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Quantum information and computation with one-dimensional anyons

Niterói

Março - 2021

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Tese apresentada ao Curso de Pós-Graduação em Física da Universidade Federal Fluminense, como requisito parcial para a obtenção do Título de Doutor em Física.

Universidade Federal Fluminense

Instituto de Física

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Niterói

Março - 2021

Ficha catalográfica automática - SDC/BIF
Gerada com informações fornecidas pelo autor

T716q Tosta, ALLAN DAVID CONY
Quantum information and computation with one-dimensional anyons / ALLAN DAVID CONY Tosta ; Daniel Jost Brod, orientador ; Ernesto Fagundes Galvão, coorientador. Niterói, 2021.
167 p. : il.

Tese (doutorado)-Universidade Federal Fluminense, Niterói, 2021.

DOI: http://dx.doi.org/10.22409/PPGF.2021.d.14483444760

1. Computação quântica. 2. Ânions. 3. Óptica quântica. 4. História da Física. 5. Produção intelectual. I. Brod, Daniel Jost, orientador. II. Galvão, Ernesto Fagundes, coorientador. III. Universidade Federal Fluminense. Instituto de Física. IV. Título.

CDD -

Bibliotecário responsável: Debora do Nascimento - CRB7/6368

I dedicate this thesis to my wife, Layla

Acknowledgements

First, I would like to thank my dearest wife, Layla, for her material, emotional, and sometimes technical support. This phase of my formation as a physicist comes at great personal cost, and without her intervention I would not be able to complete it. Next, I would like to thank my father David, and mother Madelon, for everything they had to sacrifice for me in order to provide the possibilities for my growth as a person and as a scientist.

Next, I would like to thank my mentors, Prof. Daniel Brod and Prof. Ernesto Galvão, for their unwavering support with everything related to the profession, the job market, and for suggesting to me the problem that is discussed in this thesis. This work would not exist without them. I would also like to thank Prof. Luis Oxman for his support with everything related to quantum field theory.

I must also thank my friend Gustavo Moreira Simões, whose years of friendship have molded my worldview and critical sense in profound ways. Last, but not least, I would like to give special thanks to Pedro Ivo Souza de Alcântara, who I've never met personally, but had profound influence in my appreciation of the humanities and, more specifically, of History. To him, I dedicate my preference for the historical approach taken in the second chapter of this thesis.

Of course, any possible mistake in this work is my full responsibility, and has no bearing whatsoever on anyone that has been mentioned here.

Financial support: I acknowledge full financial support by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).

"The first principles of the universe are atoms and empty space;
everything else is merely thought to exist."
-Diog. Laër., Democritus, Vol. IX, 44
(trans. by Robert Drew Hicks 1925)

Abstract

We can study the relation between reversible computational models and the conservative dynamics of identical particles in order to shed light on the physical constraints of computation. From a theoretical perspective, it can help determine what types of physical resources are necessary for universal computation, in both classical and quantum models. This relation has been investigated since the nineteen-eighties and, in the quantum case, has led to the formulation of restricted models of computation that stem from the behavior of both fermionic and bosonic systems of free, identical particles.

This behavior is, in both cases, modelled by sequences of unitary maps, generated from Hamiltonians that are quadratic on creation and annihilation operators, also called Gaussian Hamiltonians. It is believed that, in both cases, computational models that use Gaussian unitaries to process information are not universal for quantum computation. For fermions, non-Gaussian unitaries are required for universality while, for bosons, Gaussian unitaries are enough if we allow for adaptive protocols. These unitaries are also used to model passive, linear optical devices in photonics, meaning that sequences of Gaussian unitaries model networks of optical devices.

In this thesis, I study computational models based on the use of sequences of Gaussian unitaries defined for one-dimensional systems of identical particles possessing fractional exchange statistics. Particles having fractional exchange statistics are, in general, called *anyons*, regardless of any specific particle model. Here, I investigate two specific families of one-dimensional anyons, called fermionic anyons and bosonic anyons, which are related to standard fermions and bosons by a Jordan-Wigner transformation. First, I propose the use of unitaries generated by quadratic Hamiltonians over anyonic oscillator algebras in order to define anyonic equivalents of optical devices. And later, I show how to calculate the action of some simple anyonic devices on the Fock-states of both anyon types.

The principal result in this work is the proof of universality for quantum computing with anyonic optical networks, for both anyon types, without requiring non-Gaussian unitaries nor adaptive protocols. I prove this by devising networks that exploit a one-dimensional analogue of the Aharonov-Bohm effect for both types of anyons. The secondary result is a theory of coherent states for bosonic anyons, together with a study of their behavior under optical networks and an application to the generation of cat-states, which are hard to obtain for coherent states of regular bosons.

Resumo

Podemos estudar a relação entre modelos de computação reversível e a dinâmica conservativa de partículas idênticas para ajudar a esclarecer os limites físicos da computação. Do ponto de vista teórico, isto pode ajudar a determinar que tipos de recursos físicos são necessários para fazer computação universal, tanto para modelos clássicos quanto para quânticos. Esse relação têm sido investigada desde a década de oitenta e, no caso quântico, levou à formulação de modelos de computação restrita provenientes do comportamento de sistemas tanto de férmions quanto de bósons livres não-interagentes.

Esse comportamento é, em ambos os casos, modelado por sequências de mapas unitários gerados por Hamiltonianas quadráticas nos operadores de criação e aniquilação, também chamadas de Hamiltonianas Gaussianas. Acredita-se que, em ambos os casos, modelos de computação que se utilizam de mapas unitários Gaussianos para processar informação não são universais para a computação quântica. Para férmions, unitários não-Gaussianos são necessários para obter a universalidade, enquanto que para bósons, os unitários Gaussianos são suficientes apenas se permitirmos protocolos adaptativos. Tais unitários também são utilizados para modelar elementos ópticos passivos e lineares em fotônica, o que implica que sequências de unitários Gaussianos são um modelo para redes de elementos ópticos.

Nesta tese, eu estudo modelos computacionais baseados no uso de sequências de unitários Gaussianos definidos para sistemas unidimensionais de partículas idênticas que têm estatística de troca fracionária. Partículas que têm estatística de troca fracionária são, em geral, chamadas de ânyons, independentemente de qualquer modelo. Aqui, eu estudo duas famílias específicas de ânyons unidimensionais, chamados de ânyons fermiônicos e anyons bosônicos, que estão relacionados a férmions e bósons comuns através de uma transformação de Jordan-Wigner. Primeiro, eu proponho o uso de unitários gerados por Hamiltonianas quadráticas nas álgebras de osciladores anyônicos a fim de definir os equivalentes anyônicos de elementos ópticos. E, depois, eu mostro como calcular a ação de elementos ópticos anyônicos simples sobre os estados de Fock de ambos os tipos de ânyon.

O resultado principal deste trabalho é a prova de universalidade para a computação quântica com redes ópticas anyônicas, para ambos os tipos de ânyon, sem requerer o uso de unitários não-Gaussianos nem protocolos adaptativos. Eu provei isso fornecendo duas redes que explorem um análogo unidimensional do efeito Aharonov-Bohm para ambos os tipos de ânyon. O resultado secundário é uma teoria de estados coerentes para ânyons bosônicos, junto com o estudo do comportamentos deles sobre a ação de redes ópticas e uma aplicação na geração de cat-states, que são difíceis de obter com estados coerentes de bósons comuns.

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1 Introduction and thesis outline

Quantum computing is a rapidly developing field at the intersection of physics, mathematics, and computer science, which aims to engineer and control quantum phenomena in order to solve computational problems. By quantum phenomena I mean something very specific, since, although the miniaturization of circuit components is reaching its quantum limits, this does not imply that standard computers will become "quantum" any time soon. Instead, the quantum part of quantum computing refers to the way information is stored and processed in such devices, which is fundamentally different from how it is done in both commercial and industrial-grade computers.

The idea of quantum computing first came around in the 80's, most notably in a work of Feynman [76]. In it, he proposed that a suitably-controllable quantum system could simulate another quantum system, in a way that is much more efficient than any classical computer. This was due to (but not only to) the prohibitive memory and processing-time requirements for solving Schrodinger's equations for a system of many quantum particles. Such suitably-controllable systems became known as quantum simulators, and they provided the first example of a quantum machine that could, in principle, have a computational advantage over any classical computer in a particular task. In fact, quantum simulators might be the first type of special-purpose quantum computers to have real-world applications, given the current level of technological development [93].

General-purpose quantum computers were first theorized by Deutsch in [57], where he generalized a classical computation model called the *Turing machine*, which was used to build the foundations of modern computer science [221]. The first proposed applications of a general quantum computer that were not related to physics were Grover's [115] and Shor's [219] algorithms. The first is an algorithm for the problem of searching in an unstructured database, and it has a polynomial advantage over the best possible classical algorithm, in terms of running time as a function of the database size. The second is an algorithm for prime factorization[1] that has an exponential advantage over the best known classical algorithm, in terms of running time as a function of the input's number of digits. These two algorithms showed that quantum computers could have very relevant, real-world applications, and they sparked the interest on quantum computing as a whole.

Nevertheless, both algorithms required a very high degree of control over each individual quantum system used in them. In fact, such necessity for control would make quantum computing as a whole unfeasible. Soon after his algorithm, Shor [220] showed how to encode quantum information in order detect and correct errors during computation.

[1] Prime factorization plays an important role in commercial encryption protocols.

This discovery sparked the possibility of doing fault-tolerant quantum computing, which was confirmed for a broad class of systems by the threshold theorem in [4]. Since then, many quantum systems were proposed as platforms for quantum computing, such as neutral atoms [210], nitrogen-vacancy centers in diamonds [49], superconducting circuits [152] and trapped ions [40], just to name a few.

At the time this thesis is written, the field of quantum computing is assumed to be in what is called its NISQ (short for noisy intermediate-scale quantum) era. It is assumed that quantum devices are advanced enough to be tested for specific tasks where they might offer some advantage over classical computer, but are still too crude for sustained, general use. Some examples of devices that are claimed to have achieved this type of specific advantage[2] are the Sycamore chip, made of superconducting circuits [14], and the Jiuzhang sampler, built in an optical setup [265]. There has been a wave of excitement in the field due to some of the previously mentioned developments. Nevertheless, even with all of these advancements, there are some fundamental questions regarding the connection between quantum computing and physics that have not been fully addressed.

One of the fundamental questions lying in the intersection of physics and computer science is what exactly defines the computational properties of models based on the dynamics of identical particle systems. For classical systems, the prototypical example of such a model is the billiard ball computer [85] proposed by Fredkin and Toffoli, which was proven to be able execute any algorithm using elastic (and therefore conservative) collisions between identical classical particles. The consequences of this formalism was best put by them in the abstract of [85]:

> *Quite literally, the functional behavior of a general-purpose digital computer can be reproduced by a perfect gas placed in a suitably shaped container and given appropriate initial conditions.*

In the quantum case, linear optical computers, for both standard bosons [151] and fermions [62], can be regarded as the quantum versions of the billiard ball model, where the identical classical particles are replaced by identical quantum particles, and the "suitably shaped container" is replaced by sequences of linear optical devices, such as beam splitters and phase shifters.

However, unlike billiard balls, computers built on the basis of linear dynamical maps acting over systems of fermions and bosons are not believed to be able to execute all possible quantum computational processes. In other words, perfect classical gases can mimic classical computers, but perfect fermionic or bosonic gases do not seem to mimic quantum computers. Investigations into the nature of these models [35, 149, 258] brings credence to the belief that the statistical properties of bosons and fermions are largely

[2] Also called quantum supremacy.

the reason why they are not usable for general purpose computation without introducing complicated interactions or measurement protocols.

This thesis aims to generalize those billiard-like models of quantum computing to non-standard models of identical particles, which are generically named *anyons*. The concept of anyon is commonly attributed to Wilczek [250, 251, 253], due to his coinage of the term, but the idea itself goes back to the works of Leinaas and Myrheim [159, 160, 162]. Basically, anyons are particles, or particle-like objects such as solitons or monopoles, where the many-body wave-functions describing identical anyons acquires non-trivial phase factors under particle permutations. Anyons can be broadly classified into two types, those that are obtained from topological interactions in two dimensions, and those that are obtained in exactly solvable models in one dimension.

Two-dimensional anyons play a major role in explaining some of the most important phenomena in condensed matter physics, such as the fractional quantum Hall effect [228, 231], and are related to frontier subjects in mathematical physics, such as topological quantum field theories [209, 256] and topological order [17, 18]. But perhaps the most relevant role of two dimensional anyons is their possible application in intrinsically fault-tolerant models of quantum computing [86, 88, 146], which has been the major drive for experimental research in the subject [19, 184, 211].

One-dimensional anyons can also be broadly divided into two classes, those arising from two-body interactions, and those arising from three-body interactions. The two-body, one-dimensional anyons have been closely related to two-dimensional anyons as boundary theories [118]. They are also connected to Calogero-Sutherland-Moser systems of exactly solvable models, and are intimately connected to the algebra of observables in such systems [200]. Finally, three-body, one-dimensional anyons were first given by Kundu [155], as an exactly solvable system generalizing the well known Lieb-Liniger model [165, 166]. Gases of such particles have been studied for a long time [20, 21, 42, 122, 196], and are examples of systems whose dynamics is interesting to simulate in optical lattice devices [46, 112, 114, 168].

This last type of one-dimensional anyon is the one for which we generalize the linear-optical quantum computing models defined for fermions and bosons. They can be divided into *fermionic anyons*, which obey the Pauli exclusion principle, and *bosonic anyons*, which do not. Fermionic and bosonic anyons are related to fermions and bosons, respectively, by what is known as a fractional Jordan-Wigner transformation [79, 178], which was shown to simplify many of our results. The results of our research showed that by generalizing the linear optical model from fermions to fermionic anyons, we are able to explain their computing power in terms of the anyonic character of the particle, more specifically, on the intensity of a one-dimensional analogue of the Aharonov-Bohm effect [5]. In particular, this shows that quantum computing with qubits, which is equivalent to quantum computing with hard-core bosons [258], can also be seen as deriving their power from an inherent Aharonov-Bohm effect. This is proven by obtaining a protocol for a two-qubit entangling gate that takes advantage of this effect for both fermionic and bosonic anyons. Our research also shows that we can generalize the notion of coherent states from standard bosons to bosonic anyons, and exploit the intrinsic Aharonov-Bohm effect for building cat-states [182, 206], which are resource states for quantum computing in continuous variables.

1.1 Thesis outline

By necessity, this thesis contains a great deal of material reviewing previous results. Chapters 2 and 3 contain revision material pertaining to the physics and history of alternate theories of identical particle statistics, quantum computing, bosonic and fermionic linear optics, and matchgates. Chapter 4 is based on my co-authored paper [238], published in *Physical Review A*, as well as [239], currently submitted for publication.

In chapter 2, I make a historical account of the development of the contemporary theories on the statistics of identical particles. In section 2.1, I do a brief review of the standard theory of quantum statistics. In section 2.2, I review the first theories of non-standard quantum statistics, made by Gentile [91] and Green [103]. In section 2.3, I review the theories of quantum statistics that appeared after the development of the topological theory, by Leinaas and Myrheim in [162]. Finally, in section 2.4, I introduce the type of non-standard statistics that is used for the new computing models in chapter 4. The main purpose of chapter 2 is to be a comprehensive, historical introduction to the subject of non-standard quantum statistics, in order to dispel any confusion with regard to the type of particle that is studied in this work.

