcal{I} \otimes C \xrightarrow{\ e\ } C$$

which maps $0 \otimes c_n$ and $1 \otimes c_n$ to c_n and maps $\iota \otimes c_n$ to the identity at tc_n in C_{n+1}. Thus if $f : C \to C$ is a crossed complex homomorphism, the composite of e with f defines a homotopy $f \simeq f$ which we will write as 0_f.

For symmetry we use the non-trivial automorphism of $\mathcal{I}$ which induces a homomorphism

$$\mathcal{I} \otimes C \xrightarrow{\ s\ } \mathcal{I} \otimes C$$

mapping $\iota \otimes c$ to $\iota^{-1} \otimes c$. Thus if h is a homotopy $f \simeq g$, the composite of s with h defines a homotopy $g \simeq f$ which we will write as $\overline{h}$.

For transitivity we consider (vertical) composition of homotopies. Let $\mathcal{J}$ be the groupoid

$$0 \underset{\jmath^{-1}}{\overset{\jmath}{\rightleftarrows}} 1 \underset{\kappa^{-1}}{\overset{\kappa}{\rightleftarrows}} 2$$

with three objects $0, 1, 2$ and non-identity arrows $\jmath : 0 \to 1$ and $\kappa : 1 \to 2$ together with their inverses and composites. As usual $\mathcal{J}$ may be regarded as a crossed complex which is trivial in dimensions ≥ 2. Given crossed complex homomorphisms $f_0, f_1, f_2 : C \to D$ and homotopies $h_1: f_0 \simeq f_1$ and $h_2: f_1 \simeq f_2$ their vertical composite $h_1 \circ h_2$ is a homotopy $f_0 \simeq f_2$ defined by

$$\mathcal{I} \otimes C \xrightarrow{\ t \otimes \mathrm{id}\ } \mathcal{J} \otimes C \xrightarrow{\ h_1 \vee h_2\ } D$$

where t is given by $\iota \mapsto \jmath \cdot \kappa$ and $h_1 \vee h_2$ is given by $\jmath \otimes c \mapsto h_1(\iota \otimes c)$ and $\kappa \otimes c \mapsto h_2(\iota \otimes c)$. □

Moreover, $\simeq$ is a *congruence*. Suppose f is a crossed complex morphism $C \to D$ and k is a homotopy $g_0 \simeq g_1 \colon D \to E$. Then we get $f \cdot g_0 \simeq f \cdot g_1$ by considering the 'horizontal' composite homotopy ${}^f k$ defined by

$$\mathcal{I} \otimes C \xrightarrow{\text{id} \otimes f} \mathcal{I} \otimes D \xrightarrow{k} E$$

Similarly if h is a homotopy $f_0 \simeq f_1 \colon C \to D$ and g is a morphism $D \to E$ we can define a homotopy h^g from $f_0 \cdot g$ to $f_1 \cdot g$ by

$$\mathcal{I} \otimes C \xrightarrow{h} D \xrightarrow{g} E$$

Using these definitions, we can define the horizontal composite $h \cdot k$ of the homotopies h and k as the vertical composite of $h^{g_0} \colon f_0 \cdot g_0 \simeq f_1 \cdot g_0$ and ${}^{f_1}k \colon f_1 \cdot g_0 \simeq f_1 \cdot g_1$. Equivalently, let d be the map

$$\mathcal{I} \xrightarrow{d} \mathcal{I} \otimes \mathcal{I}$$

defined by $\iota \mapsto (0 \otimes \iota) \cdot (\iota \otimes 1)$. Then $h \cdot k$ may be defined directly as the homotopy

$$\mathcal{I} \otimes C \xrightarrow{d \otimes \text{id}} \mathcal{I} \otimes \mathcal{I} \otimes C \xrightarrow{\text{id} \otimes h} \mathcal{I} \otimes D \xrightarrow{k} E$$

Proposition 2.1.3 *The homotopy constructions described above satisfy the following relations:*

1. $h_1 \circ (h_2 \circ h_3) = (h_1 \circ h_2) \circ h_3$
2. $h_1 \cdot (h_2 \cdot h_3) = (h_1 \cdot h_2) \cdot h_3$
3. $0_{f_0} \circ h = h \circ 0_{f_1} = h$
4. ${}^f h = 0_f \cdot h$ *and* $h^g = h \cdot 0_g$
5. $h \circ \overline{h} = 0_{f_0}$ *and* $\overline{h} \circ h = 0_{f_1}$
6. $\overline{h \circ k} = \overline{k} \circ \overline{h}$.
7. $h \cdot (k_1 \circ k_2) = (h \cdot k_1) \circ (0_{f_1} \cdot k_2)$ *and* $(h_1 \circ h_2) \cdot k = (h_1 \cdot 0_{g_0}) \circ (h_2 \cdot k)$.

Proof: Clear. □

Note that the full interchange law between the horizontal and vertical compositions does not hold in general and neither does $\overline{h \cdot k} = \overline{h} \cdot \overline{k}$. This is because there are actually two choices for the definition of horizontal composition of homotopies, given by $h^{g_0} \circ {}^{f_1}k$ and ${}^{f_0}k \circ h^{g_1}$. These are not in general equal, although as morphisms they are themselves homotopic. Similarly there are two 'diagonal approximations' $d: \mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$ given by $\iota \mapsto (0 \otimes \iota) \cdot (\iota \otimes 1)$ and $\iota \mapsto (\iota \otimes 0) \cdot (1 \otimes \iota)$. The non-trivial homotopy between these possible choices is what leads to Steenrod squares, etc.

The notion of homotopy may also be translated into statements about the elements of C and D. The formulæ which result date back to J.H.C. Whitehead [42].

Proposition 2.1.4 *Specifying a homotopy $h: f \simeq g$ is equivalent to specifying the morphism g together with a degree one map $(\phi_n: C_n \to D_{n+1})$ which satisfies the following*

$$\begin{aligned} t(\phi_0 c_0) &= g c_0 \\ t(\phi_n c_n) &= t(g c_n) \quad \textit{for } n \geq 1 \\ \phi_n({c_n}^{c_1}) &= (\phi_n c_n)^{g c_1} \quad \textit{for } n \geq 2 \\ \phi_1(c_1 \cdot c_1') &= (\phi_1 c_1)^{g c_1'} \cdot \phi_1 c_1' \\ \phi_n(c_n \cdot c_n') &= \phi_n c_n \cdot \phi_n c_n' \quad \textit{for } n \geq 2 \end{aligned}$$

The morphism f is then completely determined by

$$\begin{aligned} s(\phi_0 c_0) &= f c_0 \\ \delta_2(\phi_1 c_1) &= (g c_1)^{-1} \cdot (\phi_0 s c_1)^{-1} \cdot f c_1 \cdot \phi_0 t c_1 \\ \delta_{n+1}(\phi_n c_n) &= (g c_n)^{-1} \cdot (f c_n)^{\phi_0 t c_n} \cdot (\phi_{n-1} \delta_n c_n)^{-1} \quad \textit{for } n \geq 2 \end{aligned}$$

Proof: Consider an arbitrary homomorphism $\mathcal{I} \otimes C \xrightarrow{h} D$. The relations of proposition 1.2.5 imply that for all $c_n, c_n' \in C_n$, h must satisfy the following

1. $$\begin{aligned} sh(\iota \otimes c_0) &= h(0 \otimes c_0) \\ th(\iota \otimes c_0) &= h(1 \otimes c_0) \\ th(\iota \otimes c_n) &= h(1 \otimes t c_n) \end{aligned}$$

2. $$\begin{aligned} \delta_2 h(\iota \otimes c_1) &= h(1 \otimes c_1)^{-1} \cdot h(\iota \otimes s c_1)^{-1} \cdot h(0 \otimes c_1) \cdot h(\iota \otimes t c_1) \\ \delta_{n+1} h(\iota \otimes c_n) &= h(1 \otimes c_n)^{-1} \cdot h(0 \otimes c_n)^{h(\iota \otimes t c_n)} \cdot h(\iota \otimes \delta_n c_n)^{-1} \quad \text{for } n \geq 2 \end{aligned}$$

3. $$h(\iota \otimes {c_n}^{c_1}) = h(\iota \otimes c_n)^{h(1 \otimes c_1)} \quad \text{for } n \geq 2$$

4. $$\begin{aligned} h(\iota \otimes (c_1 \cdot c_1')) &= h(\iota \otimes c_1)^{h(1 \otimes c_1')} \cdot h(\iota \otimes c_1') \\ h(\iota \otimes (c_n \cdot c_n')) &= h(\iota \otimes c_n) \cdot h(\iota \otimes c_n') \quad \text{for } n \geq 2 \end{aligned}$$

The proposition then follows by writing f, g for the homomorphisms

$$c_n \overset{f}{\longmapsto} h(0 \otimes c_n) \qquad c_n \overset{g}{\longmapsto} h(1 \otimes c_n)$$

and ϕ for the degree one map $c_n \overset{\phi_n}{\longmapsto} h(\iota \otimes c_n)$ ◻

The definitions of vertical and horizontal composition of homotopies may be similarly translated by considering the expansion of the expression $(\jmath \cdot \kappa) \otimes c_n$.

2.1.2 Strong deformation retractions and splitting homotopies

Definition 2.1.5 *Two crossed complexes C, D are* homotopy equivalent *if there exist homomorphisms $f : C \to D$ and $g : D \to C$ together with homotopies $h : f \cdot g \simeq \mathrm{id}_C$ and $k : g \cdot f \simeq \mathrm{id}_D$.*

Since the notion of a homotopy from an endomorphism to the identity plays such a large rôle, we make the following definition.

Definition 2.1.6 *A* derivation *$\phi\colon C \to C$ is a degree one map ($\phi_n\colon C_n \to C_{n+1}$) which satisfies the following*

$$\begin{aligned} t(\phi_0 c_0) &= c_0 \\ t(\phi_n c_n) &= t(c_n) \quad \text{for } n \geq 1 \\ \phi_n({c_n}^{c_1}) &= (\phi_n c_n)^{c_1} \quad \text{for } n \geq 2 \\ \phi_1(c_1 \cdot c_1') &= (\phi_1 c_1)^{c_1'} \cdot \phi_1 c_1' \\ \phi_n(c_n \cdot c_n') &= \phi_n c_n \cdot \phi_n c_n' \quad \text{for } n \geq 2 \end{aligned}$$

Corollary 2.1.7 *Given $f\colon C \to C$, a homotopy h from f to the identity is given by a derivation $\phi\colon C \to C$ such that*

$$\begin{aligned} f c_0 &= s\phi_0 c_0 \\ f c_1 &= \phi_0 s c_1 \cdot c_1 \cdot \delta_2 \phi_1 c_1 \cdot (\phi_0 t c_1)^{-1} \\ f c_n &= (c_n \cdot \delta_{n+1}\phi_n c_n \cdot \phi_{n-1}\delta_n c_n)^{(\phi_0 t c_n)^{-1}} \quad \text{for } n \geq 2 \end{aligned}$$

Proof: Follows by substituting $g = \mathrm{id}$ into proposition 2.1.4 and by definition of a derivation. □

Most of the derivations and homotopies we meet will be of a special kind, satisfying certain 'side-conditions'.

Proposition 2.1.8 *Let f be an endomorphism of a crossed complex C and h a homotopy $f \simeq \mathrm{id}_C$ corresponding to a derivation ϕ. Suppose further that $\phi_1\phi_0 c_0 = e_{c_0}$ and $\phi_{n+1}\phi_n c_n = e_{tc_n}$ for $n \geq 1$. Then*

$$\begin{aligned} f\phi_0 c_0 &= \phi_0 f c_0 = \phi_0 s \phi_0 c_0 \\ \text{and } f\phi_n c_n &= \phi_n f c_n = (\phi_n c_n \cdot \phi_n \delta_{n+1}\phi_n c_n)^{(\phi_0 t c_n)^{-1}} \quad \text{for } n \geq 1 \end{aligned}$$

Thus if any one of

1. $f\phi_0 c_0 = e_{fc_0}$ *and* $f\phi_n c_n = e_{tfc_n}$ *for* $n \geq 1$
2. $\phi_0 f c_0 = e_{fc_0}$ *and* $\phi_n f c_n = e_{tfc_n}$ *for* $n \geq 1$
3. $\phi_0 s \phi_0 c_0 = e_{fc_0}$ *and* $\phi_n \delta_{n+1}\phi_n c_n = (\phi_n c_n)^{-1}$ *for* $n \geq 1$

hold, then all three hold, and furthermore f is idempotent.

Proof: From the formulæ for f in corollary 2.1.7 we get

$$\begin{aligned}
f\phi_0 c_0 &= \phi_0 s\phi_0 c_0 \cdot \phi_0 c_0 \cdot \delta_2\phi_1\phi_0 c_1 \cdot (\phi_0 c_0)^{-1} \\
f\phi_n c_n &= (\phi_n c_n \cdot \delta_{n+2}\phi_{n+1}\phi_n c_n \cdot \phi_n\delta_{n+1}\phi_n c_n)^{(\phi_0 t c_n)^{-1}} \\
\phi_0 f c_0 &= \phi_0 s\phi_0 c_0 \\
\phi_1 f c_1 &= (\phi_1\phi_0 s c_1)^{c_1\cdot\delta_2\phi_1 c_1\cdot(\phi_0 t c_1)^{-1}} \cdot (\phi_1 c_1)^{\delta_2\phi_1 c_1\cdot(\phi_0 t c_1)^{-1}} \\
&\quad \cdot (\phi_1\delta_2\phi_1 c_1)^{(\phi_0 t c_1)^{-1}} \cdot ((\phi_1\phi_0 t c_1)^{-1})^{(\phi_0 t c_1)^{-1}} \\
\phi_n f c_n &= (\phi_n c_n \cdot \phi_n\delta_{n+1}\phi_n c_n \cdot \phi_n\phi_{n-1}\delta_n c_n)^{(\phi_0 t c_n)^{-1}}
\end{aligned}$$

Since the ϕ^2 terms disappear we get the first four equalities as required, and from these the equivalence of the three conditions is clear. Under such conditions $f^2 = f$ follows by some further routine manipulation of the formulæ of the corollary. □

Definition 2.1.9 *A* splitting homotopy *is a homotopy h: $f \simeq \mathrm{id}$ for which the associated derivation ϕ satisfies*

$$\begin{aligned}
\phi_1\phi_0 c_0 &= e_{c_0} \quad \textit{and} \quad & \phi_{n+1}\phi_n c_n &= e_{tc_n} \ \textit{for } n \geq 1 \\
\phi_0 s\phi_0 c_0 &= e_{fc_0} \quad \textit{and} \quad & \phi_n\delta_{n+1}\phi_n c_n &= (\phi_n c_n)^{-1} \ \textit{for } n \geq 1
\end{aligned}$$

As a consequence of proposition 2.1.8, the additional relations ${}^f h = 0_f$, $h^f = 0_f$ and $f \cdot f = f$ hold automatically for a splitting homotopy.

Proposition 2.1.10 *Suppose h is a homotopy $f \simeq \mathrm{id}$ which satisfies ${}^f h = 0_f$ and $h^f = 0_f$. Then the corresponding derivation ϕ satisfies*

$$\begin{aligned}
(\phi_1\delta_2\phi_1 c_1)^{-1} &= \phi_1 c_1 \cdot \delta_3\phi_2\phi_1 c_1 &&= (\phi_1\phi_0 t c_1)^{-1} \cdot (\phi_1\phi_0 s c_1)^{c_1\cdot\delta_2\phi_1 c_1} \cdot \phi_1 c_1 \\
(\phi_n\delta_{n+1}\phi_n c_n)^{-1} &= \phi_n c_n \cdot \delta_{n+2}\phi_{n+1}\phi_n c_n &&= \phi_n\phi_{n-1}\delta_n c_n \cdot \phi_n c_n \quad \textit{for } n \geq 2
\end{aligned}$$

Furthermore, the degree one map ϕ' defined by

$$\begin{aligned}
\phi'_0(c_0) &= \phi_0(c_0) \\
\phi'_n(c_n) &= (\phi_n\delta_{n+1}\phi_n c_n)^{-1} \quad \textit{for } n \geq 1
\end{aligned}$$

is a derivation corresponding to a splitting homotopy h': $f \simeq \mathrm{id}$.

Proof: The equalities of the first part follow from the formulæ in the proof of proposition 2.1.8 and the triviality of $f\phi$ and ϕf. The functions ϕ' clearly define a derivation $f' \simeq \mathrm{id}$, where f' is given by

$$\begin{aligned}
f' c_0 &= s\phi_0 c_0 \\
f' c_1 &= \phi_0 s c_1 \cdot c_1 \cdot (\delta_2\phi_1\delta_2\phi_1 c_1)^{-1} \cdot (\phi_0 t c_1)^{-1} \\
f' c_n &= (c_n \cdot (\delta_{n+1}\phi_n\delta_{n+1}\phi_n c_n \cdot \phi_{n-1}\delta_n\phi_{n-1}\delta_n c_n)^{-1})^{(\phi_0 t c_n)^{-1}}
\end{aligned}$$

But $(\delta_{n+1}\phi_n\delta_{n+1}\phi_n c_n)^{-1} = \delta_{n+1}\phi_n c_n$ and $(\phi_{n-1}\delta_n\phi_{n-1}\delta_n c_n)^{-1} = \phi_{n-1}\delta_n c_n$ follow from the equalities of the first part, so $f' = f$. Also $\phi' f$ is trivial, so to show that ϕ' gives a splitting homotopy it only remains to prove that ϕ'^2 vanishishes. We can write

$$\begin{aligned}
\phi'_1\phi'_0 c_0 &= (\phi_1\delta_2\phi_1(\phi_0 c_0))^{-1} \\
&= (\phi_1\phi_0 c_0)^{-1}\cdot(\phi_1\phi_0 s\phi_0 c_0)^{\phi_0 c_0\cdot\delta_2\phi_1\phi_0 c_0}\cdot\phi_1\phi_0 c_0 \\
\phi'_{n+1}\phi'_n c_n &= (\phi_{n+1}\delta_{n+2}\phi_{n+1}(\phi_n\delta_{n+1}\phi_n c_n)^{-1})^{-1} \\
&= \phi_{n+1}\phi_n\delta_{n+1}(\phi_n c_n\cdot\delta_{n+2}\phi_{n+1}\phi_n c_n)\cdot\phi_{n+1}(\phi_n\delta_{n+1}\phi_n c_n)^{-1}
\end{aligned}$$

and so the result follows by the vanishing of δ^2 and of $\phi_0 s\phi_0 c_0 = \phi_0 f c_0$. □

Theorem 2.1.11 *Suppose f is an idempotent endomorphism of a crossed complex C, and k a homotopy between f and the identity on C. Then there exists a splitting homotopy $h : f \simeq \mathrm{id}$.*

Proof: Consider the homotopies ${}^f k$, ${}^f k^f$ and k^f. Since f is idempotent and k is a homotopy $f \simeq \mathrm{id}$, these are all homotopies $f \simeq f$, and we can consider the homotopy $f \simeq \mathrm{id}$ given by the vertical composite

$$k' = \overline{{}^f k} \circ {}^f k^f \circ \overline{k^f} \circ k$$

We now have ${}^f k' = 0_f$ and $k'^f = 0_f$, and so the result follows from proposition 2.1.10. □

A homotopy equivalence $f\colon C \longleftrightarrow D\colon g$ in which $g\cdot f = \mathrm{id}_D$ is known as a *deformation retraction*. The endomorphism $f\cdot g$ of C is now idempotent, and so the homotopy $h : (f\cdot g) \simeq \mathrm{id}_C$ may be replaced by a splitting homotopy.

Definition 2.1.12 *A deformation retraction given by $f\colon C \longleftrightarrow D\colon g$ with $g\cdot f = \mathrm{id}_D$ and a homotopy $h : (f\cdot g) \simeq \mathrm{id}_C$ corresponding to a derivation ϕ is said to be a* strong deformation retraction (SDR) *if the following side-conditions are satisfied*

$$\begin{array}{rclcrcll}
\phi_1\phi_0 c_0 &=& e_{c_0} & \textit{and} & \phi_{n+1}\phi_n c_n &=& e_{tc_n} & \textit{for } n \geq 1 \\
\phi_0 g d_0 &=& e_{gd_0} & \textit{and} & \phi_n g d_n &=& e_{tgc_n} & \textit{for } n \geq 1 \\
f\phi_0 c_0 &=& e_{fc_0} & \textit{and} & f\phi_n c_n &=& e_{tfc_n} & \textit{for } n \geq 1 \\
\phi_0 s\phi_0 c_0 &=& e_{gfc_0} & \textit{and} & \phi_n\delta_{n+1}\phi_n c_n &=& (\phi_n c_n)^{-1} & \textit{for } n \geq 1
\end{array}$$

We will write these as $h^2 = 0$, ${}^g h = 0$, $h^f = 0$ and $h\delta h = -h$ respectively.

Theorem 2.1.13 *Any deformation retraction may be replaced by a strong deformation retraction.* □

In the chain complex case, analogous side conditions on chain homotopies have been very useful in homological perturbation theory, and the result which corresponds to theorem 2.1.13 may be found in [30]. It is expected that there will also be a 'non-abelian' homological perturbation theory for crossed complexes.

2.2 Diagonal Approximation and Shuffles

2.2.1 The Artin-Mazur diagonal

We recall from [1] that the Artin-Mazur diagonal $\nabla(X)$ of a bisimplicial set X is defined as follows. Each set $\nabla(X)_n$ is given by the following subset of $\prod_{p+q=n} X_{p,q}$

$$\nabla(X)_n = \Big\{(x_0, x_1, \ldots, x_n) : \ x_i \in X_{i,n-i}, \ d_0^{\mathrm{v}} x_i = d_{i+1}^{\mathrm{h}} x_{i+1} \, (0 \leq i \leq n-1)\Big\}$$

where d_i^{h} and d_i^{v} are the horizontal and vertical face maps of X. Geometrically the elements of $X_{p,q}$ should be thought of as generalised prisms given by products of a p-simplex with a q-simplex, and the $(n+1)$-tuples which define elements of $\nabla(X)_n$ should be thought of as connected unions of these with the first vertical face of one prism identified with the last horizontal face of the next.

For $0 \leq i \leq n$ the faces and degeneracies of an element of $\nabla(X)_n$ are given by

$$\begin{array}{rcl} d_i(x_0, x_1, \ldots, x_n) & = & (d_i^{\mathrm{v}} x_0, d_{i-1}^{\mathrm{v}} x_1, \ldots, d_1^{\mathrm{v}} x_{i-1}, d_i^{\mathrm{h}} x_{i+1}, d_i^{\mathrm{h}} x_{i+2}, \ldots, d_i^{\mathrm{h}} x_n) \\ s_i(x_0, x_1, \ldots, x_n) & = & (s_i^{\mathrm{v}} x_0, s_{i-1}^{\mathrm{v}} x_1, \ldots, s_0^{\mathrm{v}} x_i, s_i^{\mathrm{h}} x_i, s_i^{\mathrm{h}} x_{i+1}, \ldots, s_i^{\mathrm{h}} x_n) \end{array}$$

where s_i^{h} and s_i^{v} are the horizontal and vertical degeneracy maps of X. That is, the ith face map acts on the $(n+1)$-tuple (x_k) by applying d_{i-k}^{v} to the components with $k < i$, applying d_i^{h} to the components with $k > i$, and deleting the ith component. Similarly the ith degeneracy repeats the ith component and acts via s_{i-k}^{v} or s_i^{h} on the components of the result.

In section 1.3.1 the fundamental crossed complex $\pi(K)$ of a simplicial set K was defined, and it was shown how this leads to a definition of the fundamental double crossed complex of a bisimplicial set. Thus we have the following diagram of categories and functors

$$\begin{array}{ccc} \mathbf{BiSimpSet} & \xrightarrow{\nabla} & \mathbf{SimpSet} \\ \Big\downarrow{\scriptstyle \pi^{(2)}} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbf{Crs}^{(2)} & \xrightarrow{\mathrm{Tot}} & \mathbf{Crs} \end{array}$$

where Tot is the total crossed complex functor.

In dimension n, generators of $\mathrm{Tot}\pi^{(2)}X$ are given by elements of $X_{p,q}$ where $p+q = n$, and generators of $\pi\nabla X$ are given by certain $(n+1)$-tuples of these. We can construct a natural transformation from $\pi\nabla$ to $\mathrm{Tot}\,\pi^{(2)}$, but this will not be an isomorphism in general. Intuitively, the comparison map $\pi\nabla X \to \mathrm{Tot}\,\pi^{(2)}X$ will send each $(n+1)$-tuple to the (non-abelian) sum of its components.

Proposition 2.2.1 *For X a bisimplicial set, there is a natural map*

$$\pi\nabla X \xrightarrow{\ \theta_X\ } \mathrm{Tot}\,\pi^{(2)}X$$

which is defined on the usual generators by

$$\begin{array}{rcl}(x_0) & \mapsto & x_0 \\ (x_0, x_1) & \mapsto & x_0 \cdot x_1 \\ (x_0, x_1, x_2) & \mapsto & x_1^{d_0^{\mathrm{h}} x_2} \cdot x_2 \cdot x_0^{d_1^{\mathrm{h}} x_2} \\ (x_0, x_1, \ldots, x_n) & \mapsto & \prod_{i=0}^{n} x_i^{y_i(x_n)}\end{array}$$

where $y_i(x) \in \mathrm{Tot}\,\pi^{(2)}(X)_1$ is given by $d_0^{\mathrm{h}} d_1^{\mathrm{h}} \ldots d_{i-1}^{\mathrm{h}} d_{i+1}^{\mathrm{h}} d_{i+2}^{\mathrm{h}} \ldots d_{n-1}^{\mathrm{h}} x \in X_{1,0}$ or by the identity at $d_0^{\mathrm{h}} d_1^{\mathrm{h}} \ldots d_{n-1}^{\mathrm{h}} x$ if $i = n$.

Proof: We need to check that θ_X is well-defined on $\pi\nabla X$, i.e. that θ_X respects the relations between the generators. In dimension one, $s\theta_1(x_0, x_1)$ and $\theta_0 s(x_0, x_1)$ are both given by $d_1^{\mathrm{v}} x_0$, and $t\theta_1(x_0, x_1)$ and $\theta_0 t(x_0, x_1)$ are given by $d_0^{\mathrm{h}} x_1$, and in dimensions ≥ 2 the y_i ensure that $t(x_i^{y_i(x_n)}) = t x_n = \theta_0 t(x_0, \ldots, x_n)$ for all i. Thus the products on the right hand side are defined and the functions respect the base points. Also θ_X maps degenerate generators to the appropriate identity elements in $\mathrm{Tot}\,\pi^{(2)}X$, since if $(x_0, \ldots, x_n) = s_i(y_0, \ldots, y_{n-1})$ then each x_k is $s_{i-k}^{\mathrm{v}} y_k$ or $s_i^{\mathrm{h}} y_k$ and gives an identity in $\pi^{(2)}X$.

For the boundary relations, $\delta_2\theta_2(x_0, x_1, x_2)$ is given by

$$\begin{array}{l}(d_0^{\mathrm{h}} x_2)^{-1} \cdot \delta_2 x_1 \cdot d_0^{\mathrm{h}} x_2 \ \cdot\ \delta_2 x_2 \ \cdot\ (d_1^{\mathrm{h}} x_2)^{-1} \cdot \delta_2 x_0 \cdot d_1^{\mathrm{h}} x_2 \ = \\ (d_0^{\mathrm{h}} x_2)^{-1} \cdot (d_0^{\mathrm{h}} x_1)^{-1} \cdot (d_1^{\mathrm{v}} x_1)^{-1} \cdot d_1^{\mathrm{h}} x_1 \cdot d_0^{\mathrm{v}} x_1 \cdot (d_2^{\mathrm{h}} x_2)^{-1} \cdot (d_0^{\mathrm{v}} x_0)^{-1} \cdot (d_2^{\mathrm{v}} x_0)^{-1} \cdot d_1^{\mathrm{v}} x_0 \cdot d_1^{\mathrm{h}} x_2\end{array}$$

But since $d_0^{\mathrm{v}} x_1 = d_2^{\mathrm{h}} x_2$ and $d_0^{\mathrm{v}} x_0 = d_1^{\mathrm{h}} x_1$ four of these terms cancel leaving

$$\theta_1(d_0^{\mathrm{h}} x_1, d_0^{\mathrm{h}} x_2)^{-1} \cdot \theta_1(d_2^{\mathrm{v}} x_0, d_1^{\mathrm{v}} x_1)^{-1} \cdot \theta_1(d_1^{\mathrm{v}} x_0, d_1^{\mathrm{h}} x_2)$$

which is just $\theta_1\delta_2(x_0, x_1, x_2)$.

For $n \geq 3$, $x \in X_0$, the groups $(\mathrm{Tot}\,\pi^{(2)}X)_n(x)$ are abelian. In $\mathrm{Tot}\,\pi^{(2)}X$ the boundary relations on generators $x \in X_{p,q}$, $p + q \geq 4$, may be written as

$$\delta_{p+q} x \ = \ \prod_{j=0}^{p} \left((d_j^{\mathrm{h}} x)^{(-1)^{j+1}}\right)^{z_j^{\mathrm{h}}(x)} \ \cdot\ \prod_{k=0}^{q} \left((d_k^{\mathrm{v}} x)^{(-1)^{p+k+1}}\right)^{z_k^{\mathrm{v}}(x)}$$

(or only one of these products if p or q is zero) where the $z(x)$ are identities unless $j = p$ or $k = q$ when they are given by the one-cells

$$z_p^{\mathrm{h}}(x) \ = \ {d_0^{\mathrm{h}}}^{p-1} {d_0^{\mathrm{v}}}^{q}(x) \qquad z_q^{\mathrm{v}}(x) \ = \ {d_0^{\mathrm{v}}}^{q-1} {d_0^{\mathrm{h}}}^{p}(x)$$

Thus $\delta_n\theta_n(x_0,\ldots,x_n)$ is given by

$$\prod_{i=0}^{n}\left(\prod_{j=0}^{i}((d^{\mathrm{h}}_j x_i)^{(-1)^{j+1}})^{z^{\mathrm{h}}_j(x_i)} \cdot \prod_{k=0}^{n-i}((d^{\mathrm{v}}_k x_i)^{(-1)^{i+k+1}})^{z^{\mathrm{v}}_k(x_i)}\right)^{y_i(x_n)}$$

Some of these terms cancel, since $d^{\mathrm{v}}_0 x_i = d^{\mathrm{h}}_{i+1}x_{i+1}$. Also since the groups are abelian we can rewrite $\prod_{i=0}^{n}\prod_{j=0}^{i-1}$ as $\prod_{j=0}^{n}\prod_{i=j+1}^{n}$, and $\prod_{i=0}^{n}\prod_{k=1}^{n-i}$ as $\prod_{j=0}^{n}\prod_{i=0}^{j-1}$ by putting $j=i+k$. Thus we obtain

$$\prod_{j=0}^{n}\left(\prod_{i=0}^{j-1}(d^{\mathrm{v}}_{j-i}x_i)^{z^{\mathrm{v}}_{j-i}(x_i)\cdot y_i(x_n)} \cdot \prod_{i=j+1}^{n}(d^{\mathrm{h}}_j x_i)^{z^{\mathrm{h}}_j(x_i)\cdot y_i(x_n)}\right)^{(-1)^{j+1}}$$

From the boundary relations in $\pi\nabla X$, we have

$$\begin{aligned}\theta_{n-1}d_j(x_0,\ldots,x_n) &= \prod_{i=0}^{j-1}(d^{\mathrm{v}}_{j-i}x_i)^{y_i(x_n)} \cdot \prod_{i=j+1}^{n}(d^{\mathrm{h}}_j x_i)^{y_i(x_n)} \quad (j\neq n)\\ \theta_{n-1}d_n(x_0,\ldots,x_n) &= \prod_{i=0}^{n-1}(d^{\mathrm{v}}_{n-i}x_i)^{y_i(d^{\mathrm{v}}_1 x_{n-1})}\end{aligned}$$

On comparing terms, we need to show that

$$(d^{\mathrm{v}}_{n-i}x_i)^{z^{\mathrm{v}}_{n-i}(x_i)\cdot y_i(x_n)} = (d^{\mathrm{v}}_{n-i}x_i)^{y_i(d^{\mathrm{v}}_1 x_{n-1})\cdot\theta_1 d_0^{n-1}(x_0,\ldots,x_n)}$$

for $0\le i\le n-1$. Noting that

$$\begin{aligned} z^{\mathrm{v}}_{n-i}(x_i) &= {d^{\mathrm{v}}_0}^{n-i-1}{d^{\mathrm{h}}_0}^{i}x_i = {d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-1}x_{n-1}\\ \text{and}\ \ \theta_1 d_0^{n-1}(x_0,\ldots,x_n) &= {d^{\mathrm{h}}_0}^{n-1}x_{n-1}\cdot {d^{\mathrm{h}}_0}^{n-1}x_n\end{aligned}$$

the result holds for $i=n-1$ since $y_i x_n = {d^{\mathrm{h}}_0}^{n-1}x_n$ and $y_i(d^{\mathrm{v}}_1 x_{n-1})$ disappears. Otherwise we must compare the terms

$$\begin{aligned} &{d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-1}x_{n-1}\cdot {d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-1}x_n\\ \text{and}\quad &{d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-2}d^{\mathrm{v}}_1 x_{n-1}\cdot {d^{\mathrm{h}}_0}^{n-1}x_{n-1}\cdot {d^{\mathrm{h}}_0}^{n-1}x_n\end{aligned}$$

The difference between these is precisely the boundary of the element w_i given by

$$\left({d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-2}x_{n-1}\right)^{{d^{\mathrm{h}}_0}^{n-1}x_n}\cdot {d^{\mathrm{h}}_0}^{i}{d^{\mathrm{h}}_{i+1}}^{n-i-2}x_n$$

in $\mathrm{Tot}\pi^{(2)}(X)_2$. Since $n\ge 4$, $\delta_2 w_i$ acts trivially on $d^{\mathrm{v}}_{n-i}x_i$ and we have $\delta_n\theta_n(x_0,\ldots,x_n) = \theta_{n-1}\delta_n(x_0,\ldots,x_n)$ as required.

It only remains to prove that $\delta_3\theta_3(x_0,x_1,x_2,x_3)=\theta_2\delta_3(x_0,x_1,x_2,x_3)$. The boundary relations in $\mathrm{Tot}\,\pi^{(2)}(X)_3$ are

$$\begin{aligned}\delta_{0,3}(x_0) &= d^{\mathrm{v}}_3 x_0^{{d^{\mathrm{v}}_0}^2 x_0}\cdot d^{\mathrm{v}}_1 x_0\cdot(d^{\mathrm{v}}_2 x_0)^{-1}\cdot(d^{\mathrm{v}}_0 x_0)^{-1}\\ \delta_{1,2}(x_1) &= d^{\mathrm{h}}_1 x_1^{{d^{\mathrm{v}}_0}^2 x_1}\cdot(d^{\mathrm{v}}_1 x_1)^{-1}\cdot(d^{\mathrm{h}}_0 x_1)^{-1}\cdot d^{\mathrm{v}}_2 x_1^{d^{\mathrm{h}}_0 d^{\mathrm{v}}_0 x_1}\cdot d^{\mathrm{v}}_0 x_1\\ \delta_{2,1}(x_2) &= (d^{\mathrm{h}}_2 x_2^{d^{\mathrm{h}}_0 d^{\mathrm{v}}_0 x_2})^{-1}\cdot(d^{\mathrm{h}}_0 x_2)^{-1}\cdot d^{\mathrm{v}}_1 x_2^{{d^{\mathrm{h}}_0}^2 x_2}\cdot d^{\mathrm{h}}_1 x_2\cdot(d^{\mathrm{v}}_0 x_2)^{-1}\\ \delta_{3,0}(x_3) &= d^{\mathrm{h}}_1 x_3\cdot(d^{\mathrm{h}}_2 x_3)^{-1}\cdot(d^{\mathrm{h}}_0 x_3)^{-1}\cdot d^{\mathrm{h}}_3 x_3^{{d^{\mathrm{h}}_0}^2 x_3}\end{aligned}$$

Using the relations $d_0^{\mathrm{v}} x_i = d_{i+1}^{\mathrm{h}} x_{i+1}$ together with $u^{-1} \cdot v \cdot u = v^{\delta u}$ and $\delta_3 w \cdot v = v \cdot \delta_3 w$ in dimension 2, we can write $\delta_3\theta(x_0, x_1, x_2, x_3)$ as

$$\begin{aligned} & d_1^{\mathrm{h}} x_3 \cdot \delta x_0^{y_0} \cdot (d_2^{\mathrm{h}} x_3)^{-1} \cdot \delta x_1^{y_1} \cdot (d_0^{\mathrm{h}} x_3)^{-1} \cdot \delta x_2^{y_2} \cdot d_3^{\mathrm{h}} x_3^{{d_0^{\mathrm{h}}}^2 x_3} \\ = \;& d_1^{\mathrm{h}} x_3 \cdot (d_3^{\mathrm{v}} x_0^{{d_0^{\mathrm{v}}}^2 x_0} \cdot d_1^{\mathrm{v}} x_0 \cdot (d_2^{\mathrm{v}} x_0)^{-1})^{y_0} \\ \cdot \;& (d_2^{\mathrm{h}} x_3)^{-1} \cdot ((d_1^{\mathrm{v}} x_1)^{-1} \cdot (d_0^{\mathrm{h}} x_1)^{-1} \cdot d_2^{\mathrm{v}} x_1^{d_0^{\mathrm{h}} d_0^{\mathrm{v}} x_1})^{y_1} \\ \cdot \;& (d_0^{\mathrm{h}} x_3)^{-1} \cdot ((d_0^{\mathrm{h}} x_2)^{-1} \cdot d_1^{\mathrm{v}} x_2^{{d_0^{\mathrm{h}}}^2 x_2} \cdot d_1^{\mathrm{h}} x_2)^{y_2} \end{aligned}$$

On permuting these terms cyclically and moving $d_3^{\mathrm{v}} x_0$ two terms to the left and $d_2^{\mathrm{v}} x_1$ two terms to the right, by adding the appropriate actions, we get

$$\begin{aligned} & \left(d_2^{\mathrm{v}} x_1^{d_0^{\mathrm{h}} d_1^{\mathrm{v}} x_2} \cdot d_1^{\mathrm{v}} x_2 \cdot d_3^{\mathrm{v}} x_0^{d_1^{\mathrm{h}} d_1^{\mathrm{v}} x_2} \right)^{{d_0^{\mathrm{h}}}^2 x_2 \cdot {d_0^{\mathrm{h}}}^2 x_3} \cdot d_1^{\mathrm{h}} x_2^{y_2} \cdot d_1^{\mathrm{h}} x_3 \cdot d_1^{\mathrm{v}} x_0^{y_0} \\ \cdot \;& (d_1^{\mathrm{v}} x_1^{y_1} \cdot d_2^{\mathrm{h}} x_3 \cdot d_2^{\mathrm{v}} x_0^{y_0})^{-1} \cdot (d_0^{\mathrm{h}} x_2^{y_2} \cdot d_0^{\mathrm{h}} x_3 \cdot d_0^{\mathrm{h}} x_1^{y_1})^{-1} \end{aligned}$$

which is precisely $\theta_2\delta_3(x_0, x_1, x_2, x_3)$. □

2.2.2 The Alexander-Whitney diagonal approximation

Suppose K, L are simplicial sets. In this section we define the natural comparison map

$$\pi(K \times L) \xrightarrow{a_{K,L}} \pi K \otimes \pi L$$

between the fundamental crossed complex of a cartesian product and the tensor product of the fundamental crossed complexes. This is a 'slightly non-abelian' version of the classical diagonal approximation map for chain complexes on a simplicial set [21].

In fact we will define $a_{K,L}$ via the natural transformation θ of the previous section. Suppose K, L are simplicial sets and X is the bisimplicial set $K \times^{(2)} L$. Then $\pi^{(2)} X$ is just $\pi K \otimes^{(2)} \pi L$ and $\operatorname{Tot} \pi^{(2)} X$ is $\pi K \otimes \pi L$. Thus θ_X gives a comparision map

$$\pi \nabla X \xrightarrow{\theta_X} \pi K \otimes \pi L$$

Proposition 2.2.2 *Suppose K, L are simplicial sets and $X = K \times^{(2)} L$ as above. Then the Artin-Mazur diagonal ∇X of X is naturally isomorphic to the diagonal of X, that is, to the cartesian product of K and L.*

Proof: Elements σ_n of ∇X are given by $(n+1)$-tuples of pairs $(k_i, l_{n-i})_{0 \le i \le n}$. Since these must satisfy $(k_i, d_0 l_{n-i}) = (d_{i+1} k_{i+1}, l_{n-i-1})$, σ_n is completely determined by the pair (k_n, l_n) of $K \times L$, and conversely any pair (k_n, l_n) gives an element $(d_{i+1}^{n-i} k_n, d_0^i l_n)_{0 \le i \le n}$ of ∇X. This correspondence clearly respects the face and degeneracy maps, and so we have the result. □

We thus have

Proposition 2.2.3 *For K, L simplicial sets, there is a natural comparison map*

$$\pi(K \times L) \xrightarrow{a_{K,L}} \pi K \otimes \pi L$$

defined by $\theta_{K \times^{(2)} L}$.

By the definition of θ in proposition 2.2.1 and the description of the isomorphism $K \times L \cong \nabla(K \times^{(2)} L)$ in the proposition above, the diagonal approximation map a may be given explicitly as follows:

Proposition 2.2.4 *Given simplicial sets K, L, the crossed complex homomorphism*

$$\pi(K \times L) \xrightarrow{a_{K,L}} \pi K \otimes \pi L$$

is given by the homomorphism which acts on the generators of $\pi(K \times L)$ by

$$\begin{aligned}
(x_0, y_0) &\mapsto x_0 \otimes y_0 \\
(x_1, y_1) &\mapsto d_1 x_1 \otimes y_1 \cdot x_1 \otimes d_0 y_1 \\
(x_2, y_2) &\mapsto (d_2 x_2 \otimes d_0 y_2)^{d_0 x_2 \otimes d_0^2 y_2} \cdot x_2 \otimes d_0^2 y_2 \cdot (d_1 d_2 x_2 \otimes y_2)^{d_1 x_2 \otimes d_0^2 y_2} \\
(x_n, y_n) &\mapsto \prod_{i=0}^{n} (d_{i+1}^{n-i} x_n \otimes d_0^i y_n)^{c_i(x_n) \otimes d_0^n y_n}
\end{aligned}$$

where $c_i(x)$ is given by the one-cell $d_0^i d_{i+1}^{n-i-1} x$ or by the identity at $d_0^n x$ if $i = n$.

The following proposition gives the associativity of a.

Proposition 2.2.5 *For simplicial sets K, L, M, the following diagram commutes.*

$$\begin{array}{ccc}
\pi(K \times L \times M) & \xrightarrow{a_{K \times L, M}} & \pi(K \times L) \otimes \pi M \\
\Big\downarrow{\scriptstyle a_{K, L \times M}} & & \Big\downarrow{\scriptstyle a_{K,L} \otimes \mathrm{id}} \\
\pi K \otimes \pi(L \times M) & \xrightarrow{\mathrm{id} \otimes a_{L,M}} & \pi K \otimes \pi L \otimes \pi M
\end{array}$$

Proof: It is only necessary to check the result on generators $w_n = (x_n, y_n, z_n) \in \pi(K \times L \times M)$. For $n = 0$ the result is clear. For $n = 1$ it holds since both

$$\begin{aligned}
& d_1 x_1 \otimes (d_1 y_1 \otimes z_1 \cdot y_1 \otimes d_0 z_1) \cdot x_1 \otimes d_0 y_1 \otimes d_0 z_1 \\
\text{and} \quad & d_1 x_1 \otimes d_1 y_1 \otimes z_1 \cdot (d_1 x_1 \otimes y_1 \cdot x_1 \otimes d_0 y_1) \otimes d_0 z_1
\end{aligned}$$

are equal to

$$d_1 x_1 \otimes d_1 y_1 \otimes z_1 \cdot d_1 x_1 \otimes y_1 \otimes d_0 z_1 \cdot x_1 \otimes d_0 y_1 \otimes d_0 z_1$$

For $n \geq 3$ consider

$$(\mathrm{id} \otimes a_{L,M})(a_{K,L\times M} w_n) = \prod_{i=0}^{n} (d_{i+1}^{n-i} x_n \otimes a(d_0^i y_n, d_0^i z_n))^{c_i(x_n)\otimes d_0^n y_n \otimes d_0^n z_n}$$

Consider the term for $n - i = 1$.

$$\begin{aligned}
&\Big(d_n x_n \otimes (d_1 d_0^{n-1} y_n \otimes d_0^{n-1} z_n \cdot d_0^{n-1} y_n \otimes d_0 d_0^{n-1} z_n)\Big)^{c_{n-1}(x_n)\otimes d_0^n y_n \otimes d_0^n z_n} \\
= \ & \Big((d_n x_n \otimes d_1 d_0^{n-1} y_n \otimes d_0^{n-1} z_n)^{d_0^{n-1} d_n x_n \otimes d_0^{n-1} y_n \otimes d_0 d_0^{n-1} z_n} \\
& \cdot \ d_n x_n \otimes d_0^{n-1} y_n \otimes d_0 d_0^{n-1} z_n\Big)^{c_{n-1}(x_n)\otimes d_0^n y_n \otimes d_0^n z_n}
\end{aligned}$$

The terms for all i can be put in this form, and the product may be written as

$$\prod_{i=0}^{n} \prod_{j=0}^{n-i} \Big(d_{i+1}^{n-i} x_n \otimes d_{j+1}^{n-i-j} d_0^i y_n \otimes d_0^j d_0^i z_n\Big)^{c_{i,j}}$$

where $c_{i,j} = d_0^i d_{i+1}^{n-i} x_n \otimes c_j(d_0^i y_n) \otimes d_0^{n-i} d_0^i z_n \cdot c_i(x_n) \otimes d_0^n y_n \otimes d_0^n z_n$. Similarly

$$(a_{K,L} \otimes \mathrm{id})(a_{K\times L,M} w_n) = \prod_{k=0}^{n} (a(d_{k+1}^{n-k} x_n, d_{k+1}^{n-k} y_n) \otimes d_0^k z_n)^{a(c_k(x_n,y_n))\otimes d_0^n z_n}$$

may be written as

$$\prod_{k=0}^{n} \prod_{i=0}^{k} \Big(d_{i+1}^{k-i} d_{k+1}^{n-k} x_n \otimes d_0^i d_{k+1}^{n-k} y_n \otimes d_0^k z_n\Big)^{c'_{k,i}}$$

where $c'_{k,i} = c_i(d_{k+1}^{n-k} x_n) \otimes d_0^k d_{k+1}^{n-k} y_n \otimes d_0^{n-k} d_0^k z_n \cdot a(c_k(x_n, y_n)) \otimes d_0^n z_n$. Putting $k = i + j$ we have $\prod_{i=0}^{n} \prod_{j=0}^{n-i} = \prod_{k=0}^{n} \prod_{i=0}^{k}$ and

$$d_{i+1}^{n-i} = d_{i+1}^{k-i} d_{k+1}^{n-k}, \qquad d_{j+1}^{n-i-j} d_0^i = d_0^i d_{k+1}^{n-k}, \qquad d_0^j d_0^i = d_0^k$$

and so it only remains to check that the actions of $c_{i,j}$ and $c'_{i+j,i}$ agree. But as usual $c'_{i+j,i} \cdot c_{i,j}^{-1}$ is a loop and must be δ_2 of some term generated by the $x' \otimes y' \otimes z'$ for x', y', z' faces of x_n, y_n, z_n. Thus $c'_{i+j,i} \cdot c_{i,j}^{-1}$ acts trivially, since $n \geq 3$, and the result follows.

For $n = 2$ we have

$$\begin{aligned}
&(\mathrm{id} \otimes a_{L,M})(a_{K,L\times M} w_2) \\
&= d_2 x_2 \otimes a(d_0 y_2, d_0 z_2)^{d_0 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \cdot x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2 \cdot d_1^2 x_2 \otimes a(y_2, z_2)^{d_1 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \\
&= \Big((d_2 x_2 \otimes d_0 d_2 y_2 \otimes d_0 z_2)^{d_0 d_2 x_2 \otimes d_0 y_2 \otimes d_0^2 z_2} \cdot d_2 x_2 \otimes d_0 y_2 \otimes d_0^2 z_2\Big)^{d_0 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \cdot x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2 \cdot \\
&\Big((d_1^2 x_2 \otimes d_2 y_2 \otimes d_0 z_2)^{d_1^2 x_2 \otimes d_0 y_2 \otimes d_0^2 z_2} \cdot d_1^2 x_2 \otimes y_2 \otimes d_0^2 z_2 \cdot (d_1^2 x_2 \otimes d_1^2 y_2 \otimes z_2)^{d_1^2 x_2 \otimes d_1 y_2 \otimes d_0^2 z_2}\Big)^{d_1 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2}
\end{aligned}$$

On moving the fourth term two terms to the left, using $u \cdot v = v \cdot u^{\delta_2 v}$, this gives

$$\begin{aligned}
&\Big(d_2 x_2 \otimes d_0 d_2 y_2 \otimes d_0 z_2 \cdot (d_1^2 x_2 \otimes d_2 y_2 \otimes d_0 z_2)^{d_2 x_2 \otimes d_0 d_2 y_2 \otimes d_0^2 z_2}\Big)^{d_0 d_2 x_2 \otimes d_0 y_2 \otimes d_0^2 z_2 \cdot d_0 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \\
\cdot \quad & (d_2 x_2 \otimes d_0 y_2 \otimes d_0^2 z_2)^{d_0 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \cdot x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2 \cdot (d_1^2 x_2 \otimes y_2 \otimes d_0^2 z_2)^{d_1 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \\
\cdot \quad & (d_1^2 x_2 \otimes d_1^2 y_2 \otimes z_2)^{d_1^2 x_2 \otimes d_1 y_2 \otimes d_0^2 z_2 \cdot d_1 x_2 \otimes d_0^2 y_2 \otimes d_0^2 z_2} \\
= \quad & (a(d_2 x_2, d_2 y_2) \otimes d_0 z_2)^{a(d_0 x_2, d_0 y_2) \otimes d_0^2 z_2} \cdot a(x_2, y_2) \otimes d_0^2 z_2 \cdot (d_1^2 x_2 \otimes d_1^2 y^2 \otimes z_2)^{a(d_1 x_2, d_1 y_2) \otimes d_0^2 z_2}
\end{aligned}$$

which is just $(a_{K,L} \otimes \mathrm{id})(a_{K\times L,M} w_2)$. □

2.2.3 Crossed differential graded algebras

In this section we will introduce an example application of the diagonal approximation map discussed above, and define the notions of crossed differential graded algebras and coalgebras, which are the translations of differential graded algebras and coalgebras from the chain complex to the crossed complex situation.

First we define the crossed complex version of the approximation to the diagonal, which is the natural transformation given by the composite homomorphisms

$$\begin{array}{ccc} \pi K & \dashrightarrow & \pi K \otimes \pi K \\ & {\scriptstyle \pi(d)} \searrow \quad \nearrow {\scriptstyle a_{K,K}} & \\ & \pi(K \times K) & \end{array}$$

for each simplicial set K.

From proposition 2.2.4 the approximation to the diagonal has the following explicit description.

Proposition 2.2.6 *Given a simplicial set K, the crossed complex approximation to the diagonal*

$$\pi K \longrightarrow \pi K \otimes \pi K$$

is given by the homomorphism which acts on the generators of πK by

$$\begin{aligned} x_0 &\mapsto x_0 \otimes x_0 \\ x_1 &\mapsto d_1 x_1 \otimes x_1 \cdot x_1 \otimes d_0 x_1 \\ x_2 &\mapsto (d_2 x_2 \otimes d_0 x_2)^{d_0 x_2 \otimes d_0^2 x_2} \cdot x_2 \otimes d_0^2 x_2 \cdot (d_1 d_2 x_2 \otimes x_2)^{d_1 x_2 \otimes d_0^2 x_2} \\ x_n &\mapsto \prod_{i=0}^{n} (d_{i+1}^{n-i} x_n \otimes d_0^i x_n)^{c_i(x_n) \otimes d_0^n x_n} \end{aligned}$$

where $c_i(x)$ is given by the one-cell $d_0^i d_{i+1}^{n-i-1} x$ or by the identity at $d_0^n x$ if $i = n$.

Definition 2.2.7 *A* crossed differential graded algebra *is a crossed complex C together with a homomorphism $C \otimes C \xrightarrow{m} C$, termed the* multiplication *map, which makes the associativity diagram*

$$\begin{array}{ccc} C \otimes C \otimes C & \xrightarrow{m \otimes \mathrm{id}} & C \otimes C \\ {\scriptstyle \mathrm{id} \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\ C \otimes C & \xrightarrow{m} & C \end{array}$$

commute. Dually, a crossed differential graded coalgebra *is a crossed complex C together with a homomorphism $C \xrightarrow{w} C \otimes C$, termed the* comultiplication *map, which makes the coassociativity diagram*

$$\begin{array}{ccc} C & \xrightarrow{w} & C \otimes C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes w} \\ C \otimes C & \xrightarrow{w \otimes \mathrm{id}} & C \otimes C \otimes C \end{array}$$

commute.

Our fundamental example of a crossed differential graded coalgebra will be the following. Suppose that K is a simplicial set. Then the approximation to the diagonal map $\pi K \longrightarrow \pi K \otimes \pi K$ is coassociative by proposition 2.2.5. Thus $\pi(K)$ has a crossed differential graded coalgebra structure. Also we have naturality of this construction in K and hence we have a functor from simplicial sets to the category of crossed differential graded coalgebras.

$$\mathbf{SimpSet} \longrightarrow \mathbf{CDGcA}$$

In particular, consider the case where K is the representable simplicial set $\triangle^n$. Then we have a crossed differential graded coalgebra $\pi[n]$ for each n, together with the coface and codegeneracy homomorphisms induced between these as n varies. In fact we have

Proposition 2.2.8 *The collection of crossed complexes $\pi[n]$ together with the homomorphisms*

$$\pi[n] \longrightarrow \pi[n] \otimes \pi[n]$$

and the coface and codegeneracy maps

$$\pi[n-1] \xrightarrow{\pi(d(i))} \pi[n] \qquad\qquad \pi[n+1] \xrightarrow{\pi(s(i))} \pi[n]$$

define a cosimplicial crossed differential graded coalgebra $\pi(\triangle^\bullet)$.

This idea may be used to give insight into a construction of Brown and Gilbert in [6]. In this work the notion of a braided regular crossed module is defined, and the category of such is shown to be equivalent to that of simplicial groups with Moore complex trivial above dimension two. A braided regular crossed module C may be thought of as a crossed differential graded algebra $m : C \otimes C \longrightarrow C$ such that C is trivial in dimensions ≥ 3 together with a unit $e : 0 \longrightarrow C$ such that $m_0 : C_0 \times C_0 \longrightarrow C_0$ gives C_0 a group structure. It is not pointed out, however, that the construction of a simplicial group from C may be regarded in the general context of the Eilenberg-Zilber theorem.

We will consider a more general situation, and show how to form a simplicial semi-group from an arbitrary crossed differential graded algebra. Consider the nerve functor from crossed complexes to simplicial sets, given by $(NC)_n = \mathbf{Crs}(\pi[n], C)$. Then an algebra structure on C together with the coalgebra structure on each $\pi[n]$ induce an associative multiplication structure on the nerve. Explicitly, we have

Proposition 2.2.9 *Suppose C is a crossed differential graded algebra. If f, g are n-simplices of NC given by homomorphisms $\pi[n] \longrightarrow C$, then define $f \cdot g$ by the convolution product:*

$$\begin{array}{ccc} \pi[n] & \xrightarrow{f \cdot g} & C \\ {\scriptstyle w}\downarrow & & \uparrow{\scriptstyle m} \\ \pi[n] \otimes \pi[n] & \xrightarrow[f \otimes g]{} & C \otimes C \end{array}$$

This gives a simplicial map

$$NC \times NC \longrightarrow NC$$

which is associative.

In the same way, any homomorphism of crossed complexes $C \otimes D \longrightarrow E$ will induce a simplicial map $NC \times ND \longrightarrow NE$ via the cosimplicial coalgebra $\pi(\triangle^\bullet)$ and the convolution product. In particular, considering the identity map on $C \otimes D$ leads to a natural comparison map

$$NC \times ND \longrightarrow N(C \otimes D)$$

We will return to this idea in section 4.1.

2.2.4 Shuffles and the Eilenberg-MacLane map

In this section we recall the notion of *shuffles* and hence define the natural maps

$$\pi K \otimes \pi L \xrightarrow{b_{K,L}} \pi(K \times L)$$

This was originally carried out in the chain complex situation by Eilenberg and MacLane in [20].

Let us write $\underline{k}$ for the set $\{0, 1, \ldots, k-1\}$, and i_0, i_1 for the functions

$$\begin{array}{ccc} \underline{q} & \xrightarrow{i_0} & \underline{p+q} \\ r & \longmapsto & p+r \end{array} \qquad \begin{array}{ccc} \underline{p} & \xrightarrow{i_1} & \underline{p+q} \\ r & \longmapsto & r \end{array}$$

for $p, q \geq 0$. Then a (p,q)-shuffle is any permutation σ of the set $\underline{p+q}$ such that the functions $\sigma_0 = i_0 \circ \sigma$ and $\sigma_1 = i_1 \circ \sigma$

$$\underline{q} \xrightarrow{i_0} \underline{p+q} \xrightarrow{\sigma} \underline{p+q} \qquad\qquad \underline{p} \xrightarrow{i_1} \underline{p+q} \xrightarrow{\sigma} \underline{p+q}$$

are both monotonic increasing. We write $\mathrm{Shuff}(p,q)$ for the set of such shuffles and

$$\mathrm{Shuff}(p,q) \xrightarrow{\mathrm{sg}} \{-1, 1\}$$

for the function which gives the signature of each permutation σ.

Consider the representable simplicial sets $\triangle^p$ and $\triangle^q$, and write x_p and y_q for the top-dimensional non-degenerate simplices of each. Their cartesian product $\triangle^p \times \triangle^q$ has no non-degenerate simplices in dimensions $\geq p+q+1$, and in dimension $p+q$ there is a non-degenerate simplex for each $\sigma \in \mathrm{Shuff}(p,q)$ given by $(s_{\sigma_0} x_p, s_{\sigma_1} y_q)$, where the maps s_{σ_0} and s_{σ_1} are composites of degeneracy maps as follows:

$$s_{\sigma_0} \;=\; s_{\sigma(p+q-1)} s_{\sigma(p+q-2)} \ldots s_{\sigma(p)} \qquad\qquad s_{\sigma_1} \;=\; s_{\sigma(p-1)} s_{\sigma(p-2)} \ldots s_{\sigma(0)}$$

Proposition 2.2.10 *For simplicial sets K, L there is a natural homomorphism*

$$\pi K \otimes \pi L \xrightarrow{b_{K,L}} \pi(K \times L)$$

which is defined on the usual generators by

$$\begin{array}{rcl}
x_0 \otimes y_q & \mapsto & (s_0^q x_0, y_q) \\
x_p \otimes y_0 & \mapsto & (x_p, s_0^p y_0) \\
x_1 \otimes y_1 & \mapsto & (s_1 x_1, s_0 y_1) \cdot (s_0 x_1, s_1 y_1)^{-1} \\
x_p \otimes y_q & \mapsto & \displaystyle\prod_{\sigma \in \mathrm{Shuff}(p,q)} (s_{\sigma_0} x_p, s_{\sigma_1} y_q)^{\mathrm{sg}(\sigma)}
\end{array}$$

Proof: These composites are all defined, since $t(u) = (d_0^p x_p, d_0^q y_q)$ for each term u on the left hand side, and the functions respect the source and target maps. It is clear that they respect the degeneracies, for if $x_p = s_k x_{p-1}$, say, then x_{p-1} and y_q generate no non-degenerate cells in dimension $p+q$. Explicitly for each (p,q)-shuffle σ define a $(p-1,q)$-shuffle τ by

$$\tau_0(i) \;=\; \left\{ \begin{array}{ll} \sigma_0(i) & \text{for } \sigma_0(i) < \sigma_1(k) \\ \sigma_0(i) - 1 & \text{for } \sigma_0(i) > \sigma_1(k) \end{array} \right. \qquad\qquad \tau_1(i) \;=\; \left\{ \begin{array}{ll} \sigma_1(i) & \text{for } i < k \\ \sigma_1(i+1) - 1 & \text{for } i \geq k \end{array} \right.$$

Then $s_{\sigma_1(k)} s_{\tau_1} = s_{\sigma_1}$ and $s_{\sigma_1(k)} s_{\tau_0} = s_{\sigma_0} s_{\sigma_1(k)-j}$ where j is the number of values of σ_0 which are less than $\sigma_1(k)$. But there are k values of σ_1 and $\sigma_1(k)$ values of σ less than $\sigma_1(k)$, so $j = \sigma_1(k) - k$ and

$$(s_{\sigma_0}(s_k x_{p-1}), s_{\sigma_1}(y_q)) \;=\; s_{\sigma_1(k)}(s_{\tau_0} x_{p-1}, s_{\tau_1} y_q)$$

For the boundary relations, the case $p=0$ or $q=0$ is clear. In the case $p=q=1$, we have

$$\begin{aligned}\delta_2 b(x_1\otimes y_1) &= \delta_2(s_1x_1, s_0y_1)\cdot\delta_2(s_0x_1, s_1y_1)^{-1}\\ &= (d_0s_1x_1, d_0s_0y_1)^{-1}\cdot(d_2s_1x_1, d_2s_0y_1)^{-1}\cdot(d_1s_1x_1, d_1s_0y_1)\\ &\cdot\ (d_1s_0x_1, d_1s_1y_1)^{-1}\cdot(d_2s_0x_1, d_2s_1y_1)\cdot(d_0s_0x_1, d_0s_1y_1)\\ &= (s_0d_0x_1, y_1)^{-1}\cdot(x_1, s_0d_1y_1)^{-1}\cdot(s_0d_1x_1, y_1)\cdot(x_1, s_0d_0y_1) \quad = \quad b\delta_2(x_1\otimes y_1)\end{aligned}$$

In the general case, note that for $0\le i\le p+q$ any (p,q)-shuffle satisfies precisely one of the following

1. $\{i-1, i\}\subset\{-1\}\cup\mathrm{Im}(\sigma_1)\cup\{p+q\}$
2. $\{i-1, i\}\subset\{-1\}\cup\mathrm{Im}(\sigma_0)\cup\{p+q\}$
3. $i-1\in\mathrm{Im}(\sigma_1)$ and $i\in\mathrm{Im}(\sigma_0)$
4. $i-1\in\mathrm{Im}(\sigma_0)$ and $i\in\mathrm{Im}(\sigma_1)$

and we thus have a partition $\mathrm{Shuff}(p,q) \;=\; \bigcup_{r=1}^{4}\mathrm{S}_r^{(i)}(p,q)$.

There is a bijection $\gamma: S_3^{(i)}(p,q)\cong S_4^{(i)}(p,q)$ where $\gamma\sigma$ is given by the permutation

$$(\gamma\sigma)(j) \;=\; \begin{cases} i-1 & \text{if } \sigma(j)=i\\ i & \text{if } \sigma(j)=i-1\\ \sigma(j) & \text{otherwise}\end{cases}$$

and this satisfies $d_i(s_{\gamma(\sigma)_0}x_p, s_{\gamma(\sigma)_1}y_q) = d_i(s_{\sigma_0}x_p, s_{\sigma_1}y_q)$ and $\mathrm{sg}(\gamma\sigma) = -\mathrm{sg}(\sigma)$.

For $\sigma\in S_1^{(i)}(p,q)$ let $t(\sigma,i)$ be the integer such that $\sigma_1(t)=i$, with $t(\sigma,p+q)=p$, and let $\tau(\sigma,i)$ be the $(p-1,q)$-shuffle defined by

$$\tau(\sigma,i)_0(j) = \begin{cases}\sigma_0(j) & \text{if } \sigma_0(j)<i\\ \sigma_0(j)-1 & \text{if } \sigma_0(j)>i\end{cases} \qquad \tau(\sigma,i)_1(j) = \begin{cases}\sigma_1(j) & \text{if } \sigma_1(j)<i\\ \sigma_1(j+1)-1 & \text{if } \sigma_1(j)\ge i\end{cases}$$

Then $d_i(s_{\sigma_0}x_p, s_{\sigma_1}y_q) \;=\; \left(s_{\tau(\sigma,i)_0}(d_{t(\sigma,i)}x_p), s_{\tau(\sigma,i)_1}(y_q)\right)$ and $\mathrm{sg}(\tau(\sigma,i)) \;=\; (-1)^{i+t(\sigma,i)}\cdot\mathrm{sg}(\sigma)$. Also i and σ are completely determined by $t(\sigma,i)$ and $\tau(\sigma,i)$, and we have a bijection

$$\bigcup_{i=0}^{p+q}\left(S_1^{(i)}(p,q)\times\{i\}\right) \;\cong\; \mathrm{Shuff}(p-1,q)\times\{0,1,\ldots,p\}$$

A similar relationship holds between the $S_2^{(i)}(p,q)$ and $\mathrm{Shuff}(p,q-1)$. Combining all these results for $p+q\ge 4$ gives

$$\prod_{i=0}^{p+q}\ \prod_{\sigma\in\mathrm{Shuff}(p,q)}\left(d_i(s_{\sigma_0}x_p, s_{\sigma_1}y_q)^{(-1)^{i+1}\cdot\mathrm{sg}(\sigma)}\right)^{z_i(s_{\sigma_0}x_p, s_{\sigma_1}y_q)}$$

$$= \prod_{\tau \in \mathrm{Shuff}(p-1,q)} \prod_{t=0}^{p} \left((s_{\tau_0}(d_t x_p), s_{\tau_1}(y_q))^{(-1)^{t+1} \cdot \mathrm{sg}(\tau)} \right)^{b(z_t x_p \otimes d_0^q y_q)}$$

$$\cdot \prod_{\tau \in \mathrm{Shuff}(p,q-1)} \prod_{t=0}^{q} \left((s_{\tau_0}(x_p), s_{\tau_1}(d_t y_q))^{(-1)^{t+p+1} \cdot \mathrm{sg}(\tau)} \right)^{b(d_0^p x_p \otimes z_t y_q)}$$

which is precisely $\delta_{p+q} b(x_p \otimes y_q) = b\delta_{p+q}(x_p \otimes y_q)$.

There remain the non-abelian cases $\{p, q\} = \{1, 2\}$. We will verify the result for $p = 1$, $q = 2$; the other case is similar. Now $\delta_3 b(x_1 \otimes y_2)$ may be written as

$$\begin{aligned}
&\delta_3(s_1 s_0 x_1, s_2 y_2) \cdot \delta_3(s_2 s_0 x_1, s_1 y_2)^{-1} \cdot \delta_3(s_2 s_1 x_1, s_0 y_2) \\
&= d_0(s_1 s_0 x_1, s_2 y_2)^{-1} \cdot d_3(s_1 s_0 x_1, s_2 y_2)^{(x_1, d_0^2 y_2)} \cdot d_1(s_1 s_0 x_1, s_2 y_2) \cdot d_2(s_1 s_0 x_1, s_2 y_2)^{-1} \\
&\cdot\ d_0(s_2 s_0 x_1, s_1 y_2) \cdot d_2(s_2 s_0 x_1, s_1 y_2) \cdot d_1(s_2 s_0 x_1, s_1 y_2)^{-1} \cdot (d_3(s_2 s_0 x_1, s_1 y_2)^{-1})^{(d_0 x_1, d_0 y_2)} \\
&\cdot\ d_1(s_2 s_1 x_1, s_0 y_2) d_2(s_2 s_1 x_1, s_0 y_2)^{-1} d_0(s_2 s_1 x_1, s_0 y_2)^{-1} d_3(s_2 s_1 x_1, s_0 y_2)^{(d_0 x_1, d_0 y_2)}
\end{aligned}$$

The fifth term can be moved left four places and the eighth right four places, since the image of δ_3 is central, and some cancelation now occurs.

$$\begin{aligned}
&= (s_1 x_1, s_0 d_0 y_2) \cdot (s_0 x_1, s_1 d_0 y_2)^{-1} \cdot (s_0^2 d_1 x_1, y_2)^{(x_1, d_0^2 y_2)} \cdot (s_0 x_1, s_1 d_1 y_2) \\
&\cdot\ (s_1 x_1, s_0 d_1 y_2)^{-1} \cdot (s_0^2 d_0 x_1, y_2)^{-1} \left((s_1 x_1, s_0 d_2 y_2) \cdot (s_0 x_1, s_1 d_2 y_2)^{-1} \right)^{(d_0 x_1, d_0 y_2)} \\
&= b(x_1 \otimes d_0 y_2) \cdot b(d_1 x_1 \otimes y_2)^{(x_1, d_0^2 y_2)} \cdot b(x_1 \otimes d_1 y_2)^{-1} \\
&\qquad \cdot b(d_0 x_1 \otimes y_2)^{-1} \cdot b(x_1 \otimes d_2 y_2)^{(d_0 x_1, d_0 y_2)}
\end{aligned}$$

which is $b\delta_3(x_1 \otimes y_2)$. □

The following proposition gives the associativity of b.

Proposition 2.2.11 *For simplicial sets K, L, M, the following diagram commutes.*

$$\begin{array}{ccc}
\pi K \otimes \pi L \otimes \pi M & \xrightarrow{b_{K,L} \otimes \mathrm{id}} & \pi(K \times L) \otimes \pi M \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes b_{L,M}} & & \Big\downarrow{\scriptstyle b_{K \times L, M}} \\
\pi K \otimes \pi(L \times M) & \xrightarrow{b_{K, L \times M}} & \pi(K \times L \times M)
\end{array}$$

Proof: As usual the result needs only to be checked on generators $w_n = x_p \otimes y_q \otimes z_r$ for $x_p \in K$, $y_q \in L$, $z_r \in M$. Note that the result is straightforward if any of p, q or r are zero. Thus we may suppose $p + q + r \geq 3$, and so everything is abelian.

Consider the three sets $\underline{p}$, $\underline{q}$, $\underline{r}$ and the maps j_0, j_1, j_2 into $\underline{p+q+r}$ given by $k \mapsto k$, $k \mapsto p + k$, $k \mapsto p + q + k$ respectively. Then we define a (p, q, r)-shuffle to be a permutation σ of $\underline{p+q+r}$ such that each composite $j_\alpha \circ \sigma$ is monotonic increasing.

Consider also the map i_0 from $\underline{q+r}$ into $\underline{p+q+r}$ given by $k \mapsto p+k$. It is clear that the composite $i_2 \circ \sigma$ factors uniquely into a (q,r)-shuffle followed by a monotonic map from $\underline{q+r}$ into $\underline{p+q+r}$. We thus have a bijection

$$\begin{array}{ccc} \mathrm{Shuff}(p,q,r) & \xrightarrow{\cong} & \mathrm{Shuff}(p,q+r) \times \mathrm{Shuff}(q,r) \\ \sigma & \longmapsto & (\omega, \tau) \end{array}$$

where ω and τ are defined by the diagrams

$$\begin{array}{ccc} \underline{q+r} & \xrightarrow{i_0} & \underline{p+q+r} \\ \tau \downarrow & & \downarrow \sigma \\ \underline{q+r} & \xrightarrow[\omega_0]{} & \underline{p+q+r} \end{array} \qquad \begin{array}{ccc} \underline{p} & \xrightarrow{j_0} & \underline{p+q+r} \\ \| & & \downarrow \sigma \\ \underline{p} & \xrightarrow[\omega_1]{} & \underline{p+q+r} \end{array}$$

Note that

$$\begin{aligned} (s_{\omega_0} x_p, s_{\omega_1} s_{\tau_0} y_q, s_{\omega_1} s_{\tau_1} z_r) &= (s_{\sigma_0} x_p, s_{\sigma_1} y_q, s_{\sigma_2} z_r) \\ \text{and } \mathrm{sg}(\omega) \cdot \mathrm{sg}(\tau) &= \mathrm{sg}(\sigma) \end{aligned}$$

where the monotonic functions σ_0, σ_1, σ_2 are defined from σ in a similar manner to ω_0 above.

A similar relationship holds between $\mathrm{Shuff}(p,q,r)$ and $\mathrm{Shuff}(p+q,r) \times \mathrm{Shuff}(p,q)$. Combining these results gives

$$\begin{aligned} & \prod_{\omega \in \mathrm{Shuff}(p,q+r)} \prod_{\tau \in \mathrm{Shuff}(q,r)} (s_{\omega_0} x_p, s_{\omega_1} s_{\tau_0} y_q, s_{\omega_1} s_{\tau_1} z_r)^{\mathrm{sg}(\omega)\cdot \mathrm{sg}(\tau)} \\ = & \prod_{\omega \in \mathrm{Shuff}(p+q,r)} \prod_{\tau \in \mathrm{Shuff}(p,q)} (s_{\omega_0} s_{\tau_0} x_p, s_{\omega_0} s_{\tau_1} y_q, s_{\omega_1} z_r)^{\mathrm{sg}(\omega)\cdot \mathrm{sg}(\tau)} \end{aligned}$$

and we have associativity of b as required. □

As is well known in the chain complex case, the shuffle map b is a one-sided inverse to the diagonal approximation map a introduced in section 2.2.2.

Proposition 2.2.12 *Given simplicial sets K, L, the composite homomorphism*

$$\pi K \otimes \pi L \xrightarrow{b_{K,L}} \pi(K \times L) \xrightarrow{a_{K,L}} \pi K \otimes \pi L$$

is the identity.

Proof: Suppose $x_p \otimes y_q$ is a generator of $\pi K \otimes \pi L$ for $x_p \in K_p$, $y_q \in L_q$. Then $a(b(x_p \otimes y_q))$ is given by a composite of terms each of the form

$$\left((d_{i+1}^{p+q-i} s_{\sigma_0} x_p \otimes d_0^i s_{\sigma_1} y_q)^{\mathrm{sg}(\sigma)}\right)^{c_i(s_{\sigma_0} x_p) \otimes d_0^n y_q}$$

for $\sigma \in \mathrm{Shuff}(p,q)$ and $0 \leq i \leq p+q$. Now for $d_0^i s_{\sigma_1} y_q$ to be non-degenerate requires $\sigma(k) \leq i-1$ for $k \leq p-1$, and for $d_{i+1}^{p+q-i} s_{\sigma_0} x_p$ to be non-degenerate requires $\sigma(k) \geq i$ for $k \geq p$. Thus for the whole term to be non-degenerate it is necessary to have $\sigma = \mathrm{id}$ and $i = p$. In this case $\mathrm{sg}(\sigma) = 1$, $c_i(s_{\sigma_0} x_p)$ is degenerate and the term becomes $x_p \otimes y_q$. Thus $b \circ a = \mathrm{id}$. □

Furthermore the maps a and b satisfy a kind of commutativity or interchange relation as follows.

Proposition 2.2.13 *For simplicial sets K, L, M, the following diagrams commute.*

$$\begin{array}{ccc}
\pi(K \times L) \otimes \pi M & \xrightarrow{b_{K \times L, M}} & \pi(K \times L \times M) \\
\downarrow{\scriptstyle a_{K,L} \otimes \mathrm{id}} & & \downarrow{\scriptstyle a_{K, L \times M}} \\
\pi K \otimes \pi L \otimes \pi M & \xrightarrow{\mathrm{id} \otimes b_{L,M}} & \pi K \otimes \pi(L \times M)
\end{array}$$

$$\begin{array}{ccc}
\pi K \otimes \pi(L \times M) & \xrightarrow{\mathrm{id} \otimes a_{L,M}} & \pi K \otimes \pi L \otimes \pi M \\
\downarrow{\scriptstyle b_{K, L \times M}} & & \downarrow{\scriptstyle b_{K,L} \otimes \mathrm{id}} \\
\pi(K \times L \times M) & \xrightarrow{a_{K \times L, M}} & \pi(K \times L) \otimes \pi M
\end{array}$$

Proof: We will prove the first of these two results; the second is similar.

Let $w_n = (x_p, y_p) \otimes z_q$ be a generator of $\pi(K \times L) \otimes \pi M$ for $x_p \in K_p$, $y_p \in L_p$, $z_q \in M_q$, $n = p+q$. If p or q is zero then the result is straightforward. If $p = q = 1$ we have

$$\begin{aligned}
&a_{K, L \times M}(b_{K \times L, M} w_2) \\
&\quad = a_{K, L \times M}(s_1 x_1, s_1 y_1, s_0 z_1) \cdot a_{K, L \times M}(s_0 x_1, s_0 y_1, s_1 z_1)^{-1} \\
&\quad = x_1 \otimes (s_0 d_0 y_1, z_1) \cdot (d_1 x_1 \otimes (s_1 y_1, s_0 z_1))^{x_1 \otimes d_0(y_1, z_1)} \\
&\qquad \cdot \left(d_1 x_1 \otimes (s_0 y_1, s_1 z_1)^{-1}\right)^{x_1 \otimes d_0(y_1, z_1)}
\end{aligned}$$

neglecting degenerate terms. Also

$$\begin{aligned}
&(\mathrm{id} \otimes b)(a(x_1, y_1) \otimes z_1) \\
&\quad = (\mathrm{id} \otimes b)\left(x_1 \otimes d_0 y_1 \otimes z_1 \cdot (d_1 x_1 \otimes y_1 \otimes z_1)^{x_1 \otimes d_0 y_1 \otimes d_0 z_1}\right) \\
&\quad = x_1 \otimes (s_0 d_0 y_1, z_1) \cdot \left(d_1 x_1 \otimes ((s_1 y_1, s_0 z_1) \cdot (s_0 y_1, s_1 z_1)^{-1})\right)^{x_1 \otimes d_0(y_1, z_1)}
\end{aligned}$$

Thus we have the result for $p = q = 1$.

For $n \geq 3$ we have

$$a_{K,L\times M}\left(b_{K\times L,M}((x_p, y_p)\otimes z_q)\right) \;=\; a_{K,L\times M}\left(\prod_{\sigma\in\mathrm{Shuff}(p,q)} (s_{\sigma_0}x_p, s_{\sigma_0}y_p, s_{\sigma_1}z_q)^{\mathrm{sg}(\sigma)}\right)$$

$$= \prod_{\sigma\in\mathrm{Shuff}(p,q)} \prod_{i=0}^{p+q} \left((d_{i+1}^{p+q-i}s_{\sigma_0}x_p \otimes d_0^i(s_{\sigma_0}y_p, s_{\sigma_1}z_q))^{\mathrm{sg}(\sigma)}\right)^{c_i(s_{\sigma_0}x_p)\otimes(d_0^p y_p, d_0^q z_q)}$$

Now for $d_{i+1}^{p+q-i}s_{\sigma_0}x_p$ not to be degenerate requires $\sigma(k) \geq i$ for $k \geq p$. There are no (p,q)-shuffles which satisfy this condition for $i > p$, and for $i \leq p$ the (p,q)-shuffles which satisfy it are precisely those σ defined by

$$\sigma(k) \;=\; \begin{cases} k & \text{if } k < i \\ \tau(k-i)+i & \text{if } k \geq i \end{cases}$$

for each $(p-i, q)$-shuffle τ. Thus the above expression becomes

$$\prod_{i=0}^{p} \prod_{\tau\in\mathrm{Shuff}(p-i,q)} \left((d_{i+1}^{p-i}x_p \otimes (s_{\tau_0}d_0^i y_p, s_{\tau_1}z_q))^{\mathrm{sg}(\tau)}\right)^{c_i(x_p)\otimes(d_0^p y_p, d_0^q z_q)}$$

$$= (\mathrm{id}\otimes b_{L,M})\left(\left(\prod_{i=0}^{p}(d_{i+1}^{p-i}x_p, d_0^i y_p)^{c_i(x_p)\otimes d_0^p y_p}\right)\otimes z_q\right)$$

which is $(\mathrm{id}\otimes b_{L,M})\,(a(x_p, y_p)\otimes z_q))$ as required. □

2.3 The Eilenberg-Zilber Theorem

2.3.1 The Homotopy Equivalence

In this section we prove a version of the classical Eilenberg-Zilber theorem for the fundamental crossed complex functor

$$\mathbf{SimpSet} \xrightarrow{\;\pi\;} \mathbf{Crs}$$

Theorem 2.3.1 *For simplicial sets K and L the composite*

$$\pi(K\times L) \xrightarrow{\;a\;} \pi K\otimes\pi L \xrightarrow{\;b\;} \pi(K\times L)$$

is homotopic to the identity on $\pi(K\times L)$ via a splitting homotopy

$$\mathcal{I}\otimes\pi(K\times L) \xrightarrow{\;h_{K,L}\;} \pi(K\times L)$$

Thus $\pi K\otimes\pi L$ is a strong deformation retract of $\pi(K\times L)$.

Proof: We give the derivation ϕ corresponding to the homotopy $h : a \circ b \simeq \mathrm{id}_{\pi(K\times L)}$. For each $n \geq 0$, suppose $z_n = (x_n, y_n)$ is a generator in $\pi(K \times L)_n$, with corresponding $x_n \in K_n$ and $y_n \in L_n$. We will also write C for the crossed complex $\pi(K \times L)$ and f for the idempotent endomorphism $a \circ b$ of C.