In chapter 3, I review some basic definitions, which are standard in quantum computing research, and the computational models based on the physics of standard identical particles. In section 3.1, I review the basics of classical and quantum circuits, focusing on the notion of circuit universality, as well as quantum entanglement and its role in universal gate constructions. In section 3.2, I review the optical network formalism for standard particles, how they are used in quantum computing, and the differences between bosons and fermions in this regard. Finally, in section 3.3, I review the equivalence between quantum circuits for qubits and optical networks for hard-core bosons, which is a non-standard type of particle, as well as their relationship with fermionic optical networks and a set of quantum circuits known as matchgates.

In chapter 4, I present our original results concerning quantum computing with fermionic and bosonic anyons, defined in section 2.4. In section 4.1, I present the exact solutions to the Heisenberg equations of motion for quadratic Hamiltonians of both bosonic and fermionic anyons. In section 4.2, I present a generalization of the optical networks model for fermionic and bosonic anyons. I also provide two protocols for building two-qubit entangling gates without using non-linear devices, nor adaptive measurements, based on the presence of a one-dimensional analogue of the Aharonov-Bohm effect. Finally, In section 4.3, I present a study about generalized coherent states for bosonic anyons, and their behavior under anyonic optical networks, proving that Gaussian devices for anyons generate cat-states, which cannot be done do using Gaussian devices for bosons.

Finally, chapter 5 is devoted to concluding remarks. I make a brief summary of the results obtained here, and how they fit into the larger picture of current research in quantum computing. I also describe a few more questions that are left open, as well as directions in which our results can be expanded.

1.2 Notations and conventions

In this section, I summarize the basic notation and conventions that are consistently used throughout this Thesis. Since I touch upon many different subjects notational conflicts appear, and it is necessary to depart from them to maintain consistency. States of classical systems, bit strings, and vector quantities in general are given by lowercase or uppercase Latin letters in *vector bold* $\mathbf{A}, \mathbf{x}$. Quantum states are always be written using both Dirac's notation or wave-functions depending on the context. Operators are written using a "hat", (like in $\hat{O}$) over their corresponding classical version, including field operators, except in the case of qubit or optical element operators.

Some specific notation for operators are as follows. The symbol "$\hat{a}$" is used for creation and annihilation operators in general, while "$\hat{f}$" and "$\hat{b}$" denote operators for *standard* fermions and bosons, respectively. The "$\hat{\xi}$" and "$\hat{\beta}$" symbols denote creation and annihilation operators for *anyonic* fermions and bosons respectively. The "$\hat{x}$" symbol is used to denote creation and annihilation operators for either *standard* bosons or fermions whenever an expression or identity is valid for both types of particles, while the "$\hat{\chi}$" symbol is used when the same happens for both *anyonic* bosons and fermions instead. Finally, the symbol "$\hat{q}$" is used to denote creation and annihilation operators for *second quantized qubits.*

Single-qubit operators have their circuit representation as in figure 2(a), where the lines indicate the qubit it acts on. Some examples of unitary gates are the following. The *Pauli operators* are represented by X, Y and Z. Their matrix representation on the Hilbert space basis generated by the vectors $|0\rangle, |1\rangle$ is

$$[X] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad [Y] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad [Z] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1.1}$$

The *phase*, *hadamard* and "$\pi/8$" gates are represented by the symbols P, H and T. Their matrix representation on the Hilbert space basis generated by the vectors $|0\rangle$, $|1\rangle$ is

$$[P] = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad [H] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \tag{1.2}$$

The two most relevant two-qubit gates are the CNOT and SWAP gates, with their implementations given in the figures 1(a,b,d) and 1(c) respectively. Their matrix representation over the Hilbert space generated by the vector basis $|00\rangle$,$|01\rangle$,$|10\rangle$,$|11\rangle$ is

$$[\text{CNOT}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad [\text{SWAP}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

Similarly, the most relevant three-qubit gate gate is the TOFFOLI gate, with implementations given in the figures 1(e,f). Its matrix representation over the Hilbert space generated by the vector basis $|000\rangle$,$|001\rangle$,$|010\rangle$,$|011\rangle$,$|100\rangle$,$|101\rangle$,$|110\rangle$,$|111\rangle$ is

$$[\text{TOFFOLI}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.4}$$

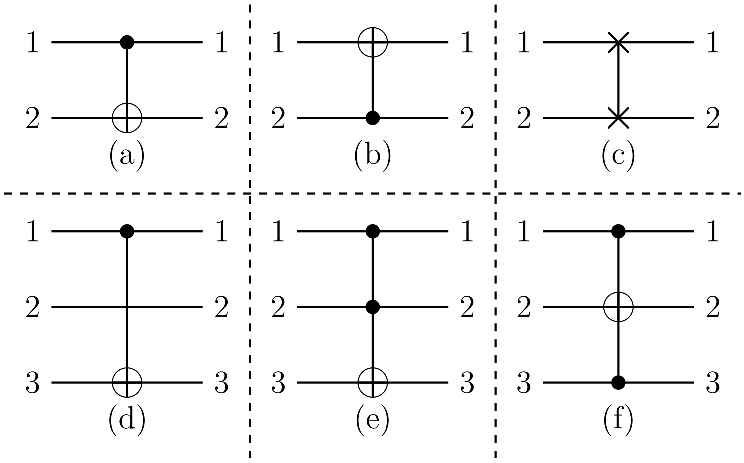


Figure 1 – Notation for classical gates. (a) Controlled-not gate $\text{CNOT}_{1,2}$, with control bit 1 and target bit 2 . (b) Controlled-not gate $\text{CNOT}_{2,1}$, with control bit 2 and target bit 1. (c) $\text{SWAP}_{1,2}$ gate between bits 1 and 2. (d) Controlled-not gate $\text{CNOT}_{1,3}$, with control bit 1 and target bit 3. (e) Toffoli gate $\text{TOFFOLI}_{1,2,3}$, with control bits $1, 2$ and target bit 3 . (f) Toffoli gate $\text{TOFFOLI}_{1,3,2}$, with control bits $1, 3$ and target bit 2.

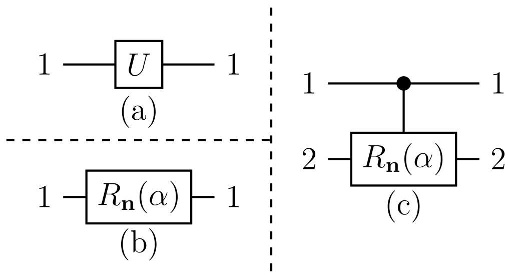


Figure 2 – Notation for quantum gates. (a) Arbitrary single-qubit gate. (b) General rotation gate of axis $\mathbf{n}$ and angle α. (c) Controlled rotation gate $\Lambda(R_{\mathbf{n}}(\alpha))_{1,2}$, with control qubit 1 and target qubit 2.

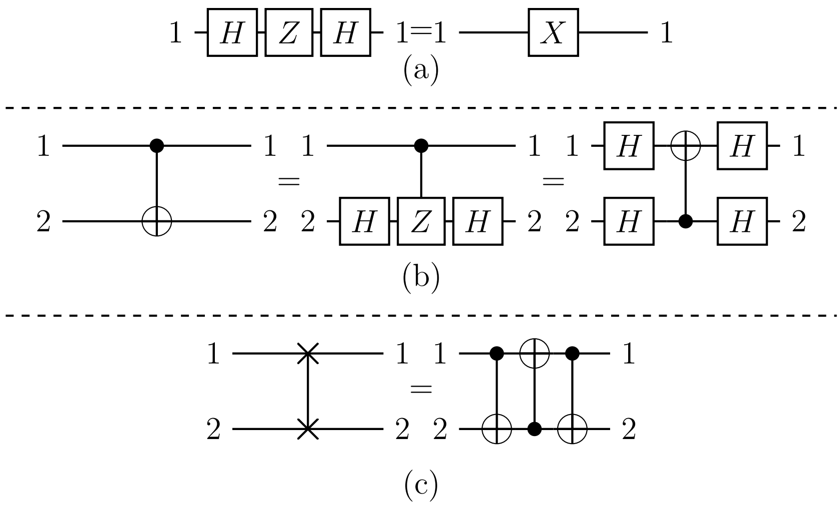


Figure 3 – Some circuit identities: (a) $HZH = X$. (b) $\text{CNOT}_{1,2} = H_2\Lambda(Z)_{1,2}H_2 = H_1H_2\text{CNOT}_{2,1}H_1H_2$. (c)$\text{SWAP}_{1,2} = \text{CNOT}_{1,2}\text{CNOT}_{2,1}\text{CNOT}_{1,2}$

2 Review: The theories of non-standard quantum statistics

The concept of quantum statistics originates from Bose's discovery that the correct statistical distribution of Planck's light quanta is not the Maxwell-Boltzmann distribution [32]. At that time, neither Schrödinger's formulation of quantum mechanics [213] nor Heisenberg's matrix mechanics [29, 30, 127] had been developed. So, even before proper quantum particle models were built, both Bose-Einstein [32, 69] or Fermi-Dirac [61, 75] distributions were required as ad hoc assumptions of semiclassical explanations for both atomic and solid state systems.

The experimental success obtained from the models that used these ad hoc distributions established them as the standard, expected statistical behavior of many-particle systems without strong, long-range interactions. In this work, the statistical behavior defined by Bose-Einstein or Fermi-Dirac distributions are called *standard quantum statistics*. After quantum mechanics was formalized, it became necessary to express these ad hoc behaviors in terms of quantum theory, and in this process new forms of quantum statistics arose, which here we call *non-standard quantum statistics*.

To understand non-standard quantum statistics we require concepts and methods borrowed from many areas of physics and mathematics. So much, in fact, that it would be impossible to cover all of the different definitions of "non-standard" statistics without going outside the scope of this work. At the same time, I must also introduce specific concepts and methods of quantum computation and information theory which are not common, and require attention on their own. This chapter's purpose, then, is to lay out a historical/conceptual narrative about the development of non-standard quantum statistics, culminating in the definition of the kind of statistics that will be the actual subject of this work.

This chapter is organized as follows. First, I briefly review standard quantum statistics from the semiclassical, many-body and field theory points of view in section 2.1. Then, I proceed with the history of non-standard quantum statistics and divide it in two periods. The first period, which I call the *early period* (1940-1971), is treated in section 2.2 and begins with Gentile's intermediate statistics [91], the first known proposal of a non-standard quantum statistics. This period also contains the history of the rise and fall of Green's Parastatistics [103].

The second period, which I call the *modern period* (1971-), is treated in section 2.3. It is set to begin in 1971, because it was the year when it was demonstrated that

Green's Parastatistics could be understood in terms of standard quantum statistics [63]. It is also the same year when the configuration space model of classical identical particles was proposed [156], which eventually lead to the form of non-standard quantum statistics that would later be called *fractional statistics* [162].

The chapter ends in section 2.4, where I depart from the historical narrative and introduce the general family of non-standard quantum statistics that include our model. In this section I also define what is a *particle interpretation* of an arbitrary quantum system. Section 2.4 is the most important section of the chapter, and will be referred to frequently throughout this work.

2.1 Review on standard quantum statistics

As stated previously, in this section we discuss standard quantum statistics from three different points of view. I call these the *semiclassical formalism*, the *many-body formalism*, and the *field formalism*[1] of standard quantum statistics. Each is discussed on its own terms in subsection 2.1.1, and in subsection 2.1.2 we show why all formalisms give equivalent definitions of standard quantum statistical behavior.

This section is not intended to be a historical account of any of these formalisms, but the division itself will be used as part to build the arguments in sections 2.2 and 2.3. All assertions in this section are proved in traditional textbooks about statistical mechanics/thermodynamics, such as [111, 195]. Whenever I use more complex mathematical methods additional references will be given.

2.1.1 Three definitions of quantum statistics

The first point of view on the theory of standard quantum statistics it the one I called the *semiclassical formalism*. In it, particles obey classical equations of motion, but only some trajectories, indexed by a set of quantum numbers taking discrete values, is allowed, as in the Bohr atomic model. From now on, we refer to this description as the *isolated particle model* of the semiclassical formalism. For simplicity, we will describe this formalism for a system with N semiclassical particles having only the single-particle energy ϵ as the relevant quantum number, where the energy spectrum is $\{\epsilon_i\}_{i\in I}$ for some index set $I \in \mathbb{N}$ with the lowest energy state being ϵ_0.

A semiclassical N-particle system is said to satisfy the *indistinguishability assumption* if all particles have the same energy spectrum. This same system is said to satisfy the *no-interaction assumption* if it is completely described by specifying the number of particles n_i (i.e., occupation numbers) that are in a classical state with energy ϵ_i. The se-

[1] This subdivision in three points of view was made specifically for the argument in this section and for historical comparisons in the next two sections, it is not standard in the literature.

quence of numbers $(n_i)_{i\in I}$, is the set of allowed *microstates* of the N-particle system, and its total energy is $E = \sum_{i\in I} \epsilon_i$. The properties just described are called the *fundamental assumptions* of the semiclassical formalism.

A particle system satisfying these fundamental assumptions is called *fermionic* if the occupation numbers n_i are restricted to be either 0 or 1 for all $i \in I$. When there is no restriction on the image set of any of the n_i, the system is called *bosonic*. It is a fact from statistical mechanics, then, that the occupation number distribution for a fermionic semiclassical system is the Fermi-Dirac distribution, and the Bose-Einstein distribution for a bosonic one.

The second point of view is called the *many-body formalism*. In it, individual particles are represented by a Hilbert space $\mathscr{H}$ with basis labelled by the particle's quantum numbers. Assuming the same convention about quantum numbers made in our description of the semiclassical formalism, the Hilbert space of a quantum particle $\mathbb{H}$ will be given by the set of vectors $\{|\epsilon_i\rangle\}_{i\in I}$. I call this quantum description the *isolated particle model* of the many-body formalism.

A system of N particles, each described by the Hilbert space $\mathscr{H}$ defined in the last paragraph, satisfied the *no-interaction assumption*, is the N-particle system as a whole is described by the tensor product space $\mathscr{H}_N = \mathscr{H} \otimes \overset{N-1}{\cdots} \otimes \mathscr{H}$ whose basis is given by the set

$$\left\{ \bigotimes_{\rho=1}^{N} \left|\epsilon_{i_\rho}\right\rangle_\rho, \text{ such that } \rho \in \{1, \cdots N\} \right\}. \tag{2.1}$$

The index ρ labels the order of the tensor factors in $\mathscr{H}_N$ and each ϵ_{i_ρ} is an arbitrary state of the basis of $\mathscr{H}$. The basis states of $\mathscr{H}_N$ will also be written as $|\epsilon_{i_1}, \ldots, \epsilon_{i_N}\rangle$.

One example of N-particle system satisfying the no-interaction assumption is the one described by the Schrödinger equation

$$i\hbar\frac{\partial\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t)}{\partial t} = -\sum_{\rho=1}^{N}\left\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla}_\rho^2 + V(\mathbf{x}_\rho,t)\right\}\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t), \tag{2.2}$$

where $V(\mathbf{x},t)$ is an external potential, and the coordinate labels $\mathbf{x}_\rho$ with $\rho = 1,\ldots,N$, describe the position of each individual particle. The lack of an inter-particle potential is the hallmark of the no-interaction assumption of the many-body formalism.

Indistinguishability in the many-body formalism is expressed by a type of symmetry. To see this, take for example a three particle system in a state $|u\rangle_1 \otimes |v\rangle_2 \otimes |w\rangle_3$, where all of the $|u\rangle, |v\rangle$ and $|w\rangle$ are orthonormal. The state $|u\rangle_1 \otimes |v\rangle_2 \otimes |w\rangle_3$ implies that measurement of each particle state would determine that particle 1 is in state $|u\rangle$, particle 2 in state $|v\rangle$ and particle 3 is in state $|w\rangle$.