In dimension zero, f is the identity function on C_0, so we define ϕ_0 by

$$\phi_0 z_0 \ = \ e_{z_0} \ \text{ in } C_1$$

In dimension one, f acts on the generators by

$$(x_1, y_1) \ \mapsto \ (s_0 d_1 x_1, y_1) \cdot (x_1, s_0 d_0 y_1)$$

and we define ϕ_1 on the generators by

$$\phi_1 z_1 \ = \ (s_0 x_1, s_1 y_1)^{-1}$$

Note that this satisfies $t\phi_1 z_1 = t z_1$ and that if z_1 is a 'degenerate' generator, $(x_1, y_1) = s_0(x_0, y_0)$ say, then $\phi_1 z_1$ is also degenerate. Thus we can extend ϕ_1 to a function $C_1 \to C_2$ inductively by

$$\begin{aligned} \phi_1 e_{z_0} &= e_{z_0} \\ \phi_1(w_1^{-1}) &= \left((\phi_1 w_1)^{-1}\right)^{w_1^{-1}} \\ \phi_1(z_1 \cdot w_1) &= (\phi_1 z_1)^{w_1} \cdot \phi_1 w_1 \end{aligned}$$

for any $w_1 \in C_1$. On the generators we have also

$$\begin{aligned} & z_1 \cdot \delta_2 \phi_1 z_1 \\ &= \ (x_1, y_1) \cdot (d_1 s_0 x_1, d_1 s_1 y_1)^{-1} \cdot (d_2 s_0 x_1, d_2 s_1 y_1) \cdot (d_0 s_0 x_1, d_0 s_1 y_1) \\ &= \ f_1 z_1 \end{aligned}$$

as required by corollary 2.1.7, with $\phi_0 = e$. This relation extends to all of C_1 since

$$\begin{aligned} z_1 \cdot w_1 \ \cdot \ \delta_2 \phi_1 (z_1 \cdot w_1) &= z_1 \cdot w_1 \cdot \delta_2\left((\phi_1 z_1)^{w_1}\right) \cdot \delta_2 \phi_1 w_1 \\ &= z_1 \cdot \delta_2 \phi_1 z_1 \ \cdot \ w_1 \cdot \delta_2 \phi_1 w_1 \end{aligned}$$

To define ϕ in dimensions ≥ 2 we can use the notion of simplicial and derived operators as in [20, 21]. Consider first a (finite, possibly zero) formal sum

$$F_p^q \ = \ \sum_{i \in I} r_i(\mu_i, \nu_i)$$

of distinct pairs (μ_i, ν_i) of monotonic functions $[p] \to [q]$, with integral coefficients r_i. We will call such a sum a *simplicial operator* of dimension (p, q), and say that it is *frontal* if $\mu_i(0) = \nu_i(0) = 0$ for all $i \in I$.

Clearly morphisms $\lambda : [r] \to [p]$ or $\rho : [q] \to [s]$ of Δ will act on such an F, by composition with the μ_i and ν_i and collecting together like terms, to produce formal sums λF or $F\rho$ respectively. The general composites $F_p^q G_q^r$ can also be defined, as can sums $F_p^q + H_p^q$. In fact the collection of all such simplicial operators forms the free ringoid (abelian-group enriched category) over the category $\Delta^{(2)}$, where $\Delta^{(2)}$ is the full subcategory of $\Delta \times \Delta$ on the objects of the form $([n],[n])$.

If each term (μ_i, ν_i) of F with $r_i \neq 0$ can be written as $(\lambda_i\sigma_i, \lambda_i\tau_i)$ for some $\lambda_i : [p] \to [p-1]$ then we say F is *degenerate*. We will write $F \equiv G$ if $F - G$ is degenerate, and say that F *preserves degeneracies* if the composite $F\rho$ is degenerate for each $\rho : [q] \to [q-1]$.

Suppose F is a simplicial operator of dimension (p, q) as above. Then we define the corresponding *derived* simplicial operator F' of dimension $(p+1, q+1)$ by

$$F' \;=\; \sum_{i \in I} r_i(\mu_i', \nu_i')$$

where the monotonic functions $\mu_i', \nu_i' : [p+1] \to [q+1]$ are given by

$$\mu_i'(0) = 0, \quad \mu_i'(n+1) = \mu_i(n) + 1$$
$$\nu_i'(0) = 0, \quad \nu_i'(n+1) = \nu_i(n) + 1$$

Clearly taking derived operators respects the addition and composition structure. All derived operators are frontal, and if an operator is degenerate then so is the corresponding derived operator. The most important property of taking derived operators is the behaviour on composing with the zeroth coface and codegeneracy maps:

Lemma 2.3.2 *Suppose F is a simplicial operator and F' the corresponding derived operator. Then*

1. $d(0)\, F' \;=\; F\, d(0)$
2. *If F is frontal, then $s(0)\, F \;=\; F'\, s(0)$.*

Now consider the simplicial operators ∂_p of dimension $(p-1, p)$ defined by

$$\partial_p \;=\; \sum_{i=0}^{p} (-1)^{i+1}\, (d(i), d(i))$$

and note the relation

$$\partial_p + \partial_{p-1}' + (d(0), d(0)) \;=\; 0$$

Proposition 2.3.3 *Suppose that $n_0 \geq 1$ and $(F_n)_{n \geq n_0}$ is a sequence of operators of dimensions (n, n) which satisfy*

$$\partial_n F_n \;= F_{n-1}\partial_n$$

and Φ_{n_0-1}, Φ_{n_0} *are frontal operators of dimensions* (n_0, n_0-1), (n_0+1, n_0) *respectively which satisfy*

$$\Phi_{n_0} + \Phi'_{n_0-1} + F'_{n_0}s(0) \ = \ 0$$

and

$$F_{n_0} \ \equiv \ \mathrm{id}_{n_0} + \partial_{n_0+1}\Phi_{n_0} + \Phi_{n_0-1}\partial_{n_0}$$

Then the operators Φ_n *of dimensions* $(n+1, n)$ *defined inductively by*

$$\Phi_n + \Phi'_{n-1} + F'_n s(0) \ = \ 0$$

are all frontal and satisfy

$$F_n \ \equiv \ \mathrm{id}_n + \partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n$$

for $n \geq n_0$. *Furthermore if* Φ_{n_0-1} *and all the* F_n *preserve degeneracies, then so do all the* Φ_n.

Proof: Since s_0 is frontal and any derived operator or sum or composite of frontal operators is frontal, it follows from their definition that the Φ_n are all frontal. Thus by the lemma and the relations $\partial + \partial' + d(0) = 0$ and $\Phi + \Phi' + F's(0) = 0$ we may rewrite $\partial\Phi$ and $\Phi\partial$ as follows:

$$\begin{aligned}
\partial_{n+1}\Phi_n &= (d(0) + \partial'_n)\Phi'_{n-1} - \partial_{n+1}F'_n s(0) \\
&= \Phi_{n-1}d(0) + \partial'_n\Phi'_{n-1} - \partial_{n+1}F'_n s(0) \\
\Phi_{n-1}\partial_n &= -\Phi_{n-1}d(0) + (\Phi'_{n-2} + F'_{n-1}s(0))\partial'_{n-1} \\
&= -\Phi_{n-1}d(0) + \Phi'_{n-2}\partial'_{n-1} + F'_{n-1}\partial''_{n-1}s(0)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&-\partial_{n+1}F'_n s(0) + F'_{n-1}\partial''_{n-1}s(0) \\
&= (d(0) + \partial'_n)F'_n s(0) - F'_{n-1}(\partial_n + d(0))'s(0) \\
&= d(0)s(0)F_n + \partial'_n F'_n s(0) - F'_{n-1}\partial'_n s_0 - F'_{n-1}d(1)s(0) \\
&= F_n - F'_{n-1}
\end{aligned}$$

and so

$$\partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n \ = \ \partial'_n\Phi'_{n-1} + \Phi'_{n-2}\partial'_{n-1} + F_n - F'_{n-1}$$

Thus taking the derivative of the relation

$$F_{n-1} \ \equiv \ \mathrm{id}_{n-1} + \partial_n\Phi_{n-1} + \Phi_{n-2}\partial_{n-1}$$

implies the relation for F_n, and so it holds for all $n \geq n_0$ by induction.

For the last part, suppose inductively that Φ_{n-1} preserves degeneracies. Then $\Phi'_{n-1}s(i)$ can be written as $(\Phi_{n-1}s(i-1))'$ if $i \geq 1$ or as $s(0)\Phi_{n-1}$ if $i = 0$ since

Φ_{n-1} is frontal. Also for all i we have $F'_n s(0)s(i) = F'_n s(i+1)s(0) = (F_n s(i))' s(0)$. Therefore $\Phi_n = -\Phi'_{n-1} - F'_n s(0)$ preserves degeneracies also. □

By regarding the monotonic functions μ_i, ν_i as corresponding to functions $\mu_i^* : K_q \to K_p$, $\nu_i^* : L_q \to L_p$ respectively, we note that in sufficiently high dimensions a simplicial operator defines a map on C.

Proposition 2.3.4 *Suppose $F = \sum_I r_i(\mu_i, \nu_i)$ is a simplicial operator of dimension $(p \geq 3, q \geq 2)$ which preserves degeneracies. Then F induces a homomorphism of groupoids-with-C_1-action*

$$C_q \xrightarrow{\overline{F}} C_p$$

which is given on the generators by

$$\overline{F}(x_q, y_q) \;=\; \prod_{i \in I} ((\mu_i^*(x_q), \nu_i^*(y_q))^{r_i})^{(\sigma_i^*(x_q), \tau_i^*(y_q))}$$

where the monotonic functions $\sigma_i, \tau_i : [1] \to [q]$ are given by

$$\sigma_i(0) = \mu_i(p), \quad \sigma_i(1) = q$$
$$\tau_i(0) = \nu_i(p), \quad \tau_i(1) = q$$

If G is another simplicial operator of dimension $(r \geq 3, s \geq 2)$ which preserves degeneracies and $\overline{G}$ the corresponding homomorphism, then the following relations hold for $w_n \in C_n$

1. *If $p = r$ and $q = s$ then* $\overline{F \pm G}(w_r) \;=\; \overline{F}(w_r) \cdot \left(\overline{G}(w_r)\right)^{\pm 1}$,

2. *If $q = r$ then* $\overline{F}\left(\overline{G}(w_s)\right) \;=\; \overline{FG}(w_s)$,

3. $\overline{F\partial_p}(w_p) \;=\; \overline{F}(\delta_p w_p)$.

Proof: Note that C_q and C_p are both totally disconnected and that each group $C_p(z_0)$ is abelian. Since F preserves degeneracies and the (σ_i, τ_i) ensure that $t(u) = t(z_q)$ for each term u in the product, $\overline{F}$ is well defined on the generators and may be extended to a C_1-homomorphism on C_q inductively by

$$\begin{aligned} \overline{F}(e_{z_0}) &= e_{z_0} \\ \overline{F}\left(w_q^{w_1}\right) &= \left(\overline{F} w_q\right)^{w_1} \\ \overline{F}(z_q \cdot w_q) &= \overline{F} z_q \cdot \overline{F} w_q \end{aligned}$$

for any $w_1 \in C_1$, $w_q \in C_q$.

The first relation follows trivially from the above. In the second, corresponding elements of C_p on the left and right hand sides are given by

$$(\mu_i^*\mu_j^* x_s, \nu_i^*\nu_j^* y_s)^{d_1(\sigma_{i,j}^* x_q, \tau_{i,j}^* y_q)}$$
$$\text{and} \quad (\mu_i^*\mu_j^* x_s, \nu_i^*\nu_j^* y_s)^{d_0(\sigma_{i,j}^* x_q, \tau_{i,j}^* y_q)\cdot d_2(\sigma_{i,j}^* x_q, \tau_{i,j}^* y_q)}$$

for $i \in I_F$, $j \in I_G$, where the monotonic functions $\sigma_{i,j}, \tau_{i,j} : [2] \to [s]$ are given by

$$\sigma_{i,j}(0) = \mu_i\mu_j(p), \quad \sigma_{i,j}(1) = \mu_i(q), \quad \sigma_{i,j}(2) = s$$
$$\tau_{i,j}(0) = \nu_i\nu_j(p), \quad \tau_{i,j}(1) = \nu_i(q), \quad \tau_{i,j}(2) = s$$

But $\delta_2 C_2$ acts trivially on C_p, so the above elements are equal.

For the third relation, note that for $p \geq 4$ we have $\overline{\partial_p}(w_p) = \delta_p w_p$ and the result follows from the previous relation. In fact the result for $p = 3$ holds by the same reasoning, since it is only the intermediate values that lie in the non-abelian C_2. □

Since $\overline{F} = \overline{G}$ for $F \equiv G$, we have

Corollary 2.3.5 *Suppose $n_0 = 1$ and Φ_0, Φ_1 and $(F_n)_{n\geq 1}$ are simplicial operators as in proposition 2.3.3, with Φ_0 and the F_n preserving degeneracies. Then the resulting homomorphisms $f_n = \overline{F_n}$ for $n \geq 3$ and $\phi_n = \overline{\Phi_n}$ for $n \geq 2$ satisfy*

$$\begin{aligned} t(\phi_n w_n) &= t(w_n) \\ \phi_n({w_n}^{w_1}) &= (\phi_n w_n)^{w_1} \\ \phi_n(z_n \cdot w_n) &= \phi_n z_n \cdot \phi_n w_n \\ \textit{and} \;\; f_n w_n &= w_n \cdot \delta_{n+1}\phi_n w_n \cdot \phi_{n-1}\delta_n w_n \end{aligned}$$

Returning at last to the proof of theorem 2.3.1, define operators Φ_0 of dimension $(1,0)$, Φ_1 of dimension $(2,1)$, and $(F_n)_{n\geq 1}$ of dimensions (n,n) as follows:

$$\begin{aligned} \Phi_0 &= (s(0), s(0)) \\ \Phi_1 &= -(s(0), s(1)) - (s(1), s(1)) - (s(0)^2 d(1), s(0)) \\ F_n &= \sum_{i=0}^{n} \sum_{\sigma\in \mathrm{Shuff}(i,n-i)} \mathrm{sg}(\sigma)\left(s(\sigma_0)d(i+1)^{n-i}, s(\sigma_1)d(0)^i\right) \end{aligned}$$

Clearly $\overline{F_n} = f_n$ for $n \geq 3$, Φ_0 preserves degeneracies, and $\Phi_1 + \Phi_0' + F_1' s(0) = 0$. The relation $F_1 \equiv \mathrm{id} + \partial_2\Phi_1 + \Phi_0\partial_1$ holds as for f_1 and ϕ_1 earlier. The proof that the F_n preserve degeneracies and satisfy $\partial_n F_n = F_{n-1}\partial_n$ is the same as that for the homomorphisms a and b. Thus we have ϕ_n for $n \geq 2$ from the corollary above with all the required relations for a derivation $\phi : f \simeq \mathrm{id}$ except

$$f_2 w_2 = w_2 \cdot \delta_3\phi_2 w_2 \cdot \phi_1\delta_2 w_2$$

for $w_2 \in C_2$. In fact it is only necessary to check this relation on the generators since

$$\begin{aligned}
&(z_2 \cdot w_2) \cdot \delta_3\phi_2(z_2 \cdot w_2) \cdot \phi_1\delta_2(z_2 \cdot w_2) \\
&\quad = \quad z_2 \cdot \delta_3\phi_2 z_2 \cdot w_2 \cdot (\phi_1\delta_2 z_2)^{\delta_2 w_2} \cdot \delta_3\phi_2 w_2 \cdot \phi_1\delta_2 w_2 \\
&\quad = \quad z_2 \cdot \delta_3\phi_2 z_2 \cdot \phi_1\delta_2 z_2 \cdot w_2 \cdot \delta_3\phi_2 w_2 \cdot \phi_1\delta_2 w_2
\end{aligned}$$

since $\delta_2 w_2$ acts as conjugation and $\delta_3 C_3$ is central in C_2, and

$$\begin{aligned}
&{w_2}^{w_1} \cdot \delta_3\phi_2({w_2}^{w_1}) \cdot \phi_1\delta_2({w_2}^{w_1}) \\
&\quad = \quad {w_2}^{w_1} \cdot (\delta_3\phi_2 w_2)^{w_1} \cdot \phi_1(w_1^{-1} \cdot \delta_2 w_2 \cdot w_1) \\
&\quad = \quad {w_2}^{w_1} \cdot (\delta_3\phi_2 w_2)^{w_1} \cdot ((\phi_1 w_1)^{-1})^{\delta_2({w_2}^{w_1})} \cdot (\phi_1\delta_2 w_2)^{w_1} \cdot \phi_1 w_1 \\
&\quad = \quad (\phi_1 w_1)^{-1} \cdot (w_2 \cdot \delta_3\phi_2 w_2 \cdot \phi_1\delta_2 w_2)^{w_1} \cdot \phi_1 w_1 \\
&\quad = \quad (w_2 \cdot \delta_3\phi_2 w_2 \cdot \phi_1\delta_2 w_2)^{f_1(w_1)}
\end{aligned}$$

since $f_1(w_1) = w_1 \cdot \delta_2\phi_1 w_1$. So now consider

$$\begin{aligned}
&\delta_3\phi_2(z_2) \quad = \quad \delta_3\overline{(-F_2's(0) - \Phi_1')}(z_2) \\
&= \delta_3\left(\left((s_2s_0d_2x_2, s_1y_2)^{-1} \cdot (s_1s_0d_2x_2, s_2y_2)\right)^{(d_0x_2, s_0d_0^2y_2)} \cdot (s_0x_2, s_2s_1d_1y_2)^{-1} \cdot (s_1x_2, s_2y_2)\right) \\
&= \left((s_1d_2x_2, y_2)^{-1} \cdot ((s_0d_2x_2, s_1d_2y_2)^{-1})^{(s_0d_0d_2x_2, d_0y_2)} \cdot (s_1d_2x_2, s_0d_0y_2) \cdot (s_0d_2x_2, y_2)\right)^{(d_0x_2, s_0d_0^2y_2)} \\
&\cdot \left((s_0d_2x_2, y_2)^{-1} \cdot (s_0d_2x_2, s_1d_0y_2)^{-1} \cdot (s_0^2d_1^2x_2, y_2)^{(d_2x_2, s_0d_0^2y_2)} \cdot (s_0d_2x_2, s_1d_1y_2)\right)^{(d_0x_2, s_0d_0^2y_2)} \\
&\cdot \left((s_0d_2x_2, s_1d_1y_2)^{-1}\right)^{(d_0x_2, s_0d_0^2y_2)} \cdot (x_2, s_0^2d_0^2y_2) \cdot (s_0d_1x_2, s_1d_1y_2) \cdot (x_2, s_1d_1y_2)^{-1} \\
&\cdot (x_2, s_1d_1y_2) \cdot (x_2, y_2)^{-1} \cdot (s_0d_0x_2, s_1d_0y_2)^{-1} \cdot (s_1d_2x_2, y_2)^{(d_0x_2, s_0d_0^2y_2)} \\
&= \left((s_1d_2x_2, s_0d_0y_2) \cdot (s_0d_2x_2, s_1d_0y_2)^{-1} \cdot (s_0^2d_1^2x_2, y_2)^{(d_2x_2, s_0d_0^2y_2)}\right)^{(d_0x_2, s_0d_0^2y_2)} \cdot (x_2, s_0^2d_0^2y_2) \\
&\cdot (s_0d_1x_2, s_1d_1y_2) \cdot (x_2, y_2)^{-1} \cdot (s_0d_0x_2, s_1d_0y_2)^{-1} \cdot ((s_0d_2x_2, s_1d_2y_2)^{-1})^{(s_0d_0d_2x_2, d_0y_2)\cdot(d_0x_2, s_0d_0^2y_2)}
\end{aligned}$$

on permuting the terms cyclically by two places and cancelling. Using $u \cdot v^{\delta_2 u} = v \cdot u$ twice more, to move the third term to the right one place and the last term to the left one place, and composing with

$$\begin{aligned}
z_2 \cdot \phi_1\delta_2 z_2 \quad &= \quad z_2 \cdot \phi_1(d_0z_2^{-1} \cdot d_2z_2^{-1} \cdot d_1z_2) \\
&= z_2 \cdot (s_0d_0x_2, s_1d_0y_2)^{\delta_2 z_2} \cdot (s_0d_2x_2, s_1d_2y_2)^{d_2z_2^{-1}\cdot d_1z_2} \cdot (s_0d_1x_2, s_1d_1y_2)^{-1} \\
&= (s_0d_0x_2, s_1d_0y_2) \cdot (s_0d_2x_2, s_1d_2y_2)^{d_0z_2} \cdot z_2 \cdot (s_0d_1x_2, s_1d_1y_2)^{-1}
\end{aligned}$$

leaves

$$\left((s_1d_2x_2, s_0d_0y_2) \cdot (s_0d_2x_2, s_1d_0y_2)^{-1}\right)^{(d_0x_2, s_0d_0^2y_2)} \cdot (x_2, s_0^2d_0^2y_2) \cdot (s_0^2d_1^2x_2, y_2)^{(d_1x_2, s_0d_0^2y_2)}$$

which is just f_2z_2.

Thus we have a derivation ϕ corresponding to homotopy $h : f \simeq \mathrm{id}$. By the work of section 2.1.2, h may be replaced by a splitting homotopy, although it may be checked that the definition of ϕ here is such that it already satisfies the necessary side conditions. □

Before we move on to the next section, there are four more commutativity relations that the above homotopy h satisfies with respect to a and b. Recall from [12] that the tensor product of crossed complexes is *symmetric*, where for crossed complexes C, D the homomorphism $s : C \otimes D \to D \otimes C$ is given on the generators by

$$\begin{array}{ccc} C \otimes D & \xrightarrow{s_{C,D}} & D \otimes C \\ c_p \otimes d_q & \longmapsto & (d_q \otimes c_p)^{(-1)^{pq}} \end{array}$$

for $c_p \in C_p$, $d_q \in D_q$.

Proposition 2.3.6 *For simplicial sets K, L, M the following diagrams commute*

$$\begin{array}{ccc} \mathcal{I} \otimes \pi(K \times L \times M) & \xrightarrow{h_{K,L\times M}} & \pi(K \times L \times M) \\ \downarrow{\scriptstyle \mathrm{id} \otimes a} & & \downarrow{\scriptstyle a} \\ \mathcal{I} \otimes \pi(K \times L) \otimes \pi M & \xrightarrow{h_{K,L} \otimes \mathrm{id}} & \pi(K \times L) \otimes \pi M \end{array}$$

$$\begin{array}{ccccc} \mathcal{I} \otimes \pi(K \times L \times M) & & \xrightarrow{h_{K\times L,M}} & & \pi(K \times L \times M) \\ \downarrow{\scriptstyle \mathrm{id} \otimes a} & & & & \downarrow{\scriptstyle a} \\ \mathcal{I} \otimes \pi K \otimes \pi(L \times M) & \xrightarrow{s \otimes \mathrm{id}} & \pi K \otimes \mathcal{I} \otimes \pi(L \times M) & \xrightarrow{\mathrm{id} \otimes h_{L,M}} & \pi K \otimes \pi(L \times M) \end{array}$$

Proof: Suppose as usual that $w_n = (x_n, y_n, z_n)$ is a generator of $\pi(K \times L \times M)$. Then the results need only be checked on the generators $\iota \otimes w_n$ of $\mathcal{I} \otimes \pi(K \times L \times M)$; the commutativity for $0 \otimes w_n$ follows by $h(0 \otimes -) = a \circ b$ and propositions 2.2.5 and 2.2.13, and for $1 \otimes w_n$ is trivial. For $n = 0$ the results are also clear. For $n = 1$ we have

$$\begin{aligned} (h_{K,L\times M} \circ a_{K\times L,M})(\iota \otimes w_n) &= a_{K\times L,M}(s_0x_1, s_1y_1, s_1z_1)^{-1} \\ &= \left((s_0x_1, s_1y_1) \otimes d_0^2 s_1 z_1\right)^{-1} \\ ((\mathrm{id} \otimes a) \circ (h \otimes \mathrm{id}))(\iota \otimes w_n) &= (h \otimes \mathrm{id})\,(\iota \otimes (d_1(x_1, y_1) \otimes z_1 \cdot (x_1, y_1) \otimes d_0 z_1)) \\ &= (s_0x_1, s_1y_1)^{-1} \otimes d_0 z_1 \end{aligned}$$

neglecting the degenerate terms in $s_1 z_1$, $d_0 s_1 z_1$ and $h(\iota \otimes d_1(x_1, y_1))$. Similarly,

$$\begin{aligned}
&(h_{K\times L,M} \circ a_{K,L\times M})(\iota \otimes w_n) \;=\; a_{K,L\times M}(s_0x_1, s_0y_1, s_1z_1)^{-1} \\
&\quad = \left(d_1^2 s_0 x_1 \otimes (s_0y_1, s_1z_1)^{-1}\right)^{x_1 \otimes d_0(y_1,z_1)} \\
&((\mathrm{id}\otimes a)\circ(s\otimes \mathrm{id})\circ(\mathrm{id}\otimes h))\,(\iota\otimes w_n) \\
&\quad = ((s\otimes \mathrm{id})\circ(\mathrm{id}\otimes h))\left((\iota\otimes d_1x_1\otimes(y_1,z_1))^{1\otimes x_1\otimes d_0(y_1,z_1)}\cdot \iota\otimes x_1\otimes d_0(y_1,z_1)\right) \\
&\quad = (\mathrm{id}\otimes h)\left((d_1x_1\otimes\iota\otimes(y_1,z_1))^{1\otimes x_1\otimes d_0(y_1,z_1)}\cdot(x_1\otimes\iota)^{-1}\otimes d_0(y_1,z_1)\right) \\
&\quad = \left(d_1x_1\otimes(s_0y_1,s_1z_1)^{-1}\right)^{x_1\otimes d_0(y_1,z_1)}
\end{aligned}$$

as required.

For the case $n \geq 2$ we can again represent everything using a straightforward generalisation of the notion of simplicial operators above. Recall that h was defined via

$$\Phi_n \;=\; -\,\Phi'_{n-1} - F'_n s(0) \qquad\qquad F_n \;=\; \sum_{\substack{0\leq i\leq n \\ \sigma\in \mathrm{Shuff}(i,n-i)}} \mathrm{sg}(\sigma)\left(s(\sigma_0)d(i+1)^{n-i}, s(\sigma_1)d(0)^i\right)$$

For a simplicial operator G of dimension (p,q), let $G^{(0)}$ and $G^{(1)}$ be the formal sums given by the first and second components of the terms in G, and write $G = (G^{(0)}, G^{(1)})$ where the summation here takes place in parallel. Then the actions of $h_{K,L\times M}\circ a_{K\times L,M}$ and $(\mathrm{id}\otimes a_{K\times L,M})\circ(h_{K,L}\otimes \mathrm{id})$ on $\iota\otimes w_n$ may be represented by the formal expressions

$$\sum_{j=0}^{n+1}\left(d(j+1)^{n+1-j}\Phi_n^{(0)}, d(j+1)^{n+1-j}\Phi_n^{(1)}\right)\otimes d(0)^j\Phi_n^{(1)}$$

$$\text{and}\qquad \sum_{k=0}^{n}\left(\Phi_k^{(0)}d(k+1)^{n-k}, \Phi_k^{(1)}d(k+1)^{n-k}\right)\otimes d(0)^k$$

For $n=1$ an argument as above shows that modulo degeneracies these are both equal to $-(s_0,s_1)\otimes d_0$. Suppose inductively that the result holds for $n-1$:

$$\sum_{j=0}^{n} d(j+1)^{n-j}\Phi_{n-1}\otimes d(0)^j\Phi_{n-1}^{(1)} \;\equiv\; \sum_{k=0}^{n-1}\Phi_k d(k+1)^{n-1-k}\otimes d(0)^k$$

Taking the derivative of this expression, writing j, k for $j-1$, $k-1$, and multiplying on the left by $\mathrm{id}\otimes d(0)$, gives

$$\sum_{j=1}^{n+1} d(j+1)^{n+1-j}\Phi'_{n-1}\otimes d(0)^j{\Phi_{n-1}^{(1)}}' \;\equiv\; \sum_{k=1}^{n}\Phi'_{k-1}d(k+1)^{n-k}\otimes d(0)^k$$

Thus it remains to show that

$$\sum_{j=0}^{n+1} d(j+1)^{n+1-j}F'_n s(0)\otimes d(0)^j {F_n^{(1)}}' s(0) \;\equiv\; \sum_{k=0}^{n} F'_k s(0) d(k+1)^{n-k}\otimes d(0)^k$$

The expression $d(0)^j F_n^{(1)'} s(0)$ is degenerate for $j=0$, and if $j \geq k+1$ consists of terms of the form $d(0)^j s(\sigma_1)' d(1)^i s(0)$ for $0 \leq i \leq n$, $\sigma \in \text{Shuff}(i, n-i)$. For $d(0)^j s(\sigma_1)'$ to be non-degenerate requires σ satisfy $\sigma(r) \leq j-2$ for $0 \leq r \leq i-1$, and for each $i < j$ the restriction to $\underline{j-1}$ gives a bijection between such σ and $\text{Shuff}(i, j-1-i)$. If $G_{n,j}$ is the expression obtained from F_n when only these i and σ are considered we have

$$\begin{aligned} d(0)^j F_n^{(1)'} \quad &\equiv \quad d(0)^j G_{n,j}^{(1)'} \quad = \quad d(0)^j \\ \text{and} \quad d(j+1)^{n+1-j} G'_{n,j} \quad &= \quad F'_{j-1} d(j+1)^{n+1-j} \end{aligned}$$

Hence

$$d(j+1)^{n+1-j} F'_n s(0) \otimes d(0)^j F_n^{(1)'} s(0) \quad \equiv \quad F'_k d(k+2)^{n-k} s(0) \otimes d(0)^{k+1} s(0)$$

for $k = j-1$ and the result follows.

For the second result we must show

$$\sum_{j=0}^{n+1} d(j+1)^{n+1-j} \Phi_n^{(0)} \otimes d(0)^j \Phi_n \quad \equiv \quad \sum_{j=0}^{n} (-1)^j d(j+1)^{n-j} \otimes \Phi_{n-j} d(0)^j$$

where the $(-1)^j$ comes from the symmetry. By the definition of Φ_n we have

$$\Phi_n \quad = \quad -\Phi'_{n-1} - F'_n s(0) \quad = \quad \sum_{i=0}^{n} (-1)^{i+1} F_{n-i}^{\{i+1\}} s(0)^{\{i\}}$$

where the superscripts $\{r\}$ indicate the r-fold derived operator. Now $F^{(0)}$ is frontal, so

$$d(j+1)^{n+1-j} \Phi_n^{(0)} \quad = \quad \sum_{i=0}^{n} (-1)^i d(j+1)^{n+1-j} s(i) F_{n-i}^{\{i\}}$$

These terms are degenerate for $i < j$, and for $i \geq j$ we have

$$d(j+1)^{n+1-j} s(i) F_{n-i}^{\{i\}} \quad = \quad d(j+1)^{n-j} \quad \text{and} \quad d(0)^j F_{n-i}^{\{i+1\}} s(i) \quad = \quad F_{n-i}^{\{i-j+1\}} s(i-j) d(0)^j$$

Thus

$$\begin{aligned} d(j+1)^{n+1-j} \Phi_n^{(0)} \otimes d(0)^j \Phi_n \quad &\equiv \quad \sum_{i=j}^{n} (-1)^{i+1} d(j+1)^{n-j} \otimes F_{n-i}^{\{i-j+1\}} s(i-j) d(0)^j \\ &= \quad d(j+1)^{n-j} \otimes (-1)^j \left(\sum_{i=0}^{n-j} (-1)^{i+1} F_{n-j-i}^{\{i+1\}} s(i) \right) d(0)^j \\ &= \quad (-1)^j d(j+1)^{n-j} \otimes \Phi_{n-j} d(0)^j \end{aligned}$$

and we have the result. □

Proposition 2.3.7 *For simplicial sets K, L, M the following diagrams commute*

$$\begin{array}{ccc}
\mathcal{I}\otimes\pi(K\times L)\otimes\pi M & \xrightarrow{h_{K,L}\otimes\mathrm{id}} & \pi(K\times L)\otimes\pi M \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes b} & & \Big\downarrow{\scriptstyle b} \\
\mathcal{I}\otimes\pi(K\times L\times M) & \xrightarrow{h_{K,L\times M}} & \pi(K\times L\times M)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{I}\otimes\pi K\otimes\pi(L\times M) & \xrightarrow{s\otimes\mathrm{id}} & \pi K\otimes\mathcal{I}\otimes\pi(L\times M) & \xrightarrow{\mathrm{id}\otimes h_{L,M}} & \pi K\otimes\pi(L\times M) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes b} & & & & \Big\downarrow{\scriptstyle b} \\
\mathcal{I}\otimes\pi(K\times L\times M) & & \xrightarrow{\qquad h_{K\times L,M}\qquad} & & \pi(K\times L\times M)
\end{array}$$

Proof: Suppose $v_{p,q}$ and $w_{p,q}$ are generators in dimension $p+q$ of $\pi(K\times L)\otimes\pi M$ and $\pi K\otimes\pi(L\times M)$ as usual. Then the results hold for the generators $\alpha\otimes v_{p,q}$ and $\alpha\otimes w_{p,q}$ for $\alpha=0,1$ in $\mathcal{I}_0$ by propositions 2.2.11 and 2.2.13. Also the results are clear for $\iota\otimes v_{p,q}$ and $\iota\otimes w_{p,q}$ if p or q are zero. Thus we may assume these generators have dimension at least three and work in terms of simplicial operators as before.

Let $B_{p,q}=(B^{(0)}_{p,q},B^{(1)}_{p,q})$ be the simplicial operators representing the shuffle homomorphism b in each dimension. Then for the first result we show inductively that

$$\left(\Phi_{p+q}B^{(0)}_{p,q},\Phi^{(1)}_{p+q}B^{(1)}_{p,q}\right) \;\equiv\; \left(B^{(0)}_{p+1,q}\Phi_p, B^{(1)}_{p+1,q}\right)$$

Partitioning the set of (p,q)-shuffles σ according to whether zero is in the image of σ_1 or σ_0, we obtain the following recursive formula for B:

$$B_{p,q} \;=\; \left({B^{(0)}_{p-1,q}}',{B^{(1)}_{p-1,q}}'s(0)\right)+(-1)^p\left({B^{(0)}_{p,q-1}}'s(0),{B^{(1)}_{p,q-1}}'\right)$$

Together with the inductive hypothesis, this gives

$$\begin{aligned}
&\left(\Phi'_{p+q-1}B^{(0)}_{p,q},{\Phi^{(1)}_{p+q-1}}'B^{(1)}_{p,q}\right)\\
&\quad=\;\left(\Phi'_{p+q-1}{B^{(0)}_{p-1,q}}',{\Phi^{(1)}_{p+q-1}}'{B^{(1)}_{p-1,q}}'s(0)\right)+(-1)^p\left(\Phi'_{p+q-1}{B^{(0)}_{p,q-1}}'s(0),{\Phi^{(1)}_{p+q-1}}'{B^{(1)}_{p,q-1}}'\right)\\
&\quad\equiv\;\left({B^{(0)}_{p,q}}'\Phi'_{p-1},{B^{(1)}_{p,q}}'s(0)\right)+(-1)^p\left({B^{(0)}_{p+1,q-1}}'\Phi'_p s(0),{B^{(1)}_{p+1,q-1}}'\right)
\end{aligned}$$

Also propositions 2.2.11 and 2.2.13 imply the following commutativity relation between B and F:

$$\left(F_{p+q}B^{(0)}_{p,q},F^{(1)}_{p+q}B^{(1)}_{p,q}\right) \;\equiv\; \left(B^{(0)}_{p,q}F_p,B^{(1)}_{p,q}\right)$$

which since B is frontal gives

$$\left(F'_{p+q}s(0)B^{(0)}_{p,q}, {F^{(1)}_{p+q}}'s(0)B^{(1)}_{p,q}\right) \;\equiv\; \left({B^{(0)}_{p,q}}'F'_p s(0), {B^{(1)}_{p,q}}'s(0)\right)$$

Combining these results using $-\Phi_n = \Phi'_{n-1} + F'_n s(0)$, we get

$$\begin{aligned} &\left(\Phi_{p+q}B^{(0)}_{p,q}, \Phi^{(1)}_{p+q}B^{(1)}_{p,q}\right) \\ &\quad\equiv\; \left({B^{(0)}_{p,q}}'\Phi_p, {B^{(1)}_{p,q}}'s(0)\right) + (-1)^{p+1}\left({B^{(0)}_{p+1,q-1}}'\Phi'_p s(0), {B^{(1)}_{p+1,q-1}}'\right) \end{aligned}$$

But $\Phi'_p s(0) = s(0)\Phi_p$, so using the recursive shuffle relation for $B_{p+1,q}$ gives the required result.

For the second part, we use similar arguments to show inductively that

$$\left(\Phi^{(0)}_{p+q}B^{(0)}_{p,q}, \Phi_{p+q}B^{(1)}_{p,q}\right) \;\equiv\; (-1)^p\left(B^{(0)}_{p,q+1}, B^{(1)}_{p,q+1}\Phi_q\right)$$

Using the recursive formulæ for Φ and B, and since both Φ and B are frontal, this may be expanded into

$$\begin{array}{rcl} -\left({\Phi^{(0)}_{p+q-1}}'{B^{(0)}_{p-1,q}}', \Phi'_{p+q-1}{B^{(1)}_{p-1,q}}'s(0)\right) & & (-1)^p\left({B^{(0)}_{p-1,q+1}}', {B^{(1)}_{p-1,q+1}}'\Phi'_q s(0)\right) \\ -(-1)^p\left({\Phi^{(0)}_{p+q-1}}'{B^{(0)}_{p,q-1}}'s(0), \Phi'_{p+q-1}{B^{(1)}_{p,q-1}}'\right) & \equiv & -\left({B^{(0)}_{p,q}}'s(0), {B^{(1)}_{p,q}}'\Phi'_{q-1}\right) \\ -\left({F^{(0)}_{p+q}}'{B^{(0)}_{p,q}}'s(0), F'_{p+q}{B^{(1)}_{p,q}}'s(0)\right) & & -\left({B^{(0)}_{p,q}}'s(0), {B^{(1)}_{p,q}}'F'_q s(0)\right) \end{array}$$

which holds by the inductive hypothesis and propositions 2.2.11 and 2.2.13. □

These two propositions 2.3.6 and 2.3.7 will be used in the next section to show that the Eilenberg-Zilber theorem extends to give a coherent system of higher homotopies

$$\mathcal{I}^{\otimes r} \otimes \pi(K_0 \times \ldots \times K_r) \longrightarrow \pi(K_0 \times \ldots \times K_r)$$

between the 2^r endomorphisms of $\pi(K_0 \times \ldots \times K_r)$ defined by various composites of a and b.

2.3.2 Higher Homotopies and Coherence

For simplicial sets K, L, M, there are homotopies between

$$\pi(K \times L \times M) \xrightarrow{a^2} \pi K \otimes \pi L \otimes \pi M \xrightarrow{b^2} \pi(K \times L \times M)$$

and the identity, induced either by $h_{K\times L,M}$ and $h_{K,L}$, or by $h_{K,L\times M}$ and $h_{L,M}$. These homotopies are not the same, although they are themselves homotopic via a *double homotopy*

$$\mathcal{I} \otimes \mathcal{I} \otimes \pi(K \times L \times M) \xrightarrow{h_{K,L,M}} \pi(K \times L \times M)$$

More generally we make the following definition:

Definition 2.3.8 *An* r-fold homotopy *of crossed complexes is given by a crossed complex homomorphism*

$$\mathcal{I}^{\otimes r} \otimes C \xrightarrow{\quad h \quad} D$$

where C*,* D *are crossed complexes and* $\mathcal{I}^{\otimes r}$ *is the* r*-fold tensor product of the crossed complex* $\mathcal{I}$ *with itself.*

Given a p-fold homotopy $\mathcal{I}^{\otimes p} \otimes C \xrightarrow{h} D$ and a q-fold homotopy $\mathcal{I}^{\otimes q} \otimes E \xrightarrow{k} F$ we will define $h * k$ to be the $(p+q)$-fold homotopy given by

$$\begin{array}{ccc} \mathcal{I}^{\otimes(p+q)} \otimes C \otimes E & \overset{h*k}{\dashrightarrow} & D \otimes F \\ \cong \downarrow & & \uparrow h \otimes k \\ \mathcal{I}^{\otimes p} \otimes \mathcal{I}^{\otimes q} \otimes C \otimes E & \xrightarrow{\mathrm{id} \otimes s \otimes \mathrm{id}} & \mathcal{I}^{\otimes p} \otimes C \otimes \mathcal{I}^{\otimes q} \otimes E \end{array}$$

where s is given by the symmetry of the tensor product. Also for $1 \leq i \leq p$ and $\alpha \in \{0, 1\}$ we will write $\delta_i^\alpha(h)$ for the $(p-1)$-fold homotopy defined by

$$\begin{array}{ccc} \mathcal{I}^{\otimes(p-1)} \otimes C & \overset{\delta_i^\alpha(h)}{\dashrightarrow} & D \\ & {\scriptstyle f_i^\alpha \otimes \mathrm{id}} \searrow \quad \nearrow {\scriptstyle h} & \\ & \mathcal{I}^{\otimes p} \otimes C & \end{array}$$

where f_i^α is the natural monomorphism given on generators by

$$\begin{array}{ccc} \mathcal{I}^{\otimes(p-1)} & \longrightarrow & \mathcal{I}^{\otimes p} \\ x_1 \otimes x_2 \otimes \cdots \otimes x_{p-1} & \longmapsto & x_1 \otimes \cdots \otimes x_{i-1} \otimes \alpha \otimes x_i \otimes \cdots \otimes x_{p-1} \end{array}$$

We will often use the notation δ_i^- for δ_i^0 and δ_i^+ for δ_i^1.