If the three particles just mentioned are indistinguishable, the only thing a measurement of the system's state can show is that some particle is in each one of the aforementioned states, without specifying which. Therefore, no physical process would be able to distinguish between any of the three particle states

$$\begin{aligned} &|u\rangle_1 \otimes |v\rangle_2 \otimes |w\rangle_3\,, \quad |v\rangle_1 \otimes |u\rangle_2 \otimes |w\rangle_3\,, \quad |w\rangle_1 \otimes |v\rangle_2 \otimes |u\rangle_3\,, \\ &|u\rangle_1 \otimes |w\rangle_2 \otimes |v\rangle_3\,, \quad |w\rangle_1 \otimes |u\rangle_2 \otimes |v\rangle_3\,, \quad |v\rangle_1 \otimes |w\rangle_2 \otimes |u\rangle_3\,. \end{aligned} \tag{2.3}$$

This property of indistinguishable particles is called *permutation degeneracy* and, in quantum mechanics, degeneracy is the consequence of a system's symmetry.

A non-interacting N-particle system is said to satisfy the *indistinguishability assumption* of the many-body formalism if any physical observable is invariant under permutations of particle labels. In other words, the Hilbert space of a system of N indistinguishable particles is a subspace of $\mathscr{H}_N$, invariant with respect to the action of the *symmetric group of N elements*, denoted by S^N. Such a system is said to satisfy the *fundamental assumptions* of the many-body formalism

To properly define what are fermions and bosons in the many-body formalism, we briefly discuss group representation theory. The symmetric group of N elements, S^N, is generated by special elements called *transpositions* τ_i with $i = 1, ..., N-1$, which satisfy the equations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \tag{2.4a}$$

$$\tau_i \tau_j = \tau_j \tau_i, \text{ for } |i-j| \geq 1, \tag{2.4b}$$

$$(\tau_i)^2 = e, \tag{2.4c}$$

where e is the identity element of the group. An arbitrary element $\sigma \in S^N$ is called even if it can be written as a product of an even number of transpositions, otherwise it is called odd.

The action of S^N as a symmetry group over the Hilbert space $\mathscr{H}_N$ is defined by the *fundamental representation* map $\rho_{fun} : S^N \to U(\mathscr{H}_N)$, where $U(\mathscr{H}_N)$ is the unitary group over $\mathscr{H}_N$. This map is completely determined by its action on transpositions $\rho_{fun}(\tau_i)$, which is defined by its action on the basis of $\mathscr{H}_N$, given by

$$\rho_{fun}(\tau_i) \left| \epsilon_{l_1}, \ldots, \epsilon_{l_i}, \epsilon_{l_{i+1}}, \ldots, \epsilon_{l_N} \right\rangle = \left| \epsilon_{l_1}, \ldots, \epsilon_{l_{i+1}}, \epsilon_{l_i}, \ldots, \epsilon_{l_N} \right\rangle. \tag{2.5}$$

The fundamental representation ρ_{fun} of S^N can be decomposed into a direct sum of *irreducible representations* $\rho_{irr} : S^N \to U(\mathscr{V}_{irr})$, where $\mathscr{V}_{irr} \subset \mathscr{H}_N$ is one of the many irreducible, simultaneous eigenspaces of all operators in the image set $\rho_{fun}(S^N) \subset U(\mathscr{H}_N)$. The number of irreducible representations of any finite group is finite, and in the case of S^N, each of them correspond to a specific integer partition of the number N.

Then, a system of N particles that satisfy the fundamental assumptions of the many-body formalism is called *bosonic*, if it described by the invariant subspace $S\mathscr{H}_N \subset$

$\mathscr{H}_N$ of the *symmetric representation* $\rho_S : S^N \to U(S\mathscr{H}_N)$. This subspace is generated by all non-zero vectors of the form

$$|S(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle = \frac{1}{N!} \sum_{\sigma \in S^N} \left|\epsilon_{i_{\sigma(1)}}, \ldots, \epsilon_{i_{\sigma(N)}}\right\rangle . \tag{2.6}$$

This irreducible representation is determined by

$$\rho_S(\tau_i) |S(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle = |S(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle , \tag{2.7}$$

for all transpositions. Therefore, the symmetric representation is such that the action of all permutations $\sigma \in S^N$ leave the basis vectors of $S\mathscr{H}_N$ invariant.

Similarly, a system of N particles that satisfies the fundamental assumptions of the many-body formalism is called *fermionic* if it is described by the invariant subspace $A\mathscr{H}_N \subset \mathscr{H}_N$ of the *antisymmetric representation* $\rho_A : S^N \to U(A\mathscr{H}_N)$. This subspace is generated by all non-zero vectors of the form

$$|A(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle = \frac{1}{N!} \sum_{\sigma \in S^N} sgn(\sigma) \left|\epsilon_{i_{\sigma(1)}}, \ldots, \epsilon_{i_{\sigma(N)}}\right\rangle , \tag{2.8}$$

where $sgn(\sigma) = 1$ if σ is an even permutation, and $sgn(\sigma) = -1$ if it is odd. This irreducible representation is determined by

$$\rho_A(\tau_i) |A(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle = - |A(\epsilon_{i_1}, \ldots, \epsilon_{i_N})\rangle , \tag{2.9}$$

for all transpositions. Therefore, in this representation all even permutations $\sigma \in S^N$ leave the basis vectors of $A\mathscr{H}_N$ invariant, while odd permutations multiply each basis vector by -1.

The *field formalism* is very different from both the semiclassical and many-body formalisms, mainly because the quanta of a quantum field may not be a well defined particle. Therefore, I will at first limit the definition of standard quantum statistics for quantum fields obtained from *linear field equations*.

The simplest example of quantum field is obtained by quantizing three-dimensional Schrödinger's wave function $\psi(\mathbf{x}, t)$. The *canonical quantization* procedure for this field is to postulate the existence of an operator function $\hat{\psi}(\mathbf{x}, t)$ that satisfies the same equation as the classical field $\psi(\mathbf{x}, t)$, which is the equation

$$i\hbar \frac{\partial \hat{\psi}(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \hat{\psi}(\mathbf{x}, t) + V(\mathbf{x}, t)\hat{\psi}(\mathbf{x}, t). \tag{2.10}$$

The next step is to express all physical observables in terms of this new operator field. This is done by studying the Hamiltonian formulation of the classical field theory. Schrödinger's wave-function $\psi(\mathbf{x}, t)$, obeys the classical Lagrangian density

$$\mathscr{L}(\psi, \boldsymbol{\nabla}\psi, \dot{\psi}) = i\hbar\bar{\psi}\frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \boldsymbol{\nabla}\bar{\psi} \cdot \boldsymbol{\nabla}\psi - V(\mathbf{x}, t)|\psi|^2, \tag{2.11}$$

where $\psi = \psi(\mathbf{x}, t)$ and $\bar{\psi}$ is its complex conjugate.

From this Lagrangian density, one obtains the *canonical conjugate field* of $\psi(\mathbf{x}, t)$ by the identity $\pi(\mathbf{x}, t) = \partial\mathscr{L}/\partial\psi$, giving $\pi(\mathbf{x}, t) = i\hbar\bar{\psi}(\mathbf{x}, t)$. After a Legendre transform of $\mathscr{L}$ and integration by parts, we obtain the Hamiltonian function

$$H = \int \mathrm{d}^3\mathbf{x}\, \bar{\psi}(\mathbf{x}, t) \left\{ -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\mathbf{x}, t) \right\} \psi(\mathbf{x}, t). \tag{2.12}$$

This Hamiltonian is then used to write the equations of motion for ψ and π, with the use of Poisson brackets for fields.

In single particle classical mechanics, all dynamical variables are written as functions of generalized coordinates and their associated canonical momenta. Similarly, in Hamiltonian classical field theory, all dynamical variables are written as function of the fundamental fields and their canonical conjugate fields. Then, by postulating the operator function $\hat{\psi}(\mathbf{x}, t)$, and replacing it in every function written in terms of the classical field $\psi(\mathbf{x}, t)$, we can define all physical observables.

However, there is more then one way to replace classical field variables by field operators, due to the non-commutativity of operator functions. The canonical choice is to simply replace the classical field by the quantum one in the classical Hamiltonian, leading to the Hamiltonian operator $\hat{H}$, and using this operator to fix all ordering ambiguities in all other operator functions using the equations of motion.

Enforcing this canonical choice of Hamiltonian and expanding the equations of motion,

$$\frac{\partial\hat{\psi}(\mathbf{x}, t)}{\partial t} = \frac{1}{i\hbar}[\hat{\psi}(\mathbf{x}, t); \hat{H}] \quad , \quad \frac{\partial\hat{\pi}(\mathbf{x}, t)}{\partial t} = \frac{1}{i\hbar}[\hat{\pi}(\mathbf{x}, t); \hat{H}], \tag{2.13}$$

in terms of the field variables $\hat{\psi}, \hat{\pi}$, is enough to fix the operator ordering ambiguity for all physical observables. The requirement of linearity for the equations of motion in terms of the fundamental field variables, and the imposition of a canonical choice for the Hamiltonian, are the *fundamental assumptions* of the field formalism for standard quantum statistics.

For any field $\hat{\psi}(\mathbf{x}, t)$, satisfying the fundamental assumptions of the field formalism, there are only two choices for fixing operator ordering ambiguities. The field is called *bosonic*, if it satisfy *equal time commutation relations*

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)] = \delta^3(\mathbf{x} - \mathbf{x}'), \tag{2.14a}$$

$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)] = [\psi^\dagger(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)] = 0. \tag{2.14b}$$

or *fermionic*, if it satisfies *equal time anticommutation relations*

$$\{\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)\} = \delta^3(\mathbf{x} - \mathbf{x}') \tag{2.15a}$$

$$\{\psi(\mathbf{x},t),\psi(\mathbf{x}',t)\} = \{\psi^\dagger(\mathbf{x},t),\psi^\dagger(\mathbf{x}',t)\} = 0. \tag{2.15b}$$

These are the only relations for which products of fundamental field variables maintain the form of the equations of motion given the canonical choice of Hamiltonian. Generically, we call canonical commutation and anticommutation relations as bosonic and fermionic commutation relations, respectively.

To sum up, in this section we saw three different formalisms for studying quantum statistics. The first was the semiclassical formalism, were fermionic and bosonic behavior are characterized by the ad hoc imposition of either Fermi-Dirac or Bose-Einstein statistical distributions, respectively. The second was the many-body formalism, were fermionic and bosonic behavior are characterized, respectively, by antisymmetric and symmetric representations of permutation groups, which act as a symmetry group of an identical particle system. And the last was the field formalism, were fermionic and bosonic behavior are characterized, respectively, by canonical anticommutation and commutation relations of fundamental field operators. It is worth mentioning that for relativistic field equations, locality imposes bosonic commutation relations on integer spin fields, and fermionic commutation relations on half-integer spin fields. This is known as the spin-statistics theorem, and is always valid for relativistic field equations.

2.1.2 The equivalence of the three definitions

In this section, I prove that the three different definitions of standard quantum statistics described in the last subsection are equivalent. When I say that they are equivalent, I mean that there exists at least one physical system that can be described in any of the three formalisms, and that for this system, all definitions of what are bosonic and fermionic statistics agree.

The equivalence is proven for the system of N non-interacting identical particles described by the many-body Schrödinger's equation (2.2). First we show how the many-body definitions of fermions and bosons imply the semiclassical definitions. Then we show that the quantum Schrödinger field entails the many-body description of the system, and then show how the field definition of quantum statistics implies the many-body one.

The system of interest obeys the fundamental assumptions of the many-body formalism, stated in the last subsection 2.1.1. Therefore, our system is described by either the symmetric of antisymmetric invariant subspaces of $\mathscr{H}_N$, which has basis states given by $|\epsilon_{i_1},\ldots,\epsilon_{i_N}\rangle$. Define the state

$$|\{\epsilon_i^{n_i}\}_{i\in I}\rangle = \bigotimes_{i\in I}\left(|\epsilon_i\rangle^{\otimes n_i}\right), \tag{2.16}$$

where each ϵ_i is bigger than ϵ_{i-1}, and $|\epsilon_i\rangle^{\otimes n_i}$ is the n_i-th tensor power of $|\epsilon_i\rangle$. This state is called a *primitive state* of the coefficient set $\{n_i\}_{i\in I}$, and it is unique for each set.

Primitive states are such that the first n_0 particles have energy ϵ_0, the next n_1 have energy ϵ_1 and so on. It's easy to see that any basis state of $\mathscr{H}_N$, can be written as permutation of some primitive state. The subspace $\mathscr{H}_{\{n_i\}_{i\in I}} \subset \mathscr{H}_N$ of all states generated by the action of permutation operators in the the fundamental representation over the primitive state $|\{\epsilon_i^{n_i}\}_{i\in I}\rangle$, is called the subspace of microstates with occupation numbers $\{n_i\}_{i\in I}$.

These subspaces of microstates are invariant under the action of permutation operators in the fundamental representation, and are orthogonal to each other. These subspaces end up being the ones corresponding to the microstates defined in the semiclassical formalism in subsection 2.1.1 and, therefore, are uniquely specified by the set of occupation numbers. To show that the many-body definition of quantum statistics is the same as the semiclassical one, we need to show how to obtain from each subspace of microstates a unique microstate. Then, for the fermionic case, we need to show how the maximum occupation number for each energy level caps at 1, and show why there is no restriction on the maximum occupation number in the bosonic case.

Given the fundamental representation of S^N, one can define special operators called *symmetrizers*

$$S = \frac{1}{N!}\sum_{\sigma\in S^N} \rho_{fun}(\sigma), \tag{2.17a}$$

$$A = \frac{1}{N!}\sum_{\sigma\in S^N} sgn(\sigma)\rho_{fun}(\sigma). \tag{2.17b}$$

They are special because they are orthogonal projection operators, and satisfy both $S\rho_{fun}(\sigma) = S$ and $A\rho_{fun}(\sigma) = sgn(\sigma)A$ for all $\sigma \in S^N$.

Using these properties, it is easy to see that S projects an arbitrary state $|\psi\rangle \in \mathscr{H}_N$ into the symmetric subspace $S\mathscr{H}_N$. Similarly, A projects any $|\psi\rangle \in \mathscr{H}_N$ into the antisymmetric subspace $A\mathscr{H}_N$. Given any subspace $\mathscr{K} \subset \mathscr{H}_N$, let's define its *symmetric component* as the set $S\mathscr{K} = \{S\,|\psi\rangle\,, \text{ with } |\psi\rangle \in \mathscr{K}\}$, and its *antisymmetric component* as the set $A\mathscr{K} = \{A\,|\psi\rangle\,, \text{ with } |\psi\rangle \in \mathscr{K}\}$.

For any subspace of microstates $\mathscr{H}_{\{n_i\}_{i\in I}}$, all basis states are permutations of $|\{\epsilon_i^{n_i}\}_{i\in I}\rangle$. Therefore, applying the operator S on any basis state yields the unique vector

$$|\{n_i\}_{i\in I}\rangle_S = S\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle\,, \tag{2.18}$$

which exists for all possible sets $\{n_i\}_{i\in I}$, and we identify with the semiclassical microstate for bosonic systems.