Note that 0-fold homotopies are given by homomorphisms, and a 1-fold homotopy h is thus just an ordinary homotopy $h\colon \delta_1^-(h) \simeq \delta_1^+(h)$ as in section 2.1.1.

For convenience in later chapters we will make a change in the conventions of proposition 2.3.1, and write $h_{K,L}$ for the homotopy $\mathrm{id} \simeq a \circ b$ given by the *reverse* of the homotopy denoted $h_{K,L}$ in that section. We will also use the notation $a^{(i)}$ and $b^{(i)}$ for the homomorphisms defined by a and b at the ith factor of a product

$$\pi(K_0 \times \ldots \times K_r) \underset{b^{(i)}}{\overset{a^{(i)}}{\rightleftarrows}} \pi(K_0 \times \ldots \times K_{i-1}) \otimes \pi(K_i \times \ldots \times K_r)$$

and will write $h^{(i)}$ for the homotopy $\mathrm{id} \simeq a^{(i)} \circ b^{(i)}$.

Theorem 2.3.9 *Suppose that K_i are simplicial sets for $0 \leq i \leq r$. Then there is an r-fold homotopy*

$$\mathcal{I}^{\otimes r} \otimes \pi(K_0 \times \ldots \times K_r) \xrightarrow{h_{K_0,K_1,\ldots,K_r}} \pi(K_0 \times \ldots \times K_r)$$

These homotopies are natural in the K_i, and satisfy the cubical boundary relations

$$\begin{array}{rcll} \delta_i^-(h_{K_0,\ldots,K_r}) & = & h_{K_0,K_1,\ldots,(K_{i-1}\times K_i),\ldots,K_r} & \textit{for } r \geq 1 \\ \delta_i^+(h_{K_0,\ldots,K_r}) & = & (\mathrm{id} \otimes a^{(i)}) \circ (h_{K_0,\ldots,K_{i-1}} * h_{K_i,\ldots,K_r}) \circ b^{(i)} & \textit{for } r \geq 1 \end{array}$$

$$\begin{array}{ccc} \mathcal{I}^{\otimes(r-1)}\otimes\pi(K_0\times\ldots\times K_r) & \xrightarrow{\mathrm{id}\otimes a^{(i)}} & \mathcal{I}^{\otimes(r-1)}\otimes\pi(K_0\times\ldots\times K_{i-1})\otimes\pi(K_i\times\ldots\times K_r) \\ \Big\downarrow{\scriptstyle \delta_i^+(h_{K_0,\ldots,K_r})} & & \Big\downarrow{\scriptstyle h_{K_0,\ldots,K_{i-1}} * h_{K_i,\ldots,K_r}} \\ \pi(K_0\times\ldots\times K_r) & \xleftarrow{b^{(i)}} & \pi(K_0\times\ldots\times K_{i-1})\otimes\pi(K_i\times\ldots\times K_r) \end{array}$$

together with the relations

$$\begin{array}{rcl} h_{K_0} & = & \mathrm{id}_{\pi K_0} \\ (\mathrm{id} \otimes b^{(i)}) \circ \delta_i^-(h_{K_0,\ldots,K_r}) & = & (\mathrm{id} \otimes b^{(i)}) \circ \delta_i^+(h_{K_0,\ldots,K_r}) \\ \delta_i^-(h_{K_0,\ldots,K_r}) \circ a^{(i)} & = & \delta_i^+(h_{K_0,\ldots,K_r}) \circ a^{(i)} \end{array}$$

Proof: Suppose $K_0, K_1, \ldots, K_r$ are simplicial sets and write K_i^j for the product $K_i \times K_{i+1} \times \ldots \times K_j$ for $0 \leq i \leq j \leq r$. Then the r-fold homotopies $h_{K_0,\ldots,K_r}$ may be defined inductively by

$$h_{K_0,K_1,\ldots,K_r} = (\mathrm{id}_{\mathcal{I}} \otimes h_{K_0^1,K_2,\ldots,K_r}) \circ h_{K_0,K_1^r}$$

where $h_{K_0} = \mathrm{id}_{K_0}$ and h_{K_0,K_1} is as defined earlier. Thus $h_{K_0,K_1,\ldots,K_r}$ is the composite of the maps

$$\mathcal{I}^{\otimes i} \otimes \pi(K_0 \times \ldots \times K_r) \xrightarrow{\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes h^{(i)}} \mathcal{I}^{\otimes(i-1)} \otimes \pi(K_0 \times \ldots \times K_r)$$

for $i = r, r-1, \ldots, 1$.

To prove that the r-fold homotopies h satisfy the appropriate boundary relations, we need a lemma.

Lemma 2.3.10 *The maps a, b, h as above satisfy*

$$\begin{array}{rcl} h_{K_0^i,K_{i+1},\ldots,K_r} \circ a^{(i)} & = & (\mathrm{id}_{\mathcal{I}^{\otimes(r-i)}} \otimes a^{(i)}) \circ (s \otimes \mathrm{id}_{\pi K_i^r}) \circ (\mathrm{id}_{\pi K_0^{i-1}} \otimes h_{K_i,\ldots,K_r}) \\ h_{K_0,\ldots,K_{i-2},K_{i-1}^r} \circ a^{(i)} & = & (\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes a^{(i)}) \circ (h_{K_0,\ldots,K_{i-1}} \otimes \mathrm{id}_{\pi K_i^r}) \end{array}$$

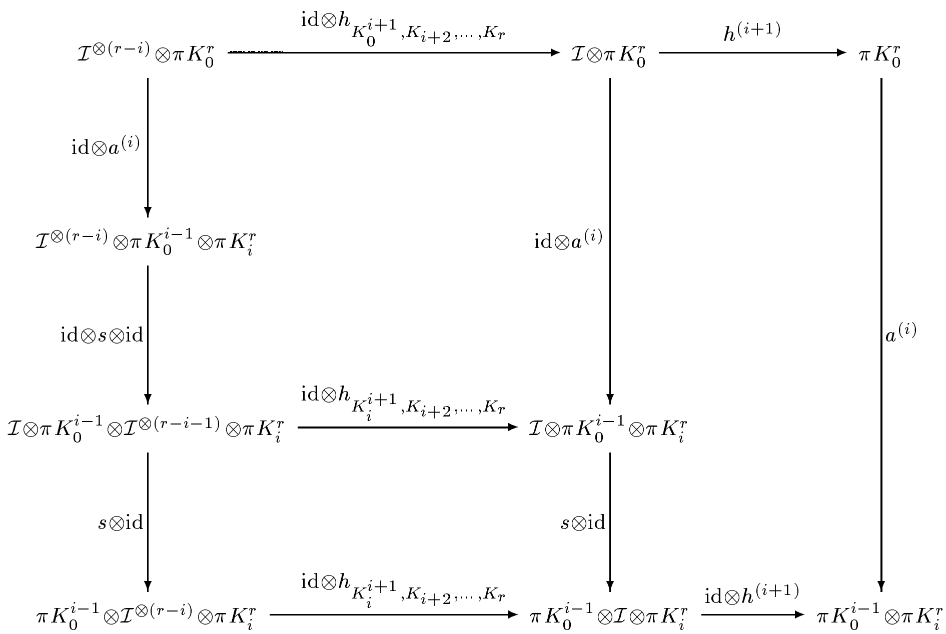
and

$$\begin{array}{rcl} (\mathrm{id}_{\mathcal{I}^{\otimes(r-i)}} \otimes b^{(i)}) \circ h_{K_0^i,K_{i+1},\ldots,K_r} & = & (s \otimes \mathrm{id}_{\pi K_i^r}) \circ (\mathrm{id}_{\pi K_0^{i-1}} \otimes h_{K_i,\ldots,K_r}) \circ b^{(i)} \\ (\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes b^{(i)}) \circ h_{K_0,\ldots,K_{i-2},K_{i-1}^r} & = & (h_{K_0,\ldots,K_{i-1}} \otimes \mathrm{id}_{\pi K_i^r}) \circ b^{(i)} \end{array}$$

Proof: We prove the first result of these four; the others are similar. Assume inductively that the result holds with K_i and K_{i+1} replaced by their product

$$h_{K_0^{i+1},K_{i+2},\ldots,K_r} \circ a^{(i)} = (\mathrm{id}_{\mathcal{I}^{\otimes(r-i-1)}} \otimes a^{(i)}) \circ (s \otimes \mathrm{id}_{\pi K_i^r}) \circ (\mathrm{id}_{\pi K_0^{i-1}} \otimes h_{K_i^{i+1},K_{i+2},\ldots,K_r})$$

and consider the diagram



This commutes by the inductive hypothesis, by naturality of s and by proposition 2.3.6. Since the horizontal composites are just the inductive definitions of $h_{K_0^i,K_{i+1},\ldots,K_r}$ and $\mathrm{id} \otimes h_{K_i,\ldots,K_r}$ we have the required result. □

Returning to the proof of the proposition, we can write

$$h_{K_0,\ldots,K_r} \quad = \quad (\mathrm{id}_{\mathcal{I}^{\otimes i}} \otimes h_{K_0^i,K_{i+1},\ldots,K_r}) \circ (\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes h^{(i)}) \circ h_{K_0,\ldots,K_{i-2},K_{i-1}^r}$$

Since $\delta_1^- h^{(i)} = \mathrm{id}$ and $\delta_1^+ h^{(i)} = a^{(i)} \circ b^{(i)}$ this gives

$$\begin{aligned}
\delta_i^- h_{K_0,\ldots,K_r} &= (\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes h_{K_0^i,K_{i+1},\ldots,K_r}) \circ h_{K_0,\ldots,K_{i-2},K_{i-1}^r} \\
\delta_i^+ h_{K_0,\ldots,K_r} &= \left(\mathrm{id}_{\mathcal{I}^{\otimes(i-1)}} \otimes (h_{K_0^i,K_{i+1},\ldots,K_r} \circ a^{(i)} \circ b^{(i)})\right) \circ h_{K_0,\ldots,K_{i-2},K_{i-1}^r} \\
&= (\mathrm{id} \otimes a^{(i)}) \circ (\mathrm{id} \otimes s \otimes \mathrm{id}) \circ (\mathrm{id} \otimes h_{K_i,\ldots,K_r}) \circ (h_{K_0,\ldots,K_{i-1}} \otimes \mathrm{id}) \circ b^{(i)}
\end{aligned}$$

by first and fourth parts of the lemma, and the $\delta_i^\pm$ boundary relations follow. The final two relations hold by the second and third parts of the lemma since $b^{(i)} \circ a^{(i)} = \mathrm{id}$. □

The following additional properties of the r-fold homotopies h are easy consequences of the relations given in the theorem.

Proposition 2.3.11 *Given simplicial sets K_i with corresponding higher homotopies h as above, the following equations hold*

$$\begin{aligned}
\delta_i^+(h_{K_0,\ldots,K_r}) &= \mathrm{id} \otimes a^{(i)} \circ \mathrm{id} \otimes b^{(i)} \circ \delta_i^+(h_{K_0,\ldots,K_r}) \\
&= \delta_i^+(h_{K_0,\ldots,K_r}) \circ a^{(i)} \circ b^{(i)} \\
\mathrm{id} \otimes b^{(i)} \circ h_{K_0,\ldots,K_{i-1}\times K_i,\ldots,K_r} &= h_{K_0,\ldots,K_{i-1}} * h_{K_i,\ldots,K_r} \circ b^{(i)} \\
h_{K_0,\ldots,K_{i-1}\times K_i,\ldots,K_r} \circ a^{(i)} &= \mathrm{id} \otimes a^{(i)} \circ h_{K_0,\ldots,K_{i-1}} * h_{K_i,\ldots,K_r}
\end{aligned}$$

Suppose that $h_{K_0,K_1,\ldots,K_r}$ is an r-fold homotopy of the theorem. Then for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, $\alpha_i \in \{0,1\}$, there is an endomorphism $h^\alpha_{K_0,K_1,\ldots,K_r}$ of $\pi(K_0 \times \ldots \times K_r)$ given by restricting the homotopy to the corner α of the r-cube $\mathcal{I}^{\otimes r}$. That is,

$$h^\alpha_{K_0,K_1,\ldots,K_r}(x) = h_{K_0,K_1,\ldots,K_r}(\alpha_1 \otimes \cdots \otimes \alpha_r \otimes x)$$

We say that the various r-fold homotopies h of the theorem provide a *coherent system* of homotopies between the homomorphisms h^α.

For simplicial sets $L_0, L_1, \ldots, L_k$ the diagonal approximation and shuffle maps give homomorphisms

$$\pi(L_0 \times L_1 \times \ldots \times L_k) \underset{b^k}{\overset{a^k}{\rightleftarrows}} \pi L_0 \otimes \pi L_1 \otimes \ldots \otimes \pi L_k$$

which are well defined by the associativity of a and b. Thus for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ as before we have homomorphisms a_α and b_α

$$\pi(K_0 \times K_1 \times \ldots \times K_r) \underset{b_\alpha}{\overset{a_\alpha}{\rightleftarrows}} \pi\left(\prod_{i=i_0}^{i_1-1} K_i\right) \otimes \pi\left(\prod_{i=i_1}^{i_2-1} K_i\right) \otimes \ldots \otimes \pi\left(\prod_{i=i_k}^{i_{k+1}-1} K_i\right)$$

where $i_1 < i_2 < \ldots < i_k$ are those i such that $\alpha_i = 1$, and $i_0 = 0$, $i_{k+1} = r+1$. In particular, $a_\alpha = b_\alpha = \mathrm{id}$ if $\alpha_i = 0$ for $1 \leq i \leq r$.

By using the boundary relations which the h satisfy, we can show that the homomorphisms a_α and b_α give an explicit description of the endomorphisms h^α.

Proposition 2.3.12 *For h an r-fold homotopy and $\alpha \in \{0,1\}^r$ as above, the endomorphism given by h^α is precisely the composite $a_\alpha \circ b_\alpha$. Thus the r-fold homotopies h provide a system of coherent homotopies between the various composites $a^k \circ b^k$ for $0 \leq k \leq r$.*

These results will be used in chapter four to make precise the statement that 'up to homotopy' there is an enriched natural adjunction between simplicial enrichments of π and N.

Chapter 3

Homotopy Colimits and Small Resolutions

3.0 Introduction

In this chapter we give a definition of homotopy colimits of diagrams of crossed complexes. It is proved that there is a strong deformation retraction

$$\operatorname{hocolim}_{\mathrm{Crs}}(F \circ \pi) \;\simeq\; \pi(\operatorname{hocolim}_S F)$$

for F a small diagram of simplicial sets and $\operatorname{hocolim}_S F$ its homotopy colimit as defined in [4]. We discuss an alternative definition of homotopy colimit in **SimpSet**, written $\operatorname{hocolim}'_S$, such that there is a natural isomorphism

$$\operatorname{hocolim}'_S(F \circ \mathrm{Ner}) \;\cong\; \mathrm{Ner}(\operatorname{hocolim}_{\mathrm{Cat}} F)$$

for F a small diagram of categories and $\operatorname{hocolim}_{\mathrm{Cat}}$ the usual homotopy colimit in **Cat** [38].

As a simple motivating example, these results are applied to a functor corresponding to a group action. This gives a free crossed resolution for a semidirect product of groups which is a strong deformation retraction of the standard resolution, and which may be written as a twisted tensor product of standard resolutions.

The structure of this chapter is as follows. In the first section we set out the motivation in terms of finding small resolutions of semidirect products of groups.

In the second section, we recall the Bousfield-Kan definition of homotopy colimit in **SimpSet** in terms of a coend and of the diagonal of a bisimplicial set Ψ. An alternative (homotopy equivalent) definition is proposed using the Artin-Mazur diagonal of the transpose Ψ' of Ψ, and it is shown that this behaves better with respect to homotopy colimits in **Cat** as defined by the Grothendieck construction.

In the third section, we propose a definition of homotopy colimits in the monoidal closed category of crossed complexes, both in terms of a coend and of a total complex of a simplicial crossed complex. The main result that we prove is that the fundamental crossed complex functor preserves homotopy colimits up to a strong deformation retraction. Finally we apply this result to obtain a small crossed resolution of a semidirect product of groups in terms of a twisted tensor product.

3.1 Motivation: Small Crossed Resolutions

3.1.1 Standard crossed resolutions

Recall the following:

Definition 3.1.1 *Suppose* $\mathbf{C}$ *is a small category. Then the* nerve *of* $\mathbf{C}$ *is the simplicial set* $\mathrm{Ner}(\mathbf{C})$ *given by strings of composable arrows in* $\mathbf{C}$*:*

$$\mathrm{Ner}(\mathbf{C})_n = \{[x_0, a_1, x_1, a_2, x_2, \ldots, a_n, x_n] \;:\; a_i \in \mathbf{C}(x_{i-1}, x_i)\}$$

The degeneracy maps are given by inserting an identity arrow:

$$s_i[x_0, a_1, x_1, \ldots, a_n, x_n] = [x_0, a_1, x_1, \ldots, a_i, x_i, ex_i, x_i, a_{i+1}, x_{i+1}, \ldots, a_n, x_n]$$

The first and last boundary maps are given by deleting the first and last arrow respectively, and the others by composing consecutive arrows:

$$\begin{aligned} d_0[x_0, a_1, x_1, \ldots, a_n, x_n] &= [x_1, a_2, x_2, \ldots, a_n, x_n] \\ d_n[x_0, a_1, x_1, \ldots, a_n, x_n] &= [x_0, a_1, x_1, \ldots, a_{n-1}, x_{n-1}] \\ d_i[x_0, a_1, x_1, \ldots, a_n, x_n] &= \\ & [x_0, a_1, x_1, \ldots, a_{i-1}, x_{i-1}, a_i \cdot a_{i+1}, x_{i+1}, a_{i+2}, x_{i+2} \ldots, a_n, x_n] \quad \text{for } 1 \le i < n \end{aligned}$$

The n-simplices $[x_0, a_1, x_1, a_2, x_2, \ldots, a_n, x_n]$ will often be written as $[a_1, a_2, \ldots, a_n]$, or as $[\,]_{x_0}$ in the zero-dimensional case.

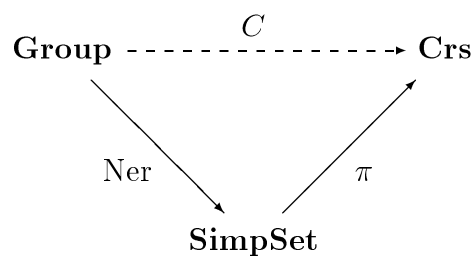
The functor Ner : $\mathbf{Cat} \to \mathbf{Simp}$ has a left adjoint c : $\mathbf{Simp} \to \mathbf{Cat}$ termed *categorisation*. The category $\mathrm{c}(K)$ has object set K_0 and is generated by arrows a_1 for each one-simplex $a_1 \in K_1$. Identity arrows are given by degenerate one-simplices, source and target maps by the boundary maps, and there are relations from the two-simplicies. Altogether the relations are thus:

$$\begin{aligned} s_0 a_0 &= e a_0 \\ d_1 a_1 &= s a_1 \\ d_0 a_1 &= t a_1 \\ d_1 a_2 &= d_2 a_2 \cdot d_0 a_2 \end{aligned}$$

The bijection of hom-sets $\mathbf{Cat}(\mathrm{c}(K), \mathbf{C}) \cong \mathbf{SimpSet}(K, \mathrm{Ner}(\mathbf{C}))$ is well known, as is the isomorphism of categories $\mathrm{c}\,(\mathrm{Ner}(\mathbf{C})) \cong \mathbf{C}$.

In the case where **C** is a group, the nerve of **C** is said to give a simplicial set which *resolves* the group structure. This simplicial set has a single zero-simplex, and has fundamental group the original group **C** and all higher homotopy groups trivial. Taking the fundamental crossed complex of the nerve thus gives a crossed complex whose homology is **C** in dimension one and trivial in higher dimensions. The fundamental crossed complex of the nerve has been proposed in [27, 11] as an algebraic resolution of the group structure.

Definition 3.1.2 *The* standard crossed resolution $C(G)$ *of a group* G *is given by the fundamental crossed complex of its nerve.*



We will also write C for the functor defined on the whole of **Cat**.

Using definition 1.3.1 we may present the functor C in terms of generators and relations.

Proposition 3.1.3 *Suppose* G *is a group. Then* $C(G)$ *is the crossed complex of groups generated by elements* $[g_1, g_2, \ldots, g_n] \in C(G)_n$ *subject to the relations*

$$\begin{aligned}
[g_1, g_2, \ldots, g_n] &= e \textit{ in } C(G)_n \textit{ if any } g_i \textit{ is the identity} \\
\delta_2[g_1, g_2] &= [g_2]^{-1} \cdot [g_1]^{-1} \cdot [g_1 g_2] \\
\delta_3[g_1, g_2, g_3] &= [g_2, g_3]^{-1} \cdot [g_1, g_2]^{[g_3]} \cdot [g_1 g_2, g_3] \cdot [g_1, g_2 g_3]^{-1} \\
\delta_n[g_1, g_2, \ldots, g_n] &= [g_2, \ldots, g_n]^{-1} \cdot \left([g_1, \ldots, g_{n-1}]^{[g_n]}\right)^{(-1)^{n+1}} \\
&\quad \cdot \prod_{i=1}^{n-1} [g_1, \ldots, g_i g_{i+1}, \ldots, g_n]^{(-1)^{i+1}} \quad \textit{for } n \geq 4
\end{aligned}$$

Note that the only relations involved are those for boundaries and degeneracies and so $C(G)$ can be regarded as free in a certain sense. Thus the standard crossed resolution of G is termed a *free aspherical resolution* for G.

If G, H are groups, then we may form the tensor product of the crossed resolutions $C(G)$ and $C(H)$. Combining propositions 1.2.5 and 3.1.3, this has the following standard presentation.

Proposition 3.1.4 *Suppose* G, H *are groups. Then the tensor product* $C(G) \otimes C(H)$ *is the crossed complex of groups given by generators* $a_p \otimes b_q$ *in dimension* $n = p + q$ *for all* $a_p = [g_1, \ldots, g_p] \in \mathrm{Ner}(G)$ *and* $b_q = [h_1, \ldots, h_q] \in \mathrm{Ner}(H)$, *subject to the relations:*

1. $a_p \otimes b_q = *$, *the identity element, if any of the g_i or h_i are identities*

$$
\begin{aligned}
\textit{2.}\quad \delta_2(a_2 \otimes b_0) &= ([g_2] \otimes [\,])^{-1} \circ ([g_1] \otimes [\,])^{-1} \circ ([g_1 g_2] \otimes [\,]) \\
\delta_2(a_0 \otimes b_2) &= ([\,] \otimes [h_2])^{-1} \circ ([\,] \otimes [h_1])^{-1} \circ ([\,] \otimes [h_1 h_2]) \\
\delta_2(a_1 \otimes b_1) &= ([\,] \otimes [h_1])^{-1} \circ ([g_1] \otimes [\,])^{-1} \circ ([\,] \otimes [h_1]) \circ ([g_1] \otimes [\,]) \\
\delta_3(a_1 \otimes b_2) &= ([g_1] \otimes [h_2]) \circ ([\,] \otimes [h_1, h_2])^{[g_1]\otimes[\,]} \circ ([g_1] \otimes [h_1 h_2])^{-1} \\
&\qquad \circ ([\,] \otimes [h_1, h_2])^{-1} \circ ([g_1] \otimes [h_1])^{[\,]\otimes[h_2]} \\
\delta_3(a_2 \otimes b_1) &= ([g_2] \otimes [h_1])^{-1} \circ ([g_1, g_2] \otimes [\,])^{[\,]\otimes[h_1]} \circ ([g_1 g_2] \otimes [h_1]) \\
&\qquad \circ ([g_1, g_2] \otimes [\,])^{-1} \circ \left(([g_1] \otimes [h_1])^{[g_2]\otimes[\,]}\right)^{-1} \\
\delta_p(a_p \otimes b_0) &= \delta^{\mathrm{h}}(a_p \otimes b_0) \quad \textit{for } p \geq 3 \\
\delta_q(a_0 \otimes b_q) &= \delta^{\mathrm{v}}(a_0 \otimes b_q) \quad \textit{for } q \geq 3 \\
\delta_{p+q}(a_p \otimes b_q) &= \delta^{\mathrm{h}}(a_p \otimes b_q) \circ (\delta^{\mathrm{v}}(a_p \otimes b_q))^{(-1)^p} \quad \textit{otherwise}
\end{aligned}
$$

where the abbreviations $\delta^{\mathrm{h}}(a_p \otimes b_q)$ and $\delta^{\mathrm{v}}(a_p \otimes b_q)$ stand for the following expressions:

$$
\begin{aligned}
\delta^{\mathrm{h}}(a_p \otimes b_q) &= ([g_2, \ldots, g_p] \otimes [h_1, \ldots, h_q])^{-1} \\
&\qquad \circ \left(([g_1, \ldots, g_{p-1}] \otimes [h_1, \ldots, h_q])^{[g_p]\otimes[\,]}\right)^{(-1)^{p+1}} \\
&\qquad \circ \prod_{k=1}^{p-1} ([g_1, \ldots, g_k g_{k+1}, \ldots, g_p] \otimes [h_1, \ldots, h_q])^{(-1)^{k+1}} \\
\delta^{\mathrm{v}}(a_p \otimes b_q) &= ([g_1, \ldots, g_p] \otimes [h_2, \ldots, h_q])^{-1} \\
&\qquad \circ \left(([g_1, \ldots, g_p] \otimes [h_1, \ldots, h_{q-1}])^{[\,]\otimes[h_q]}\right)^{(-1)^{q+1}} \\
&\qquad \circ \prod_{k=1}^{q-1} ([g_1, \ldots, g_p] \otimes [h_1, \ldots, h_k h_{k+1}, \ldots, h_q])^{(-1)^{k+1}}
\end{aligned}
$$

Now consider the standard resolution of the product $G \times H$ of the two groups G, H. Since the nerve functor commutes with products we have

$$C(G \times H) \;\cong\; \pi(\mathrm{Ner}\, G \times \mathrm{Ner}\, H)$$

Comparing this with $C(G) \otimes C(H) \cong \pi(\mathrm{Ner}\, G) \otimes \pi(\mathrm{Ner}\, H)$ we find that we may replace the standard resolution of the product by the tensor product of standard resolutions, as follows.

Theorem 3.1.5 *Suppose G, H are groups with product $G \times H$. Then the tensor product $C(G) \otimes C(H)$ defines a free aspherical resolution for $G \times H$.*

Proof: From the presentation of $C(G) \otimes C(H)$ above it can be seen that there are no relations except those given by the boundary maps and degeneracies, and so we have freeness. We also know from theorem 2.3.1 that $C(G \times H)$ and $C(G) \otimes C(H)$ are homotopy equivalent. But the former is the standard aspherical resolution for $G \times H$, and so the latter is also an aspherical resolution since homotopy equivalence implies equivalence in homology. □

3.1.2 Semidirect products and homotopy colimits in Cat

In the previous section it was shown that a resolution for a product of groups may be obtained from the tensor product of the resolutions. The important point to note is that the resulting free crossed complex is *smaller* than the standard resolution of the product group, and that the Eilenberg-Zilber theorem gives a strong deformation retraction of the larger onto the smaller. Consider now the case where the group H acts on the group G, and let E be the semidirect product of G by H. We would like to use the semidirect product decomposition to find a free aspherical crossed resolution for E which is smaller than the standard resolution $C(E)$.

An *action* of a group H on a group G is a function $H \times G \longrightarrow G$, written $(h, g) \longmapsto g^h$, satisfying

$$g^{e_H} = g, \quad g^{h_1 h_2} = (g^{h_1})^{h_2}, \quad {e_G}^h = e_G, \quad (g_1 g_2)^h = {g_1}^h {g_2}^h$$

Note that this is consistent with definition 1.1.3.

The function $H \times G \longrightarrow G$ is not a group homomorphism since we do not have $(g_1 g_2)^{h_1 h_2} = {g_1}^{h_1} {g_2}^{h_2}$. However regarding the groups G and H as categories we have the following equivalent formulation:

Proposition 3.1.6 *An action of a group H on a group G is given by a functor α from H to* **Cat** *such that $\alpha(e_H) = G$.*

Proof: The correspondence is given by $g^h = (\alpha h)(g)$. The first two axioms for a group action given above correspond to the functoriality of α, the other two to the functoriality of $\alpha(h)$ for each arrow h of H. □

The next construction we need is due to Grothendieck.

Definition 3.1.7 *Suppose I is a small category and F a functor from I to* **Cat**. *Then the* Grothendieck construction on *F, written $\int^I F$, is the category with objects the pairs (i, x) with $i \in \mathrm{Ob}(I)$ and $x \in \mathrm{Ob}(Fi)$ and arrows $(f, a) : (i_0, x_0) \to (i_1, x_1)$ for all $f \in I(i_0, i_1)$ and $a \in \mathrm{Arr}(Fi_1)$ with source $(Ff)(x_0)$ and target x_1. The composite of the arrows*

$$(i_0, x_0) \xrightarrow{\quad (f_1, a_1) \quad} (i_1, x_1) \xrightarrow{\quad (f_2, a_2) \quad} (i_2, x_2)$$

is defined by $(f_1 \cdot f_2, (Ff_2)(a_1) \cdot a_2)$.

Note that the Grothendieck construction comes equipped with a canonical projection ('opfibration') functor $p : \int^I F \to I$ defined by $(i, x) \mapsto i$ and $(f, a) \mapsto f$.

Consider again the case of a functor $\alpha : H \to$ **Cat** $: e_H \mapsto G$ corresponding to a group action as above. The Grothendieck construction $\int^H \alpha$ on this functor is the category with a single object (e_H, e_G) and set of arrows (h, g) for all $h \in H$ and $g \in G$. Composition of arrows is given by

$$(h_1, g_1)(h_2, g_2) = (h_1 h_2, g_1^{h_2} g_2)$$

Thus we have

Proposition 3.1.8 *The Grothendieck construction applied to a functor $\alpha : H \to \mathbf{Cat}$ corresponding to a group action of H on G gives the usual semidirect product E of G by H. The canonical projection p corresponds to the epimorphism $E \to H$, $(h, g) \mapsto h$, which gives the usual split short exact sequence of groups*

$$1 \to G \to E \to H \to 1$$

The following definition is due to Thomason [38].

Definition 3.1.9 *Suppose that $F : I \to \mathbf{Cat}$ is any diagram of categories and functors. Then the homotopy colimit of F,* hocolim(F)*, is defined by the Grothendieck construction on F.*

In particular suppose $\alpha : H \to \mathbf{Cat} : e_H \mapsto G$ is a functor corresponding to a group action. Then $E = \operatorname{hocolim}(\alpha)$ is the semidirect product of G by H. When considering the effect under the functor $C : \mathbf{Cat} \to \mathbf{Crs}$ we will see that there is a definition of hocolim in $\mathbf{Crs}$ such that $\operatorname{hocolim}(\alpha \circ C)$ is a strong deformation retract of $C(\operatorname{hocolim}(\alpha))$. That is, using the semidirect product decomposition we have an aspherical resolution for E which is smaller than the standard resolution. It will turn out that the small resolution has a presentation with the same generators (but different boundary relations) as that for $C(G) \otimes C(H)$. For this reason the new resolution of the semidirect product may be considered as a *perturbation* of the small resolution for the direct product, and will lead to a definition of a *twisted tensor product* $C(G) \otimes_\alpha C(H)$. Also the small resolution for the semidirect product will be free in our usual sense.

3.2 Simplicial Homotopy Colimits

3.2.1 Introduction to coends

In this section we give a brief review of the definitions and calculus of certain limits and colimits termed *ends* and *co-ends* respectively. A basic reference for this section is [32].

Definition 3.2.1 *Suppose $\mathbf{C}$ is an arbitrary complete category, I a small category, and F a functor $I^{\mathrm{op}} \times I \to \mathbf{C}$. Then the* end *of F over I, written $\int_i F(i,i)$ is given by the following equaliser in $\mathbf{C}$*

$$\int_i F(i,i) \dashrightarrow \prod_{i \in \mathrm{Ob}(I)} F(i,i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{f \in I(i_i, i_2)} F(i_1, i_2)$$

where a and b are those arrows defined componentwise by

$$a \circ \pi_f \;=\; \pi_{i_1} \circ F(i_1, f) \quad \textit{and} \quad b \circ \pi_f \;=\; \pi_{i_2} \circ F(f, i_2)$$

Often $\mathbf{C}$ will be an 'algebraically-defined' category, and in this case, working with elements, we can define the end as follows. Let A be the object of $\mathbf{C}$ formed from the $\mathrm{Ob}(I)$-indexed product of the objects $F(i,i)$, and write the elements of A as sequences $(x_i)_{i \in \mathrm{Ob}(I)}$. For f an arrow of $I(j,k)$ we write $f^i_* : F(i,j) \to F(i,k)$ for the morphism $F(i,f)$, and $f^*_i : F(k,i) \to F(j,i)$ for the morphism $F(f,i)$. Then $\int_i F(i,i)$ is the subobject of A consisting of those sequences satisfying the relation $f^j_*(x_j) = f^*_k(x_k)$ in $F(j,k)$ for all arrows $f : j \to k$ in I.

$$\begin{array}{ccc} & & F(j,j) \\ & & \Big\downarrow {\scriptstyle f^j_*} \\ F(k,k) & \xrightarrow{\;f^*_k\;} & F(j,k) \end{array}$$

Example 3.2.2 If F, G are functors from I to $\mathbf{C}$, then there is a functor

$$\begin{array}{ccc} I^{\mathrm{op}} \times I & \longrightarrow & \mathbf{Set} \\ (i,j) & \longmapsto & \mathbf{C}(F(i), G(j)) \end{array}$$

defined by the hom-sets, and the end $\displaystyle\int_i \mathbf{C}(F(i), G(i))$ is just the set of natural transformations from F to G.

Dually, there is:

Definition 3.2.3 *Suppose $\mathbf{C}$ is an arbitrary cocomplete category, I a small category, and F a functor $I^{\mathrm{op}} \times I \to \mathbf{C}$. Then the* coend *of F over I, written $\int^i F(i,i)$ is given by the following coequaliser in $\mathbf{C}$*

$$\coprod_{f \in I(i_i, i_2)} F(i_2, i_1) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{i \in \mathrm{Ob}(I)} F(i,i) \dashrightarrow \int^i F(i,i)$$

where a and b are those arrows defined componentwise by

$$\iota_f \circ a \;=\; F(i_2, f) \circ \iota_{i_2} \quad \textit{and} \quad \iota_f \circ b \;=\; F(f, i_1) \circ \iota_{i_1}$$

In suitable categories $\mathbf{C}$ we can define coends more explicitly in terms of generators and relations. Let A be the $\mathrm{Ob}(I)$-indexed free product of the objects $F(i,i)$ in $\mathbf{C}$. Then

$\int^i F(i,i)$ is the quotient object of A given by imposing the relations $f_*^k(x) = f_j^*(x)$ for each $f : j \to k$ in I and x in $F(k,j)$.

$$\begin{array}{ccc} F(k,j) & \xrightarrow{f_*^k} & F(k,k) \\ {\scriptstyle f_j^*}\downarrow & & \\ F(j,j) & & \end{array}$$

Since ends and coends may be viewed in terms of limits and colimits, they are preserved by the appropriate adjoint functors and by hom-set functors. Suppose F, G are functors from $I^{\mathrm{op}} \times I$ to $\mathbf{C}$, $\mathbf{D}$ respectively, that $L : \mathbf{C} \to \mathbf{D}$ is a functor with right adjoint R, and that C is an object of $\mathbf{C}$. Then the following natural isomorphisms hold when the appropriate ends and coends exist:

$$\begin{aligned} R\left(\int_i G(i,i)\right) &\cong \int_i R\left(G(i,i)\right) \\ L\left(\int^i F(i,i)\right) &\cong \int^i L\left(F(i,i)\right) \\ \mathbf{C}\left(C, \int_i F(i,i)\right) &\cong \int_i \mathbf{C}\left(C, F(i,i)\right) \\ \mathbf{C}\left(\int^i F(i,i), C\right) &\cong \int_i \mathbf{C}\left(F(i,i), C\right) \end{aligned}$$

Ends and coends also have nice properties with respect to natural transformations. Given functors $F, G : I^{\mathrm{op}} \times I \to \mathbf{C}$ and a natural transformation $\theta : F \Rightarrow G$ there are universal morphisms in $\mathbf{C}$

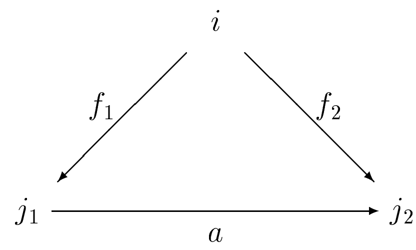
$$\int_i F(i,i) \xrightarrow{\int_i \theta_{i,i}} \int_i G(i,i) \qquad\qquad \int^i F(i,i) \xrightarrow{\int^i \theta_{i,i}} \int^i G(i,i)$$

providing the appropriate ends and coends exist. Furthermore this process is functorial in that it takes identity and composite natural transformations to the corresponding identity and composite morphisms.

3.2.2 Homotopy colimits of simplicial sets...

In this section we recall the definition of homotopy colimits in **SimpSet** from [4].

Suppose I is a small category. Recall that for any object i of I the cocomma category i/I is that category with objects given by the arrows $f : i \to j$ in I for all objects j of I, and arrows from $f_1 : i \to j_1$ to $f_2 : i \to j_2$ given by arrows $a : j_1 \to j_2$ of I such that the triangle



commutes. Composition in i/I is defined by that in I. Now an arrow $g : i_1 \to i_2$ induces a functor $g/I : i_2/I \to i_1/I$ by precomposition. Thus the cocomma construction defines a contravariant functor $(-/I) : I^{\mathrm{op}} \to \mathbf{Cat}$.

Suppose we have a small diagram of simplicial sets given by a functor $F : I \to \mathbf{SimpSet}$. Consider the functor $\mathrm{Ner}(-/I) \cdot F$ defined by

$$I^{\mathrm{op}} \times I \xrightarrow{\mathrm{Ner}(-/I) \times F} \mathbf{SimpSet} \times \mathbf{SimpSet} \xrightarrow{\times} \mathbf{SimpSet}$$

Definition 3.2.4 *The* homotopy colimit *of a diagram $F : I \to$ **SimpSet** is given by the coend of* $\mathrm{Ner}(-/I) \cdot F$ *over I:*

$$\mathrm{hocolim}(F) \cong \int^{i} \mathrm{Ner}(i/I) \cdot F(i)$$

For F a functor as above, let $\Psi(F)$ be the bisimplicial set with (p, q)-simplices given by pairs (a, b) where $a = [i_0, f_1, i_1, \ldots, f_p, i_p] \in \mathrm{Ner}(I)_p$ and $b \in F(i_0)_q$. The vertical face and degeneracy maps are defined by those of the simplicial sets $F(i_0)$ and the horizontal face and degeneracy maps by those of $\mathrm{Ner}(I)$, except that d_0^{h} is defined by

$$(a, b) \mapsto (d_0 a, b^{f_1})$$

where we are writing b^{f_1} for $(F(f_1))(b)$. That this does define a bisimplicial set is clear; $d_0^{\mathrm{h}} d_0^{\mathrm{h}} = d_0^{\mathrm{h}} d_1^{\mathrm{h}}$ follows from the functoriality of F, and the vertical face and degeneracy functions commute with d_0^{h} since each $F(f_1)$ is a morphism of simplicial sets.