Similarly, applying A to any state in $\mathscr{H}_{\{n_i\}_{i\in I}}$ yields the unique vector

$$|\{n_i\}_{i\in I}\rangle_A = A\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle\,. \tag{2.19}$$

However in this case, if some n_i is bigger than 1, there is some odd permutation $\sigma \in S^N$ such that

$$A\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle = A(\rho_{fun}(\sigma)\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle) = (A\rho_{fun}(\sigma))\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle = -A\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle\,, \quad (2.20)$$

implying that $A\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle = 0$ in those cases.

Therefore, $A\mathscr{H}_{\{n_i\}_{i\in I}}$ is one-dimensional only when all n_i are either 0 or 1, and in all other cases $A\,|\{\epsilon_i^{n_i}\}_{i\in I}\rangle$ contains only the zero vector. This allows us to identify $|\{n_i\}_{i\in I}\rangle_A$ with the semiclassical microstate for fermionic systems. Now that I showed how to write the definition of quantum statistics in the many-body formalism in terms of the semiclassical one, we only have to show how the system's many-body description can be obtained from the Schrödinger field, and how their respective definitions of quantum statistics align.

To show why the Schrödinger field is indeed the correct field, we must find its *particle representation*. Since this field is linear, we can expand it as a sum of energy eigenstates

$$\hat{\psi}(\mathbf{x},t) = \sum_{i\in I} \hat{b}_i u_i(\mathbf{x}) e^{i\epsilon_i t}, \quad (2.21)$$

where the operators $\{\hat{b}_i\}$ are the coefficients of this expansion, the set $\{\epsilon_i\}_{i\in I}$ is the energy spectrum of the field, and the functions $u_i(\mathbf{x})$, are the spatial part of the energy eigenfunctions of the Schrödinger equation[2].

Assuming bosonic commutation relations, we can see that the operator coefficients $\{\hat{b}_i\}$, as well as their conjugates $\{\hat{b}_i^\dagger\}$, must also obey bosonic commutation relations

$$[\hat{b}_i;\hat{b}_j^\dagger] = \delta_{ij}, \quad [\hat{b}_i;\hat{b}_j] = [\hat{b}_i^\dagger;\hat{b}_j^\dagger] = 0. \quad (2.22)$$

Then, by the canonical prescription defined in subsection 2.1.1, and the field expansion just defined in equation (2.21), the Hamiltonian operator can be written as the sum

$$\hat{H} = \sum_{i\in I} \epsilon_i \hat{b}_i^\dagger \hat{b}_i. \quad (2.23)$$

The Hamiltonian $\hat{H}$ is a sum of terms of the form $\epsilon_i\hat{n}_i$, where the operator $\hat{n}_i$ is given by $\hat{b}_i^\dagger\hat{b}_i$. Then, applying the field expansion and the recently obtained commutation relations for the operator coefficients eq. (2.22) in the equations of motion for the field variables, we can show that

$$[\hat{n}_i;\hat{b}_i^\dagger] = \hat{b}_i^\dagger, \quad [\hat{n}_i;\hat{b}_i] = -\hat{b}_i \quad \textit{and} \quad [\hat{n}_i;\hat{n}_j] = 0. \quad (2.24)$$

Therefore, the set $\{\hat{n}_i\}_{i\in I}$ forms a commuting set of operators that generate the Hamiltonian. We can use this fact to build the bosonic *Fock space representation* for the

[2] For the Maxwell field, the $u_i(\mathbf{x})$ are eigenfunctions of the spatial part of wave equations obtained from Maxwell's equations in the vacuum.

algebra of physical observables $\hat{\mathscr{O}}_{Sch}$ of the Schrödinger field, which is the algebra of all convergent Taylor series over the field variables.

The bosonic Fock representation is a function $\rho^b_{Fock} : \hat{\mathscr{O}}_{Sch} \to GL(\mathscr{F}^b_{Sch})$ that takes an arbitrary observable into an arbitrary linear operator over the Hilbert space $\mathscr{F}^b_{Sch}$, called the bosonic *Fock space*. This space is generated by the bosonic *Fock basis*

$$|n_1, \ldots, n_i, \ldots\rangle_b = \frac{(\hat{b}_1^\dagger)^{n_1} \ldots (\hat{b}_i^\dagger)^{n_i} \ldots}{\sqrt{n_1! \ldots n_i! \ldots}} |0\rangle_b \,, \tag{2.25}$$

where I may, or may not have infinite size (i.e., infinite energy levels), and the numbers n_i are the eigenvalues of the operators $\hat{n}_i$ whose representations can be calculated using equations (2.24).

The state $|0\rangle_b$ is called the bosonic *vacuum state*, and it satisfies the equations $\hat{b}_i |0\rangle_b = 0$ for all $i \in I$. Its physical interpretation comes from the fact that

$$\hat{H} |n_1, \ldots, n_i, \ldots\rangle_b = \sum_{i \in I} \epsilon_i n_i |n_1, \ldots, n_i, \ldots\rangle_b \,, \tag{2.26}$$

which implies that the states in the Fock basis are eigenstates of $\hat{H}$ with total energy $E = \sum_{i \in I} \epsilon_i n_i$. This property allows us to interpret the eigenvalues n_i are the occupation numbers for bosonic microstates, fixing the interpretation of the vacuum state as a state without any particles.

The actions of $\hat{b}_i$ and $\hat{b}_i^\dagger$ in the Fock basis are calculated using the bosonic commutation relations in eq. (2.22), and are given by

$$\hat{b}_i |n_1, n_2, ..., n_i, ...\rangle = \sqrt{n_i} |n_1, n_2, ..., n_i - 1, ...\rangle \,, \tag{2.27a}$$

$$\hat{b}_i^\dagger |n_1, n_2, ..., n_i, ...\rangle = \sqrt{n_i + 1} |n_1, n_2, ..., n_i + 1, ...\rangle \,. \tag{2.27b}$$

Therefore, the expansion operator coefficients $\{\hat{b}_i^\dagger\}$ can be interpreted as *creation operators* that create a particle with energy ϵ_i. Similarly, the operators $\{\hat{b}_i\}$ can be interpreted as *annihilation operators*, that destroy a particle with energy ϵ_i. What we just showed is that the bosonic Fock basis contain states that can be identified with the bosonic microstates of the many-body formalism.

In the fermionic case we do exactly the same process. First, we expand the Schrödinger field in a basis of energy eigenfunctions

$$\hat{\psi}(\mathbf{x}, t) = \sum_{i \in I} \hat{f}_i u_i(\mathbf{x}) e^{i\epsilon_i t}, \tag{2.28}$$

where this time we use $\{\hat{f}_i\}$ for the operator coefficients of this expansion, and the functions $u_i(\mathbf{x})$, are the spatial part of the energy eigenfunctions of the Schrödinger equation.

However, now we impose fermionic commutation relations on the field, which forces the operator coefficients to have the same fermionic relations

$$\{\hat{f}_i; \hat{f}_j^\dagger\} = \delta_{ij}, \quad \{\hat{f}_i; \hat{f}_j\} = \{\hat{f}_i^\dagger; \hat{f}_j^\dagger\} = 0. \tag{2.29}$$

If we still assume the canonical choice of Hamiltonian, apply the field expansion and use the fermionic commutation relations on the field equations of motion, we find that the Hamiltonian operator is written as a sum of number operators $\hat{n}_i = \hat{f}_i^\dagger \hat{f}_i$, satisfying the same relations as eq. (2.24).

Therefore, we can build a fermionic Fock representation for the operator algebra, with a fermionic Fock basis given by the simultaneous eigenstates of number operators. However, due to the fermionic commutation relations, we have that $(\hat{f}_i^\dagger)^2 = 0$ for all i, implying that $(\hat{n}_i)^2 = \hat{n}_i$ and that the only possible eigenvalues of the number operators are 0 or 1. So what we just showed is that the fermionic Fock basis contain states that can be identified with the fermionic microstates of the many-body formalism. And this finished the proof of equivalence.

2.2 A history of non-standard quantum statistics: The early period (1940-1971)

In the last section, we showed three different formalisms for defining standard quantum statistics. They were, respectively, the semiclassical, the many-body and the field formalisms. I have also shown that all of them give the same concept of fermionic and bosonic behavior when applied to the equivalent descriptions of the same physical system. In this section and the next, I use these formalisms to illustrate the differences between standard quantum statistics, and the multitude of non-standard quantum statistics we will see in this work.

In this section I begin by describing what I have decided to call the *early period (1940-1971)* of the history of non-standard definitions of quantum statistics. It is further divided into two sub-periods. The sub-period of *Gentile's intermediate statistics (1940-1952)*, marks the beginning of the history of non-standard quantum statistics with the 1940 paper of Gentile's showcasing his semiclassical model. The sub-period of *Green's parastatistics (1953-1971)*, marks the transition from semiclassical to quantum models of non-standard statistics with Green's 1953 paper introducing *parastatistics*, the most general type of field quantum statistics compatible with Hamiltonian equations of motion of linear fields.

As we see in the next section, and as foreshadowed in this chapter's introduction, the year of 1971 also marks the beginning of a new type of formalism for quantum statistics, which eventually led to the concept of *fractional statistics*. The main purpose of this section, then, is to lay the groundwork for understanding how this transition happened, and the motivations behind it.

2.2.1 Gentile's intermediate statistics (1940-1952)

Gentile's 1940 paper [91] details a new model of quantum statistics based directly on Bose's approach to the distribution of light quanta [32]. Essentially, it presupposes all of the fundamental assumptions of the semiclassical formalism for standard quantum statistics described in subsection 2.1.1. However, instead of assuming Pauli's exclusion principle for the occupation numbers n_i or choosing to impose no restriction on their domain, he introduces the ad hoc assumption that n_i can take values from 0 up to some integer d.

He called the assumption by the name of *generalized exclusion principle*, and the statistical theory derived from it *intermediate statistics*. In this same paper, he used the technique of Bose's 1924 paper [32] to deduce the associated distribution function

$$\bar{n}_i^G = \left\{ \frac{1}{e^{-(\mu-\epsilon_i)\beta} - 1} - \frac{d+1}{e^{-(\mu-\epsilon_i)(d+1)\beta} - 1} \right\}, \tag{2.30}$$

where $\bar{n}_i^G$ is Gentile's mean value of the occupation number n_i, β is the inverse temperature and μ is the chemical potential[3].

It is easy to see, from Gentile's distribution, that when $d = 1$ it reduces to the Fermi-Dirac distribution

$$\bar{n}_i^{FD} = \frac{1}{e^{-(\mu-\epsilon_i)\beta} + 1}, \tag{2.31a}$$

and in the limit $d \to \infty$, it becomes the Bose-Einstein distribution

$$\bar{n}_i^{BE} = \frac{1}{e^{-(\mu-\epsilon_i)\beta} - 1}. \tag{2.31b}$$

Two years latter, in [92], Gentile proposed that, formally, the condensed state of a Bose-Einstein gas of N particles is a system described by his statistics with $d = N$. He proposed that this could explain the superfluidity of Helium II. It was Caldirola's paper [43] however, that made the case for using Gentile's statistics for a real gas, instead of an ideal gas, to model Helium II, showing how it was better than the best real gas model at the time.

There were two main criticisms of Gentile's statistics. First, it was realized that Bose's method for obtaining the distribution function for occupation numbers is a very inaccurate approximation in the low-temperature regime [222], where Bose-Einstein condensation occurs. In fact, even the best method known at the time, the Darwin-Fowler method, also failed in this regime [214, 215]. This method was also proven to be incompatible with intermediate statistics.

The second main criticism was the incompatibility of Gentile's intermediate statistics with many-body quantum mechanics. It was argued in Borsellino's paper [31] that a

[3] This calculation assumes variable total energy and particle number, but is done in the microcanonical ensemble

system of particles obeying a generalized exclusion principle would be described by a wave function with mixed symmetry type, since a totally symmetric wave-function is bosonic, and a totally antisymmetric one is fermionic (see subsection 2.1.1). In the representation theory of the symmetric group, there are many different representations with mixed symmetry, but they either are not irreducible, which means that a system can be taken from one mixed symmetry representation to another by even small interactions, or they are multidimensional, and do not describe scalar particles. That meant that there were no stable definition of quantum statistics in the many-body formalism compatible with Gentile's statistics for general d.

These criticisms were summarized in Ter Haar's Letter *"Gentile's intermediate statistics"* [116], and eventually led to it being sidelined as a theory for Helium II [58]. The idea of generalizing the exclusion principle would remain, however, given that the quantum statistics is still ad hoc. In the next subsection we see that generalized exclusion principles tend to reappear in other models of non-standard quantum statistics.

2.2.2 Green's parastatistics (1953-1971)

As we saw in the previous subsection, one of the main criticisms of Gentile's intermediate statistics was its incompatibility with quantum mechanics. The semiclassical treatment was just too coarse to deal with the subtleties implied by the many-body formalism. It is natural then to expect that the next models of non-standard quantum statistics would be done in either the many-body or field formalisms.

The first theory of non-standard quantum statistics made in one of the two quantum formalisms was proposed by Okayama in (1952-1953) [187, 188]. As a non-standard theory of statistics in the many-body formalism, his main concern was to build a generalized exclusion principle. Okayama's statistics avoided the problems with Gentile's statistics described in subsection 2.2.1 by generalizing the multiparticle wave-function to a square matrix $[\Psi]$ instead of a scalar, with probability amplitude given by $\mathrm{Tr}\left\{\Psi^\dagger\Psi\right\}$. Then, instead of representing permutations in the symmetric or antisymmetric representations, he allowed irreducible representations of higher order to act on the wave-matrices, and deduced conditions for them to obey generalized exclusion principles. From this new formalism, he also deduced a second quantized, equivalent formulation of his statistical theory in the field formalism.

Okayama's statistics, however, was overshadowed by its most famous cousin, *Green's parastatistics*. In his 1953 paper [103], Green proposed a solution to what was known at the time as *Wigner's problem*. This problem was defined in a 1950 paper [249] by Wigner, where he showed that the form of the classical equations of motion in Hamiltonian mechanics was not enough to determine the form of the quantum Hamiltonian in the Heisenberg picture [29, 30, 127].

Wigner's problem then, is basically the problem of how to choose a unique quantization prescription for classical Hamiltonian mechanics. Green took a non-canonical prescription for the field Hamiltonian, and used it to find more general commutation relations compatible with the equations of motion for field operators. In other words, he developed a theory of non-standard statistics in the field formalism, by forgoing the assumption of a canonical Hamiltonian. Green's theory gives not just new types of quantum statistics, but also gives standard quantum statistics in particular limits.

To make all of the previous points clear, let us follow Green's argument more closely. If we take a relativistic free field $\hat{\psi}(x^0, \mathbf{x})$ (or a non-linear relativistic field in the interaction picture), the equations of motion for the fields will be linear and given by

$$\partial_\mu \hat{\psi} = i[P^\mu; \hat{\psi}], \tag{2.32}$$

where $\hat{P}^\mu$ is the energy-momentum 4-vector operator, with $\hat{P}^0$ acting as the Hamiltonian.

If we ignore the canonical quantization prescription for the $\hat{P}^\mu$ operators, and instead chose the *paraquantization prescriptions*

$$\hat{P}^\mu = \sum_i p_i^\mu \frac{1}{2}[\hat{a}_i^\dagger; \hat{a}_i], \tag{2.33a}$$

for half-integer spin fields, and

$$\hat{P}^\mu = \sum_i p_i^\mu \frac{1}{2}\{\hat{a}_i^\dagger; \hat{a}_i\}, \tag{2.33b}$$

for integer spin fields, the equations of motion could still be satisfied if, given an energy eigenstate expansion of the field operator $\hat{\psi}(x^0, \mathbf{x})$, its operator coefficients $\hat{a}_i$ obeyed the *trilinear commutation relations*

$$[\hat{a}_i; [\hat{a}_j^\dagger; \hat{a}_k]] = 2\delta_{ij}\hat{a}_k, \quad [\hat{a}_i; [\hat{a}_j; \hat{a}_k]] = 0, \tag{2.34a}$$

for the half-spin case, and

$$[\hat{a}_i; \{\hat{a}_j^\dagger; \hat{a}_k\}] = 2\delta_{ij}\hat{a}_k, \quad [\hat{a}_i; \{\hat{a}_j; \hat{a}_k\}] = 0, \tag{2.34b}$$

for the integer-spin case.

Green also showed that the trilinear commutation relations had a particle representation with number operators defined by

$$\hat{n}_i = \frac{1}{2}[\hat{a}_i^\dagger; \hat{a}_i], \tag{2.35a}$$

for half-integer spin fields, and

$$\hat{n}_i = \frac{1}{2}\{\hat{a}_i^\dagger; \hat{a}_i\}, \tag{2.35b}$$

for integer spin fields. However these trilinear commutation relations are not enough to determine a field statistics. In fact, there are many algebraic relations over the parastatistical operators compatible with them. In particular, fermionic and bosonic commutation relations satisfy eqs. (2.34a,2.34b) respectively.

Green shows examples of non-standard statistics using an ansatz for solving the trilinear commutation relations, obtaining operators with bilinear commutation relations. Green's ansatz consists on choosing particular forms for the matrix representations of the operators $\{\hat{a}_i\}$. In the half-spin case, this leads to theories of non-standard statistics with a generalized exclusion principle. These are called *para-Fermi statistics* of order d, with d being the maximum eigenvalue of the number operators. Green's ansatz for para-Fermi fields can also be given as a decomposition of the operators $\hat{a}_i$ as a sum of components $\hat{a}_i^{(r)}$ with $r = 1, \ldots, k$ that satisfy

$$\begin{gathered}[\hat{a}_i^{(r)}; \hat{a}_j^{(s)}] = [\hat{a}_i^{\dagger(r)}; \hat{a}_j^{(s)}] = 0 \text{ for all } i, j \text{ with } r \neq s \\ \{\hat{a}_i^{(r)}; \hat{a}_j^{(r)}\} = 0, \quad \{\hat{a}_i^{(r)}; \hat{a}_j^{\dagger(r)}\} = \delta_{ij}.\end{gathered} \tag{2.36}$$

He then uses the same strategy to obtain the theories of non-standard statistics for integer spin fields. In that case, the decomposition of $\hat{a}_i$ into a sum of k operators $\hat{a}_i^{(r)}$ with $r = 1, \ldots, k$ leads to relations

$$\begin{gathered}\{\hat{a}_i^{(r)}; \hat{a}_j^{(s)}\} = \{\hat{a}^\dagger(r)_i; \hat{a}_j^{(s)}\} = 0 \text{ for all } i, j \text{ with } r \neq s \\ [\hat{a}_i^{(r)}; \hat{a}_j^{(r)}] = 0, \quad [\hat{a}_i^{(r)}; \hat{a}_j^{\dagger(r)}] = \delta_{ij}.\end{gathered} \tag{2.37}$$

These relations determine what is called *para-Bose statistics* of order k. Para-Bose fields are interpreted as models of particles that organize into k different groups that fit any integer number of particles, but with particles on different groups being anti-symmetric with respect to one another.

At the time, it was not known if there were other solutions for the trilinear commutation relations besides Green's ansatz. In fact, the operator components in equations eqs. (2.36,2.37) do not have a particle representation since the number operators exist only for the para-fields themselves, not their components. This was seen as a problem because even if the para-fields had a proper particle representation, no specific algebraic relations were known for them besides the examples given in Green's paper. Even worse, since these were not known, the many-body formulation of Green's parastatistics was impossible to obtain.

A proposal for finding specific algebraic relations for para-fields was first made in Kamefuchi and Takahashi's 1962 paper [141, 142]. They found an algorithm to calculate these algebraic relations by studying the Lie algebra structure of the infinitesimal linear transformations leaving para-particle Hamiltonians invariant. These transformations have the form

$$\hat{a}_k \to \hat{a'}_k = \hat{a}_k - i \sum_{m=1}^{\infty} \hat{a}_m \xi_{km} - i \sum_{m=1}^{\infty} \hat{a}_m^\dagger \eta_{km}, \tag{2.38a}$$

$$\hat{a}_k^\dagger \to \hat{a'}_k^\dagger = \hat{a}_k^\dagger + i \sum_{m=1}^{\infty} \hat{a}_m^\dagger \xi_{mk}^* + i \sum_{m=1}^{\infty} \hat{a}_m \zeta_{mk}, \tag{2.38b}$$

which can be written as $\hat{a'}_k = G^{-1}\hat{a}_k G$ for the operator G given by

$$G = 1 - i \sum_{l,m=1}^{\infty} \hat{N}_{lm}\xi_{lm} - i\frac{1}{2} \sum_{l,m=1}^{\infty} \hat{L}_{lm}\eta_{lm} - i\frac{1}{2} \sum_{l,m=1}^{\infty} \hat{M}_{lm}\zeta_{lm}. \tag{2.39}$$

The operators $\hat{N}_{lm}$, $\hat{L}_{lm}$ and $\hat{M}_{lm}$ end up being the generators of such Lie-algebras.

The transformations that leave the para-Fermi Hamiltonian invariant are said to be of R-type, because they form the Lie algebra of a rotation group, while the ones preserving the para-Bose Hamiltonian are of S-type, and form the Lie algebra of a symplectic group. In each case, the generators $\hat{N}_k = \hat{N}_{kk}$ are shown to obey characteristic equations that allowed them to develop the algorithm responsible for producing the algebraic relations for para-fields.

Kamefuchi's theory of parastatistics still could not answer if the algebraic relations they found for para-field operators are different from the ones produced from Green's ansatz. This question was answered in the negative by Bialynicki-Birula's 1963 paper [24], proving that the field operators obtained from Green's ansatz obeyed exactly the same algebraic relations as the ones found by Kamefuchi's theory. Therefore, Green's theory of parastatistics was already complete, and no other form of parastatistics was possible.

One question still remained, and that was how to find the many-body version para-Fermi and para-Bose statistics for orders above 2. Many authors have proposed solutions to this problem. The first, decisive step was taken by Messiah and Greenberg's 1964 paper [179]. Before this paper, the many-body formalism introduced the definition of bosonic and fermionic behavior by imposing the symmetry type of multiparticle states directly. This ad hoc imposition was named the *symmetrization postulate*, and was believed to be an necessary extra postulate of quantum mechanics to deal with identical particle systems found in nature.

Messiah and Greenberg where the first to, instead, impose the permutation invariance of physical observables as the foundational postulate of the many-body theory of quantum statistics. In this perspective (also used in sections 2.1.1 and 2.1.2), the permutation operators $\hat{\sigma} \in \rho_{fun}(S^N)$, acting on the Hilbert space $\mathscr{H}_N$, will commute not just with the Hamiltonian, but with all observables of a system of N indistinguishable non-interacting particles. Therefore, no physical process can change a system from one symmetry type to another, imposing what is called a *superselection rule* on the types of symmetry allowed on nature.

Using the superselection theory of standard quantum statistics, Messiah and Greenberg showed that entire invariant subspaces of irreducible representations of S^N should represent the same physically observable state, which generalizes the global phase invariance of scalar wave-functions. The next step in building the many-body theory of parastatistics came from the work of Landshoff and Stapp. In their 1967 paper [157], they provided a response to another paper, by Steinmann [226], that argued for the impossibility of parastatistics. Steinmann showed that particle systems described by non-trivial irreducible representations of the symmetric group cannot be decomposed into two smaller systems with independent descriptions, which is a necessary condition for a local theory.

To counter this assertion, Landshoff argued that the *label permutations*, the kinds of permutations used in Messiah's approach, are not the same type of permutation that should act over the Fock states obtained from para-fields. Given a general basis state $|\{\epsilon_i\}_{i\in I}\rangle$ of the Hilbert space $\mathscr{H}_N$ for an N-particle system with energy spectrum $\{\epsilon_j\}_{j\in I}$, a *label transposition* τ_i^L will act on all basis states by taking whatever energy label is in position i, and exchanging it by the energy label in $i+1$. A *particle transposition* τ_i^P, as defined by Landshoff and Stapp, exchanges the specific label ϵ_i, with the label ϵ_{i+1} regardless of their position in the state vector. They argued that this type of transposition is the correct one, and then developed a theory of many-body statistics that encompasses both Steinmann's approach and the Fock representation of para-fields.

However, Landshoff's work did not address the question of whether the Fock space representation of para-fields and the superselection approach to a many-body theory of quantum statistics are compatible. In a 1968 paper, Yamada [262] provided an argument in the negative, but the question was only settled by Stolt and Taylor's 1970 paper [230]. In this paper they showed that one could choose a specific vector[4] inside any subspace invariant under the action of label permutations, and build the representation of particle permutations inside of them, giving a equivalent representation to the Fock-space representation of para-fields.

In parallel to the development of the statistical theory itself, there have been some applications of parastatistical models in physical problems. The most important one was the alternative model of particle physics developed by Greenberg [106] to explain problems with the quark model. Throughout the 70's there was a dispute between the para-quark model and the $SU(3)$ global gauge theory quarks. The final resolution was given by Doplicher, Haag and Roberts [63, 64], proving that para-Fermi or para-Bose theory is equivalent to theory with bosonic and fermionic fields together with a non-abelian global gauge symmetry.

[4] This "choice of vector" technique is essentially equivalent to the definition of a "primitive vector" I have made in proving the equivalence of many-body and field formalisms for bosons and fermions in section 2.1.2

The fact that quarks could be used to build a local gauge theory of the strong force, while para-quarks could not [90], made the para-quark model, and parastatistics itself, go out of favor. After these events, parastatistics was still studied in the mathematical physics community [9, 23, 163], and even a local gauge theory based on para-fields was found [107], but the attention of the physics community in general faded considerably.

To sum up, parastatistics was a quantum statistical theory developed in the field formalism, which tried to explore the freedom of quantization prescriptions consistent with linear equations of motion. The equivalent many-body formalism only came up after the superselection approach to the theory of standard quantum statistics was developed, introducing a new type of permutation symmetry generated by the so-called particle transpositions. However, it was proved that theories with para-fields were completely equivalent to theories with standard statistics under the action of global gauge field.

2.3 A history of non-standard quantum statistics: The modern period (1971-)

As seen in the last section, the main characteristic of theories of non-standard statistics in the early period was the attempt at finding consistent generalizations of Pauli's exclusion principle. These attempts led to many general results regarding the freedom of choice in quantization methods, as seen with parastatistical models. Nonetheless, all forms of quantum statistics up to this point have been introduced by assuming that particle identity was a quantum effect.

What characterizes the part of history the I chose to call the *modern period (1971-)* is the development of a classical model for particle identity. As we will see, this model allowed the creation of theories of quantum statistics not just for classical particles, but also for special particle-like field configurations collectively known as *solitons*. The myriad of applications of the theory of solitons in many different areas of physics is the main reason why the prototypical type of non-standard quantum statistics today is taken to be the one developed in the modern period, known as *fractional statistics*.

This section is divided in three subsections. The first one (2.3.1), comprises the period from the first appearance of the classical model of particle identity, made by Laidlaw and DeWitt in 1971 [156], up to the first use of this model to define a new theory of quantum statistics by Leinaas and Myrheim in 1977 [162]. The second and the third ones comprise two simultaneous timelines where the development of the concepts can be taken in parallel.

The second subsection (2.3.2) deals with theories of non-standard statistics developed in the context of so-called *topological solitons* and *topological defects*. In this context, non-trivial statistics arise from the topological properties of defects coupled to gauge interactions. We say that these theories comprise the *topological trend* (1977-) of non-standard quantum statistics. The modern concept of *anyons* comes from this trend, and it will be analysed in detail.

The third, and last subsection (2.3.3) deals with the theories of non-standard quantum statistics arising from the theory of quantum integrable models in one dimension. We say that these theories comprise the *non-topological trend* (1977-) of non-standard quantum statistics in the modern period. This is the trend from where the model of non-standard statistics we study in subsequent chapters comes from, which are also considered a model of *anyonic particles*.

The main purpose here is to clarify possible confusions arising from the use of the word *anyons*, by the theories of quantum statistics coming from the two different trends. We see that while in statistical terms, all types of anyons obey fractional statistics, the particle models themselves are very different depending on what trends they belong to.

2.3.1 The classical model of identical particles and its quantizations (1971-1977)

As pointed out earlier in this section's introduction, what marks the beginning of the modern period of non-standard statistics is the development of a classical model for identical particle systems. In a 1971 paper [156], Laidlaw and DeWitt showed that the configuration space of N distinguishable classical particles could be used to build a configuration space that models a system of N indistinguishable classical particles.

An intuitive description of Laidlaw-DeWitt's construction goes roughly as follows. The set of allowed classical configurations of N non-interacting distinguishable particles moving in m dimensions is the set $Y(N, m) = \mathbb{R}^{mN} - \Delta(N, m)$, where $\Delta(N, m)$ is the set of configurations where at least one pair of particles occupy the same position. If the configuration space $Y(N, m)$ were used to model indistinguishable particles, then the configurations $(\mathbf{x}_1, ..., \mathbf{x}_N)$ and $(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)})$, where σ is some permutation of the N particles, should be physically the same, and therefore this pair of configurations would be redundant. This implies that the space $Y(N, m)$ contains $N!$ copies of a configuration space, called $Y(N, m)/S^N$, where all points $(\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_N)$ identify a unique configuration of a system with N identical particles.

In the configuration space $Y(N, m)$, all allowed classical trajectories taking some $(\mathbf{x}_1, ..., \mathbf{x}_N)$ to some of its $N!$ permuted counterparts $(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)})$, are represented by continuous curves γ, beginning at the point $(\mathbf{x}_1, ..., \mathbf{x}_N)$ and ending at the point $(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)})$. But if the particles are identical, any such path γ in $Y(N, m)$ can be reduced to a path $\tilde{\gamma}$ in $Y(N, m)/S^N$ that begins and ends at the same point $(\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_N)$.

Therefore, in contrast to the many-body formalism of subsection 2.1.1, the identity of the particles is imposed before any quantization, by going from the description in terms of the configuration space $Y(N,m)$ to one in terms of $Y(N,m)/S^N$. A partial description of the $Y(N,m)/S^N$ spaces for $N=2$, with $m=1,2$ and 3 is given in Figs.4,5 and 6.

The spaces $Y(N,m)$ have the property of being *simply connected*, meaning that any closed paths γ and γ' based at the same endpoint $(\mathbf{x}_1,...,\mathbf{x}_N)$, can be continuously mapped to one another, without finding any singularities. However, $Y(N,m)/S^N$ do not have this property, and are what we call *multiply connected spaces*. Such spaces possess spatial singularities that obstruct the path of maps trying to continuously deform one closed curve into another. The structure of singularities of multiply connected spaces divide all paths, not just the closed ones, into classes such that paths in one class cannot be continuously mapped onto paths in another. These classes of paths are called *homotopy classes*, and in the case of closed paths, an operation of path-composition can be defined such that the set of homotopy classes of closed paths becomes a group, called the *fundamental group* of the space.