Proposition 3.2.5 *Suppose F is a functor $I \to$ **SimpSet** as above. Then there is a natural isomorphism*

$$\Psi(F) \cong \int^{i} \mathrm{Ner}(i/I) \times^{(2)} F(i)$$

between the bisimplicial set $\Psi(F)$ and the coend of

$$I^{\mathrm{op}} \times I \xrightarrow{\mathrm{Ner}(-/I) \times F} \mathbf{SimpSet} \times \mathbf{SimpSet} \xrightarrow{\times^{(2)}} \mathbf{BiSimpSet}$$

Proof: Elements of $\mathrm{Ner}(i/I)_p$ may be written as pairs (f_0, a) for $f_0 : i \to i_0$ an arrow of I and $a = [i_0, f_1, i_1, \ldots, f_p, i_p]$ in $\mathrm{Ner}(I)_p$, and the face and degeneracy maps act by

$$s_r(f_0, a) = (f_0, s_r a), \quad d_r(f_0, a) = (f_0, d_r a), \ (r > 0), \quad d_0(f_0, a) = (f_0 \circ f_1, d_0 a)$$

Let A be the disjoint union of the $\mathrm{Ner}(i/I) \times^{(2)} F(i)$. Then elements of $A_{p,q}$ are given by triples (f_0, a, b) for f_0, a as above and $b \in F(i)_q$, and the relation $f^j_*(x) \sim f^*_k(x)$ becomes

$$(f_0, a, (Ff)(b)) \;\sim\; (f \circ f_0, a, b)$$

Each (f_0, a, b) is thus related to a unique element of the form (e, a, b') with e the identity arrow at i_0 and $b' \in F(i_0)_q$, given by $(F(f_0))(b)$. The faces and degeneracies of an element of the form (e, a, b) are again of this form, except for the zeroth horizontal face for which we have

$$d^{\mathrm{h}}_0(e, a, b) \;=\; (f_1, d_0 a, b) \;\sim\; (e, d_0 a, (F(f_1))(b))$$

Thus the quotient of A by $\sim$ is naturally isomorphic to $\Psi(F)$, and we have the result. $\square$

Corollary 3.2.6 *The homotopy colimit of a diagram F as above is naturally isomorphic to the diagonal of the bisimplicial set $\Psi(F)$.*

Proof: The functor $\mathrm{Diag} : \mathbf{BiSimpSet} \to \mathbf{SimpSet}$ has a right adjoint, which takes a simplicial set K to the bisimplicial set X with $X_{p,q} = \mathbf{SimpSet}(\triangle^p \times \triangle^q, K)$. Thus Diag commutes with coends and we have

$$\begin{aligned} \int^i \mathrm{Ner}(i/I) \cdot F(i) \;&\cong\; \int^i \mathrm{Diag}\left(\mathrm{Ner}(i/I) \times^{(2)} F(i)\right) \\ &\cong\; \mathrm{Diag}\left(\int^i \mathrm{Ner}(i/I) \times^{(2)} F(i)\right) \end{aligned}$$

that is, $\mathrm{hocolim}(F) \cong \mathrm{Diag}\,\Psi(F)$. $\square$

3.2.3 . . . using the Artin-Mazur diagonal

In this section we will introduce an alternative definition of homotopy colimits in **SimpSet** which has slightly nicer properties with respect to the nerve functor from **Cat**.

We considered in section 2.2.1 the Artin-Mazur diagonal $\mathbf{BiSimpSet} \xrightarrow{\nabla} \mathbf{SimpSet}$. Zisman has shown [16, *loc. cit.*] that the comparison map $\mathrm{Diag}(X) \longrightarrow \nabla(X)$ given by

$$x_{n,n} \longmapsto \left((d^{\mathrm{h}}_1)^n x_{n,n}, (d^{\mathrm{h}}_2)^{n-1} d^{\mathrm{v}}_0 x_{n,n}, \ldots, (d^{\mathrm{h}}_{i+1})^{n-i} (d^{\mathrm{v}}_0)^i x_{n,n}, \ldots, (d^{\mathrm{v}}_0)^n x_{n,n}\right)$$

induces a weak homotopy equivalence between Diag and ∇. For the bisimplicial sets which arose in the previous section, we have the following stronger result.

Proposition 3.2.7 *Given a functor $F : I \to \mathbf{SimpSet}$, the simplicial sets $\mathrm{Diag}\,\Psi(F)$ and $\nabla\Psi(F)$ are naturally isomorphic.*

Proof: Elements of $\nabla\Psi(F)_n$ are given by those $(n+1)$-tuples of pairs $(a_k, b_k)_{0\leq k\leq n}$ with $a_k \in \mathrm{Ner}(I)_k$ and $b_k \in F(d_1^k a_k)_{n-k}$, and satisfying $d_0^{\mathrm{v}}(a_k, b_k) = d_{k+1}^{\mathrm{h}}(a_{k+1}, b_{k+1})$ in $\Psi(F)$.

Writing $a_k = [f_{k,1}, f_{k,2}, \ldots, f_{k,k}]$ the conditions $(a_k, d_0 b_k) = (d_{k+1}a_{k+1}, b_{k+1})$ that the elements must satisfy become $f_{j,k} = f_{n,k}$ and $b_k = d_0^k b_0$. Thus an n-simplex of $\nabla\Psi(F)$ is completely determined by the n-simplices $a_n = [f_1, f_2, \ldots, f_n]$ and $b_0 \in F(sf_1)_n$. Conversely any pair (a, b) with $b \in F(d_1^n a)_n$ gives an n-simplex $(d_{k+1}^{n-k}a, d_0^k b)_{0\leq k\leq n}$ of $\nabla\Psi(F)$. Under this correspondence the face and degeneracy maps in $\nabla\Psi(F)$ become $d_0(a,b) = (d_0 a, d_0 b^{f_1})$, $d_i(a,b) = (d_i a, d_i b)$ for $i \geq 1$, and $s_i(a,b) = (s_i a, s_i b)$.

But this is precisely a description of the elements and the face and degeneracy maps of $\mathrm{Diag}\,\Psi(F)_n$. □

Corollary 3.2.8 *The homotopy colimit of a diagram F as above is naturally isomorphic to the Artin-Mazur diagonal of the bisimplicial set $\Psi(F)$.*

We note that this isomorphism may be thought of as an extension of the result that $\mathrm{Diag}(K \times^{(2)} L) \cong K \times L \cong \nabla(K \times^{(2)} L)$ to a result for a twisted cartesian product. The existence of the extended result is mainly due to the fact that the twisting only appears in d_0^{h} which does not occur in the relation $d_0^{\mathrm{v}} x_k = d_{k+1}^{\mathrm{h}} x_{k+1}$ used to define the Artin-Mazur diagonal.

Suppose instead of the bisimplicial set $\Psi(F)$ we consider its transpose $\Psi'(F)$ obtained by interchanging the rôles of horizontal and vertical. Clearly $\Psi'(F)$ and $\Psi(F)$ are weakly homotopy equivalent. Also $\Psi'(F)$ may be defined as the coend of the composite of $F(-) \times^{(2)} \mathrm{Ner}(-/I)$ and the symmetry functor $I^{\mathrm{op}} \times I \to I \times I^{\mathrm{op}}$. Note that although $\mathrm{Diag}\,\Psi(F) \cong \mathrm{Diag}\,\Psi'(F)$, it is not in general true that $\mathrm{Diag}\,\Psi'(F) \cong \nabla\Psi'(F)$ since now the twisting of d_0^{v} interacts with the definition of ∇.

We make the following alternative definition of homotopy colimits in **SimpSet**.

Definition 3.2.9 *For F a diagram $I \to$ **SimpSet**, $\mathrm{hocolim}'(F)$ is the simplicial set given by $\nabla\Psi'(F)$.*

Proposition 3.2.10 *For F a functor $I \to$ **SimpSet**, there is a natural comparison map θ' from $\mathrm{Diag}\,\Psi'(F)$ to $\nabla\Psi'(F)$ defined by*

$$(b, a) \;\mapsto\; \Big((d_1^n b, a),\ (d_2^{n-1} b^{f_1}, [f_2, \ldots, f_n]),\ (d_3^{n-2} b^{f_1 f_2}, [f_3, \ldots, f_n]), \ldots, \\ (d_{k+1}^{n-k} b^{f_1 f_2 \cdots f_k}, [f_{k+1}, f_{k+2}, \ldots, f_n]), \ldots, (b^{f_1 f_2 \cdots f_n}, [\,]_{tf_n}) \Big)$$

where $a = [f_1, f_2, \ldots, f_n] \in \mathrm{Ner}(I)_n$ and $b \in F(sf_1)_n$, and we write b^f for $(F(f))(b)$. If $F(f)$ is an isomorphism for each arrow f of I (in particular, if I is a groupoid) the comparison map becomes an isomorphism.

Proof: Since the twisted face d_0^{v} is not used in the definition of the faces or degeneracies of ∇, it is a routine check that θ' is a simplicial map. Suppose $(b_k, a_k)_{0\le k\le n}$ is an arbitrary n-simplex of $\nabla\Psi'(F)$ with $a_k = [f_{k,1}, f_{k,2}, \ldots, f_{k,n-k}]$ and $b_k \in F(sf_{k,1})_k$. Then the condition $d_0^{\mathrm{v}}(b_k, a_k) = d_{k+1}^{\mathrm{h}}(b_{k+1}, a_{k+1})$ may be written ${b_k}^{f_{k,1}} = d_{k+1}b_{k+1}$ and $[f_{k,2}, \ldots, f_{k,n-k}] = [f_{k+1,1}, \ldots, f_{k+1,n-k-1}]$. Clearly these conditions are satisfied by $\theta'(b, a)$. Also if each $F(f)$ is invertible, then the element $(b_k, a_k)_{0\le k\le n}$ is determined by b_n and a_0, and in particular θ' has a 2-sided inverse. □

As a special case we have

Corollary 3.2.11 *Suppose that $H \xrightarrow{\alpha}$ **Cat** is a functor corresponding to a group action. Then there is a natural isomorphism*

$$\mathrm{hocolim}(\alpha \circ \mathrm{Ner}) \cong \mathrm{hocolim}'(\alpha \circ \mathrm{Ner})$$

Thomason has shown in [38] that for an arbitrary diagram $F : I \to$ **Cat** in **Cat** there is a weak homotopy equivalence between the nerve of the Grothendieck construction on F and the homotopy colimit of the diagram $F \circ \mathrm{Ner}$ in **SimpSet**, and for this reason the Grothendieck construction is thought of as defining homotopy colimits in **Cat**. It is interesting that replacing hocolim by hocolim$'$ gives a natural isomorphism rather than a weak equivalence:

Theorem 3.2.12 *Suppose I is a small category and F an arbitrary functor $I \to$ **Cat**. Then the nerve of the Grothendieck construction on F and the Artin-Mazur diagonal of $\Psi'(F \circ \mathrm{Ner})$ are isomorphic.*

$$\mathrm{Ner}\left(\int^I F\right) \cong \nabla\Psi'(F \circ \mathrm{Ner})$$

Proof: An n-simplex of $\nabla\Psi'(F\circ\mathrm{Ner})$ is given by an $(n+1)$-tuple of pairs $(b_k, a_k)_{0\le k\le n}$ where

$$\begin{array}{rcll} a_k &=& [i_{k,0}, f_{k,1}, i_{k,1}, \ldots, f_{k,n-k}, i_{k,n-k}] & \in \mathrm{Ner}(I) \\ b_k &=& [x_{k,0}, g_{k,1}, x_{k,1}, \ldots, g_{k,k}, x_{k,k}] & \in \mathrm{Ner}(F(i_{k,0})) \end{array}$$

and these data must satisfy the conditions

$$\begin{array}{rcl} [i_{k,1}, f_{k,2}, i_{k,2}, \ldots, f_{k,n-k}, i_{k,n-k}] &=& [i_{k+1,0}, f_{k+1,1}, i_{k+1,1}, \ldots, f_{k+1,n-k-1}, i_{k+1,n-k-1}] \\ [x_{k,0}^{f_{k,1}}, g_{k,1}^{f_{k,1}}, x_{k,1}^{f_{k,1}}, \ldots, g_{k,k}^{f_{k,1}}, x_{k,k}^{f_{k,1}}] &=& [x_{k+1,0}, g_{k+1,1}, x_{k+1,1}, \ldots, g_{k+1,k}, x_{k+1,k}] \end{array}$$

where as usual we write the operation of the functor $F(f)$ as a right action $x \mapsto x^f$, $g \mapsto g^f$. These conditions imply that the $(n+1)$-tuple is completely determined by $a_0 = [i_{0,0}, f_{0,1}, i_{0,1}, \ldots, f_{0,n}, i_{0,n}]$ and the elements $[x_{0,0}, g_{1,1}, x_{1,1}, \ldots, g_{n,n}, x_{n,n}]$. Conversely any data $c = [i_0, f_1, i_1, \ldots, f_n, i_n]$ in $\mathrm{Ner}(I)$ and $d = [x_0, g_1, x_1, \ldots, g_n, x_n]$ with $x_k \in \mathrm{Ob}(F(i_k))$ and $g_k \in (F(i_k))(x_{k-1}^{f_k}, x_k)$ determine an n-simplex of $\nabla\Psi'(F \circ \mathrm{Ner})$ by

$$\begin{array}{rcl} i_{j,k} &=& i_{j+k} \\ f_{j,k} &=& f_{j+k} \\ x_{j,k} &=& {x_j}^{f_{j+1}\cdots f_k} \\ g_{j,k} &=& {g_j}^{f_{j+1}\cdots f_k} \end{array} \qquad (3.1)$$

and these processes are inverse. Note that if $\mathrm{hocolim}(F) = \int^I F$ is the category given by the Grothendieck construction on F then an n-simplex of Ner(hocolim(F)) is given by a string

$$[(i_0, x_0), (f_1, g_1), (i_1, x_1), (f_2, g_2), (i_2, g_2), \ldots, (f_n, g_n), (i_n, x_n)]$$

where $i_k \in \mathrm{Ob}(I)$, $f_k \in I(i_{k-1}, i_k)$, x_k is an object of $F(i_k)$, and g_k is an arrow of $(F(i_k))$ with source $(Ff_k)(x_{k-1})$ and target x_k. But this corresponds precisely to the data (c, d) above, so (3.1) gives a bijection

$$\mathrm{Ner}(\mathrm{hocolim}(F))_n \xrightarrow{\phi_n} \nabla\Psi'(F \circ \mathrm{Ner})_n$$

It is straightforward to check that this defines a morphism of simplicial sets. □

Thus if we define homotopy colimits in **Cat** by the Grothendieck construction and in **SimpSet** by hocolim$'$ rather than hocolim, we have that the nerve functor preserves homotopy colimits up to natural isomorphism, and Thomason's weak equivalence may be considered in the context of that of Zisman between Diag and ∇ and also that between a bisimplicial set and its transpose.

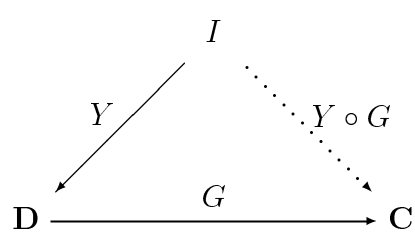
3.3 Homotopy Colimits of Crossed Complexes

3.3.1 Kan extensions and monoidal categories

We recall here a few details of the theories of (left) Kan extensions and of closed (symmetric) monoidal categories. These concepts will then be used to try to formulate a general framework for the constructions of the rest of the chapter.

The left Kan extension construction may be considered in a similar way to that of induced modules discussed in section 1.2.1. Suppose I, **C** and **D** are categories, and Y is a functor $I \longrightarrow \mathbf{D}$. Composition with Y then gives an induced functor

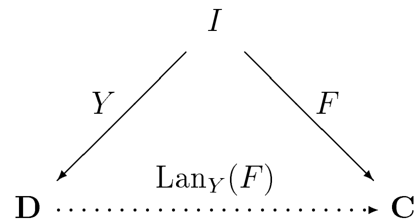
$$[\mathbf{D}, \mathbf{C}] \xrightarrow{Y^*} [I, \mathbf{C}]$$



between the functor categories. In many cases (for example if **C** is cocomplete) the functor Y^* will have a left adjoint, written Lan_Y. For a particular functor F from I to

$\mathbf{C}$, the functor $\mathrm{Lan}_Y(F)$ from $\mathbf{D}$ to $\mathbf{C}$ is termed the *left Kan extension* of F along Y.

$$[I, \mathbf{C}] \xrightarrow{\mathrm{Lan}_Y} [\mathbf{D}, \mathbf{C}]$$



The case we will be most interested in is when $\mathbf{D}$ is itself given as the functor category $[I^{\mathrm{op}}, \mathbf{Set}]$ and Y is the functor

$$\begin{array}{rcl} I & \xrightarrow{\;Y\;} & [I^{\mathrm{op}}, \mathbf{Set}] \\ i & \longmapsto & I(-, i) \end{array}$$

defined by the hom-sets and the composition in I. For any functor $I^{\mathrm{op}} \xrightarrow{G} \mathbf{Set}$ and object i of I the Yoneda lemma gives a natural bijection between elements of the set $G(i)$ and natural transformations $I(-, i) \Longrightarrow G$, and taking G to be the representable functor $I(-, j)$ shows that the functor Y is full and faithful. Thus I may be regarded as a full subcategory of $[I^{\mathrm{op}}, \mathbf{Set}]$. We note the following well-known result that Kan extensions along this embedding may be given by a coend formula.

Proposition 3.3.1 *Suppose* $\mathbf{C}$ *is cocomplete and* F *is a functor* $I \longrightarrow \mathbf{C}$*. Then there is a natural isomorphism between the left Kan extension of* F *along the Yoneda embedding* $I \longrightarrow [I^{\mathrm{op}}, \mathbf{Set}]$ *and the functor*

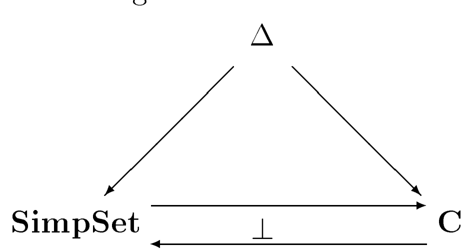
$$\begin{array}{rcl} [I^{\mathrm{op}}, \mathbf{Set}] & \longrightarrow & \mathbf{C} \\ G & \longmapsto & \int^{i} F(i) \cdot G(i) \end{array}$$

where $C \cdot S$ *denotes the coproduct of copies of* C *indexed by the elements of the set* S*. Furthermore,* $Y \circ \mathrm{Lan}_Y(F) \cong F$ *and* $\mathrm{Lan}_Y(F)$ *itself has a right adjoint given by*

$$\begin{array}{rcl} \mathbf{C} & \longrightarrow & [I^{\mathrm{op}}, \mathbf{Set}] \\ C & \longmapsto & I(F(-), C) \end{array}$$

Proof: Follows from standard manipulations with the end calculus. See for example [32, X.4]. □

In particular consider the embedding of Δ into **SimpSet**. Any functor $\Delta \longrightarrow \mathbf{C}$ then gives a diagram of the following form



Note that we could have given definition 1.3.1 in this way, since the presentation there shows that **SimpSet** $\xrightarrow{\pi}$ **Crs** is freely generated by its values on the representable functors modulo their degeneracies and common faces. The categorisation and nerve functors discussed in section 3.1.1 also fit this pattern.

Definition 3.3.2 A *monoidal* structure on a category $\mathbf{C}$ consists of

1. an object O of $\mathbf{C}$,
2. a functor $\mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$,
3. natural isomorphisms $O \otimes C \xrightarrow{l} C$ and $C \otimes O \xrightarrow{r} C$ for each object C of $\mathbf{C}$,
4. a natural isomorphism $C \otimes (D \otimes E) \xrightarrow{a} (C \otimes D) \otimes E$ for each triple of objects C, D, E of $\mathbf{C}$.

These data are required to satisfy the following commutative diagrams

$$\begin{array}{ccccc}
B \otimes (C \otimes (D \otimes E)) & \xrightarrow{a} & (B \otimes C) \otimes (D \otimes E) & \xrightarrow{a} & ((B \otimes C) \otimes D) \otimes E \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes a} & & & & \Big\uparrow {\scriptstyle a \otimes \mathrm{id}} \\
B \otimes ((C \otimes D) \otimes E) & & \xrightarrow{\qquad\quad a \qquad\quad} & & (B \otimes (C \otimes D)) \otimes E
\end{array}$$

$$\begin{array}{ccc}
C \otimes (O \otimes D) & \xrightarrow{a} & (C \otimes O) \otimes D \\
{\scriptstyle \mathrm{id} \otimes l} \searrow & & \swarrow {\scriptstyle r \otimes \mathrm{id}} \\
& C \otimes D &
\end{array}$$

Definition 3.3.3 A *symmetry* for a monoidal structure $(\mathbf{C}, O, \otimes, l, r, a)$ is given by a natural isomorphism $C \otimes D \xrightarrow{s} D \otimes C$ for each pair of objects C, D of $\mathbf{C}$, satisfying the following commutative diagrams

$$\begin{array}{ccccccc}
C \otimes (D \otimes E) & \xrightarrow{a} & (C \otimes D) \otimes E & \xrightarrow{s} & E \otimes (C \otimes D) & \xrightarrow{a} & (E \otimes C) \otimes D \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes s} & & & & & & \Big\uparrow {\scriptstyle s \otimes \mathrm{id}} \\
C \otimes (E \otimes D) & & & \xrightarrow{\qquad\quad a \qquad\quad} & & & (C \otimes E) \otimes D
\end{array}$$

$$\begin{array}{ccc}
O \otimes C & \xrightarrow{s} & C \otimes O \\
{\scriptstyle l} \searrow & & \swarrow {\scriptstyle r} \\
& C &
\end{array}
\qquad
\begin{array}{ccc}
C \otimes D & = & C \otimes D \\
{\scriptstyle s} \searrow & & \nearrow {\scriptstyle s} \\
& D \otimes C &
\end{array}$$

The commutative diagrams in definitions 3.3.2 and 3.3.3 are known collectively as the MacLane-Kelly equations. It follows from a *coherence theorem* [31, 29] that any diagram made up of instances of l, r, s and a will commute.

Note that any category with finite products has a *cartesian* symmetric monoidal structure, with $\otimes$ given by the binary product and O by the terminal object. The isomorphisms l, r, s and a are given by the universal properties of the limits.

Definition 3.3.4 A symmetric monoidal category $(\mathbf{C}, O, \otimes, l, r, a, s)$ is said to be *closed* if for each object D of $\mathbf{C}$ the functor $- \otimes D$ has a right adjoint, written $[D, -]$.

$$\mathbf{C}(C \otimes D, E) \;\cong\; \mathbf{C}(C, [D, E])$$

The counits of these adjuctions give an evaluation map $[C, D] \otimes C \xrightarrow{\text{ev}} D$, corresponding to $\mathrm{id}_{[C,D]}$. Using this, $[-, D]$ can be considered as contravariantly functorial in the first variable, where for $f\colon C \longrightarrow C'$ the morphism $[f, D]$ corresponds under the adjunction to

$$[C', D] \otimes C \xrightarrow{\quad \mathrm{id} \otimes f \quad} [C', D] \otimes C' \xrightarrow{\quad \text{ev} \quad} D$$

Also we have internal adjunction isomorphisms $[C \otimes D, E] \rightleftarrows [C, [D, E]]$ corresponding to

$$[C, [D, E]] \otimes (C \otimes D) \xrightarrow{\quad a \quad} ([C, [D, E]] \otimes C) \otimes D \xrightarrow{\text{ev} \otimes \mathrm{id}} [D, E] \otimes D \xrightarrow{\quad \text{ev} \quad} E$$

$$([C \otimes D, E] \otimes C) \otimes D \xrightarrow{\quad a^{-1} \quad} [C \otimes D, E] \otimes (C \otimes D) \xrightarrow{\quad \text{ev} \quad} E$$

and internal composition morphisms $[D, E] \otimes [C, D] \xrightarrow{\circ} [C, E]$ corresponding to

$$([D, E] \otimes [C, D]) \otimes C \xrightarrow{\quad a^{-1} \quad} [D, E] \otimes ([C, D] \otimes C) \xrightarrow{\mathrm{id} \otimes \text{ev}} [D, E] \otimes D \xrightarrow{\quad \text{ev} \quad} E$$

Our main example of a monoidal closed category is **Crs**, the category of crossed complexes of groupoids. The tensor product and internal hom were explicitly defined for this category in [12] using a natural definition of a monoidal closed structure on the equivalent category of 'cubical' ω-groupoids [9].

Other examples are given by cartesian closed categories, for example **Cat** as discussed earlier. Also note that **SimpSet** is cartesian closed. In fact for any small category $\mathbf{C}$ and functors $F, G\colon \mathbf{C} \longrightarrow \mathbf{Set}$ the product functor $F \times G$ can be defined pointwise using the cartesian product of sets, and a functor $[F, G] : \mathbf{C} \longrightarrow \mathbf{Set}$ can be defined, using the Yoneda embedding Y, by mapping an object C to the set of natural transformations $\mathrm{Nat}(Y_C \times F, G)$. This gives a cartesian closed structure on the functor category $[\mathbf{C}, \mathbf{Set}]$, since

$$\begin{aligned}\mathrm{Nat}\,(E, [F, G]) \;\cong\; \int_C \mathbf{Set}\,(EC, \mathrm{Nat}(Y_C \times F, G)) \;\cong\; \int_C \mathrm{Nat}\,(EC \cdot Y_C \times F, G) \\ \cong\; \mathrm{Nat}\left(\int^C EC \cdot Y_C \,\times\, F,\; G\right) \;\cong\; \mathrm{Nat}(E \times F, G)\end{aligned}$$

We can now state our aim: to investigate the notion of homotopy colimits in cocomplete closed monoidal categories $\mathbf{C}$ for which there is a 'good' functor

$$\Delta \xrightarrow{\ \pi\ } \mathbf{C}$$

As above, π induces a pair of adjoint functors

$$\begin{array}{ccc} & \Delta & \\ \swarrow & & \searrow^{\pi} \\ \mathbf{SimpSet} & \underset{\mathrm{N}}{\overset{\pi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathbf{C} \end{array}$$

which can be defined by

$$\pi(K) \;=\; \int^{[n]} \pi([n]) \cdot K_n \qquad \text{and} \qquad \mathrm{N}(C)_n \;=\; \mathbf{C}(\pi([n]), C)$$

We have notions of homotopy and deformation retraction in $\mathbf{C}$, defined by the tensor product and the unit interval object $\mathcal{I}$ given by $\pi([1])$, and we can make precise the word 'good' above by saying that π must satisfy an Eilenberg-Zilber theorem with repect to these notions.

We will concentrate on the case where $\mathbf{C} = \mathbf{Crs}$, the category of crossed complexes, although we believe a more general theory proceeds similarly. The category of ∞-categories is also believed to be a suitable candidate, using the orientals of Street [37] and the monoidal biclosed structure of Steiner [34]. This category has been shown by Golasiński [23] and by Kapranov and Voevodsky [28] to model all homotopy types.

3.3.2 Homotopy colimits in Crs

We now propose a definition of homotopy colimits for diagrams of crossed complexes. Consider first the functor

$$\mathbf{Crs} \times \mathbf{SimpSet} \xrightarrow{\ *\ } \mathbf{SimpCrs}$$

which takes a crossed complex C and a simplicial set K to the simplicial crossed complex $C * K$ with $(C * K)_{p,q} = C_p \times K_q$, crossed complex structures given by K_q-indexed coproducts of C and simplicial structures by C_p-indexed coproducts of K. Composing with the simplicial total functor defined in section 1.3.3 gives a functor

$$\mathbf{Crs} \times \mathbf{SimpSet} \xrightarrow{\ \overline{\otimes}\ } \mathbf{Crs}$$

However the definitions of S-Tot and $\otimes$ given in chapter 1 show that

$$\text{S-Tot}(C * K) \;\cong\; \mathrm{Tot}(C \otimes^{(2)} \pi K) \;\cong\; C \otimes \pi K$$

and we use this as a slightly more explicit definition.

Definition 3.3.5 *If C is a crossed complex and K a simplicial set, then their tensor product $C\overline{\otimes}K$ is given by the crossed complex $C \otimes \pi K$.*

$$\begin{array}{ccc} \mathbf{Crs} \times \mathbf{SimpSet} & \overset{\overline{\otimes}}{\dashrightarrow} & \mathbf{Crs} \\ & {\scriptstyle \mathrm{id} \times \pi} \searrow \quad \nearrow {\scriptstyle \otimes} & \\ & \mathbf{Crs} \times \mathbf{Crs} & \end{array}$$

Suppose I is a small category and we have a diagram of crossed complexes given by a functor $I \xrightarrow{F} \mathbf{Crs}$. Consider the functor $\mathrm{Ner}(-/I)\overline{\otimes}F$ defined by

$$I^{\mathrm{op}} \times I \xrightarrow{\cong} I \times I^{\mathrm{op}} \xrightarrow{F \times \mathrm{Ner}(-/I)} \mathbf{Crs} \times \mathbf{SimpSet} \xrightarrow{\overline{\otimes}} \mathbf{Crs}$$

We can now make the following definition

Definition 3.3.6 *The* homotopy colimit *of a diagram $I \xrightarrow{F} \mathbf{Crs}$ of crossed complexes and their homomorphisms is given by the coend of $F\overline{\otimes}\mathrm{Ner}(-/I)$ over I:*

$$\mathrm{hocolim}(F) \;\; \cong \;\; \int^{i} F(i)\overline{\otimes}\mathrm{Ner}(i/I)$$

This definition may also be given as the total complex of a 'twisted' simplicial crossed complex $\Phi(F)$. In a manner similar to the definition of Ψ (or rather Ψ') in section 3.2.2, we let $\Phi(F)$ be the simplicial crossed complex with elements in $\Phi(F)_{p,q}$ given by pairs (c, a) where $a = [i_0, f_1, i_1, \ldots, f_q, i_q] \in \mathrm{Ner}(I)_q$ and $c \in F(i_0)_p$. The (horizontal) source, target, identity, composition, action and boundary maps are defined by those of the crossed complexes $F(i)$, and the (vertical) face and degeneracy maps are defined by those of $\mathrm{Ner}(I)$, except for d_0 which repaces i_0 by i_1 and so must also translate the first component from $F(i_0)$ to $F(i_1)$:

$$(c, a) \overset{d_0}{\longmapsto} (c^{f_1}, d_0 a)$$

where we write c^{f_1} for $(F(f_1))(c)$. Clearly this defines a simplicial crossed complex.

Analogously to (the transpose of) proposition 3.2.5 we have

Proposition 3.3.7 *Suppose F is a functor $I \to \mathbf{Crs}$ as above. Then there is a natural isomorphism*

$$\Phi(F) \cong \int^{i} F(i) * \mathrm{Ner}(i/I)$$

between the simplicial crossed complex $\Phi(F)$ and the coend over I of

$$I^{\mathrm{op}} \times I \xrightarrow{\cong} I \times I^{\mathrm{op}} \xrightarrow{F \times \mathrm{Ner}(-/I)} \mathbf{Crs} \times \mathbf{SimpSet} \xrightarrow{*} \mathbf{SimpCrs}$$

Before we can show that the total complex of the simplicial crossed complex $\Phi(F)$ gives the same thing as the definition of $\mathrm{hocolim}(F)$ above we need the following result.

Proposition 3.3.8 *The simplicial total functor* S-Tot *has a right adjoint.*

Proof: First note that any simplicial crossed complex C may be written as a coend

$$C \;\cong\; \int^q C_{\bullet,q} * \triangle^q$$

of the representable crossed complexes $C_{\bullet,q} * \triangle^q$, and that proposition 1.3.5 shows that the simplicial total functor is freely generated by its values on the representables, modulo degeneracies and common faces, and so

$$\text{S-Tot}(C) \;\cong\; \int^q \text{S-Tot}(C_{\bullet,q} * \triangle^q) \;\cong\; \int^q C_{\bullet,q} \otimes \pi(\triangle^q)$$

For any crossed complex D the functor $\pi(\triangle^\bullet) : \Delta \longrightarrow \mathbf{SimpSet} \xrightarrow{\pi} \mathbf{Crs}$ defines a simplicial crossed complex $[\pi(\triangle^\bullet), D] : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Crs}$, and this gives a right adjoint to the simplicial total functor since

$$\mathbf{Crs}\,(\text{S-Tot}(C), D) \cong \mathbf{Crs}\left(\int^q C_{\bullet,q} \otimes \pi(\triangle^q), D\right) \cong \int_q \mathbf{Crs}\,(C_{\bullet,q} \otimes \pi(\triangle^q), D)$$
$$\cong \int_q \mathbf{Crs}\,(C_{\bullet,q}, [\pi(\triangle^q), D]) \cong \mathbf{SimpCrs}\,(C, [\pi(\triangle^\bullet), D])$$

using a version internal to **Crs** of the result given in example 3.2.2. □

Proposition 3.3.9 *The homotopy colimit of a diagram F of crossed complexes is naturally isomorphic to the total complex of the simplicial crossed complex $\Phi(F)$.*

Proof: By the above proposition the simplicial total functor preserves coends, so

$$\text{S-Tot}\,\Phi(F) \cong \text{S-Tot}\left(\int^i F(i) * \mathrm{Ner}(i/I)\right) \cong \int^i \text{S-Tot}\,(F(i) * \mathrm{Ner}(i/I))$$
$$\cong \int^i F(i) \overline{\otimes} \mathrm{Ner}(i/I) \cong \mathrm{hocolim}(F)$$

as required. □

Following proposition 1.3.5, we can thus give a presentation of the homotopy colimit of F in terms of generators and relations.

Proposition 3.3.10 *Suppose F is a functor from a small category I to the category of crossed complexes of groupoids. Then* $\mathrm{hocolim}(F)$ *is the crossed complex of groupoids given by generators $c_p \otimes a_q \in \mathrm{hocolim}(F)_n$ for all $a_q = [i_0, f_0, i_1, \ldots, f_q, i_q] \in \mathrm{Ner}(I)_q$ and $c_p \in F(i_0)_p$ with $n = p + q$, satisfying the following relations*

1. $c_p \otimes a_q \quad = \quad e_{t(c_p \otimes a_q)}$ *if any* f_k *is an identity arrow*

2. $$\begin{aligned} s(c_1 \otimes a_0) &= sc_1 \otimes a_0 \\ s(c_0 \otimes a_1) &= c_0 \otimes [\,]_{i_0} \\ t(c_0 \otimes a_q) &= c_0^{f_1 \cdots f_q} \otimes [\,]_{i_q} \quad \textit{for } q \geq 1 \\ t(c_p \otimes a_q) &= tc_p^{f_1 \cdots f_q} \otimes [\,]_{i_q} \quad \textit{for } p \geq 1,\ q \geq 0 \end{aligned}$$

3. $$c_p^{c_1} \otimes a_q \quad = \quad (c_p \otimes a_q)^{c_1^{f_1 \cdots f_q} \otimes [\,]_{i_q}} \quad \textit{for } p \geq 2$$

4. $$\begin{aligned} (c_1' \circ c_1) \otimes a_q &= c_1 \otimes a_q \circ (c_1' \otimes a_q)^{c_1^{f_1 \cdots f_q} \otimes [\,]_{i_q}} && \textit{for } q \geq 1 \\ (c_p \circ c_p') \otimes a_q &= c_p \otimes a_q \circ c_p' \otimes a_q && \textit{for } q = 0 \textit{ or } p \geq 2 \end{aligned}$$

5. $$\begin{aligned} \delta_2(c_0 \otimes a_2) &= (c_0 \otimes [f_2])^{-1} \circ (c_0 \otimes [f_1])^{-1} \circ (c_0 \otimes [f_1 f_2]) \\ \delta_2(c_1 \otimes a_1) &= (tc_1 \otimes [f_1])^{-1} \circ (c_1 \otimes [\,]_{i_0})^{-1} \circ (sc_1 \otimes [f_1]) \circ (c_1^{f_1} \otimes [\,]_{i_1}) \\ \delta_3(c_1 \otimes a_2) &= (c_1^{f_1} \otimes [f_2]) \circ (sc_1 \otimes [f_1, f_2])^{c_1^{f_1 f_2} \otimes [\,]_{i_2}} \circ (c_1 \otimes [f_1 f_2])^{-1} \\ &\qquad \circ (tc_1 \otimes [f_1, f_2])^{-1} \circ (c_1 \otimes [f_1])^{tc_1^{f_1} \otimes [f_2]} \\ \delta_p(c_p \otimes a_0) &= \delta^{\mathrm{h}}(c_p \otimes a_0) \quad \textit{for } p \geq 2 \\ \delta_q(c_0 \otimes a_q) &= \delta^{\mathrm{v}}(c_0 \otimes a_q) \quad \textit{for } q \geq 3 \\ \delta_{p+q}(c_p \otimes a_q) &= \delta^{\mathrm{h}}(c_p \otimes a_q) \ \circ \ (\delta^{\mathrm{v}}(c_p \otimes a_q))^{(-1)^p} \quad \textit{otherwise} \end{aligned}$$

where the abbreviations $\delta^{\mathrm{h}}(c_p \otimes a_q)$ *and* $\delta^{\mathrm{v}}(c_p \otimes a_q)$ *stand for the following expressions:*

$$\begin{aligned} \delta^{\mathrm{h}}(c_1 \otimes a_q) &= (tc_1 \otimes a_q)^{-1} \circ (sc_1 \otimes a_q)^{c_1^{f_1 \cdots f_q} \otimes [\,]_{i_q}} \\ \delta^{\mathrm{h}}(c_p \otimes a_q) &= \delta_p c_p \otimes a_q \\ \delta^{\mathrm{v}}(c_p \otimes a_1) &= (c_p^{f_1} \otimes [\,]_{i_1})^{-1} \circ (c_p \otimes [\,]_{i_0})^{tc_p \otimes a_1} \\ \delta^{\mathrm{v}}(c_p \otimes a_q) &= \left(c_p^{f_1} \otimes [f_2, \ldots, f_q]\right)^{-1} \circ \left((c_p \otimes [f_1, \ldots, f_{q-1}])^{tc_p^{f_1 \cdots f_{q-1}} \otimes [f_q]}\right)^{(-1)^{q+1}} \\ &\qquad \circ \prod_{k=1}^{q-1} (c_p \otimes [f_1, \ldots, f_k f_{k+1}, \ldots, f_q])^{(-1)^{k+1}} \end{aligned}$$

and c^f *stands for* $(F(f))(c)$ *as usual.*

The remainder of this section will be concerned with the following result, the proof of which is essentially the fact that a coend of a strong deformation retraction is also a strong deformation retraction.

Theorem 3.3.11 *The functor* **SimpSet** $\xrightarrow{\pi}$ **Crs** *preserves homotopy colimits up to strong deformation retraction.*

Proof: Given a functor $I \xrightarrow{F}$ **SimpSet** we have

$$\begin{aligned} \mathrm{hocolim}(F \circ \pi) &\cong \int^i \pi(F(i)) \otimes \pi(\mathrm{Ner}(i/I)) \\ \pi\left(\mathrm{hocolim}(F)\right) &\cong \pi\left(\int^i F(i) \times \mathrm{Ner}(i/I)\right) \\ &\cong \int^i \pi\left(F(i) \times \mathrm{Ner}(i/I)\right) \end{aligned}$$

since π preserves coends.

Consider the functors

$$I \times I^{\mathrm{op}} \underset{\pi(F(-)) \otimes \pi(\mathrm{Ner}(-/I))}{\overset{\pi(F(-) \times \mathrm{Ner}(-/I))}{\rightrightarrows}} \mathbf{Crs}$$

and note that there are natural transformations a, b between these given by the Eilenberg-Zilber theorem

$$\pi(F(j) \times \mathrm{Ner}(k/I)) \underset{b_{j,k}}{\overset{a_{j,k}}{\rightleftarrows}} \pi(F(j)) \otimes \pi(\mathrm{Ner}(k/I))$$

which satisfy $b \circ a \cong \mathrm{id}$. Taking coends over I thus gives homomorphisms

$$\int^i \pi\left(F(i) \times \mathrm{Ner}(i/I)\right) \underset{\int^i b_{i,i}}{\overset{\int^i a_{i,i}}{\rightleftarrows}} \int^i \pi(F(i)) \otimes \pi(\mathrm{Ner}(i/I))$$

which satisfy $\left(\int^i b_{i,i}\right) \circ \left(\int^i a_{i,i}\right) \cong \mathrm{id}$. That is, we have

$$\pi\left(\mathrm{hocolim}(F)\right) \underset{b}{\overset{a}{\rightleftarrows}} \mathrm{hocolim}(F \circ \pi)$$

with $b \circ a \cong \mathrm{id}$.

Similarly we have natural transformations

$$\pi(F(j) \times \mathrm{Ner}(k/I)) \underset{1_{j,k}}{\overset{0_{j,k}}{\rightrightarrows}} \mathcal{I} \otimes \pi(F(j) \times \mathrm{Ner}(k/I)) \xrightarrow{h_{j,k}} \pi(F(j) \times \mathrm{Ner}(k/I))$$

satisfying $0 \circ h \cong a \circ b$ and $1 \circ h \cong \mathrm{id}$, and hence homomorphisms

$$\int^i \pi(F(i) \times \mathrm{Ner}(i/I)) \rightrightarrows \int^i \mathcal{I} \otimes \pi(F(i) \times \mathrm{Ner}(i/I)) \longrightarrow \int^i \pi(F(i) \times \mathrm{Ner}(i/I))$$

satisfying the corresponding relations. But $\mathcal{I} \otimes -$ also preserves coends, so these may be written as

$$\mathcal{I} \otimes \pi\left(\mathrm{hocolim}(F)\right) \xrightarrow{h} \pi\left(\mathrm{hocolim}(F)\right)$$

with $h : a \circ b \simeq \mathrm{id}$. □

It is this result which justifies our definition of homotopy colimits of diagrams of crossed complexes.