The way the $Y(N,m)/S^N$ spaces are built does not change the classical dynamics of the particles in any meaningful way, apart from the redundancy of the particle coordinates. However, according to Laidlaw and DeWitt, the path-integral quantization of classical systems in multiply connected spaces has very special features. In such spaces, the propagation amplitude $K(a,t_a,b,t_b)$ from system configuration "a" at time t_a to configuration "b" at time t_b in a multiply connected configuration space T must have the form

$$K(a,t_a,b,t_b) = \sum_{[\gamma]\in\pi_1(T)} \chi([\gamma])K^{[\gamma]}(a,t_a,b,t_b), \tag{2.40}$$

where $K^{[\gamma]}$ are partial amplitudes corresponding to propagation from configuration a to b via the curves in $[\gamma]$, and $\chi([\gamma])$ are unknown weight factors.

Using the fact that for particles in three dimensions, $m=3$, one has $\pi_1(Y(N,3)/S^N) = S^N$ (see [73, 74]), Laidlaw and DeWitt showed that the weight factors $\chi[\gamma]$ in the amplitudes must correspond to one-dimensional representations of S^N. Since the only such representations are the symmetric and antisymmetric ones, this implies that the only types of quantum statistics compatible with such formalism are standard bosons and fermions. Therefore, using the classical model of identical particles and the assumptions of path-integral quantization, they have developed a new formalism of standard quantum statistics, which we hereby call the *configuration space path-integral formalism*.

The advantages of this configuration space formalism over the many-body and field formalisms are pointed out in a famous 1977 paper by Leinaas and Myrheim [162]. In it, they argue that this classical model of indistinguishability can be used, for example, to justify the $1/N!$ Gibbs correction factor to the total number of microstates of a classical ideal gas. For them, however, a more important fact is that this formalism avoids the problems pointed out in another paper, which we discuss first. In the 1973 paper [181], Mirman offers a critique of the alternative formulations of the many-body formalism presented in subsection 2.2.2. The most important of his criticisms was that neither the concept of permutation invariance nor the difference between particle and label permutations have experimental meaning, since in both cases they do not specify what kind of physical transformation is a "permutation".

But in the configuration space formalism, the physical permutations are given by curves corresponding to classical trajectories, which always have operational meaning. This prompts Leinaas and Myrheim to offer a new quantization method adapted for multiply connected spaces. In their paper [162], they proceed to discuss the topology of the spaces $Y(N,m)/S^N$, showing that for $N = 2$, one can use center-of-mass coordinates to prove that

$$Y(2,m)/S^2 = \mathbb{R}^m \times \mathbb{P}_{m-1} \times (0,\infty) \tag{2.41}$$

where the first space is the domain of the center-of-mass coordinate, and the other product is the domain of the relative coordinate, with $\mathbb{P}_{m-1}$ being the real projective space of dimension $d-1$, which end up determining the fundamental group of the $Y(2,m)/S^2$ spaces.

For $m = 1$, the associated real projective space $\mathbb{P}_0$ is a single point. Therefore $Y(2,1)/S^2$ is in fact simply connected. However, using the previous decompositions, we can see that this space is topologically equivalent to $\mathbb{R} \times (0,\infty)$, which is a half-plane, as in Fig.4. Therefore $Y(2,1)/S^2$ has a natural boundary, and its quantization must account for this fact. To be more precise, a point in $Y(2,1)/S^2$ is given by the center-of-mass coordinates (x,z)

$$x = \frac{1}{2}(x_1 + x_2), \tag{2.42a}$$

$$z = |x_1 - x_2|, \tag{2.42b}$$

where (x_1, x_2) are the Cartesian coordinates in $Y(2,1)$. In these coordinates, the boundary line $x_1 = x_2$ is now given by $z = 0$. The quantization is done in the Schrödinger picture by postulating a multiparticle wave function $\psi(x,z)$, that represents the two-particle system as a whole.

The free particle Hamiltonian acting on functions defined over $Y(2,1)$ can be written in center-of-mass coordinates, assuming the form

$$H = -\frac{\hbar^2}{m}\left(\frac{1}{4}\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 z}\right). \tag{2.43}$$

However, restricting this Hamiltonian to have as its domain the functions defined in $Y(2,1)/S^2$ forces us to pick a boundary condition. If we demand that $\psi(x,0) = 0$, we

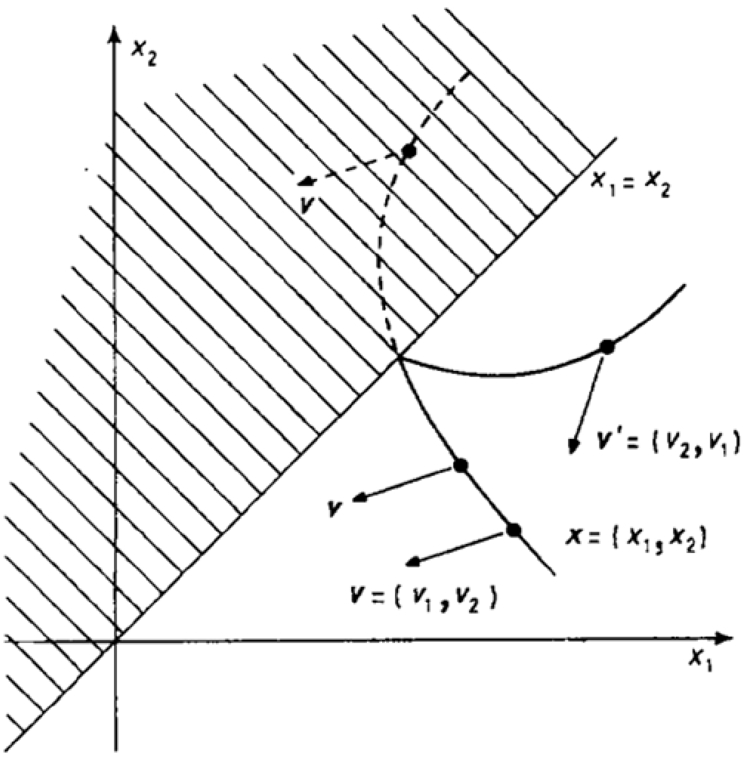


Figure 4 – Configuration space $Y(2,1)/S^2$: Solid line represents the trajectory of two identical particles in the configuration space. The dashed line represents the continuation of the particle trajectory in the upper half of boundary line $x_1 = x_2$, which is identified with the lower half. The vector $\mathbf{v}$, gets reflected to the vector $\mathbf{v}'$ when the trajectory hits the boundary $x_1 = x_2$. Figure obtained from the work [162], on page 7.

are automatically describing fermions, since this condition is equivalent to the exclusion principle. On the other hand, if we demand that $\partial_z \psi(x,0) = 0$ we automatically describing bosons, because it implies that $\partial_z \psi(x,0) = \partial_{-z}\psi(x,0)$, which can only be true if $\psi(x_1,x_2) = \psi(x_2,x_1)$.

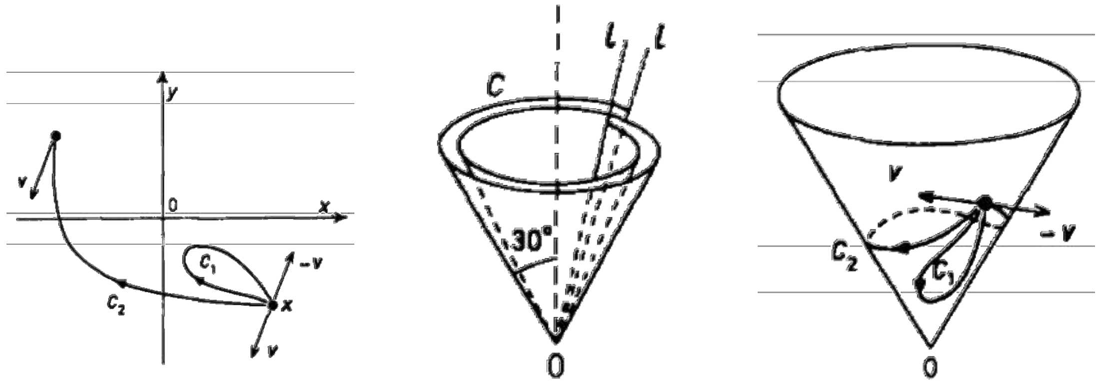
However, Leinaas and Myrheim argue that the most general boundary condition comes from demanding the global conservation of probability, which imposes the vanishing of the probability current component $J_z(x,0)$ normal to the boundary. The continuity equation for the probability current

$$J_z(x,0) = \psi^*(x,0)\frac{\partial \psi(x,0)}{\partial z} - \frac{\partial \psi^*(x,0)}{\partial z}\psi(x,0) = 0 \tag{2.44}$$

has as its general solution the boundary condition

$$\frac{\partial \psi(x,0)}{\partial z} = \eta \psi(x,0), \tag{2.45}$$

with $\eta \in \mathbb{R}$. Since this equation is linear, the value of η is independent of x, and its value describes the quantum statistic of the particle pair. For $\eta = 0$ we have bosons, for $1/\eta \to 0$ we have fermions. Therefore, for all intermediate values this system has non-standard quantum statistics.



(a) Planar view (b) Folding of the identified plane (c) Spatial view

Figure 5 – Relative coordinate space of $Y(2,2)/S^2$: (a) Planar view of $\mathbb{P}_1 \times (0, \infty)$, the points $\mathbf{x}$, and $-\mathbf{x}$ are identified. Parallel transport of the velocity vector $\mathbf{v}$ over the path C_1 leads to the vector $-\mathbf{v}$, while parallel transport over C_2 (which connects identified points) leads to $\mathbf{v}$. (b) Represents the identification of the configuration space as the folding of the planar view, l is a line whose positive part gets glued to the negative part, transforming the plane into a cone without the origin. (c) Spatial view of $\mathbb{P}_1 \times (0, \infty)$, with the paths C_1 and C_2 represented on the conical surface. Since C_2 circles around the cone without the origin, it is topologically different from C_1, which shows that $\mathbb{P}_1 \times (0, \infty)$ is not connected. Figures obtained from the work [162], on page 8.

The configuration spaces $Y(2,m)/S^2$ for $m = 2$ and $m = 3$ have as "relative coordinate spaces" (see eq.(2.41)) the spaces $\mathbb{P}_1 \times (0, \infty)$ and $\mathbb{P}_2 \times (0, \infty)$, and have as fundamental groups the integers under addition $\mathbb{Z}$, and the permutation group S^2, respectively, as seen in Figs. 5 and 6. Since neither of these spaces are simply connected, the previously used method of quantization does not apply, and a new prescription is necessary. Leinaas and Myrheim's proposal is to use those multiply connected configuration spaces as a basis for building a multiparticle Hilbert space, generalizing the many-body formalism of quantum statistics.

Since the quantization prescription makes sense for any multiply connected space T, we describe the method itself first, and then talk about the particular cases $Y(2,2)/S^2$ and $Y(2,3)/S^2$. First, assume that the particles have no internal degrees of freedom. Informally speaking, the prescription consists of "attaching" a copy $\mathscr{H}_x$ of a one-dimensional Hilbert space $\mathscr{H}$ to every single point $\mathbf{x}$ of the space T. The state of the particle system is assumed to be described by the continuum of vectors

$$|\Psi(\mathbf{x})\rangle = \psi(\mathbf{x}) \, |h\rangle_{\mathbf{x}} \,, \tag{2.46}$$

where $|h\rangle_{\mathbf{x}}$ is the basis vector of the space $\mathscr{H}_{\mathbf{x}}$, and the coefficients $\psi(\mathbf{x})$ are the system's multiparticle wave-function.

The set of basis vectors $\{|h\rangle_{\mathbf{x}}\}$ is completely arbitrary, implying that the wave-

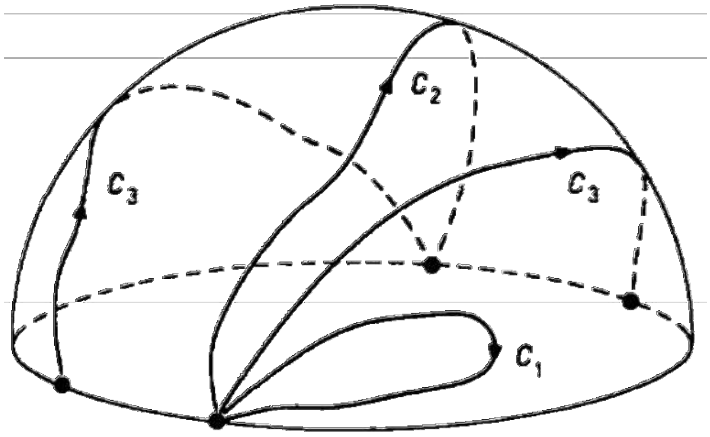


Figure 6 – Projective plane $\mathbb{P}_2$, a factor of the relative coordinate space of $Y(2,3)/S^2$: The projective plane is pictured as the upper half of a spherical surface, with opposite points in the equator identified. Path C_1 is topologically trivial, but C_2 is not, in fact $[C_2]^2$ is the identity element in $\pi_1(Y(2,3)/S^2)$, with the curve C_3 being an intermediate step in a homotopy transformation between a curve in the class $[C_2]^2$ and a single point.

function $\psi(\mathbf{x})'$ defined by the equation

$$\psi(\mathbf{x})' = \exp\{[i\varphi(\mathbf{x})]\}\psi(\mathbf{x}), \tag{2.47}$$

must be a completely equivalent description of the particle system as the wave-function $\psi(\mathbf{x})$. This fact establishes the Leinaas-Myrheim quantization prescription as a *gauge theory* of quantum statistics.

As any gauge theory, there exists a gauge field $b_k(\mathbf{x})$ that tells us how to relate the Hilbert spaces $\mathscr{H}_{\mathbf{x}}$, and $\mathscr{H}_{\mathbf{x}+d\mathbf{x}}$, which are separated by an infinitesimal displacement in the configuration space T. The basis vectors $|h\rangle_{\mathbf{x}}$ and $|h\rangle_{\mathbf{x}+d\mathbf{x}}$ are related by the *infinitesimal parallel displacement*

$$P(\mathbf{x}, \mathbf{x}+d\mathbf{x}) = (1 + ib_k(\mathbf{x})dx^k), \tag{2.48}$$

such that $|h\rangle_{\mathbf{x}+d\mathbf{x}} = P(\mathbf{x}, \mathbf{x}+d\mathbf{x})\,|h\rangle_{\mathbf{x}}$. Then, the displacement operator along an arbitrary path $\gamma : [0,1] \to T$ is given by

$$P_\gamma(\gamma(0), \gamma(1)) = P\exp\Big\{i\int_\gamma b_k(\mathbf{x})dx^k\Big\}, \tag{2.49}$$

where $P\exp$ is the path-ordered exponential.

Restricting ourselves to the gauges where the above formula of infinitesimal parallel displacement is valid, one can deduce the form of the gauge-covariant differential operator

$$D_k = \frac{\partial}{\partial x_k} - ib_k(\mathbf{x}). \tag{2.50}$$

This operator is the canonical momentum operator in the Schrödinger formalism, and it can be used to build gauge-invariant Hamiltonians. To every gauge field, there is also an associated gauge-invariant force tensor field

$$f_{l,k} = i[D_l; D_k] = \frac{\partial b_l}{\partial x_k} - \frac{\partial b_k}{\partial x_l}. \tag{2.51}$$

Given this description of a gauge theory, Leinaas and Myrheim argue in [162] that to model a theory of identical particles the gauge field $b_k(\mathbf{x})$ must always be *pure gauge*, meaning that $f_{l,k}(\mathbf{x}) = 0$ for all non-singular points $\mathbf{x} \in T$. This is because, under this condition, the displacement operator P_γ for any closed path γ is trivial if the path does not enclose a singular point. In fact, for all closed paths in an equivalence class $[\gamma]$ the associated displacement operators are the same. Because of these facts, the set of distinct operators $\{P_{[\gamma]}\}_{[\gamma]\in\pi_1(T)}$ forms a representation of the fundamental group of $\pi_1(T)$. Their conclusion is that the statistics of the quantized system is determined by the representation of $\pi_1(T)$ realized by the displacement operators of closed paths, and to each representation, one obtains a different type of statistics.

Now we are in the position to discuss the two and three dimensional two-particle systems. In the two dimensional case, the wave function is defined over the base space $Y(2,2)/S^2$. A wave-function in $Y(2,2)/S^2$ has coordinates $(\mathbf{x}_{cm}, r, \varphi)$, where $\mathbf{x}_{cm}$ are the coordinates of the system's center-of-mass and (r, φ) are the relative coordinates in polar form. Since the particles have no degrees of freedom, any operator acting on the wave-function must be a phase factor.

The physical permutations are determined by the displacement operators on closed loops. In the two-dimensional case, these form the set $\{P_{[i]}\}_{i\in\mathbb{Z}}$ since $\pi_1(Y(2,2)/S^2) = \mathbb{Z}$. Therefore, if we choose $P_{[1]} = \exp\{i\xi\}$, for some $\xi \in \mathbb{R}$, it automatically follows that $P_{[n]} = \exp\{in\xi\}$ for all integers n, meaning that the parameter ξ is what determines the type of statistics in the system. By eq. (2.49), it follows that a possible gauge field generating $P_{[1]}$ is given by $b_{\mathbf{x}_{cm}}(\mathbf{x}_{cm}, r, \varphi) = b_r(\mathbf{x}_{cm}, r, \varphi) = 0$ and $b_\varphi(\mathbf{x}, r, \varphi) = -\xi/2\pi$. Therefore the gauge-covariant Hamiltonian for a system of two free identical particles in center-of-mass, polar coordinates, must be

$$H = -\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}\left(\frac{\partial}{\partial \varphi} + i\frac{\xi}{2\pi}\right)\right), \tag{2.52}$$

where we omitted the center-of-mass part $\mathbf{x}_{cm}$, for brevity's sake.

The last calculation shows that the effect of quantum statistics in two dimensions is a shift in the angular momentum operator L. However, to connect this effect with the usual many-body formalism we must find a description where the role of statistics appears in the symmetry of the wave-function, and not on the Hamiltonian. This is done by a change of gauge, where the new gauge has $b_k(\mathbf{x}_{cm}, r, \varphi)' = 0$. In this gauge, the wave-function assumes the form

$$\psi(\mathbf{x}_{cm}, r, \varphi)' = \exp\left\{i\frac{\xi}{2\pi}\varphi\right\}\psi(\mathbf{x}_{cm}, r, \varphi). \tag{2.53}$$

One of the exchange transformations corresponding to closed paths in the class that generates $\pi_1(Y(2,2)/S^2)$, is the one taking $\psi(\mathbf{x}_{cm}, r, \varphi)'$ to $\psi(\mathbf{x}_{cm}, r, \varphi + 2\pi)'$, and because $\psi(\mathbf{x}_{cm}, r, \varphi)$ is periodic, we have that

$$\psi(\mathbf{x}_{cm}, r, \varphi + 2\pi)' = \exp\{i\xi\}\psi(\mathbf{x}_{cm}, r, \varphi)'. \tag{2.54}$$

Therefore, in this gauge, the wave-function acquires a non-trivial exchange phase when the particles are permuted along a path in class [1].

Similarly, in the three dimensional case, we have that $\pi_1(Y(2,3)/S^2) = S^2$ implies all closed paths belong to the class of trivial loops $[e]$, or the class of transpositions $[t_1]$, such that $P_{[e]}^2 = P_{[t]}^2 = P_{[e]} = 1$. Therefore, in the gauge where $b_k(\mathbf{x}) = 0$ the wave functions $\psi(\mathbf{x}_{cm}, \mathbf{z})$ are either symmetric or antisymmetric with respect to exchanges in the class of transpositions $[t]$. In fact, in the three dimensional case we end up re-obtaining the standard behaviors associated to fermions and bosons in the many-body formalism.

Therefore, in Leinaas and Myrheim's theory of quantum statistics, which we hereby call the *topological formalism*, wave-functions of two-particle systems in three-dimensions must be either bosonic or fermionic, but in one and two dimensions they can have non-standard quantum statistics. In the one-dimensional case the statistics is determined by the boundaries of the identical particle configuration spaces, while in the two and three dimensional cases, they are determined instead by the representations of the fundamental groups of those spaces. In both cases the topology of the classical model for identical particles developed by Laidlaw and DeWitt plays a crucial role in the statistical theory.

The interpretation of physical permutations in the one-dimensional case are not very clear, since there is not really a physical way to permute classical particles confined in one dimension. However, in all other cases physical permutations correspond to classical trajectories of particles, and in the two dimensional case the wave-function acquires non-trivial phase factors governed by a parameter ξ under these exchanges. When $\xi = 0$, the particles are bosons, when $\xi = \pi$ they are fermions, and for all other values of ξ they would be later called *anyons*, particle with a new type of statistics called *fractional statistics.*

2.3.2 The topological trend (1977-)

From the topological formalism, at least two new branches of theories of non-standard quantum statistics came forward. The first branch, called the *topological trend*, is the one from which most physicists learn non-standard quantum statistics, since it is the one that led to the most important applications in condensed matter and quantum computation. As stated in this section's introduction (2.3), here in subsection 2.3.2 we deal with this branch, leaving the remainder for subsection 2.3.3.

The first interesting application of the topological formalism was to solve problems related to the statistics of charge-monopole composites, and were also made by Leinaas in [159]. Before we discuss this paper, we need a little bit of context. Monopoles are a type of *topological defect*, which are singular configurations of classical fields that have been studied since Dirac's 1931 work on the subject [59], where he showed that particles bearing magnetic monopole fields can explain the apparent quantization of the electron's charge.

In 1948, Dirac [60] also gave a formal Hamiltonian description of the relativistic equation of motion for magnetic monopoles, and built a quantum theory for them. Since then, many other physicists were trying to find quantum Dirac monopoles in Nature, with no success. The importance of monopoles in this part of history is due to an alternative analysis of the physics of quantum Dirac monopoles made by Goldharber, first in 1965 [99] and latter in 1976 [98].

In his 1965 paper, Goldhaber studied the scattering of a charged particle by a magnetic monopole and showed that, when they form a bound state, the interaction shifts the angular momentum spectrum of the pair by a half-integer unit of $\hbar$ relative to the spectrum of uncharged particles. This poses a paradox, since the exchange of a pole and a charge in the bound state has bosonic behavior but half-integer total angular momentum, which violates the spin-statistics theorem. In his 1976 paper, Goldhaber showed that pairs of bound charge-monopole composites have, indeed, an antisymmetric wave-function with respect to permutations of the composites.

Leinaas' 1978 contribution was an interpretation of Goldhaber's solution in terms of the topological formalism. He models a system of two charge-monopole composites with the configuration space $Y(2,3)/S^2$ (see subsection 2.3.1), while the configuration space of the relative motion between charges and monopoles in the composites are taken as internal degrees of freedom. Then, the monopole gauge fields play the role of the statistical gauge field present in the topological formalism, which ends up changing the statistics of the composite as a whole. This was the first time the topological formalism was applied to a problem relating topological defects.

The next noteworthy application of the topological formalism was made by Leinaas in a 1980 paper, where he compared his theory of identical particles, the topological formalism, with both general topological defects and *topological solitons* in gauge field theories. Solitons are configurations of non-linear classical fields that have finite total energy and are localized in a finite region of space [77, 78] (or decay sufficiently fast). Topological solitons, in particular, are soliton solutions connecting distinct zero-energy configurations of the non-linear field and, as a result, they are acquire a new degree of freedom called *topological charge*, which is affected by a non-trivial conservation law.

At the time of Leinaas' work, topological solitons had been studied in connection to problems in cosmology [186] and particles physics [197], as well as with non-perturbative quantum field theory in general [205]. However, their role in the history of non-standard quantum statistics begins in a series of papers [124, 135] which showed that the same type of charge quantization that occurs for Dirac monopoles also occurs in some theories of topological solitons, but with a charge quantum of a fraction of the electron's charge. Even if Leinaas' 1980 paper made the first connection between topological solitons, topological singularities and identical particles, it was Wilczek who, independently, solved the mystery of charge *fractionalization* in these soliton theories.

In a series of three papers from 1982 to 1983 [250, 251, 253], Wilczek showed that, for many different kinds of soliton configurations, the rise of fractional charge, fractional angular momentum and fractional statistics [101] can be explained by the same heuristic model. In the first paper, he pointed out that a charged particle orbiting the gauge potential of a solenoid has its quantized angular momentum spectra shifted by a value proportional to its magnetic dipole moment. Then he showed how the classical derivation of this result could be used to explain the statistics of electron-vortex composites in a superconductor, the Dirac monopole quantization condition and the fractionalization of charge in fields with soliton solutions.

In his second 1982 paper [251], Wilczek showed how that composites made of tightly bound electrically charged particles and very thin, infinite solenoids carrying a constant magnetic flux behave like the particles with fractional statistics in the topological formalism. The magnetic vector potential played the role of the statistical gauge field, and the wave-function exchange phase was interpreted as coming from the Aharonov-Bohm effect generated by the statistical field of one composite interacting with the charge of the other [5]. It was in this paper where the word *"anyon"* was used for the first time, and he used it to call all particles (in a broad sense) that acquired fractional statistics via this mechanism, the he called *fictitious flux attachment*. This specific flux-charge composite model of anyons was later called a "cyon", in [136, 167], to differentiate then it from the monopole-charge composites that also behave as anyons, and were called "dyons".

Finally, in the last work of this series [253], Wilczek showed that soliton solutions for the non-linear $O(3)$ σ model in $(2+1)$ dimensions, which are not gauge theories, could also exhibit fractionalized charge, angular momentum and statistics. When interpreted as particles, the soliton spin is defined in a semiclassical path-integral approximation by adiabatically rotating the solution in the two dimensional plane by an angle of 2π. The conservation of topological charge for solitons allows the introduction of a new term in the model's Lagrangian that does not affect the equations of motion, and is described by a fictitious abelian gauge field.

The action of this new Lagrangian leaves the equations of motion unchanged, but it appears as an extra fractional spin for the solitons of the theory, as a quantum field effect. Wilczek also showed that this extra Lagrangian is a topological invariant, called the Hopf invariant, representing the linking number[5] of the trajectories associated to soliton anti-soliton[6] pair creation-annihilation events. Later, this Lagrangian would be recognized as the Chern-Simons Lagrangian, and this realization led to profound changes in the theory of quantum statistics, as we will see shortly.

After these three works by Wilczek, the topological formalism of quantum statistics became intrinsically linked with the theory of topological solitons, or defects in general. Therefore, anyons started appearing more and more in connection to problems that were previously unrelated to the theory of quantum statistics. So, from now on, we only consider works whose primary concern is fractional statistics itself, leaving applications to the end. Chronologically, the first of these works was Wu's 1984 papers [259, 260].

The first of Wu's papers [259] was primarily an extension of Laidlaw and DeWitt's path-integral quantization (see subsection 2.3.1) from N identical particles in three dimensions to two dimensions. This generalization was possible due to the fact that the fundamental group of the identical particle configuration spaces $Y(N,2)/S^N$ was already known at the time. These fundamental groups are called the *braid groups* on N letters B^N, and Wu showed that it was because they are very different from the symmetric group S^N that fractional statistics exists.

The braid groups B^N, were first described by Artin in 1947 [12] as the symmetry groups of set of N identical lines that can braid around each other by moving their endpoints, as seen in figure 7. They can be described algebraically in terms of generators

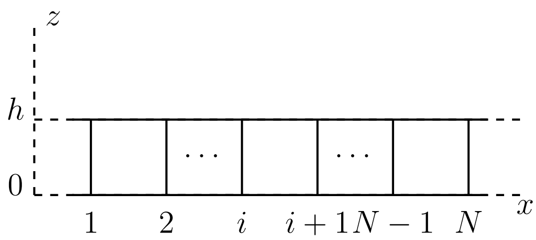


Figure 7 – Geometric description of a set of N strands in space. This diagram also represents the trivial braid **1**.

σ_i, given geometrically in figure 8, that behave according to the equations

$$\sigma_i^2 \neq \mathbf{1} \tag{2.55a}$$

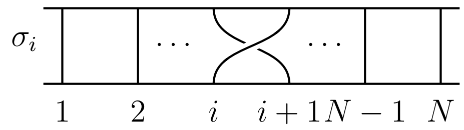
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ for } |i-j| \geq 1, \tag{2.55b}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \tag{2.55c}$$

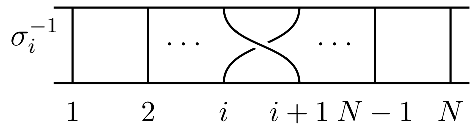
which are represented geometrically by figure 9. Notice how these conditions are similar to the ones defining transpositions in eq. (2.4).

[5] This number measures the number of crossings in a three dimensional knot.

[6] Since topological solitons are field configurations, they can interfere and annihilate each other in the same way as classical waves, meaning that when interpreted as particles, there are also solitons corresponding to anti-particles.

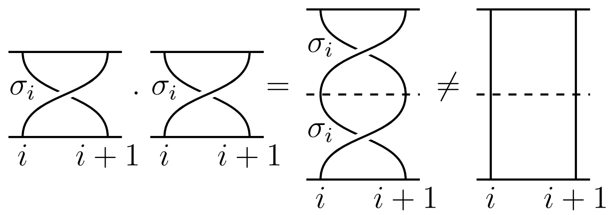


(a) Diagram of σ_i

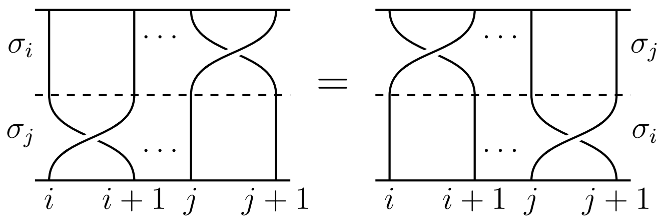


(b) Diagram of σ_i^{-1}

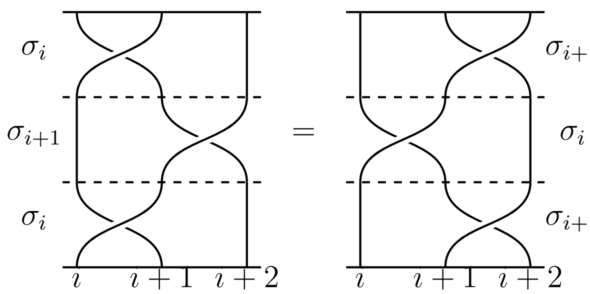
Figure 8 – Geometric description of generators.



(a) Representation of equation $\sigma_i^2 \neq \mathbf{1}$



(b) Representation for the equation $\sigma_i \sigma_j = \sigma_j \sigma_i$



(c) Representation for the equation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Figure 9 – Braid group identities for generators

In this same work [259], it was shown that the weight factors $\chi([\gamma])$ associated to homotopy classes of paths in the path-integral formalism discussed in subsection 2.3.1 were equal to the path displacement operators $P_{[\gamma]}$ of the topological formalism. It was also shown that, for the two dimensional case, the set of all $\chi([\gamma])$ are a one-dimensional representation of the braid group, with the geometric presentation of the generators σ_i corresponding to the actual classical motion of the particles under physical permutations. It was from this time onward that any type of topological defect that when quantized had their many-particle wave-functions transforming as an abelian representation of the braid group under physical permutations were called *abelian anyons*.