3.3.3 Twisted tensor products

We now apply the machinery of homotopy colimits in the category of crossed complexes to the functor

$$H \xrightarrow{\alpha} \mathbf{Cat} \xrightarrow{\mathrm{Ner}} \mathbf{SimpSet} \xrightarrow{\pi} \mathbf{Crs}$$

where $\alpha\colon H \longrightarrow \mathbf{Cat}\colon e_H \longmapsto G$ is a functor corresponding to a group action. We know by corollary 3.2.11 and theorems 3.2.12 and 3.3.11 that the result is a stong deformation retract of the standard resolution of the Grothendieck construction on α, which is just the semidirect product of G by H. Thus we have a small resolution of the semidirect product.

We can give a presentation of $\mathrm{hocolim}(\alpha \circ \mathrm{Ner} \circ \pi)$ as follows:

Proposition 3.3.12 *Suppose α is a functor corresponding to an action of a group G on a group H as above. Then the homotopy colimit of $\alpha \circ C : H \longrightarrow \mathbf{Crs}$ is the crossed complex of groups given by generators $a_p \otimes b_q$ in dimension $n = p + q$ for all $a_p = [g_1, \dots, g_p] \in \mathrm{Ner}(G)$ and $b_q = [h_1, \dots, h_q] \in \mathrm{Ner}(H)$, subject to the relations:*

1. $a_p \otimes b_q \;=\; *$*, the identity element, if any of the g_i or h_i are identities*
2. $$\begin{aligned}
\delta_2(a_2 \otimes b_0) &= ([g_2] \otimes [\,])^{-1} \circ ([g_1] \otimes [\,])^{-1} \circ ([g_1 g_2] \otimes [\,]) \\
\delta_2(a_0 \otimes b_2) &= ([\,] \otimes [h_2])^{-1} \circ ([\,] \otimes [h_1])^{-1} \circ ([\,] \otimes [h_1 h_2]) \\
\delta_2(a_1 \otimes b_1) &= ([\,] \otimes [h_1])^{-1} \circ ([g_1] \otimes [\,])^{-1} \circ ([\,] \otimes [h_1]) \circ ([{g_1}^{h_1}] \otimes [\,]) \\
\delta_3(a_1 \otimes b_2) &= ([{g_1}^{h_1}] \otimes [h_2]) \circ ([\,] \otimes [h_1, h_2])^{[{g_1}^{h_1 h_2}] \otimes [\,]} \circ ([g_1] \otimes [h_1 h_2])^{-1} \\
&\qquad \circ ([\,] \otimes [h_1, h_2])^{-1} \circ ([g_1] \otimes [h_1])^{[\,] \otimes [h_2]} \\
\delta_3(a_2 \otimes b_1) &= ([g_2] \otimes [h_1])^{-1} \circ ([g_1, g_2] \otimes [\,])^{[\,] \otimes [h_1]} \circ ([g_1 g_2] \otimes [h_1]) \\
&\qquad \circ ([{g_1}^{h_1}, {g_2}^{h_1}] \otimes [\,])^{-1} \circ \left(([g_1] \otimes [h_1])^{[{g_2}^{h_1}] \otimes [\,]}\right)^{-1} \\
\delta_p(a_p \otimes b_0) &= \delta^{\mathrm{h}}(a_p \otimes b_0) \;\; \textit{for } p \geq 3 \\
\delta_q(a_0 \otimes b_q) &= \delta^{\mathrm{v}}(a_0 \otimes b_q) \;\; \textit{for } q \geq 3 \\
\delta_{p+q}(a_p \otimes b_q) &= \delta^{\mathrm{h}}(a_p \otimes b_q) \;\circ\; (\delta^{\mathrm{v}}(a_p \otimes b_q))^{(-1)^p} \;\; \textit{otherwise}
\end{aligned}$$

where the abbreviations $\delta^{\mathrm{h}}(c_p \otimes a_q)$ and $\delta^{\mathrm{v}}(c_p \otimes a_q)$ stand for the following expressions:

$$\begin{aligned}
\delta^{\mathrm{h}}(a_p \otimes b_q) &= ([g_2, \dots, g_p] \otimes [h_1, \dots, h_q])^{-1} \\
&\qquad \circ \left(([g_1, \dots, g_{p-1}] \otimes [h_1, \dots, h_q])^{[{g_p}^{h_1 \cdots h_q}] \otimes [\,]}\right)^{(-1)^{p+1}} \\
&\qquad \circ \prod_{k=1}^{p-1} ([g_1, \dots, g_k g_{k+1}, \dots, g_p] \otimes [h_1, \dots, h_q])^{(-1)^{k+1}} \\
\delta^{\mathrm{v}}(a_p \otimes b_q) &= \left([{g_1}^{h_1}, \dots, {g_p}^{h_1}] \otimes [h_2, \dots, h_q]\right)^{-1} \\
&\qquad \circ \left(([g_1, \dots, g_p] \otimes [h_1, \dots, h_{q-1}])^{[\,] \otimes [h_q]}\right)^{(-1)^{q+1}} \\
&\qquad \circ \prod_{k=1}^{q-1} ([g_1, \dots, g_p] \otimes [h_1, \dots, h_k h_{k+1}, \dots, h_q])^{(-1)^{k+1}}
\end{aligned}$$

By comparing this presentation with that of proposition 3.1.4 it can be seen that the small resolution of a semidirect product of G by H differs from the tensor product $C(G) \otimes C(H)$ only in actions on the terms in the boundary relations, and that the presentation above reduces to the earlier one when the action is trivial. For this reason the crossed complex defined above will be termed a twisted tensor product of $C(G)$ by $C(H)$ over the action, and written as $C(G) \otimes_\alpha C(H)$.

Also it can be seen that our small resolution for the semidirect product is again *free* in that it has no relations except for those given by the degeneracies and the boundary formulae.

Chapter 4

Simplicial Enrichment for Crossed Complexes

4.0 Introduction

Much of the categorical machinery developed for homotopy theory is set in the context of simplicially enriched categories. In this chapter we begin an investigation of the extent to which such techniques apply to the category of crossed complexes. It is shown that the monoidal closed structure induces a simplicially enriched structure on **Crs**, and that the nerve functor

$$\mathbf{Crs} \xrightarrow{\mathrm{N}} \mathbf{SimpSet}$$

can then be given a simplicial enrichment. The natural extension of the fundamental crossed complex functor to the simplicial homs does not respect the enriched composition except up to homotopy, but using the results of section 2.3.2 it is shown that these homotopies satisfy appropriate coherence conditions. The extension of the π/nerve adjunction to the simplicially enriched context is also investigated.

Possible applications of the results found here include the abstract formulation of equivariant homotopy theory in **Crs** [7], and of homotopy colimits of homotopy coherent diagrams of crossed complexes analogous to the formulation for simplicially tensored categories in [3, 15, 16, 17].

The structure of this chapter is as follows. In the first section, we present a simplicial enrichment of the category of crossed complexes. In the second section, the enrichment of the nerve functor is given. The fundamental crossed complex functor is then shown to have a *simplicially coherent enrichment.* In the third section, the adjunction between these functors is extended to a deformation retraction of simplicial homs

$$\mathbf{Crs}_S(\pi K, C) \;\simeq\; \mathbf{SimpSet}_S(K, \mathrm{N}C)$$

The rest of the section is devoted to showing that this homotopy equivalence is natural in C and 'coherently' natural in K.

4.1 A simplicial enrichment for Crs

In this section we show how the Eilenberg-Zilber theorem enables a simplicially-enriched structure to be given to the category of crossed complexes.

First we use the diagonal approximation map to define a natural transformation Ω^t from $\pi(\mathrm{N}(-) \times \mathrm{N}(-))$ to $- \otimes -$ as follows:

$$\pi(\mathrm{N}C \times \mathrm{N}D) \xrightarrow{a} \pi(\mathrm{N}C) \otimes \pi(\mathrm{N}D) \xrightarrow{\varepsilon_C \otimes \varepsilon_D} C \otimes D$$

where ε_C is the counit map $\pi(\mathrm{N}(C)) \to C$ corresponding to $\mathrm{id}_{\mathrm{N}(C)}$ under the $\pi \dashv \mathrm{N}$ adjunction. Using the adjunction again, we thus obtain a natural transformation

$$\mathrm{N}C \times \mathrm{N}D \xrightarrow{\Omega_{C,D}} \mathrm{N}(C \otimes D)$$

The natural transformation Ω has the following more explicit description:

Proposition 4.1.1 *Suppose* $\pi[n] \xrightarrow{f} C$, $\pi[n] \xrightarrow{g} D$ *are elements of* $\mathrm{N}(C)_n$, $\mathrm{N}(D)_n$ *respectively. Then the image of* (f, g) *under* $\Omega_{C,D}$ *is given by the composite*

$$\begin{array}{ccc} \pi[n] & \overset{\Omega_{C,D}(f,g)}{\dashrightarrow} & C \otimes D \\ {\scriptstyle \pi(d)}\downarrow & & \uparrow{\scriptstyle f \otimes g} \\ \pi([n] \times [n]) & \xrightarrow{a} & \pi[n] \otimes \pi[n] \end{array}$$

Proof: For K a simplicial set, elements of K_n correspond to simplicial maps $[n] \to K$, and elements of $\mathrm{N}\,\pi(K)_n$ correspond to homomorphisms $\pi[n] \to \pi(K)$. The unit η of the $\pi \dashv \mathrm{N}$ adjunction may thus be considered as given by

$$\begin{array}{ccc} K & \xrightarrow{\eta_K} & \mathrm{N}\pi(K) \\ \left([n] \xrightarrow{k_n} K\right) & \longmapsto & \left(\pi[n] \xrightarrow{\pi(k_n)} \pi(K)\right) \end{array}$$

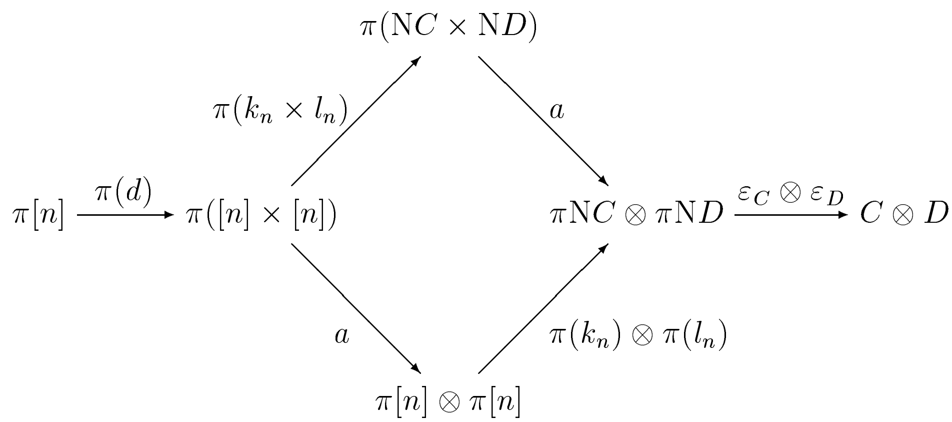
Since Ω is given by Ω^t under the adjunction, we have $\Omega \cong \eta \circ \mathrm{N}\Omega^t$, that is

$$\begin{array}{ccc} \mathrm{N}C \times \mathrm{N}D & \overset{\Omega_{C,D}}{\dashrightarrow} & \mathrm{N}(C \otimes D) \\ {\scriptstyle \eta_{\mathrm{N}C \times \mathrm{N}D}}\downarrow & & \uparrow{\scriptstyle \mathrm{N}(\varepsilon_C \otimes \varepsilon_D)} \\ \mathrm{N}\pi(\mathrm{N}C \times \mathrm{N}D) & \xrightarrow{\mathrm{N}(a)} & \mathrm{N}(\pi\mathrm{N}C \otimes \pi\mathrm{N}D) \end{array}$$

where a is the diagonal approximation map. Given an element $\sigma = (f, g)$ of $(\mathrm{N}C \times \mathrm{N}D)_n$ corresponding to a simplicial map

$$[n] \xrightarrow{d} [n] \times [n] \xrightarrow{k_n \times l_n} \mathrm{N}C \times \mathrm{N}D$$

we have the following commutative diagram by the naturality of the diagonal approximation



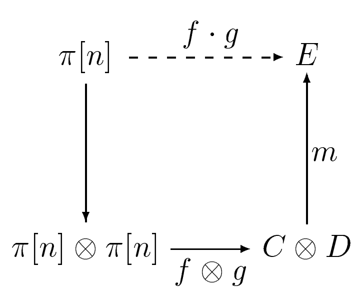
The upper path is the image of σ under $\Omega_{C,D} = \eta \circ \mathrm{N}(a) \circ \mathrm{N}(\varepsilon \otimes \varepsilon)$; the lower path is the composite of $\pi(d) \circ a$ with $(\eta_{\mathrm{N}C} \circ \mathrm{N}\varepsilon_C)(f) \otimes (\eta_{\mathrm{N}D} \circ \mathrm{N}\varepsilon_D)(g)$. But $\eta_{\mathrm{N}} \circ \mathrm{N}\varepsilon$ is the identity, so the proposition follows. □

Given any crossed complex homomorphism $C \otimes D \xrightarrow{m} E$ the natural transformation Ω defines a simplicial map from $\mathrm{N}C \times \mathrm{N}D$ to $\mathrm{N}E$ by

$$\mathrm{N}C \times \mathrm{N}D \xrightarrow{\Omega_{C,D}} \mathrm{N}(C \otimes D) \xrightarrow{\mathrm{N}m} \mathrm{N}E$$

Using the same arguments as above, it is clear that this construction agrees with that considered in section 2.2.3.

Corollary 4.1.2 *Suppose $\pi[n] \xrightarrow{f} C$, $\pi[n] \xrightarrow{g} D$ are elements of $\mathrm{N}(C)_n$, $\mathrm{N}(D)_n$ respectively and m is a homomorphism from $C \otimes D$ to E as above. Then the image of (f, g) under $\Omega_{C,D} \circ \mathrm{N}m$ is given by the composite*



The natural transformation Ω also satisfies an associative law

Proposition 4.1.3 *Given crossed complexes C, D, E, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{N}C \times \mathrm{N}D \times \mathrm{N}E & \xrightarrow{\Omega_{C,D} \times \mathrm{id}} & \mathrm{N}(C \otimes D) \times \mathrm{N}E \\ {\scriptstyle \mathrm{id} \times \Omega_{D,E}} \downarrow & & \downarrow {\scriptstyle \Omega_{C \otimes D, E}} \\ \mathrm{N}C \times \mathrm{N}(D \otimes E) & \xrightarrow{\Omega_{C, D \otimes E}} & \mathrm{N}(C \otimes D \otimes E) \end{array}$$

Proof: By the naturality of a and using $\pi\Omega \circ \varepsilon = \Omega^{\mathrm{t}} = a \circ (\varepsilon \otimes \varepsilon)$ we have the following commutative diagram

$$\begin{array}{ccccc} \pi(\mathrm{N}C \times \mathrm{N}D \times \mathrm{N}E) & \xrightarrow{a} & \pi\mathrm{N}C \otimes \pi(\mathrm{N}D \times \mathrm{N}E) & \xrightarrow{\mathrm{id} \otimes a} & \pi\mathrm{N}C \otimes \pi\mathrm{N}D \otimes \pi\mathrm{N}E \\ {\scriptstyle \pi(\mathrm{id} \times \Omega)} \downarrow & & {\scriptstyle \mathrm{id} \otimes \pi\Omega} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes \varepsilon \otimes \varepsilon} \\ \pi(\mathrm{N}C \times \mathrm{N}(D \otimes E)) & \xrightarrow{a} & \pi\mathrm{N}C \otimes \pi\mathrm{N}(D \otimes E) & \xrightarrow{\mathrm{id} \otimes \varepsilon} & \pi\mathrm{N}C \otimes D \otimes E \\ & & & & \downarrow {\scriptstyle \varepsilon \otimes \mathrm{id} \otimes \mathrm{id}} \\ & & & & C \otimes D \otimes E \end{array}$$

in which the lower path corresponds to $(\mathrm{id} \times \Omega) \circ \Omega$ under the adjunction. There is a similar diagram for $(\Omega \times \mathrm{id}) \circ \Omega$ and so the result follows by the associativity of a and $\otimes$. $\square$

We now use these results together with the internal hom structure to define a simplicial enrichment of **Crs**.

There are natural homomorphisms

$$[D, E] \otimes [C, D] \otimes C \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} [D, E] \otimes D \xrightarrow{\mathrm{ev}} E$$

where the evaluation map $\mathrm{ev}_{C,D}$ is the counit map $[C, D] \otimes C \to D$ corresponding to $\mathrm{id}_{[C,D]}$ under the tensor product-internal hom adjunction in **Crs**. These give the internal composition maps of the monoidal closed structure on **Crs**

$$[D, E] \otimes [C, D] \xrightarrow{\circ_{\mathrm{Crs}}} [C, E]$$

Definition 4.1.4 *For crossed complexes C, D the simplicial hom-set $\mathbf{Crs}_S(C, D)$ is defined by*

$$\mathbf{Crs}_S(C, D) \quad = \quad \mathrm{N}[C, D]$$

and for crossed complexes C, D, E the enriched composition is defined by

$$\begin{array}{ccc} \mathrm{N}[D,E] \times \mathrm{N}[C,D] & \dashrightarrow & \mathrm{N}[C,E] \\ & \searrow^{\Omega} \quad \nearrow_{\mathrm{N}(\circ_{\mathrm{Crs}})} & \\ & \mathrm{N}([D,E] \otimes [C,D]) & \end{array}$$

Note that this does indeed define a simplicial enrichment for **Crs**, since both Ω and $\circ_{\mathrm{Crs}}$ satisfy an associative law and Ω is a natural bijection in dimension zero.

A more explicit definition of the enriched composition may be given using the following natural bijections of hom-sets

$$\mathbf{Crs}_S(C,D)_n \;\cong\; \mathbf{Crs}(\pi[n],[C,D]) \;\cong\; \mathbf{Crs}(\pi[n] \otimes C, D)$$

Proposition 4.1.5 *Under the correspondence above, the enriched composition in* $\mathbf{Crs}_S$ *takes a pair of homomorphisms* $(\pi[n] \otimes D \xrightarrow{y} E, \pi[n] \otimes C \xrightarrow{x} D)$ *to the homomorphism* $x \cdot y$ *given by the composite*

$$\pi[n] \otimes C \xrightarrow{\lambda_C} \pi[n] \otimes \pi[n] \otimes C \xrightarrow{\mathrm{id} \otimes x} \pi[n] \otimes D \xrightarrow{\;y\;} E$$

where λ is the (ordinary) natural transformation whose components λ_C are defined using the diagonal approximation map a as follows:

$$\pi[n] \otimes C \xrightarrow{\pi(d) \otimes \mathrm{id}} \pi([n] \times [n]) \otimes C \xrightarrow{a \otimes \mathrm{id}} \pi[n] \otimes \pi[n] \otimes C$$

Proof: Let x, y correspond to the homomorphisms $\pi[n] \xrightarrow{f} [C,D]$, $\pi[n] \xrightarrow{g} [D,E]$ respectively. Then $x \cdot y$ corresponds to the homomorphism

$$\pi[n] \longrightarrow \pi[n] \otimes \pi[n] \xrightarrow{g \otimes f} [D,E] \otimes [C,D] \xrightarrow{\circ_{\mathrm{Crs}}} [C,E]$$

by corollary 4.1.2. Thus $x \cdot y$ may be written as the upper path around the following diagram.

$$\begin{array}{ccccc} \pi[n] \otimes C \xrightarrow{\lambda_C} \pi[n] \otimes \pi[n] \otimes C & & & & \\ \downarrow{\scriptstyle \mathrm{id} \otimes f \otimes \mathrm{id}} & \searrow{\scriptstyle g \otimes f \otimes \mathrm{id}} & & & \\ \pi[n] \otimes [C,D] \otimes C & \xrightarrow[g \otimes \mathrm{id} \otimes \mathrm{id}]{} & [D,E] \otimes [C,D] \otimes C & & \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{ev}} & & \\ \pi[n] \otimes D & \xrightarrow[g \otimes \mathrm{id}]{} & [D,E] \otimes D & \xrightarrow[\mathrm{ev}]{} & E \end{array}$$

By the identities $(f \otimes \mathrm{id}_C) \circ \mathrm{ev}_{C,D} = x$ and $(g \otimes \mathrm{id}_E) \circ \mathrm{ev}_{D,E} = y$, the lower path around this diagram is $\lambda_C \circ (\mathrm{id} \otimes x) \circ y$, and we have the result as required. □

Note that for $n = 1$ this description of the enriched composition is identical to the description of horizontal composition of homotopies given in section 2.1.1.

4.2 Enrichment of π and Nerve

In this section we discuss how the fundamental crossed complex and nerve functors between **SimpSet** and **Crs** can be extended to the corresponding simplicially enriched categories. For π this will not work 'on the nose' but will involve the coherent systems of higher homotopies of theorem 2.3.9.

Consider first the nerve functor from crossed complexes to simplicial sets.

Proposition 4.2.1 *The nerve functor extends to a simplicial functor*

$$\mathbf{Crs}_S \xrightarrow{\quad \mathrm{N}_S \quad} \mathbf{SimpSet}_S$$

$$\mathbf{Crs}(\pi[n] \otimes C, D) \xrightarrow{\quad \mathrm{N}_n \quad} \mathbf{SimpSet}([n] \times \mathrm{N}C, \mathrm{N}D)$$

where N_n *takes a homomorphism* $\pi[n] \otimes C \xrightarrow{f} D$ *to the simplicial map*

$$[n] \times \mathrm{N}C \xrightarrow{\zeta_{[n],C}} \mathrm{N}(\pi[n] \otimes C) \xrightarrow{\mathrm{N}(f)} \mathrm{N}D$$

and ζ *is the natural transformation with* $\zeta_{K,C}$ *given by*

$$K \times \mathrm{N}C \xrightarrow{\eta} \mathrm{N}\pi(K \times \mathrm{N}C) \xrightarrow{\mathrm{N}(a)} \mathrm{N}(\pi K \otimes \pi \mathrm{N}C) \xrightarrow{\mathrm{N}(\mathrm{id} \otimes \varepsilon)} \mathrm{N}(\pi K \otimes C)$$

Proof: Clearly N_S defines a simplicial map on each hom-object. Also since a corresponds to the identity if either component is of dimension zero, we have

$$\zeta_{[0],C} \cong \eta_{\mathrm{N}C} \circ a \circ \mathrm{N}(\varepsilon_C) = \mathrm{id}$$

using the triangle identity. Thus $N_0 \cong N$. It remains to show that N_S respects the enriched composition structures in $\mathbf{Crs}_S$ and $\mathbf{SimpSet}_S$, and for this we will need the fact that ζ satisfies a type of associativity condition.

Lemma 4.2.2 *For simplicial sets* K, L *and crossed complexes* C *the following diagram commutes*

$$\begin{array}{ccc} K \times L \times \mathrm{N}C & \xrightarrow{\mathrm{id} \times \zeta_{L,C}} & K \times \mathrm{N}(\pi L \otimes C) \\ \downarrow{\scriptstyle \zeta_{K \times L, C}} & & \downarrow{\scriptstyle \zeta_{K, \pi L \otimes C}} \\ \mathrm{N}(\pi(K \times L) \otimes C) & \xrightarrow{\mathrm{N}(a \otimes \mathrm{id})} & \mathrm{N}(\pi K \otimes \pi L \otimes C) \end{array}$$

Proof: Consider $(\mathrm{id} \times \zeta) \circ \zeta$. From the definition of ζ and naturality we have the following commutative diagram:

$$\begin{array}{ccccc}
K \times L \times \mathrm{N}C & \xrightarrow{\eta \circ \mathrm{N}(a)} & \mathrm{N}(\pi K \otimes \pi(L \times \mathrm{N}C)) & & \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}(\mathrm{id} \otimes \eta)} & & \\
 & & \mathrm{N}(\pi K \otimes \pi \mathrm{N}\pi(L \times \mathrm{N}C)) & \xrightarrow{\mathrm{N}(\mathrm{id} \otimes \varepsilon)} & \mathrm{N}(\pi K \otimes \pi(L \times \mathrm{N}C)) \\
{\scriptstyle \mathrm{id} \times \zeta}\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}(\mathrm{id} \otimes a)} \\
 & & \mathrm{N}(\pi K \otimes \pi \mathrm{N}(\pi L \otimes \pi \mathrm{N}C)) & & \mathrm{N}(\pi K \otimes \pi L \otimes \pi \mathrm{N}C) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)} \\
K \times \mathrm{N}(\pi L \otimes C) & \xrightarrow[\eta \circ \mathrm{N}(a)]{} & \mathrm{N}(\pi K \otimes \pi \mathrm{N}(\pi L \otimes C)) & \xrightarrow[\mathrm{N}(\mathrm{id} \otimes \varepsilon)]{} & \mathrm{N}(\pi K \otimes \pi L \otimes C)
\end{array}$$

Thus $(\mathrm{id} \times \zeta) \circ \zeta \;=\; \eta \circ \mathrm{N}(a) \circ \mathrm{N}(a \otimes \mathrm{id}) \circ \mathrm{N}(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)$ by the triangle identity and the associativity of a, and the result follows by the naturality of ε. $\square$

Returning to the proof of proposition 4.2.1, suppose we have homomorphisms

$$\pi[n] \otimes C \xrightarrow{f} D \qquad \pi[n] \otimes D \xrightarrow{g} E$$

Then the result $\mathrm{N}_n(f \circ g) \;=\; \mathrm{N}_n(f) \circ \mathrm{N}_n(g)$ may be seen by the commutativity of the following diagram

$$\begin{array}{ccccccc}
[n] \times \mathrm{N}C & \xrightarrow{d \times \mathrm{id}} & [n] \times [n] \times \mathrm{N}C & \xrightarrow{\mathrm{id} \times \zeta} & [n] \times \mathrm{N}(\pi[n] \otimes C) & \xrightarrow{\mathrm{id} \times \mathrm{N}(f)} & [n] \times \mathrm{N}D \\
{\scriptstyle \zeta}\Big\downarrow & & {\scriptstyle \zeta}\Big\downarrow & & {\scriptstyle \zeta}\Big\downarrow & & {\scriptstyle \zeta}\Big\downarrow \\
\mathrm{N}(\pi[n] \otimes C) & \xrightarrow{\mathrm{N}(\pi d \otimes \mathrm{id})} & \mathrm{N}(\pi([n] \times [n]) \otimes C) & \xrightarrow{\mathrm{N}(a \otimes \mathrm{id})} & \mathrm{N}(\pi[n] \otimes \pi[n] \otimes C) & \xrightarrow{\mathrm{N}(\mathrm{id} \otimes f)} & \mathrm{N}(\pi[n] \otimes D) \\
 & & & & & & {\scriptstyle \mathrm{N}(g)}\Big\downarrow \\
 & & & & & & \mathrm{N}E
\end{array}$$

and so N_S defines a simplicial enrichment of the nerve functor. $\square$

For the fundamental crossed complex functor, **SimpSet** $\xrightarrow{\pi}$ **Crs**, the situation is more complicated. We can still extend π to a collection of simplicial maps on the hom objects

$$\mathbf{SimpSet}_S \xrightarrow{\quad \pi_S \quad} \mathbf{Crs}_S$$

$$\mathbf{SimpSet}([n] \times K, L) \xrightarrow{\quad \pi_n \quad} \mathbf{Crs}(\pi[n] \otimes \pi K, \pi L)$$

by defining $\pi_n([n] \times K \xrightarrow{f} L)$ to be the homomorphism

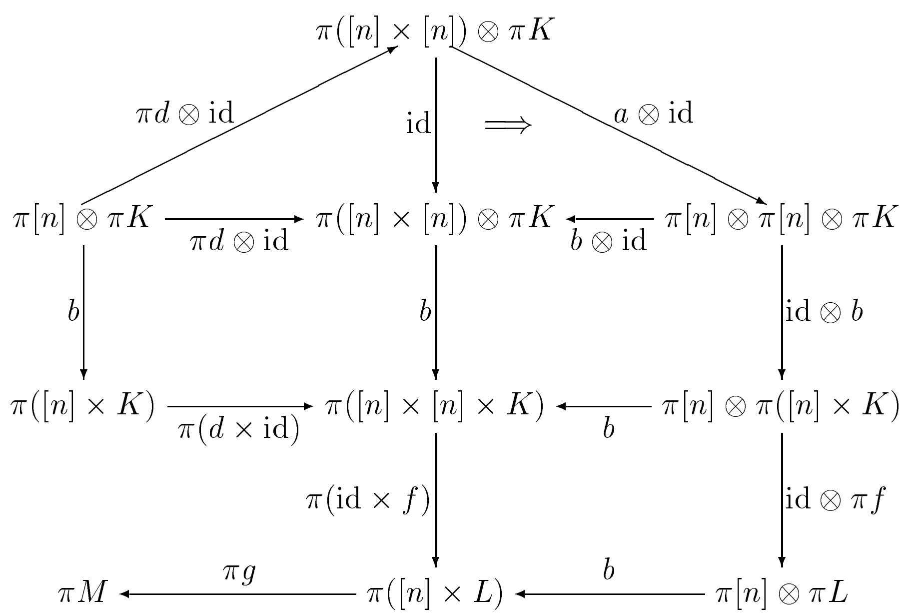
$$\pi[n] \otimes \pi K \xrightarrow{b} \pi([n] \times K) \xrightarrow{\pi f} \pi L$$

where b is given by the shuffle map, the homotopy inverse to the diagonal approximation a in the Eilenberg-Zilber theorem. However the maps π_n do not respect the enriched composition structures. For simplicial maps

$$[n] \times K \xrightarrow{f} L \qquad\qquad [n] \times L \xrightarrow{g} M$$

we have

$$\begin{aligned} \pi_n(f \circ g) &= b \circ \pi(d \times \mathrm{id}) \circ \pi(\mathrm{id} \times f) \circ \pi(g) \\ \pi_n(f) \circ \pi_n(g) &= \pi d \otimes \mathrm{id} \circ a \otimes \mathrm{id} \circ \mathrm{id} \otimes (b \circ \pi f) \circ b \circ \pi g \end{aligned}$$



The 'squares' in this diagram commute by the naturality and associativity of b; the double arrow in the upper-right triangle is given by the homotopy

$$\mathcal{I} \otimes \pi([n] \times [n]) \xrightarrow{h_{[n],[n]}} \pi([n] \times [n])$$

from the identity to $a \circ b$. We thus have a natural homotopy $h(f, g) : \pi_n(f \circ g) \simeq \pi_n(f) \circ \pi_n(g)$ given by

$$\mathcal{I} \otimes \pi[n] \otimes \pi K \xrightarrow{{}^{\pi d \otimes \mathrm{id}}(h_{[n],[n]} \otimes \mathrm{id}_{\pi K})^{b \cdot \pi(\mathrm{id} \times f \cdot g)}} \pi M$$

We will show that these homotopies form a coherent system, where the coherence information is given by the higher homotopies of the Eilenberg-Zilber theorem 2.3.9. For example given simplicial maps $[n] \times K_{i-1} \xrightarrow{f_i} K_i$ for $1 \leq i \leq 3$, we can form the composite homotopies

$$\begin{array}{rl} & \pi(f_1 \circ f_2 \circ f_3) \;\simeq\; \pi(f_1) \circ \pi(f_2 \circ f_3) \;\simeq\; \pi(f_1) \circ \pi(f_2) \circ \pi(f_3) \\ \text{and} & \pi(f_1 \circ f_2 \circ f_3) \;\simeq\; \pi(f_1 \circ f_2) \circ \pi(f_3) \;\simeq\; \pi(f_1) \circ \pi(f_2) \circ \pi(f_3) \end{array}$$

These are not equal, although they are themselves homotopic via a double homotopy

$$\mathcal{I} \otimes \mathcal{I} \otimes \pi[n] \otimes \pi K_0 \xrightarrow{h(f_1, f_2, f_3)} \pi K_3$$

Let us generalise the notion of an r-fold homotopy to that of an (r, n)-homotopy, where an (r, n)-homotopy h is a crossed complex homomorphism

$$\mathcal{I}^{\otimes r} \otimes \pi[n] \otimes C \xrightarrow{\quad h \quad} D$$

where C, D are crossed complexes.

Clearly there are $(r-1, n)$-homotopies $\delta_i^{\pm}(h)$ and $(r, n-1)$-homotopies $d_i(h)$ induced from h by considering the $2r$ faces of the r-cube and the n faces of the n-simplex. Also given a (p, n)-homotopy k_1 and a (q, n)-homotopy k_2 as follows

$$\mathcal{I}^{\otimes p} \otimes \pi[n] \otimes C \xrightarrow{k_1} D \qquad\qquad \mathcal{I}^{\otimes q} \otimes \pi[n] \otimes D \xrightarrow{k_2} E$$

then we can define a $(p + q, n)$-homotopy $k_1 \circ k_2$ by the following diagram

$$\begin{array}{ccc}
\mathcal{I}^{\otimes(p+q)} \otimes \pi[n] \otimes C & \xrightarrow{\;k_1 \circ k_2\;} & E \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes \pi(d) \otimes \mathrm{id}} & & \Big\uparrow{\scriptstyle k_2} \\
\mathcal{I}^{\otimes(p+q)} \otimes \pi([n] \times [n]) \otimes C & & \mathcal{I}^{\otimes q} \otimes \pi[n] \otimes D \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes a \otimes \mathrm{id}} & & \Big\uparrow{\scriptstyle \mathrm{id} \otimes k_1} \\
\mathcal{I}^{\otimes(p+q)} \otimes \pi[n] \otimes \pi[n] \otimes C & \xrightarrow{\mathrm{id} \otimes s \otimes \mathrm{id}} & \mathcal{I}^{\otimes q} \otimes \pi[n] \otimes \mathcal{I}^{\otimes p} \otimes \pi[n] \otimes C
\end{array}$$

Note that composition of two $(0,n)$-homotopies in this way agrees with the definition of the enriched composition of degree n maps in $\mathbf{Crs}_S$, since the symmetry of the tensor product acts as the identity when one of the factors has dimension zero.

Using these notions, we make the following definition of what we mean by a simplicially coherent (or *lax*) functor from $\mathbf{SimpSet}_S$ to $\mathbf{Crs}_S$.

Definition 4.2.3 *A simplicially coherent functor* $\mathbf{SimpSet}_S \xrightarrow{F} \mathbf{Crs}_S$ *is given by the following data:*

- *A crossed complex $F(K)$ for each simplicial set K*
- *An $(r-1,n)$-homotopy*

$$\mathcal{I}^{\otimes(r-1)} \otimes \pi[n] \otimes F(K_0) \xrightarrow{F_n(f_1, f_2, \ldots, f_r)} F(K_r)$$

for each $n \geq 0$ and each r-tuple $f = (f_1, f_2, \ldots, f_r)$, $f_i \in \mathbf{SimpSet}_S(K_{i-1}, K_i)_n$.

such that the F_n commute with the simplicial face and degeneracy operators, and the following cubical boundary relations hold:

$$\begin{aligned} \partial_i^-(F_n(f_1, f_2, \ldots, f_r)) &= F_n(f_1, f_2, \ldots, (f_{r-i} \circ f_{r-i+1}), \ldots, f_r) \\ \partial_i^+(F_n(f_1, f_2, \ldots, f_r)) &= F_n(f_1, f_2, \ldots, f_{r-i}) \circ F_n(f_{r-i+1}, \ldots, f_r) \end{aligned}$$

where $\circ$ here means enriched composition and composition of (k,n)-homotopies respectively.
The simplicially coherent functor F is said to provide a simplicially coherent enrichment *of an ordinary functor* $\mathbf{SimpSet} \xrightarrow{G} \mathbf{Crs}$ *if the following conditions hold:*

- *$F(K) = G(K)$ for each simplicial set K*
- *every $(r-1,0)$-homotopy $F_0(f_1, f_2, \ldots, f_r)$ factors through the corresponding homomorphism $G(f_1 \circ f_2 \circ \cdots \circ f_r)$*

$$\begin{array}{ccc} \mathcal{I}^{\otimes(r-1)} \otimes \pi[0] \otimes F(K_0) & \xrightarrow{F_0(f_1, f_2, \ldots, f_r)} & F(K_r) = G(K_r) \\ \Big\downarrow {\scriptstyle 0 \otimes \mathrm{id} \otimes \mathrm{id}} & & \Big\uparrow {\scriptstyle G(f_1 \circ f_2 \circ \cdots \circ f_r)} \\ \mathcal{I}^{\otimes(0)} \otimes \pi[0] \otimes F(K_0) & \xrightarrow{\cong} & F(K_0) = G(K_0) \end{array}$$

Suppose f is an r-tuple $(f_1, f_2, \ldots, f_r)$ of degree n maps and F is a simplicially coherent functor as above. Then the enriched composition in $\mathbf{SimpSet}_S$ and $\mathbf{Crs}_S$ give for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{r-1}) \in \{0,1\}^{r-1}$ an element $F_\alpha(f)$ of $\mathbf{Crs}_S(F(K_0), F(K_r))_n$ defined by

$$F_n(f_{i_0+1} \circ f_{i_0+2} \circ \ldots \circ f_{i_1}) \circ F_n(f_{i_1+1} \circ \ldots \circ f_{i_2}) \circ \ldots \circ F_n(f_{i_k+1} \circ \ldots \circ f_{i_{k+1}})$$

where $i_1 < i_2 < \ldots < i_k$ are those i such that $\alpha_{r-i} = 1$, and $i_0 = 0$, $i_{k+1} = r$. Also there is a $(0, n)$-homotopy $F'_\alpha(f)$ given by the $(r-1, n)$-homotopy $F_n(f_1, f_2, \ldots, f_r)$ at the corner of the $(r-1)$-cube given by α. The following proposition follows from the cubical boundary relations satisfied by F.

Proposition 4.2.4 *The degree n map of $\mathbf{Crs}_S$ corresponding to $F'_\alpha(f)$ is precisely $F_\alpha(f)$. Thus, the $(r-1, n)$-homotopies $F_n(f_1, f_2, \ldots, f_r)$ given by a simplicially coherent functor F for $r \geq 2$ record all the coherent homotopy information between the various enriched composites of its values on 1-tuples.*

We now use the coherent system of homotopies of theorem 2.3.9 to define a simplicially coherent enrichment of π which on 1-tuples agrees with the definition of π_S above. We write $[n]^r$ for the r-fold (cartesian) product of the representable simplicial set $[n]$ with itself, and h_r for the r-fold homotopy obtained from theorem 2.3.9 by setting $K_0 = K_1 = \ldots = K_r = [n]$.

Theorem 4.2.5 *There is a simplicially coherent enrichment $\pi : \mathbf{SimpSet}_S \longrightarrow \mathbf{Crs}_S$ of the fundamental crossed complex functor with $\pi_n(f_1, f_2, \ldots, f_r)$ given by the following commutative diagram:*

$$
\begin{array}{ccccc}
\mathcal{I}^{\otimes(r-1)} \otimes \pi[n] \otimes \pi K_0 & & \xrightarrow{\quad \pi_n(f_1, f_2, \ldots, f_r) \quad} & & \pi K_r \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \pi(d^r) \otimes \mathrm{id}} & & & & \Big\uparrow {\scriptstyle \pi[f]_1^r} \\
\mathcal{I}^{\otimes(r-1)} \otimes \pi\left([n]^r\right) \otimes \pi K_0 & \xrightarrow{h_{r-1} \otimes \mathrm{id}} & \pi\left([n]^r\right) \otimes \pi K_0 & \xrightarrow{b} & \pi\left([n]^r \times K_0\right)
\end{array}
$$

where $d^r \colon [n] \to [n]^r$ is the r-fold diagonal and $[f]_1^r$ is the simplicial map given by

$$
[n]^r \times K_0 \xrightarrow{\mathrm{id}^{r-1} \times f_1} [n]^{r-1} \times K_1 \cdots\cdots\cdots \to [n]^2 \times K_{r-2} \xrightarrow{\mathrm{id} \times f_{r-1}} [n] \times K_{r-1} \xrightarrow{f_r} K_r
$$

Proof: We have to show that the cubical boundary relations of definition 4.2.3 hold as a consequence of the relations on the n-fold homotopies h in theorem 2.3.9. First consider the δ_i^- boundaries, and let $d^{(i)} \colon [n]^{r-1} \longrightarrow [n]^r$ be the map induced by the diagonal on the ith factor. By the relation $\delta_i^- h_{[n],\ldots,[n]} = h_{[n],\ldots,[n]^2,\ldots,[n]}$ and the naturality of h and b we have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{I}^{\otimes(r-2)} \otimes \pi[n] \otimes \pi K_0 & & & & \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi(d^{r-1})\otimes\mathrm{id}} & & & & \\
\mathcal{I}^{\otimes(r-2)} \otimes \pi([n]^{r-1}) \otimes \pi K_0 & \xrightarrow{h_{r-2}\otimes\mathrm{id}} & \pi([n]^{r-1}) \otimes \pi K_0 & \xrightarrow{b} & \pi([n]^{r-1} \times K_0) \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes\pi(d^{(i)})\otimes\mathrm{id}} & & \Big\downarrow {\scriptstyle \pi(d^{(i)})\otimes\mathrm{id}} & & \Big\downarrow {\scriptstyle \pi(d^{(i)}\times\mathrm{id})} \\
\mathcal{I}^{\otimes(r-2)} \otimes \pi([n]^{r}) \otimes \pi K_0 & \xrightarrow{\delta_i^-(h_{r-1})\otimes\mathrm{id}} & \pi([n]^{r} \times K_0) & \xrightarrow{b} & \pi([n]^{r} \times K_0) \\
 & & & & \Big\downarrow {\scriptstyle \pi[f]_1^r} \\
 & & & & \pi K_r
\end{array}$$

The vertical composite on the right of the above diagram may be written as $\pi[g]_1^{r-1}$, where g is the $r-1$-tuple obtained from f by replacing f_{r-i} and f_{r-i+1} by their enriched composite. Thus the upper path through the diagram gives $\pi_n(f_1,\ldots,f_{r-i}\circ f_{r-i+1},\ldots,f_r)$. Also the lower path is $\delta_i^-\pi_n(f_1,\ldots,f_r)$, since $d^{r-1}\circ d^{(i)} = d^r$, so we have the required relation.

The relations for δ_i^+ are slightly more complicated to show. Consider the diagram in figure 4.1, which commutes by naturality of a and s, by the boundary relation for $\delta_i^+ h$, by the definition of $*$, and by associativity and naturality of b. By naturality of $\otimes$, the composite from the "top right" to the "bottom left" of the diagram is just the composite of $\mathrm{id}\otimes\mathrm{id}\otimes\pi_n(f_1,\ldots,f_{r-i})$ with $\pi_n(f_{r-i+1},\ldots,f_r)$. Thus the long path around the diagram is $\pi_n(f_1,\ldots,f_{r-i})\circ\pi_n(f_{r-i+1},\ldots,f_r)$. The short "vertical" path is $\delta_i^+\pi_n(f_1,\ldots,f_r)$ and so the relation follows. □

4.3 The coherent adjunction $\pi \dashv \mathrm{N}$

The adjunction between the nerve and the fundamental crossed complex functors takes place at the level of unenriched categories. In this section we will see that when considering **SimpSet** and **Crs** as simplicially-enriched categories the adjunction does not respect the enrichment precisely, but only up to a system of coherent homotopies.

For all simplicial sets K and crossed complexes C the ordinary adjunction gives a natural bijection of hom-sets

$$\mathbf{Crs}(\pi K, C) \;\cong\; \mathbf{SimpSet}(K, \mathrm{N}C)$$

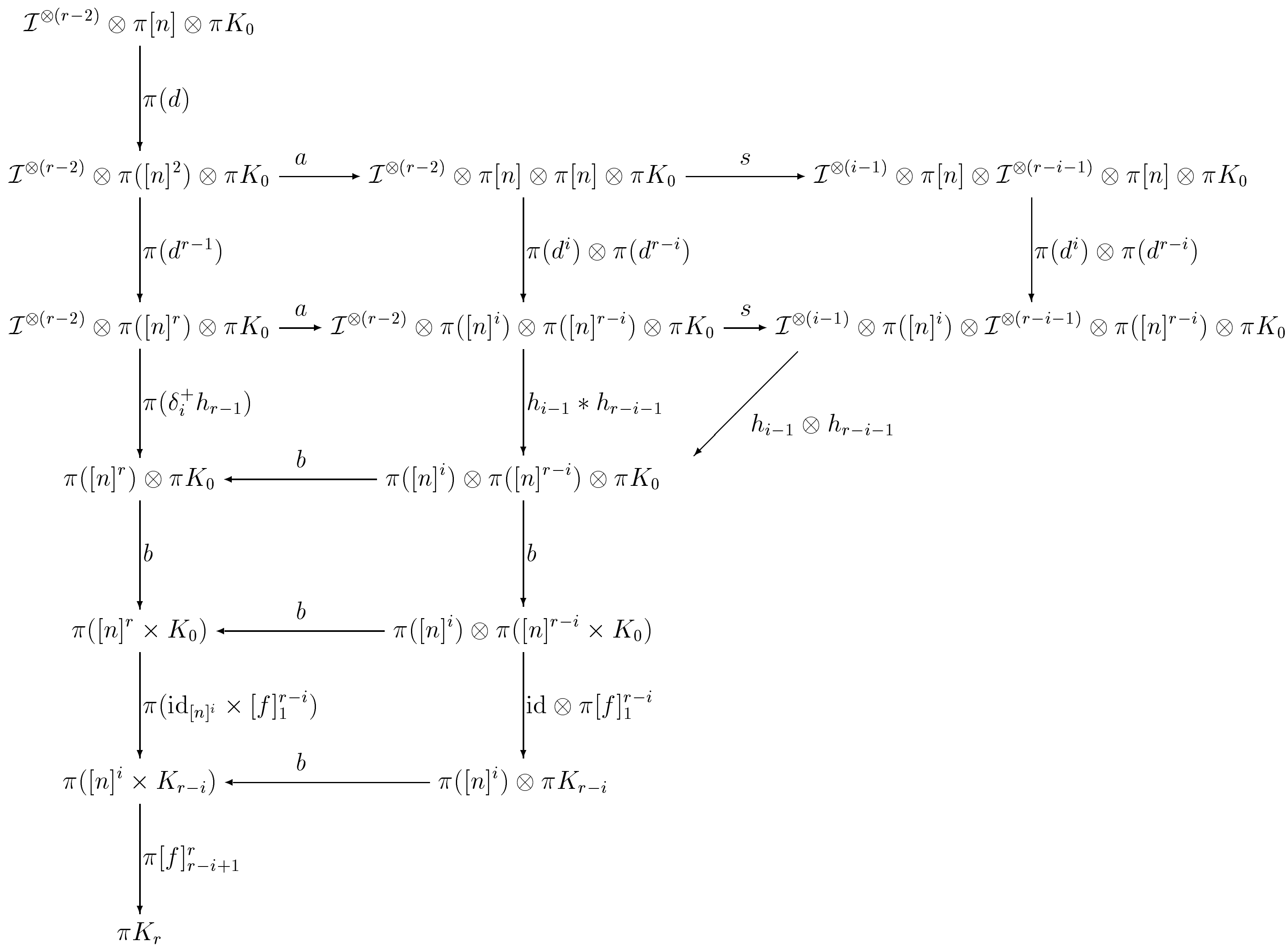


Figure 4.1: The δ_i^+ relation for π.

It is quite easy by using the diagonal approximation and shuffle maps to extend this to the simplicial hom objects. Recall that there are natural bijections of sets giving us the following representations of the enriched homs

$$\begin{array}{rcccl} \mathbf{Crs}_S(\pi K, C)_n & \cong & \mathbf{Crs}(\pi[n], [\pi K, C]) & \cong & \mathbf{Crs}(\pi[n] \otimes \pi K, C) \\ \mathbf{SimpSet}_S(K, \mathrm{N}C)_n & \cong & \mathbf{SimpSet}([n] \times K, \mathrm{N}C) & \cong & \mathbf{Crs}(\pi([n] \times K), C) \end{array}$$

Proposition 4.3.1 *Given a crossed complex C and a simplicial set K there is a homotopy equivalence between the simplicial sets $\mathbf{Crs}_S(\pi K, C)$ and $\mathbf{SimpSet}_S(K, \mathrm{N}C)$ which is a natural bijection in dimension zero. Moreover, the homotopy is a deformation retraction.*

Proof: The simplicial maps a^* and b^* between the enriched homs are given in each dimension by

$$\mathbf{Crs}_S(\pi K, C)_n \cong \mathbf{Crs}(\pi[n] \otimes \pi K, C) \underset{b_n^*}{\overset{a_n^*}{\rightleftarrows}} \mathbf{Crs}(\pi([n] \times K), C) \cong \mathbf{SimpSet}_S(K, \mathrm{N}C)_n$$

These are defined by precomposing the representing homomorphisms with the maps a and b of the Eilenberg-Zilber theorem.

$$\left(\pi[n] \otimes \pi K \xrightarrow{f} C\right) \overset{a_n^*}{\longmapsto} \left(\pi([n] \times K) \xrightarrow{a} \pi[n] \otimes \pi K \xrightarrow{f} C\right)$$
$$\left(\pi([n] \times K) \xrightarrow{g} C\right) \overset{b_n^*}{\longmapsto} \left(\pi[n] \otimes \pi K \xrightarrow{b} \pi([n] \times K) \xrightarrow{g} C\right)$$

Clearly the composite simplicial map $a^* \circ b^*$ is the identity on $\mathbf{Crs}_S(\pi K, C)$, since $b \circ a$ is the identity on each $\pi[n] \otimes \pi K$. The simplicial homotopy between $b^* \circ a^*$ and the identity on $\mathbf{SimpSet}(K, \mathrm{N}C)$

$$[1] \times \mathbf{SimpSet}_S(K, \mathrm{N}C) \xrightarrow{H} \mathbf{SimpSet}_S(K, \mathrm{N}C)$$

is defined as follows. Suppose (x, f) represents an element of dimension n of the right hand side, where x is a simplicial map $[n] \to [1]$ and f is a homomorphism $\pi([n] \times K) \to C$. Then $H_n(x, f)$ is the homomorphism given by

$$\begin{array}{ccccccccc} \pi([n] \times K) & \dashrightarrow & & & & & & & C \\ \Big\downarrow{\scriptstyle \pi(d \times \mathrm{id})} & & & & & & & & \Big\uparrow{\scriptstyle f} \\ \pi([n] \times [n] \times K) & \xrightarrow{a} & \pi[n] \otimes \pi([n] \times K) & & \xrightarrow{\pi(x) \otimes \mathrm{id}} & & \pi[1] \otimes \pi([n] \times K) & \xrightarrow{h} & \pi([n] \times K) \end{array}$$

using the diagonal approximation again, together with the homotopy h of theorem 2.3.1.
□

In the unenriched setting, the natural bijection of an adjunction $F \dashv G$ may be defined in terms of the functors F and G and the unit and counit maps. We will see in the next proposition that this is also true in our enriched situation.

For crossed complexes C, D and simplicial sets K, L we will use the notation η^*, η_*, ε^* and ε_* for the four simplicial maps

$$\begin{gathered}
\mathbf{SimpSet}_S(\mathrm{N}\pi K, L) \xrightarrow{\quad \eta^* \quad} \mathbf{SimpSet}_S(K, L) \\
\mathbf{SimpSet}_S(K, L) \xrightarrow{\quad \eta_* \quad} \mathbf{SimpSet}_S(K, \mathrm{N}\pi L) \\
\mathbf{Crs}_S(C, D) \xrightarrow{\quad \varepsilon^* \quad} \mathbf{Crs}_S(\pi \mathrm{N} C, D) \\
\mathbf{Crs}_S(C, \pi \mathrm{N} D) \xrightarrow{\quad \varepsilon_* \quad} \mathbf{Crs}_S(C, D)
\end{gathered}$$

induced by the unit η and the counit ε on the enriched homs. For example, η^* is the map which in each dimension n is given by

$$\begin{array}{ccc}
\mathbf{SimpSet}_S(\mathrm{N}\pi K, L)_n & \xrightarrow{\quad \eta^*_n \quad} & \mathbf{SimpSet}_S(K, L)_n \\
\left([n] \times \mathrm{N}\pi K \xrightarrow{f} L\right) & \longmapsto & \left([n] \times K \xrightarrow{\mathrm{id} \times \eta} [n] \times \mathrm{N}\pi K \xrightarrow{f} L\right)
\end{array}$$

We can now state the proposition.

Proposition 4.3.2 *The adjunction maps a^* and b^* are precisely the simplicial maps given by the composites*

$$\mathbf{Crs}_S(\pi K, C) \xrightarrow{N_S} \mathbf{SimpSet}_S(\mathrm{N}\pi K, \mathrm{N}C) \xrightarrow{\eta^*} \mathbf{SimpSet}_S(K, \mathrm{N}C)$$

$$\text{and} \quad \mathbf{SimpSet}_S(K, \mathrm{N}C) \xrightarrow{\pi_S} \mathbf{Crs}_S(\pi K, \pi \mathrm{N}C) \xrightarrow{\varepsilon_*} \mathbf{Crs}_S(\pi K, C)$$

respectively.

Proof: Suppose $\pi[n] \otimes \pi K \xrightarrow{f} C$ represents an element of $\mathbf{Crs}_S(\pi K, C)_n$. Then from proposition 4.2.1 we have $\eta^*(\mathrm{N}_S(f)) = (\mathrm{id} \times \eta) \circ \zeta_{[n], \pi K} \circ \mathrm{N}(f)$. But $(\mathrm{id} \times \eta) \circ \zeta_{[n], \pi K} = \eta \circ \mathrm{N}(a)$ by naturality and the triangle identity:

$$\begin{array}{ccccccc}
[n] \times K & \xrightarrow{\eta} & \mathrm{N}\pi([n] \times K) & \xrightarrow{\mathrm{N}(a)} & \mathrm{N}(\pi[n] \otimes \pi K) & = & \mathrm{N}(\pi[n] \otimes \pi K) \\
\downarrow{\scriptstyle \mathrm{id} \times \eta} & & & & \downarrow{\scriptstyle \mathrm{N}(\mathrm{id} \otimes \pi\eta)} & \nearrow{\scriptstyle \mathrm{N}(\mathrm{id} \otimes \varepsilon_\pi)} & \\
[n] \times \mathrm{N}\pi K & \xrightarrow{\eta} & \mathrm{N}\pi([n] \times \mathrm{N}\pi K) & \xrightarrow{\mathrm{N}(a)} & \mathrm{N}(\pi[n] \otimes \pi \mathrm{N}\pi K) & &
\end{array}$$

Thus $\eta^*(\mathrm{N}_S(f)) = \eta \circ \mathrm{N}(a \circ f)$, which is the simplicial map $[n] \times K \longrightarrow \mathrm{N}C$ representing $a^*(f)$ as required.

For $[n] \times K \xrightarrow{g} \mathrm{N}C$ representing an element of $\mathbf{SimpSet}_S(K, \mathrm{N}C)$, the homomorphism $b^*(g)$ is given by

$$\pi[n] \otimes \pi K \xrightarrow{\quad b \quad} \pi([n] \times K) \xrightarrow{\quad \pi g \quad} \pi\mathrm{N}C \xrightarrow{\qquad \varepsilon \qquad} C$$

Which is just $\pi_n(g) \circ \varepsilon$. □

Conversely we can reconstruct the definitions of N_S and π_S from the adjunction maps a^* and b^*.

Proposition 4.3.3 *The maps N_S and π_S are given precisely by the composite simplicial maps*

$$\mathbf{Crs}_S(C, D) \xrightarrow{\quad \varepsilon^* \quad} \mathbf{Crs}_S(\pi\mathrm{N}C, D) \xrightarrow{\quad a^* \quad} \mathbf{SimpSet}_S(\mathrm{N}C, \mathrm{N}D)$$

$$\text{and} \qquad \mathbf{SimpSet}_S(K, L) \xrightarrow{\quad \eta_* \quad} \mathbf{SimpSet}_S(K, \mathrm{N}\pi L) \xrightarrow{\quad b^* \quad} \mathbf{Crs}_S(\pi K, \pi L)$$

respectively.

Proof: For $\pi[n] \otimes C \xrightarrow{f} D$ representing an element of $\mathbf{Crs}_S(C, D)_n$, the homomorphism $\pi([n] \times \mathrm{N}C) \xrightarrow{a^*\varepsilon^* f} D$ given by $a \circ (\mathrm{id} \otimes \varepsilon) \circ f$ corresponds to the simplicial map

$$[n] \times \mathrm{N}C \xrightarrow{\eta} \mathrm{N}\pi([n] \times \mathrm{N}C) \xrightarrow{\mathrm{N}(a)} \mathrm{N}(\pi[n] \otimes \pi\mathrm{N}C) \xrightarrow{\mathrm{N}(\mathrm{id} \otimes \varepsilon)} \mathrm{N}(\pi[n] \otimes C) \xrightarrow{\mathrm{N}(f)} \mathrm{N}D$$

which is $N_S(f)$ as required.

For $[n] \times K \xrightarrow{g} L$ representing an element of $\mathbf{SimpSet}_S(K, L)_n$, the simplicial map $[n] \times K \xrightarrow{g \circ \eta} \mathrm{N}\pi L$ corresponds to a homomorphism

$$\pi([n] \times K) \xrightarrow{\quad \pi g \quad} \pi L \xrightarrow{\quad \pi\eta \quad} \pi\mathrm{N}\pi L \xrightarrow{\quad \varepsilon_\pi \quad} \pi L$$

which is just $\pi(g)$ by the triangle identity. Thus $b^*(\eta_*(g)) = b \circ \pi(g)$, which is precisely $\pi_S(g)$. □

Since N_S is a simplicially enriched functor, we have for each simplicial set K a pair of simplicially enriched functors

$$\mathbf{Crs}_S \underset{\mathrm{SimpSet}_S(K, \mathrm{N}(\cdot))}{\overset{\mathrm{Crs}_S(\pi K, \cdot)}{\rightrightarrows}} \mathbf{SimpSet}_S$$

The following proposition follows from the relation between N_S and a^*.

Proposition 4.3.4 *Let K be a simplicial set. Then a^* defines a simplicially enriched natural transformation from $\mathbf{Crs}_S(\pi K, \cdot)$ to $\mathbf{SimpSet}_S(K, \mathrm{N}(\cdot))$.*

Proof: The enriched functoriality of N_S and the definition of η^* give the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{Crs}_S(C,D)\times\mathbf{Crs}_S(\pi K,C) & \xrightarrow{\quad\circ\quad} & \mathbf{Crs}_S(\pi K,D) \\
\Big\downarrow{\scriptstyle \mathrm{N}_S\times\mathrm{N}_S} & & \Big\downarrow{\scriptstyle \mathrm{N}_S} \\
\mathbf{SimpSet}_S(\mathrm{N}C,\mathrm{N}D)\times\mathbf{SimpSet}_S(\mathrm{N}\pi K,\mathrm{N}C) & \xrightarrow{\quad\circ\quad} & \mathbf{SimpSet}(\mathrm{N}\pi K,\mathrm{N}D) \\
\Big\downarrow{\scriptstyle \mathrm{id}\times\eta^*} & & \Big\downarrow{\scriptstyle \eta^*} \\
\mathbf{SimpSet}_S(\mathrm{N}C,\mathrm{N}D)\times\mathbf{SimpSet}_S(K,\mathrm{N}C) & \xrightarrow{\quad\circ\quad} & \mathbf{SimpSet}(K,\mathrm{N}D)
\end{array}$$

By proposition 4.3.2 the vertical composites are $\mathrm{N}_S\times a^*$ and a^*, so we have $a^*(f\circ g) = a^*f\circ\mathrm{N}_S g$ as required. □

There is similar argument for b^* and π_S in the coherent rather than the strict setting.

Proposition 4.3.5 *The maps*

$$\mathbf{SimpSet}_S(K,\mathrm{N}C)\xrightarrow{b^*_{K,C}}\mathbf{Crs}_S(\pi K,C)$$

of proposition 4.3.1 can be given the structure of a coherent natural transformation in K.
That is, given a crossed complex C, simplicial sets $K_0,K_1,\ldots,K_{r-1},K_r=\mathrm{N}C$, and maps $f_i\in\mathbf{SimpSet}_S(K_{i-1},K_i)_n$ for $1\le i\le r$ there is an $(r-1,n)$-homotopy

$$\mathcal{I}^{\otimes(r-1)}\otimes\pi[n]\otimes\pi K_0\xrightarrow{b^*_n(f_1,\ldots,f_r)}C$$

which for $r=1$ agrees with the definition of b^* above, and which satisfies the cubical boundary relations

$$\begin{aligned}
\partial_i^-(b^*_n(f_1,f_2,\ldots,f_r)) &= b^*_n(f_1,f_2,\ldots,(f_{r-i}\circ f_{r-i+1}),\ldots,f_r)\\
\partial_i^+(b^*_n(f_1,f_2,\ldots,f_r)) &= \pi_n(f_1,f_2,\ldots,f_{r-i})\ \circ\ b^*_n(f_{r-i+1},\ldots,f_r)
\end{aligned}$$

Proof: We extend the relation $b^*=\pi_s\circ\varepsilon_*$ of proposition 4.3.3 and define $b^*_n(f_1,\ldots,f_r)$ to be the $(r-1,n)$-homotopy

$$\mathcal{I}^{\otimes(r-1)}\otimes\pi[n]\otimes\pi K_0\xrightarrow{\pi_n(f_1,\ldots,f_r)}\pi\mathrm{N}C\xrightarrow{\quad\varepsilon\quad}C$$

using the simplicial coherence of π_S given in theorem 4.2.5. The boundary relations follow. □

Thus in particular for $f \in \mathbf{SimpSet}(K, L)$ and $g \in \mathbf{SimpSet}(L, \mathrm{N}C)$ of the same degree we have a homotopy between $b^*(f \circ g)$ and $\pi_S f \circ b^* g$. In the general case the 2^{r-1} 'corners' of the above homotopies give all the possible results of applying b^* to f_r before or after composing with the other f_i.

The two cases left to deal with now are the naturality (or otherwise) of a^* in K and b^* in C. We approach these from the following intermediate result, in which it is necessary to use the 'commutativity' of a and b as shown in proposition 2.2.13.

Lemma 4.3.6 *Given maps $f \in \mathbf{SimpSet}_S(K, L)_n$ and $g \in \mathbf{Crs}_S(\pi L, D)_n$, the maps $\pi_S f \circ g$ and $b^*(f \circ a^* g)$ in $\mathbf{Crs}_S(\pi K, D)_n$ are equal.*

Proof: Suppose f and g are given by

$$[n] \times K \xrightarrow{\ f\ } L \qquad\qquad \pi[n] \otimes \pi L \xrightarrow{\ g\ } D$$

then $\pi_s \circ g$ and $b^*(f \circ a^* g)$ correspond to the two paths around the following diagram

$$\begin{array}{ccccc}
\pi[n] \otimes \pi K & \xrightarrow{\pi d \otimes \mathrm{id}} & \pi([n] \times [n]) \otimes \pi K & \xrightarrow{a \otimes \mathrm{id}} & \pi[n] \otimes \pi[n] \otimes \pi K \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathrm{id} \otimes b} \\
\pi([n] \times K) & \xrightarrow{\pi(d \times \mathrm{id})} & \pi([n] \times [n] \times K) & \xrightarrow{a} & \pi[n] \otimes \pi([n] \times K) \\
 & & \downarrow{\scriptstyle \pi(\mathrm{id} \times f)} & & \downarrow{\scriptstyle \mathrm{id} \otimes \pi f} \\
 & & \pi([n] \times L) & \xrightarrow{a} & \pi[n] \otimes \pi L \\
 & & \downarrow{\scriptstyle \pi(a^* g)} & & \downarrow{\scriptstyle g} \\
 & & \pi \mathrm{N} D & \xrightarrow{\varepsilon} & D
\end{array}$$

The bottom square of this diagram commutes since both paths correspond to the map $[n] \times L \longrightarrow \mathrm{N}D$ representing $a^* g$. The other squares commute by naturality of a and b and by the commutativity relation between a and b given in proposition 2.2.13, and so we have the result. $\square$

It follows from this that the maps b^* are natural in C.

Proposition 4.3.7 *Let K be a simplicial set. Then b^* defines a simplicially enriched natural transformation from $\mathbf{SimpSet}_S(K, \mathrm{N}(\cdot))$ to $\mathbf{Crs}_S(\pi K, \cdot)$.*

Proof: Suppose we have crossed complexes C, D and maps $x \in \mathbf{SimpSet}_S(K, \mathrm{N}C)_n$, $y \in \mathbf{Crs}_S(C, D)_n$. Then taking $L = \mathrm{N}C$ and applying the lemma to the maps x and $\varepsilon^*(y)$ gives

$$\pi_S x \,\circ\, \varepsilon^* y \;=\; b^* \left(x \,\circ\, a^*(\varepsilon^* y)\right)$$

It is clear from the definition of ε^* and ε_* that this may be written as

$$\varepsilon_*(\pi_S x) \,\circ\, y \;=\; b^* \left(x \,\circ\, a^*(\varepsilon^* y)\right)$$

By propositions 4.3.2 and 4.3.3 this is

$$b^* x \,\circ\, y \;=\; b^* \left(x \,\circ\, \mathrm{N}_S y\right)$$

and so b^* is natural in C as required. $\Box$

Now suppose C is a crossed complex and consider the maps

$$\mathbf{Crs}_S(\pi K, C) \xrightarrow{a^*_{K,C}} \mathbf{SimpSet}_S(K, \mathrm{N}C)$$

for varying K. Note that $\mathbf{SimpSet}_S(K, \mathrm{N}C)$ extends to a simplicially enriched functor in K, but that $\mathbf{Crs}_S(\pi K, C)$ does not since π gives only a simplicially coherent functor. We will show however that a^* may be given a coherent enriched structure such that it defines a kind of coherently natural enriched transformation between these functors.

Suppose that K_i, $1 \le i \le r$, are simplicial sets and that $f_i \in \mathbf{SimpSet}_S(K_{i-1}, K_i)$ and $g \in \mathbf{Crs}_S(\pi K_r, C)$ are maps given by

$$[n] \times K_{i-1} \xrightarrow{f_i} K_i \qquad\qquad \pi[n] \otimes \pi K_r \xrightarrow{g} C$$

Then we define a homomorphism $a^*_n(f_1, \ldots, f_r;\, g)$ from $\mathcal{I}^{\otimes r} \otimes \pi([n] \times K_0)$ to C by

$$\begin{array}{ccccc}
\mathcal{I}^{\otimes r} \otimes \pi([n] \times K_0) & \longrightarrow \mathcal{I}^{\otimes r} \otimes \pi([n]^{r+1} \times K_0) & \xrightarrow{h^{r,n}_{2,K_0}} & & \pi([n]^{r+1} \times K_0) \\
{\scriptstyle a^*_n(f_1, \ldots, f_r;\; g)} \Big\downarrow & & & & \Big\downarrow {\scriptstyle a^{(1)}} \\
C & \xleftarrow{\;\;g\;\;} \pi[n] \otimes \pi K_r & \xleftarrow{\mathrm{id} \otimes \pi[f]^r_1} & & \pi[n] \otimes \pi([n]^r \times K_r)
\end{array}$$

where $[f]^r_1$ is the simplicial map $[n]^r \times K_0 \longrightarrow K_r$ defined by the f_i as in theorem 4.2.5, and $h^{r,n}_{k,K}$ is the r-fold homotopy $h_{[n]^k,[n],\ldots,[n],K}$ as defined by theorem 2.3.9. On taking $r = 0$ we note that $a^*_n(\,;\, g)$ reduces to $a^*_n(g)$ as defined in proposition 4.3.1.

By considering the boundary relations satisfied by these maps we will show that the 'corners' correspond to all the simplicial maps $[n] \times K_0 \longrightarrow \mathrm{N}C$ obtained by applying a^* to g before or after composing with the f_i. The δ^-_i boundaries are quite clear:

Proposition 4.3.8 *The homomorphisms $a_n^*(f_1, \dots, f_r;\ g)$ defined above satisfy*

$$\begin{aligned} \delta_i^-(a_n^*(f_1,\dots,f_r;\ g)) &= a_n^*(f_1,\dots,f_{r-i}\circ f_{r-i+1},\dots,f_r;\ g) \quad \textit{for } 1 \le i \le r-1 \\ \delta_r^-(a_n^*(f_1,f_2,\dots,f_r;\ g)) &= f_1 \circ a_n^*(f_2,\dots,f_r;\ g) \end{aligned}$$

where the second equation is shorthand for the commutativity of

$$\begin{array}{ccc} \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times K_0) & \xrightarrow{\ \delta_r^- a_n^*(f_1,\dots,f_r;\ g)\ } & C \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi(d\times\mathrm{id})} & & \Big\uparrow{\scriptstyle a_n^*(f_2,\dots,f_r;\ g)} \\ \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times[n]\times K_0) & \xrightarrow{\ \mathrm{id}\otimes\pi(\mathrm{id}\times f_1)\ } & \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times K_1) \end{array}$$

Proof: The proof is analogous to that for the δ_i^- in proposition 4.2.5. For $1 \le i \le r-1$, we use the naturality of h with the diagonal $d^{(i+1)} : [n]^r \longrightarrow [n]^{r+1}$ and get the following diagram

$$\begin{array}{ccccc} \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times K_0) & & & & \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi(d^r\times\mathrm{id})} & & & & \\ \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]^r\times K_0) & \xrightarrow{h^{r-1,n}_{2,K_0}} & \pi([n]^r\times K_0) & \xrightarrow{a^{(1)}} & \pi[n]\otimes\pi([n]^{r-1}\times K_0) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi(d^{(i+1)}\times\mathrm{id})} & & \Big\downarrow{\scriptstyle \pi(d^{(i+1)}\times\mathrm{id})} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi(d^{(i)}\times\mathrm{id})} \\ \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]^{r+1}\times K_0) & \xrightarrow{\delta_i^- h^{r,n}_{2,K_0}} & \pi([n]^{r+1}\times K_0) & \xrightarrow{a^{(1)}} & \pi[n]\otimes\pi([n]^r\times K_0) \\ & & & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\pi[f]_1^r} \\ & & C & \xleftarrow{\ \ g\ \ } & \pi[n]\times\pi K_r \end{array}$$

The right hand vertical composite may be written as $\mathrm{id}\otimes\pi[f']_1^{r-1}$ where f' is the $(r-1)$-tuple obtained from f by replacing f_{r-i} and f_{r-i+1} by their enriched composite. Thus the δ_i^- relation is given by the two paths around the diagram.

For $i = r$, we have a similar argument for the map $[n]\times K_0 \xrightarrow{f_1} K_1$, and we get the diagram

$$
\begin{array}{ccccccc}
\mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times K_0) & & & & & & \\
\downarrow & & & & & & \\
\mathcal{I}^{\otimes(r-1)}\otimes\pi([n]^2\times K_0) & \longrightarrow & \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]^{r+1}\times K_0) & \xrightarrow{\delta_r^- h^{r,n}_{2,K_0}} & \pi([n]\times K_0) & \xrightarrow{a^{(1)}} & \pi[n]\otimes\pi([n]^r\times K_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{I}^{\otimes(r-1)}\otimes\pi([n]\times K_1) & \longrightarrow & \mathcal{I}^{\otimes(r-1)}\otimes\pi([n]^r\times K_1) & \xrightarrow{h^{r-1,n}_{2,K_1}} & \pi([n]^r\times K_1) & \xrightarrow{a^{(1)}} & \pi[n]\times\pi([n]^{r-1}\times K_1) \\
 & & & & & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi[f]^r_2} \\
 & & & & C & \xleftarrow{g} & \pi[n]\otimes\pi K_r
\end{array}
$$

The two paths around this diagram give precisely the δ_r^- relation required. □

Before discussing the δ_i^+ relations we need to extend our notation. Suppose we are given $f_i \in \mathbf{SimpSet}_S(K_{i-1}, K_i)_n$ for $1 \leq i \leq p$ and that Z is a (q, n)-homotopy

$$\mathcal{I}^{\otimes q} \otimes \pi[n] \otimes \pi K_p \xrightarrow{\quad Z \quad} C$$

Then we define the homomorphism $a_n^*(f_1, \ldots, f_p;\ Z)$ from $\mathcal{I}^{\otimes(q+p)} \otimes \pi([n] \times K_0)$ to C by

$$
\begin{array}{ccccc}
\mathcal{I}^{\otimes(q+p)} \otimes \pi([n] \times K_0) & \longrightarrow & \mathcal{I}^{\otimes(q+p)} \otimes \pi([n]^{p+1} \times K_0) & \xrightarrow{\mathrm{id} \otimes h^{p,n}_{2,K_0}} & \mathcal{I}^{\otimes q} \otimes \pi([n]^{p+1} \times K_0) \\
\Big\downarrow{\scriptstyle a_n^*(f_1, \ldots, f_p;\ Z)} & & & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes a^{(1)}} \\
C & \xleftarrow{\quad Z \quad} & \mathcal{I}^{\otimes q} \otimes \pi[n] \otimes \pi K_p & \xleftarrow{\mathrm{id} \otimes \pi[f]^p_1} & \mathcal{I}^{\otimes q} \otimes \pi[n] \otimes \pi([n]^p \times K_0)
\end{array}
$$

Note that for $q = 0$, $Z = g$ this reduces to the previous definition. In the next proposition we use the coherence of π_S and take Z to be composite of the $(i-1, n)$-homotopy given by

$$\mathcal{I}^{\otimes(i-1)} \otimes \pi[n] \otimes \pi K_{r-i} \xrightarrow{\pi_n(f_{r-i+1}, \ldots, f_r)} \pi K_r$$

with the $(0, n)$-homotopy given by $\pi[n] \otimes \pi K_r \xrightarrow{g} C$.

Proposition 4.3.9 *For maps f_i and g as above, the homomorphisms $a_n^*(f_1, \ldots, f_r;\ g)$ satisfy*

$$\delta_i^+(a_n^*(f_1, \ldots, f_r);\ g) \ = \ a_n^*(f_1, \ldots, f_{r-i};\ \pi_n(f_{r-i+1}, \ldots, f_r) \circ g)$$

Proof: For $1 \leq i \leq r$, $h^{r,n}_{2,K_0}$ may be written as the composite

$$\mathcal{I}^{\otimes r} \otimes \pi([n]^{r+1} \times K_0) \xrightarrow{\mathrm{id} \otimes h^{r-i,n}_{i+2,K_0}} \mathcal{I}^{\otimes i} \otimes \pi([n]^{r+1} \times K_0) \xrightarrow{h^{i,n}_{2,[n]^{r-i} \times K_0}} \pi([n]^{r+1} \times K_0)$$

Thus $\delta^+_i h^{r,n}_{2,K_0} = \mathrm{id} \otimes h^{r-i,n}_{i+2,K_0} \circ \delta^+_i h^{i,n}_{2,[n]^{r-i} \times K_0}$, and this second term may in turn be written as the composite

$$(\mathrm{id} \otimes a^{(i+1)}) \circ (h_{[n]^2,[n],\ldots,[n]} \otimes \mathrm{id}) \circ b$$

Consider now the diagram in figure 4.2. The triangular region commutes by the above discussion, and the rectangles commute by naturality and by the commutativity relations between a and b and between a and h. The lower path around the diagram is just $\delta^+_i(a^*_n(f_1, \ldots, f_r);\ g)$. After a further application of naturality with $[f]^{r-i}_1$, the upper path around the diagram can be seen to be $a^*_n(f_1, \ldots, f_{r-i};\ \pi_n(f_{r-i+1}, \ldots, f_r) \circ g)$ and we have the result. □

We can summarise the findings of this section as follows

Theorem 4.3.10 *For simplicial sets K and crossed complexes C the strong deformation retraction*

$$\mathbf{Crs}_S(\pi_S K, C) \quad \simeq \quad \mathbf{SimpSet}_S(K, \mathrm{N}_S C)$$

is natural in C and coherently natural in K.

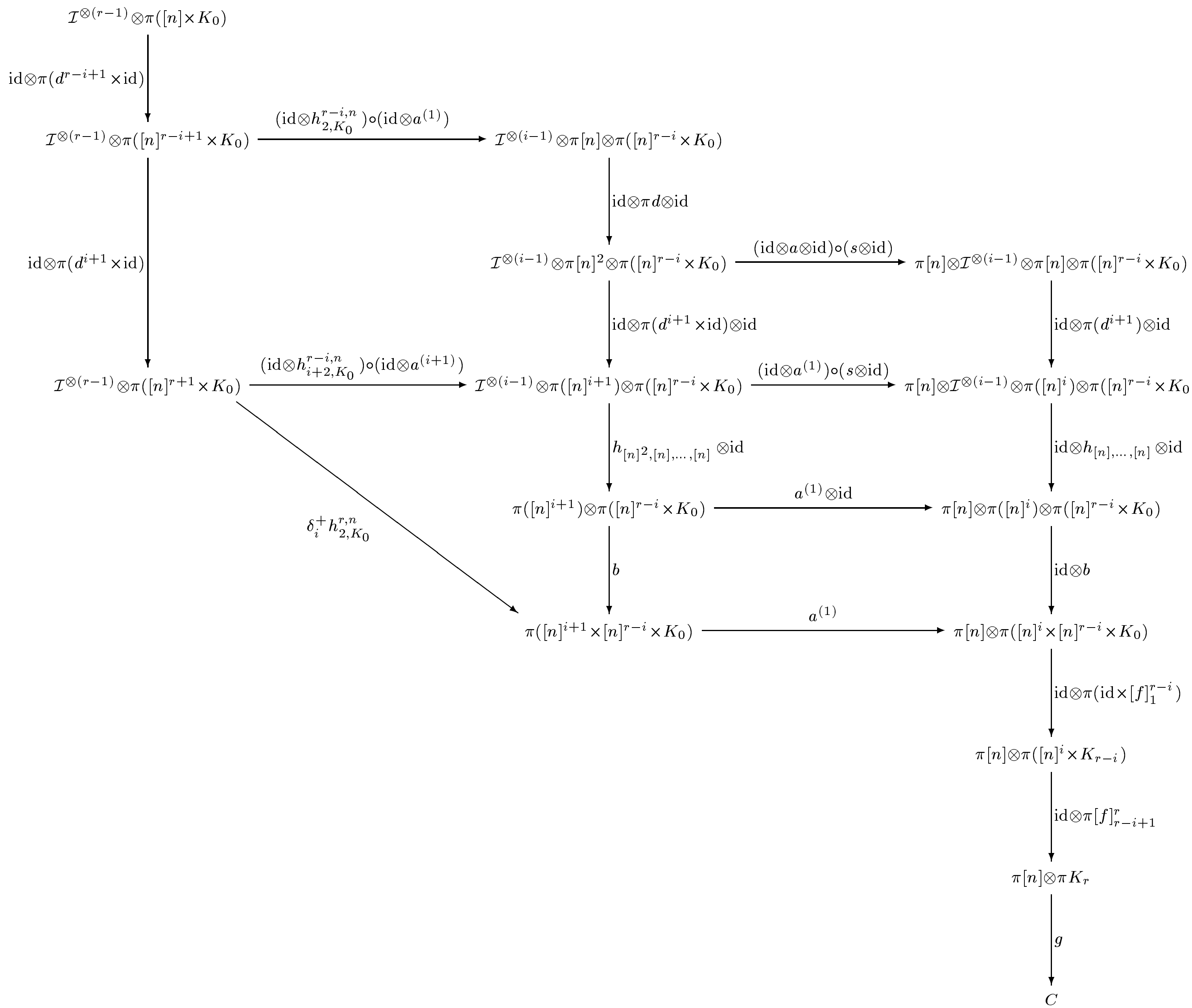


Figure 4.2: The δ_i^+ relation for a^*.

Chapter 5

Homotopy Colimits and Coherent Diagrams

5.0 Introduction

The idea of considering 'lax' functors where the functoriality only holds up to higher dimensional equivalences (satisfying appropriate associativity/coherence relations) is already well known in the categories **Cat** [2], **SimpSet** [15] and **Top** [39], and homotopy limits and colimits for diagrams of this type have been defined in [35, 38, 16, 17, 39]. In this chapter we consider a definition of homotopy coherent diagrams in the category of crossed complexes and give a tentative definition of the homotopy colimit of such a diagram. We also show how such a theory relates to crossed resolutions of extensions of groups.

The structure of this chapter is as follows. In the first section we recall the definition of homotopy colimits of lax functors in **Cat**, and show that a group extension

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

corresponds to a lax functor $H \to \mathbf{Cat}$ such that $e_H \mapsto G$. In the second section, we recall the definition of homotopy coherent diagrams in **SimpSet** and introduce a definition of homotopy coherent diagrams in **Crs**. It is shown how a lax functor in **Cat** induces a coherent functor in **SimpSet** which in turn gives a coherent functor in **Crs**. In the third section we recall the definition of homotopy colimits of coherent diagrams of simplicial sets, and discuss how this carries over to the category of crossed complexes. We end with some ideas for further development of this work.

5.1 Group Extensions and Lax Functors in Cat

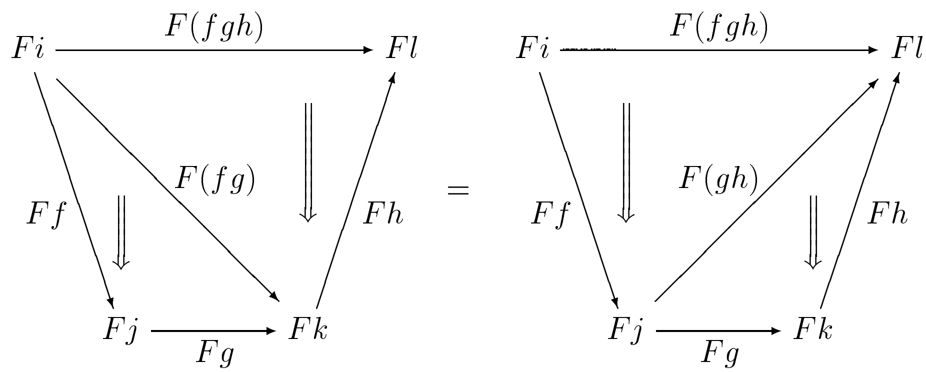
In the chapter 3 it was seen how an investigation of small models for crossed resolutions of split extensions of groups leads to a definition of homotopy colimits of functors into crossed complexes, and results in a twisted tensor product. In this chapter we will discuss a possible definition of homotopy colimits of *lax* functors into crossed complexes. In this section we provide a simple motivational example by explaining how any (not necessarily split) extension of groups corresponds to a lax functor.

Definition 5.1.1 *A* lax functor $I \xrightarrow{F} \mathbf{Cat}$ *is given by*

1. *a category $F(i)$ for each object i of I,*
2. *a functor $F(i) \xrightarrow{F(f)} F(j)$ for each arrow $i \xrightarrow{f} j$ of I, such that $F(f)$ is the identity functor if f is an identity arrow,*
3. *a natural transformation $F(f \circ g) \xRightarrow{F(f,g)} F(f) \circ F(g)$ for each pair of composable arrows (f, g) of I, such that $F(f, g)$ is the identity natural transformation if either of f, g are identity arrows, and such that for any triple (f, g, h) of composable arrows the associative law holds:*

$$F(fg, h)_a \circ (F(h))\,(F(f, g)_a) \quad = \quad F(f, gh)_a \circ F(g, h)_{(F(f))(a)} \;\; \textit{for } a \in \mathrm{Ob}(F(sf))$$

Note that there is a more general definition of a lax functor (see for example [2]) which only requires that F preserve the identity arrows up to a natural transformation, which must satisfy appropriate left and right identity relations. Also note that the associative law can be described as the equality of the following diagrams



which may also be read as asserting the commutativity of (the faces of) the obvious tetrahedron.

Suppose we have a short exact sequence of groups

$$1 \longrightarrow G \xrightarrow{\;i\;} E \xrightarrow{\;p\;} H \longrightarrow 1$$

Since p is onto we can choose a function $j\colon H \longrightarrow E$ such that $j \circ p$ is the identity on H, and then by exactness we have a function $q\colon E \longrightarrow G$ which takes $x \in E$ to the unique $g \in G$ satisfying $i(g) = x \cdot (j(px))^{-1} \in \ker(p)$. It is an old and well-known result [25] that whereas split extensions of groups are characterised by a group action, a general group extension is characterised by the pair of functions

$$H \times G \xrightarrow{k_1} G \qquad\qquad H \times H \xrightarrow{k_2} G$$
$$(h, g) \longmapsto g^h \qquad\qquad (h_1, h_2) \longmapsto \{h_1, h_2\}$$

given by

$$jh \cdot i(g^h) \;=\; ig \cdot jh \quad \text{and} \quad j(h_1 h_2) \cdot i(\{h_1, h_2\}) \;=\; jh_1 \cdot jh_2$$

In the following proposition we show how k_1 and k_2 translate into the language of lax functors.

Proposition 5.1.2 *Given a group extension E of G by H as above the assignments*

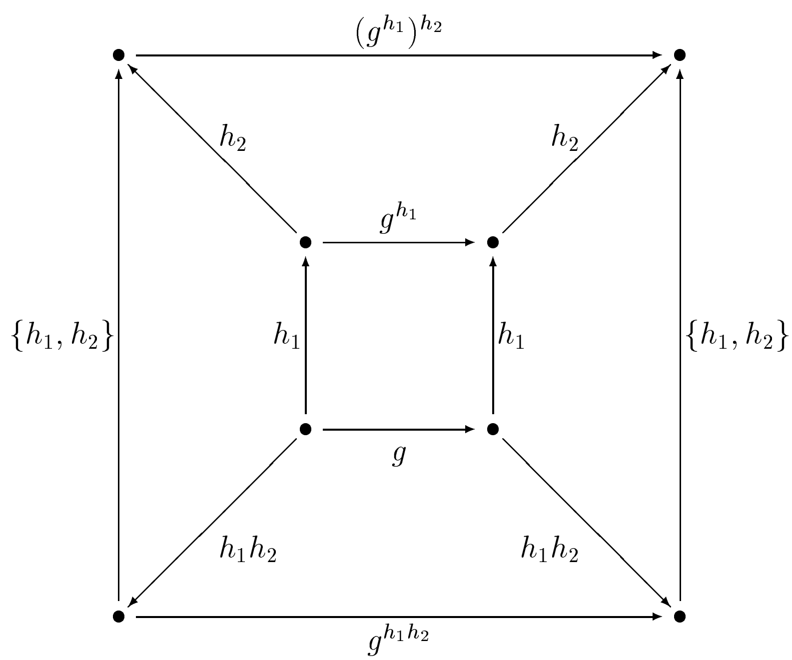
1. $\alpha(e_H) \;=\; G,$
2. $\alpha(h) \;=\; \left(g \mapsto g^h\right),$
3. $\alpha(h_1, h_2)_{e_G} \;=\; \{h_1, h_2\}$

*define a lax functor $H \xrightarrow{\alpha}$ **Cat**.*

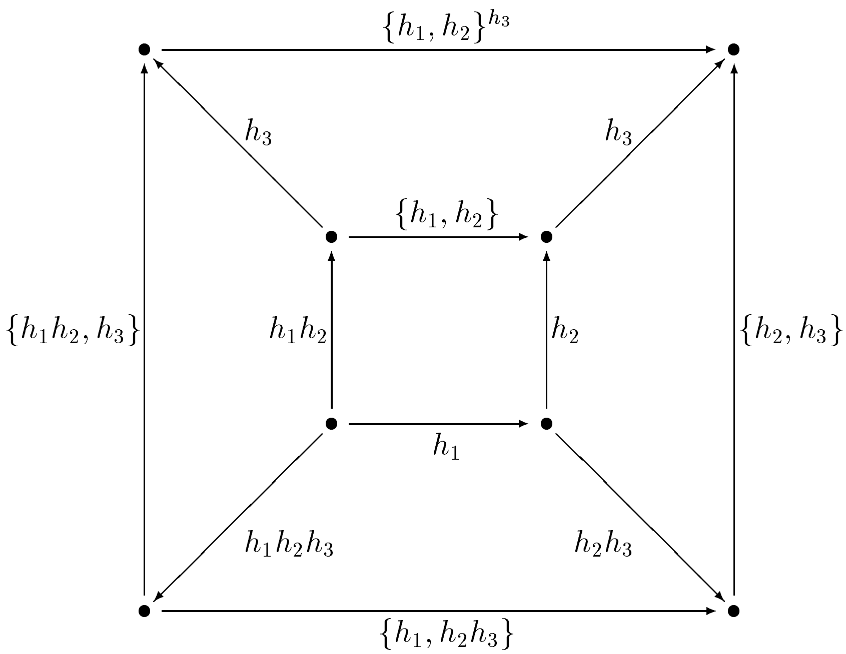
Proof: For each (arrow) $h \in H$ and $g_1, g_2 \in G$ we have

$$(jh)^{-1} \cdot i(g_1 g_2) \cdot jh = (jh)^{-1} \cdot ig_1 \cdot jh \cdot (jh)^{-1} \cdot ig_2 \cdot jh$$

and so $(g_1 g_2)^h \;=\; {g_1}^h \cdot {g_2}^h$. Thus $\alpha(h)$ defines an endofunctor of G. Consider the following diagram in E:



where we have omitted the $i(-)$ or $j(-)$ on each arrow for legibility. The diagram commutes in E by definition of g^h and $\{h_1, h_2\}$, and so its perimeter commutes in G by injectivity of i. Thus $\alpha(h_1, h_2)$ defines a natural transformation between $\alpha(h_1 h_2)$ and $\alpha(h_1) \circ \alpha(h_2)$. Similarly the required associativity of this natural transformation is shown by the commutativity in G of the perimeter of the following diagram in E:



□

In the case that the extension splits, j may be chosen to be a homomorphism and so k_2 is trivial and α reduces to an ordinary functor. Thus the above result includes that of proposition 3.1.6.

It is well known that the Grothendieck construction may also be applied to lax functors.

Definition 5.1.3 *Suppose I is a small category and F is a lax functor from I to* **Cat**. *Then the Grothendieck construction on F is the category $\int^I F$ with objects the pairs (i, x) with $i \in \mathrm{Ob}(I)$ and $x \in \mathrm{Ob}(Fi)$ and arrows $(f, a) : (i_0, x_0) \to (i_1, x_1)$ for all $f \in I(i_0, i_1)$ and $a \in \mathrm{Arr}(Fi_1)$ with source $(Ff)(x_0)$ and target x_1. The composite of the arrows*

$$(i_0, x_0) \xrightarrow{(f_1, a_1)} (i_1, x_1) \xrightarrow{(f_2, a_2)} (i_2, x_2)$$

is defined by $(f_1 \cdot f_2, F(f_1, f_2)_{x_0} \cdot (Ff_2)(a_1) \cdot a_2)$.

Following [38] we can now define homotopy colimits of lax functors in **Cat**.

Definition 5.1.4 *If $F : I \longrightarrow$ **Cat** is a lax functor, the homotopy colimit of F is the category given by the Grothendieck construction on F.*

Now return to the case where $\alpha : H \to \mathbf{Cat} : e_H \mapsto G$ is a lax functor given by a group extension E as above. Then the Grothendieck construction on α has a single object (e_H, e_G), arrows (h, g) for all $h \in H$ and $g \in G$, with composition of arrows given by

$$(h_1, g_1)(h_2, g_2) \;=\; (h_1 h_2, \{h_1, h_2\}\, g_1^{h_2} g_2)$$

This category is isomorphic to E via $(h, g) \mapsto j(h) \cdot i(g)$. Thus the homotopy colimit of α gives back the extension.

Our aim for the rest of this chapter will be as follows. Firstly, we want a suitable notion of coherent functors in **Crs** such that composing with the standard crossed resolution functor C takes a lax functor in **Cat** to a coherent functor in **Crs**, and secondly we want to define homotopy colimits of coherent functors in **Crs**. We suspect (although we do not prove) that lax/coherent homotopy colimits are preserved (up to homotopy) by C, and so we should be able to replace the standard resolution of an arbitrary group extension E by the homotopy colimit in **Crs** of the composite of C with the lax functor α corresponding to the extension.

The 'intermediate' case of coherent diagrams and homotopy colimits in **SimpSet** will also be discussed.

5.2 Coherent Functors in SimpSet and Crs

In this section we define notions of lax or homotopy coherent functors from small categories into the categories of simplicial sets and crossed complexes of groupoids. Both of these will bear some resemblence to the notion of lax functors into the category of topological spaces given by Vogt in [39].

The simplicial case is based on [15]. First we note that the representable simplicial set Δ^1 has a simplicial multiplication structure $\Gamma : \Delta^1 \times \Delta^1 \to \Delta^1$. Suppose x, y are n-simplices of Δ^1 given by monotonic functions $[n] \to [1]$. Then we define their product xy to be the n-simplex given by the monotonic function

$$(xy) : k \mapsto \max(x(k), y(k))$$

We can extend this to maps between the n-fold cartesian products of Δ^1

$$[1]^n \xrightarrow{\quad \Gamma^n_r \quad} [1]^{n-1}$$

where $\Gamma^n_r = \mathrm{id}_{[1]^{r-1}} \times \Gamma \times \mathrm{id}_{[1]^{n-r-1}}$ for $1 \leq r \leq n-1$, and we also write Γ^n_0 and Γ^n_n for the projections onto all but the first and last factor respectively.

Also we have simplicial maps

$$[1]^{n-1} \underset{\delta_r^+}{\overset{\delta_r^-}{\rightrightarrows}} [1]^n$$

for $1 \leq r \leq n$ induced by the two inclusions $\Delta^0 \to \Delta^1$.

Definition 5.2.1 *Let I be a small category. A* simplicially coherent functor *F from I to the category of simplicial sets is given by the following data*

- *a simplicial set $F(i)$ for each object i of I*
- *a simplicial map*

$$F(i_0) \times [1]^{n-1} \xrightarrow{F_{[f_k]_{k=1}^n}} F(i_n)$$

for each n-simplex $[i_0, f_1, i_1, \ldots, f_n, i_n]$ of the nerve of I

such that the following degeneracy and boundary relations are satisfied:

$$\begin{aligned}
F_{s_0([]_{i_0})} &= \mathrm{id}_{F(i_0)} \\
F_{s_r([f_k]_1^n)} &= \left(\mathrm{id}_{F(i_0)} \times \Gamma_r^n\right) \circ F_{[f_k]_1^n} \quad \textit{for } 0 \leq i \leq n \\
\left(\mathrm{id}_{F(i_0)} \times \delta_r^-\right) \circ F_{[f_k]_1^n} &= F_{d_r([f_k]_1^n)} \quad \textit{for } 1 \leq r \leq n-1 \\
\left(\mathrm{id}_{F(i_0)} \times \delta_r^+\right) \circ F_{[f_k]_1^n} &= \left(F_{[f_k]_1^r} \times \mathrm{id}_{[1]^{n-r-1}}\right) \circ F_{[f_k]_{r+1}^n} \quad \textit{for } 1 \leq r \leq n-1
\end{aligned}$$

A simplicially coherent functor in fact corresponds to a simplicially enriched functor from a certain simplicial resolution $S(I)$ of I to the category **SimpSet** regarded as being enriched over itself. The simplicially enriched category $S(I)$ was introduced in [18], and is defined as a comonadic resolution with respect to the free/forget adjoint pair between **Cat** and the category of graphs with distinguished identity loops. The degeneracy and δ_r^- relations above can be seen as arising from the definition of $S(I)$ and the δ_r^+ relations as corresponding to the enriched functoriality of $S(I) \to$ **SimpSet**. See [15] for more details.

The following result is standard.

Proposition 5.2.2 *Given two categories A, B, the nerve of the functor category $[A, B]$ is naturally isomorphic to the simplicial hom-object* $[\mathrm{Ner}\, A,\ \mathrm{Ner}\, B]$.

Proof: Since the categorisation functor is both a one-sided inverse and an adjoint to the nerve, we have

$$\mathbf{Cat}(C, D) \;\cong\; \mathbf{Cat}(c(\mathrm{Ner}\, C), D) \;\cong\; \mathbf{SimpSet}(\mathrm{Ner}\, C, \mathrm{Ner}\, D)$$

Thus there are isomorphisms

$$\mathbf{Cat}([n]\times A, B) \;\cong\; \mathbf{SimpSet}(\mathrm{Ner}([n]\times A), \mathrm{Ner}\, B) \;\cong\; \mathbf{SimpSet}(\Delta^n \times \mathrm{Ner}\, A, \mathrm{Ner}\, B)$$

which are natural in $[n]$, so the result follows. $\Box$

Now suppose $I \xrightarrow{G} \mathbf{Cat}$ is a lax functor as in definition 5.1.1. Then applying the nerve functor gives us a simplicial set $\mathrm{Ner}(Gi)$ for each object i of I and a simplicial map $\mathrm{Ner}(Gi) \to \mathrm{Ner}(Gj)$ for each arrow $i \to j$ of I. Also the natural transformation $G(f_1, f_2)$ for each pair of arrows of I corresponds by the above proposition to a simplicial map

$$\mathrm{Ner}(Gi_0) \times \Delta^1 \longrightarrow \mathrm{Ner}(Gi_2)$$

In fact this data uniquely specifies a simplicially coherent functor (cf. [36]):

Proposition 5.2.3 *Let I be a small category and $I \xrightarrow{G} \mathbf{Cat}$ a lax functor as above. Then there is a unique simplicially coherent functor*

$$I \xrightarrow{\;G\circ \mathrm{Ner}\;} \mathbf{SimpSet}$$

such that $(G\circ \mathrm{Ner})(i) = \mathrm{Ner}(Gi)$ for each object i of I, and for $n = 1$ and $n = 2$ the simplicial maps $(G\circ \mathrm{Ner})_{[f_k]_1^n}$ are defined by the $G(f_1)$ and $G(f_1, f_2)$ as above.

Proof: Suppose $I \xrightarrow{F} \mathbf{SimpSet}$ is a simplicially coherent functor such that $F(i) = \mathrm{Ner}(Gi)$ for $i \in \mathrm{Ob}(I)$. Then

$$\begin{aligned}\mathbf{SimpSet}(F(i_0)\times [1]^{n-1}, F(i_n)) &\cong \mathbf{SimpSet}([1]^{n-1}, [\mathrm{Ner}(Gi_0), \mathrm{Ner}(Gi_n)]) \\ &\cong \mathbf{Cat}(c([1]^{n-1}), [Gi_0, Gi_n])\end{aligned}$$

The category $c([1]^{n-1})$ has object set $\{0,1\}^{n-1}$ and is generated by the arrows

$$\{\ \alpha \xrightarrow{(\alpha,r,\beta)} \beta \ :\ 0 \le r \le n-1,\ \alpha_j = \beta_j \text{ for } j \ne r,\ \alpha_r = 0,\ \beta_r = 1\ \}$$

subject to the relations given by commutative diagrams of the form

$$\begin{array}{ccc} \alpha & \xrightarrow{(\alpha,r,\beta)} & \beta \\ {\scriptstyle(\alpha,r',\gamma)}\downarrow & & \downarrow{\scriptstyle(\beta,r',\delta)} \\ \gamma & \xrightarrow{(\gamma,r,\delta)} & \delta \end{array}$$

Thus specifying the simplicial maps $F_{[f_k]_1^n}$ is equivalent to specifying them on the vertices and edges of the $(n-1)$-cube — that is, to specifying functors and natural transformations

$$Gi_0 \xrightarrow{G(\alpha;[f_k]_1^n)} Gi_n \qquad G(\alpha;[f_k]_1^n) \xRightarrow{G(\alpha,r,\beta;[f_k]_1^n)} G(\beta;[f_k]_1^n)$$

satisfying the appropriate commutativity, degeneracy and boundary relations. The boundary relations here show that the data (and the degeneracy relations) for $n \geq 3$ are given by those for $n = 1, 2$. Thus the uniqueness part of the proposition holds. For existence it only remains to note that the commutativity, degeneracy and boundary relations required for $n = 1, 2$ follow from the associativity, identity and source and target relations of definition 5.1.1. □

We now turn to coherent diagrams in the category of crossed complexes of groupoids. We first define a multiplication structure on the crossed complex $\mathcal{I}$ which is given on the usual generators by

$$\mathcal{I} \otimes \mathcal{I} \xrightarrow{\quad\Gamma\quad} \mathcal{I}$$

$$\alpha \otimes \beta \longmapsto \begin{cases} \max(\alpha,\beta) & \text{if } \alpha,\beta \in \mathcal{I}_0 \\ \iota & \text{if } \{\alpha,\beta\} = \{0,\iota\} \\ e_1 \in \mathcal{I}_1 & \text{if } \{\alpha,\beta\} = \{\iota,1\} \\ e_1 \in \mathcal{I}_2 & \text{if } \alpha = \beta = \iota \end{cases}$$

Using Γ (or the projection homomorphisms for $r = 0$ or n) we obtain

$$\mathcal{I}^{\otimes n} \xrightarrow{\quad\Gamma_r^n\quad} \mathcal{I}^{\otimes(n-1)}$$

for $0 \leq r \leq n$.

Note that the homomorphisms Γ can be defined from the shuffle map b and the simplicial multiplication structure above, via

$$\mathcal{I} \otimes \mathcal{I} \;\cong\; \pi\Delta^1 \otimes \pi\Delta^1 \xrightarrow{\quad b\quad} \pi(\Delta^1 \times \Delta^1) \xrightarrow{\pi(\Gamma)} \pi(\Delta^1) \;\cong\; \mathcal{I}$$

Also we have the usual 'co-face' homomorphisms

$$\mathcal{I}^{\otimes(n-1)} \underset{\delta_r^+}{\overset{\delta_r^-}{\rightrightarrows}} \mathcal{I}^{\otimes n}$$

for $1 \leq r \leq n$.

Using these, we can define what we mean by a coherent diagram in **Crs**.

Definition 5.2.4 *Let I be a small category. A* coherent functor $I \xrightarrow{F}$ **Crs** *is given by the following data*

- *a crossed complex of groupoids $F(i)$ for each object i of I*
- *a crossed complex homomorphism*

$$F(i_0) \otimes \mathcal{I}^{\otimes(n-1)} \xrightarrow{F_{[f_k]_{k=1}^n}} F(i_n)$$

for each n-simplex $[i_0, f_1, i_1, \ldots, f_n, i_n]$ of the nerve of I

such that the following degeneracy and boundary relations are satisfied:

$$\begin{array}{rcl} F_{s_0([]_{i_0})} & = & \mathrm{id}_{F(i_0)} \\ F_{s_r([f_k]_1^n)} & = & \left(\mathrm{id}_{F(i_0)} \otimes \Gamma_r^n\right) \circ F_{[f_k]_1^n} \quad \text{for } 0 \le r \le n \\ \left(\mathrm{id}_{F(i_0)} \otimes \delta_r^-\right) \circ F_{[f_k]_1^n} & = & F_{d_r([f_k]_1^n)} \quad \text{for } 1 \le r \le n-1 \\ \left(\mathrm{id}_{F(i_0)} \otimes \delta_r^+\right) \circ F_{[f_k]_1^n} & = & \left(F_{[f_k]_1^r} \otimes \mathrm{id}_{[1]^{n-r-1}}\right) \circ F_{[f_k]_{r+1}^n} \quad \text{for } 1 \le r \le n-1 \end{array}$$

Using the shuffle homomorphism b from chapter 2, the following proposition shows that the fundamental crossed complex functor takes a simplicially coherent functor to a coherent diagram in **Crs**.

$$I \xrightarrow{G} \mathbf{SimpSet} \xrightarrow{\pi} \mathbf{Crs}$$

Proposition 5.2.5 *Suppose $I \xrightarrow{G}$ **SimpSet** is a simplicially coherent functor as in definition 5.2.1. Then there is a coherent functor $G \circ \pi$ into **Crs** with $(G \circ \pi)(i) = \pi(Gi)$ for each object i of I and with the homomorphisms $(G \circ \pi)_{[f_k]_1^n}$ for $[i_0, f_1, i_1, \ldots, f_n, i_n] \in \mathrm{Ner}(I)_n$ given by*

$$\pi(Gi_0) \otimes \mathcal{I}^{\otimes(n-1)} \xrightarrow{b^{n-1}} \pi(Gi_0 \times [1]^{n-1}) \xrightarrow{\pi(F_{[f_k]_1^n})} \pi(Gi_n)$$

Proof: The shuffle map b respects the above structure on (tensor) products of $\mathcal{I}$ and Δ^1, and we have the following commutative diagrams:

$$\begin{array}{ccc} \pi(Gi_0) \otimes \mathcal{I}^{\otimes n} & \xrightarrow{b^n} & \pi(Gi_0 \times [1]^n) \\ {\scriptstyle \mathrm{id} \otimes \Gamma_r^n}\downarrow & & \downarrow{\scriptstyle \pi(\mathrm{id} \times \Gamma_r^n)} \\ \pi(Gi_0) \otimes \mathcal{I}^{\otimes(n-1)} & \xrightarrow{b^{n-1}} & \pi(Gi_0 \times [1]^{n-1}) \end{array} \qquad \begin{array}{ccc} \pi(Gi_0) \otimes \mathcal{I}^{\otimes(n-2)} & \xrightarrow{b^{n-2}} & \pi(Gi_0 \times [1]^{n-2}) \\ {\scriptstyle \mathrm{id} \otimes \delta_r^-}\downarrow\downarrow{\scriptstyle \mathrm{id} \otimes \delta_r^+} & & {\scriptstyle \pi(\mathrm{id} \times \delta_r^-)}\downarrow\downarrow{\scriptstyle \pi(\mathrm{id} \times \delta_r^+)} \\ \pi(Gi_0) \otimes \mathcal{I}^{\otimes(n-1)} & \xrightarrow{b^{n-1}} & \pi(Gi_0 \times [1]^{n-1}) \end{array}$$

The required degeneracy and boundary relations for $G \circ \pi$ thus follow from those for G. □

This justifies our definition of coherent functors into **Crs**.

5.3 Homotopy Colimits for Coherent Functors

In chapter 3 we gave suitable algebraic models for the homotopy colimits of functors F from a small category I to the category of crossed complexes. Models had generators $a_p \otimes b_q$ in dimension $p+q$, for a_p an element of an object in the image of F and b_q a q-simplex of the nerve of I, and the boundaries of these generators were given by elements of the form $\delta a_p \otimes b_q$, ${a_p}^{f_1} \otimes d_0 b_q$ and $a_p \otimes d_i b_q$. In this chapter we will see that this description may be extended to give models for homotopy colimits of coherent functors as described in the previous section. The effect on the models of replacing strict functors by homotopy coherent ones will be that the shape $- \otimes \pi\Delta^q$ for the generators will become $- \otimes \mathcal{I}^{\otimes q}$, each element b_q of the nerve now indexing a q-dimensional cube rather than a q-simplex. Similarly, instead of having just a twisted d_0 face, the generators will now have the δ_i^+ faces of the cube 'twisted' to varying degrees by the higher coherence data. (Note however that if the coherence data is all trivial, i.e. the functor *is* strict, then a standard embedding of simplices into cubes with degenerate δ_i^+ faces shows that our model for the homotopy colimit will be *isomorphic* to that of chapter 3).

We will discuss the simplicial case first. Suppose we have a simplicially coherent functor

$$I \xrightarrow{\quad F \quad} \mathbf{SimpSet}$$

given by simplicial sets Fi for $i \in \mathrm{Ob}(I)$ together with simplicial maps

$$Fi_0 \times [1]^{n-1} \xrightarrow{\quad [f_k]_1^n \quad} Fi_n$$

for $[i_0, f_1, i_1, \ldots, f_n, i_n] \in \mathrm{Ner}(i)_n$.

Definition 5.3.1 *The homotopy colimit* $\mathrm{hocolim}(F)$ *of a simplicially coherent functor* F *is given by the* $\mathrm{Ner}(I)$*-indexed coproduct of simplicial sets*

$$\coprod_n \coprod_{[i_0, f_1, i_1, \ldots, f_n, i_n]} Fi_0 \times [1]^n$$

(whose elements we will write as

$$(a,\ (x_1, \ldots, x_n);\ [f_k]_1^n)$$

for $a \in Fi_0$, $x_k \in \Delta^1$*), quotiented by the relations*

$$\begin{aligned}
\Big(a,\ (x_1,\ldots,x_n);\ s_r([f_k]_1^{n-1})\Big) &= \Big(a,\ \Gamma_r^n(x_1,\ldots,x_n);\ [f_k]_1^{n-1}\Big) \\
\Big(a,\ \delta_r^-(x_1,\ldots,x_n);\ [f_k]_1^{n+1}\Big) &= \Big(a,\ (x_1,\ldots,x_n);\ d_r([f_k]_1^{n+1})\Big) \\
\Big(a,\ \delta_r^+(x_1,\ldots,x_n);\ [f_k]_1^{n+1}\Big) &= \Big(F_{[f_k]_1^r}(a, x_1,\ldots,x_{r-1}),\ (x_r,\ldots,x_n);\ [f_k]_{r+1}^{n+1}\Big)
\end{aligned}$$

Note that this is essentially the definition of homotopy colimits of homotopy coherent functors in **Top**-enriched categories given by Vogt in [39], which has been presented for simplicially enriched categories, in a much more categorical framework, by Cordier in [16].

For the crossed complex case we do not have a simplicially enriched structure except up to higher homotopies, as made precise in chapter 4, and so the indexed-limit machinery of [16, 3, 24] does not give a definition for homotopy colimits of coherent functors in **Crs**. In the rest of this chapter we will suggest a 'bare-hands' definition, and leave the necessary generalisation of the work of Cordier *et al.* as a subject which requires further investigation.

Suppose we have a coherent functor

$$I \xrightarrow{\quad F \quad} \mathbf{Crs}$$

given by crossed complexes Fi for $i \in \mathrm{Ob}(I)$ together with homomorphisms

$$Fi_0 \otimes \mathcal{I}^{\otimes(n-1)} \xrightarrow{[f_k]_1^n} Fi_n$$

for $[i_0, f_1, i_1, \ldots, f_n, i_n] \in \mathrm{Ner}(i)_n$.

Definition 5.3.2 *The homotopy colimit* $\mathrm{hocolim}(F)$ *of a coherent diagram* F *of crossed complexes is given by the* $\mathrm{Ner}(I)$*-indexed coproduct*

$$\coprod_n \coprod_{[i_0,f_1,i_1,\ldots,f_n,i_n]} Fi_0 \otimes \mathcal{I}^{\otimes n}$$

(whose elements we will write as

$$(a \otimes x_1 \otimes \cdots \otimes x_n;\ [f_k]_1^n)$$

for $a \in Fi_0$, $x_k \in \mathcal{I}$*), quotiented by the relations*

$$\begin{aligned}
\left(a \otimes x_1 \otimes \cdots \otimes x_n;\ s_r([f_k]_1^{n-1})\right) &= \left(a \otimes \Gamma_r^n(x_1 \otimes \cdots \otimes x_n);\ [f_k]_1^{n-1}\right) \\
\left(a \otimes \delta_r^-(x_1 \otimes \cdots \otimes x_n);\ [f_k]_1^{n+1}\right) &= \left(a \otimes x_1 \otimes \cdots \otimes x_n;\ d_r([f_k]_1^{n+1})\right) \\
\left(a \otimes \delta_r^+(x_1 \otimes \cdots \otimes x_n);\ [f_k]_1^{n+1}\right) &= \left(F_{[f_k]_1^r}(a \otimes x_1 \otimes \cdots \otimes x_{r-1}) \otimes x_r \otimes \cdots \otimes x_n;\ [f_k]_{r+1}^{n+1}\right)
\end{aligned}$$

Recall that a group extension

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

corresponds by proposition 5.1.2 (and by the Grothendieck construction) to a lax functor α, and hence gives a coherent functor

$$F = \alpha \circ \mathrm{Ner} \circ \pi : H \longrightarrow \mathbf{Crs}$$

which takes the unique object e_H of the category H to the crossed complex $C(G)$. The reader is invited to consider the two diagrams used in the proof of proposition 5.1.2 as cubes depicting elements in dimension three of the crossed complex hocolim(F). Since n-tuples of elements of G and H index n-simplices and n-cubes respectively in the homotopy colimit, the diagrams can be thought of as representing generators given by

$$[g] \otimes [h_1, h_2] \quad \text{and} \quad [] \otimes [h_1, h_2, h_3]$$

together with their boundary relations. Thus although we have not proved that $C(E)$ and hocolim(F) are homotopy equivalent, the latter certainly contains all the composition and associativity information in E and we have some justification for calling hocolim(F) a small resolution of E and thinking of it as a twisted tensor product of $C(G)$ by $C(H)$.

We end by giving a comparison map between the homotopy colimits of coherent diagrams of simplicial sets and of crossed complexes. We suspect that the fundamental crossed complex functor preserves these homotopy colimits (up to equivalence in homology, at least) although this is only a conjecture at the present time.

Proposition 5.3.3 *Suppose* $I \xrightarrow{F}$ **SimpSet** *is a simplicially coherent functor, with* $F \circ \pi$ *the corresponding coherent functor into* **Crs** *given by proposition 5.2.5. Then there is a natural comparison map*

$$\operatorname{hocolim}(F \circ \pi) \longrightarrow \pi(\operatorname{hocolim} F)$$

Proof: Consider the shuffle homomorphisms

$$\pi(Fi_0) \otimes \mathcal{I}^{\otimes n} \xrightarrow{\;b^n\;} \pi(Fi_0 \times [1]^n)$$

for i_0 an object of I. Since π preserves coproducts, we get a homomorphism

$$\coprod_n \coprod_{[i_0, f_1, i_1, \ldots, f_n, i_n]} \pi(Fi_0) \otimes \mathcal{I}^{\otimes n} \xrightarrow{\;\;\theta\;\;} \pi\left(\coprod_n \coprod_{[i_0, f_1, i_1, \ldots, f_n, i_n]} Fi_0 \times [1]^n\right)$$

To show that this defines a comparison map between the homotopy colimits, we must prove that θ respects the degeneracy and boundary relations. In fact we show that θ maps each side of each relation on hocolim$(F \circ \pi)$ to the corresponding side of a corresponding relation on π(hocolim F). For the degeneracy and δ_r^- relations, and for the left hand side of the δ_r^+ relation, this follows from the commutativity of the following diagrams:

$$\begin{array}{ccc}
\pi(Fi_0) \otimes \mathcal{I}^{\otimes n} & \xrightarrow{\;b^n\;} & \pi(Fi_0 \times [1]^n) \\
{\scriptstyle \mathrm{id} \otimes \Gamma_r^n} \downarrow & & \downarrow {\scriptstyle \pi(\mathrm{id} \times \Gamma_r^n)} \\
\pi(Fi_0) \otimes \mathcal{I}^{\otimes (n-1)} & \xrightarrow{\;b^{n-1}\;} & \pi(Fi_0 \times [1]^{n-1})
\end{array}
\qquad
\begin{array}{ccc}
\pi(Fi_0) \otimes \mathcal{I}^{\otimes n} & \xrightarrow{\;b^n\;} & \pi(Fi_0 \times [1]^n) \\
{\scriptstyle \mathrm{id} \otimes \delta_r^-} \downarrow\downarrow {\scriptstyle \mathrm{id} \otimes \delta_r^+} & & {\scriptstyle \pi(\mathrm{id} \times \delta_r^-)} \downarrow\downarrow {\scriptstyle \pi(\mathrm{id} \times \delta_r^+)} \\
\pi(Fi_0) \otimes \mathcal{I}^{\otimes (n+1)} & \xrightarrow{\;b^{n+1}\;} & \pi(Fi_0 \times [1]^{n+1})
\end{array}$$

For the right hand side of the δ_r^+ relation we have the following diagram

$$\begin{CD}
\pi(Fi_0) \otimes \mathcal{I}^{\otimes n} @>{b^n}>> \pi(Fi_0 \times [1]^n) \\
@V{(F \circ \pi)_{[f_k]_1^r} \otimes \mathrm{id}}VV @VV{\pi(F_{[f_k]_1^r} \times \mathrm{id})}V \\
\pi(Fi_r) \otimes \mathcal{I}^{\otimes(n-r+1)} @>{b^{n-r+1}}>> \pi(Fi_r \times [1]^{n-r+1})
\end{CD}$$

which commutes by definition of $(F \circ \pi)_{[f_k]_1^r}$ as $b^{r-1} \circ \pi(F_{[f_k]_1^r})$. □

In the reverse direction, we have the diagonal approximation maps

$$\pi(Fi_0 \times [1]^n) \xrightarrow{a^n} \pi(Fi_0) \otimes \mathcal{I}^{\otimes n}$$

and so we get a homomorphism

$$\pi\left(\coprod_n \coprod_{[i_0, f_1, i_1, \ldots, f_n, i_n]} Fi_0 \times [1]^n\right) \xrightarrow{\varphi} \coprod_n \coprod_{[i_0, f_1, i_1, \ldots, f_n, i_n]} \pi(Fi_0) \otimes \mathcal{I}^{\otimes n}$$

However this does not define a homomorphism between $\pi(\operatorname{hocolim} F)$ and $\operatorname{hocolim}(F \circ \pi)$ since the diagram

$$\begin{CD}
\pi(Fi_0 \times [1]^n) @>{a^n}>> \pi(Fi_0) \otimes \mathcal{I}^{\otimes n} \\
@V{\pi(F_{[f_k]_1^r} \times \mathrm{id})}VV @VV{b^{r-1} \otimes \mathrm{id}}V \\
@. @. \pi(Fi_0 \times [1]^{r-1}) \otimes \mathcal{I}^{\otimes(n-r+1)} \\
@VVV @VV{\pi(F_{[f_k]_1^r}) \otimes \mathrm{id}}V \\
\pi(Fi_r \times [1]^{n-r+1}) @>{a^{n-r+1}}>> \pi(Fi_r) \otimes \mathcal{I}^{\otimes(n-r+1)}
\end{CD}$$

does not commute and so φ does not respect the δ_r^+ relations. However the diagram does commute up to the system of higher homotopies between the composites $a^k \circ b^k$, and it would be interesting if the results of section 2.3.2 and chapter 4 could be used here.

5.4 Conclusions

In this thesis we have presented some new ways in which the algebraic structure of crossed complexes of groupoids can be used for modelling various situations in topology.

We have seen that a version of the Eilenberg-Zilber theorem for crossed complexes holds in a very similar way to the classical theorem for chain complexes, and have developed the notions of a double crossed complex and of crossed complex models for homotopy colimits. As an example of their use in non-abelian homological algebra we have explained how the crossed resolution of a group which arises as a product, a semidirect product or an extension may be replaced by a smaller model which does not have the 'diagonal cells'. One of the basic aims has been to work out some of the consequences of using tensor products instead of cartesian products wherever the Eilenberg-Zilber theorem makes this possible.

In this section we would like to give a few ideas, some of quite a speculative nature, for possible future developments of the work of this thesis. These possible developments are in two directions, which we may call the abstract development and the topological application.

Beginning with the applications, we would first like to extend the Eilenberg-Zilber theorem to a crossed complex version of the twisted Eilenberg-Zilber theorem, as proved for chain complexes in [5]. This could then be used to develop a non-abelian homological perturbation theory as mentioned in chapter 2, leading to specific calculations.

Secondly we would like to be able to find a small crossed complex model of the total space E of a Kan fibration of simplicial sets

$$F \longrightarrow E \longrightarrow B$$

The model should have the form of a twisted tensor product of πF by πB, and may arise as an application of the twisted Eilenberg-Zilber theorem or by development of the theory we have seen for small resolutions of an extension of groups.

Also we would like to investigate further the rôle in algebraic topology which might be played by crossed differential graded algebras.

The general aim here is to carry over much of the work which is regarded as 'main-stream' for chain complexes (and which seems to be regarded as *only* possible by making all spaces simply-connected and all groups abelian) to crossed complexes. The category of crossed complexes shares a lot of the formal properties of that of chain complexes, such as the monoidal closed structure, and may be seen as a quotient of the category of simplicial groupoids [19]. Thus on the one hand crossed complexes provide finer information on homotopy types than do chain complexes, including the action of the fundamental groupoid, but on the other hand they may be regarded as simply one step towards a good algebraic structure which models *all* homotopy types.

From the abstract point of view, we believe that the material presented in chapters 4 and 5, together with section 2.3.2, should admit a more categorical treatment. For example, the extension of the Eilenberg-Zilber homotopy $h_{K,L}$ to the system of coherent homotopies $h_{K_1,\dots,K_n}$ in theorem 2.3.9 has the same form as the extension of a lax functor to a simplicially coherent functor in proposition 5.2.3, except that the latter is carried out in the context of much more 'high-tech' machinery. Similarly we feel that there is more underlying the result of theorem 4.3.10 than the pages preceeding it make clear.

Cordier and others in [3, 16, 17] define homotopy colimits for homotopy coherent functors in the setting of simplicially tensored enriched categories. That is, they assume that they are working with a simplicially enriched category $\mathbf{C}$ together with an enriched functor

$$\mathbf{SimpSet} \times \mathbf{C} \xrightarrow{\overline{\otimes}} \mathbf{C}$$

such that there is a natural isomorphism of simplicial homs

$$[K\overline{\otimes}C, D] \;\;\cong\;\; [K,[C,D]]$$

for each simplicial set K and objects C, D of $\mathbf{C}$. They can then define homotopy colimits of homotopy coherent functors by a simplicially-enriched coend

$$\operatorname{hocolim}\left(S(I) \xrightarrow{F} \mathbf{C}\right) \;\;=\;\; \int^i \operatorname{Diag}\left(Y(i)\right)\overline{\otimes}F(i)$$

where $Y(i)$ is the following bisimplicial set defined using the enriched homs of the simplicial resolution $S(I)$ of I:

$$Y(i)_{n,\bullet} \;\;=\;\; \coprod_{i_0,\dots,i_n} [i,i_0] \times \ldots \times [i_{n-1},i_n]$$

Now suppose instead that $\mathbf{C}$ is a monoidal closed category and π is a functor from **SimpSet** to $\mathbf{C}$ which has a right adjoint Ner and which satisfies an Eilenberg-Zilber type theorem. Then we may define an ordinary functor $\overline{\otimes}$ by

$$K\overline{\otimes}C \;\;=\;\; \pi K \otimes C$$

With respect to the simplicially enriched structure on $\mathbf{C}$ defined by applying Ner to the internal hom, neither π or $\otimes$ become enriched functors except up to some form of homotopy coherence, and so we do not get an enriched functor $\overline{\otimes}$. Furthermore the isomorphism of simplicial homs above is only a coherently-natural homotopy equivalence. However since we are trying to define *homotopy* colimits, it is nice to imagine that there is an extension of the theory such that homotopy colimits of coherent functors into $\mathbf{C}$ may be defined in terms of some form of homotopy coherent coend of homotopy coherent functors.

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