Then, in his second work [260], Wu showed that because of this correspondence between weight factors and physical permutations, the $\chi([\gamma])$ can be written as

$$\chi([\gamma]) = \exp\left\{ i \int_{\gamma} dt L_{sta} \right\}, \tag{2.56}$$

with the *statistical Lagrangian* L_{sta} being

$$L_{sta} = -\frac{\xi}{\pi}\frac{d}{dt}\sum_{i<j}(\varphi_i(t) - \varphi_j(t)). \tag{2.57}$$

The parameter ξ is called the *statistical parameter* of the anyon, and $\varphi_i(t)$ is the polar angle of the i-th particle with respect to the origin. Since this Lagrangian is a total time derivative, it does not affect the classical equations of motion for the particles in any way, but in path integral quantization, it is the origin of fractional statistics, and it was used to build a many-body theory of non-interacting anyons.

The multiparticle theory of anyons was further developed by Arovas et al in 1985 [11], where they showed that a gas of N free anyons ("cyons") has a non-trivial virial coefficient, making it a non-ideal gas. In the same paper they have realized that the $U(1)$ statistical gauge field $A_\mu(\mathbf{x})$ of the σ model described by Wilczek in 1983 [253] obeys the action

$$S_{CS} = \frac{\xi}{2\pi}\int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{2.58}$$

called the *Chern-Simons action*. And they showed, using this action, that a soliton gas with topological charge $Q = +1$ in the limit of "point solitons" has exactly the same virial coefficient as N free anyon gas, in the semiclassical path-integral quantization.

Later this result would be generalized by Bowick et al in 1986 [33]. They described the canonical quantization of the $O(3)$ non-linear σ model with a Hopf term and showed that the spin fractionalization happens not just for solitons with topological charge $Q = +1$, but for any charge. Another work, by Wu and Zee in the same year [261], extended this result to $3+1$ dimensions where instead of the Hopf term, the topological action had the Wess-Zumino form, a type of action studied in conformal field theories. However, the connection between conformal field theories and Chern-Simons fields was only completely revealed by Witten in 1989 [255]. This work was so groundbreaking that it lead to a new area of mathematical physics called *quantum topology*.

For our purposes, another important work was done by Krauss and Wilczek in [154], where they studied a special type symmetry called *discrete local gauge symmetry*. These symmetries were important for understanding special soliton configurations called *non-abelian vortexes* that arise from partial symmetry breaking of a continuous gauge group to a discrete local non-abelian gauge symmetry. Discrete gauge theories are also relevant for models of quantum gravity, and to the study of symmetries of black holes

[202]. However the most important property of theories with discrete gauge fields was that non-abelian vortex could have another type of quantum statistics, as described by Wilczek in 1990 [252].

In that paper, Wilczek provided a simple generic particle model for non-abelian vortexes called *non-abelian flux-charge composites*. These are classical objects labelled by (a, ζ), where a is an element of the discrete gauge group G representing a generalized magnetic flux and ζ is the representation space that transforms under the action of G, playing the role of a generalized electric charge. Then, suppose that two such composites labelled, by (a, ζ) and (b, η) undergo scattering. The probability amplitude of such a process can be decomposed into a sum of terms, each of which corresponding to successive exchanges of the particle pair. For a single counterclockwise winding of composite (a, ζ) over (b, β), the non-abelian fluxes interact, and after a full turn they get transformed to

$$(a^{(1)}, \zeta^{(1)}) = ((ba)a(ba)^{-1}, D^R(ba)D^R(a^{-1})\zeta), \tag{2.59}$$

$$(b^{(1)}, \eta^{(1)}) = ((ba)b(ba)^{-1}, D^R(ba)D^R(b^{-1})\eta), \tag{2.60}$$

where $D^R(g)$ is the matrix representation of g in the representation R acting over ζ and η.

The point is that the flux label is not gauge invariant, so even if we fix a gauge and label each flux-tube with a unique element of G, it seems that the effect of winding two composites with fluxes a and b such that there is an $g \in G$ with $a = gbg^{-1}$ (i.e., a and b are conjugates) is trivial. However this is not the case, because a gauge transformation that transforms the flux labels $a^{(1)}$ and $b^{(1)}$ into their unwound configurations a and b, would deform the flux lines and force test charges to feel the transformed flux instead. Therefore, the transformed flux is physically different, but it is not detectable by local observables with support near either of the composites. This implies that such objects are indistinguishable without being identical, which appears to be a paradox.

This problem was solved with further analysis by Lo and Preskill in 1993 [170], where the behavior of non-abelian flux-charge vortexes (assumed to be solitons) was established via a series of mental experiments. First, they showed that flux-charge vortexes (a, ζ) passing trough a double-slit over the region of another vortex (b, η) can only feel the effect of b over ζ if a and b commute. This implies that R must be a representation of the subgroup of G that commutes with all a, called the *centralizer* $N(a)$ of a.

Therefore, throwing a vortex with a flux over a double-slit onto the region of another a vortex yields an interference pattern of abelian anyons since $D^{R^{(a)}}(a)$, where $R^{(a)}$ is a representation of $N(a)$, must be proportional to the identity due to a being in the center of its centralizer $N(a)$[7]. This property implies that two vortexes with flux labels a

[7] Arbitrary elements of $N(a)$ are not required to commute with each other but only with a. Therefore being an element of the center of $N(a)$ is a non-trivial property, and the only element we know is there for certain is a itself.

and b are indistinguishable only if they are conjugate, if $N(a)$ is isomorphic to $N(b)$ and if they are in the same charge representation $R^{(a)} = R^{(b)}$. However, these indistinguishable particles are not *identical*, since a vortex with flux a^{-1} can annihilate one with flux a, but not one with flux b, even if a and b are conjugate.

Lo and Preskill argue then that this occurs because the indistinguishable labelling scheme does not correspond to irreducible representations of a symmetry group, but to the irreducible representations of the *quantum double algebra* $D(G)$ of the gauge group G, which is a quantum group. The first connection between non-abelian vortexes and quantum groups was made by Bais in [15], but it was Lo and Preskill that saw how this structure solved the paradox of the existence of non-identical indistinguishable objects. An intuitive definition of quantum groups, together with the first construction of the quantum double algebra of a finite groups. was made in [65]. The intuitive definition goes roughly as follows.

A classical system is thought of as a mathematical object made of states and observables, usually with the space of states being a manifold $\mathscr{M}$ with a phase-space structure, and the observables being the algebra of dynamical functions on this manifold. When we quantize a classical system, we are essentially substituting the observable algebra over the phase space, which is a commutative algebra, by a non-commutative algebra that in some sense preserves the "equations of motion" of the theory. Then, a non-commutative algebra becomes the quantum algebra of observables and their Hilbert space representations become the spaces of quantum states.

It so happens that one can do a mathematical procedure that is formally identical to this kind of quantization with mathematical objects that are not manifolds and their functions. If one chooses a non-abelian finite group G as a "phase space", where the "dynamical structure" is given by the group multiplication, the associated space of "dynamical observables" that is a special type of algebra, and the quantization of this algebra gives the *quantum double algebra* $D(G)$.

Therefore, in the analysis of the statistics of non-abelian vortexes, the internal Hilbert spaces $\mathscr{V}_{\mathbf{a}}$, associated to the types of vortexes labelled by $\mathbf{a} = ([a], R^a)$, with $[a]$ being a flux conjugacy class and $R^{(a)}$ being the charge label, are such that when two vortexes $\mathbf{a}$ and $\mathbf{b}$ are brought into contact to "fuse" with one another, the resulting vector space is decomposed as

$$\mathscr{V}_{\mathbf{a}} \otimes \mathscr{V}_{\mathbf{b}} = \bigoplus_{\mathbf{c}} N^{\mathbf{c}}_{\mathbf{a},\mathbf{b}} \mathscr{V}_{\mathbf{c}}, \tag{2.61}$$

where the coefficients $N^{\mathbf{c}}_{\mathbf{a},\mathbf{b}}$ are called the *fusion coefficients*, and are positive integers. These equations are called the *fusion rules* of the types of indistinguishable vortexes, and the labels "**a**" themselves are the *topological charges* of the theory in the soliton sense.

The wave-functions of particles with these properties transform under non-abelian irreducible representations of the braid group, and for that reason are called *non-abelian anyons*. The existence of such type of statistics was shown to be an expression of the general fact that the soliton-like sectors of any local field theory in $(2+1)$ dimensions have their statistical properties completely determined by an abstract mathematical structure, first described by Fredenhagen et al in [83, 84], called *unitary modular tensor categories* [208, 209].

This structure was latter exploited in a series of works by Freedman [86–89] to create a model of quantum computation using non-abelian anyons. This fact, together with Kitaev's proposal for a fault-tolerant quantum memory [145, 146, 184] and the role in explaining *topological phases of matter*, which include the fractional quantum Hall states and spin liquids [245, 246], is largely the reason for why the topological trend is the most studied form of non-standard quantum statistics.

2.3.3 The non-topological trend (1977-)

In this subsection, we deal with the other, minor trends on theories of non-standard quantum statistics which are not, at least directly, related to the topological trend. In contrast to the previous subsection, a historical periodization is less meaningful due to the sub-trends themselves being less correlated. This is the reason why I refer to sub-trends instead of sub-periods.

In the sub-trend of 2.3.3.1, I deal with theories of non-standard statistics arising from the physics of integrable systems, showing as best as possible its connection with statistical theories of the topological trend when they exist. I finish this subsection in 2.3.3.2 with the sub-trend of theories of non-standard statistics arising from *deformed commutation relations*, which play a minor role in the models of the topological trend, but has an importance of its own, specially for this work.

2.3.3.1 Statistics coming from many-body quantum integrable systems

As shown before in subsection 2.3.1, the classical configuration space of two one-dimensional identical particles are equivalent to the half-plane $\mathbb{R} \times [0, \infty)$. Here, the space singularity is not localized, but an infinite boundary, and therefore, it makes sense to canonically quantize the particles and defined their statistics to a choice of boundary condition. This is a very unusual way of defining quantum statistics, since it has nothing to do with particle exchange, and will be referred to as the *boundary formalism*.

In fact, the notion of quantum statistics itself is not so well defined in one-dimensional systems, as noted by Girardeau in 1960 [95]. In this work, he showed the existence of a one-to-one mapping between one-dimensional fermionic models and one-dimensional bosonic models with hard-core interactions of any kind. Calling the hard-core bosonic wave-function for N particles by $\psi_B((x_1, ..., x_N))$ and the fermionic one by $\psi_F(x_1, ..., x_N)$ this map is expressed by

$$\psi_B(x_1, ..., x_N) = \left(\prod_{i<j}^{N} sign(x_j - x_i)\right) \psi_F(x_1, ..., x_N), \tag{2.62}$$

or in other words, the two functions are the same up to a sign, which appears whenever there is a odd permutation of the coordinates of the particles relative to the standard crescent order $x_1 < ... < x_N$.

In this work, Girardeau also proved that this map is spectrum preserving, which allowed him to give an exact solution to the interacting boson problem with extreme hard-core interactions. This was latter recognized as an early example of what is called *bosonization*, where theories of local free fermionic fields can be described by soliton solutions of bosonic fields. Soon after this, Lieb and Liniger [165, 166] generalized Girardeau's result, giving the exact solution to less intense two-particle hard-core repulsive potential in a system of N bosons in one-dimension, given by the Hamiltonian

$$H_{Lieb} = \sum_{i=1}^{N} \frac{\partial^2}{\partial^2 x_i} + 2\eta \sum_{<i,j>} \delta(x_i - x_j), \tag{2.63}$$

where η measures the intensity of repulsion.

The case $\eta \to \infty$ gives exactly the same result as Girardeau's model, while all other cases can be solved by replacing H_{Lieb} with a free particle Hamiltonian and introducing the delta potentials as boundary conditions on the multiparticle wave-function $\psi(x_1, ..., x_N)$. The form of those boundary conditions is given in the equations bellow

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) \psi|_{x_{i+1}=x_i} = \eta\psi|_{x_{i+1}=x_i}. \tag{2.64}$$

Later, after the development of the concept of anyon it will be pointed out by Aneziris et. al [8], that such boundary conditions can be used to define multiparticle statistics, being an application of the boundary formalism. In fact, a version of what could be considered an anyonic phase factor already appeared on Lieb's ansatz for the multiparticle wave-functions, where the angle φ appears as a function of η regulating the exchange of momentum coordinates.

If we believe in Leinaas and Myrheim, and take the choice of value for η as the definition of quantum statistics in one-dimension, them we are forced to agree that local, two-particle interactions are able to change the quantum statistics of those particles. Therefore, they fulfill a role analogous to the fictitious gauge potential generated by topological charges in two-dimensional models of fractional statistics, without being gauge interactions themselves. These models also exhibit fractional statistics in the sense that multiparticle wave-functions acquire non-trivial exchange phases under permutations of particle coordinates. In these cases, however, the exchange operation is not geometric, but an abstract permutation of the order of the coordinates on the line.

This transmutation of quantum statistics made by local potentials was proven to be a feature not just of the Lieb-Liniger model, but of class of two-body potentials collectively known as Calogero-Sutherland-Moser (CSM) systems [189]. These are one-dimensional, exactly solvable models of the form

$$H_{CSM} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial^2 x_i} + \sum_{i<j}\vartheta(x_i - x_j), \tag{2.65}$$

where $\vartheta(\zeta)$ are functions proportional to $\delta(\zeta)$ (Lieb-Liniger), $\zeta^{-2}+\nu^2\zeta^2$ (Calogero [44,45]), or $\sin^{-2}(\zeta)$ (Sutherland [233]).

The connection of these systems to fractional statistics in the boundary formalism was proven by Polychronakos in [198–200]. His method was successfully applied to the Calogero Hamiltonian by Brink et al [36,37] and to CSM systems in general by himself in 1992 [200]. The method itself consists of expanding the canonical algebra $[x_i; p_j] = i\delta_{i,j}$, where $p_j = -i\partial/\partial x_j$, to include operators $M_{i,j}$ that satisfy

$$M_{i,j} = M_{j,i} = M_{i,j}^{\dagger}, \quad M_{i,j}^2 = 1, \tag{2.66a}$$

$$M_{i,j}x_i = x_j M_{i,j}, \quad M_{i,j}p_i = p_j M_{i,j}, \tag{2.66b}$$

$$[M_{i,j}; x_k] = [M_{i,j}; p_k] = 0, \quad \text{for } k \neq i,j. \tag{2.66c}$$

In this extended algebra it is possible to define an effective momentum operator with the form

$$\pi_i = -i\left(\frac{\partial}{\partial x_i} - \sum_{i\neq j} v(x_i - x_j)M_{i,j}\right), \tag{2.67}$$

where the function $v(\zeta)$ is chosen such that H_{CSM} assumes the form of a harmonic oscillator

$$H_{CSM} = -\frac{1}{2}\left(\sum_{i=1}^{N}\pi_i^2 + \sum_{i=1}^{N}x_i^2\right). \tag{2.68}$$

Then, the statistical character of the particles are determined by the action of $M_{i,j}$ on the multiparticle wave-functions.

In the case of the Calogero model, it was already known that its states are in a one-to-one mapping to anyons states in the fractional quantum Hall effect [161]. In fact, the effective momentum has the form of an analogue of the minimal coupling prescription for gauge fields, even though the interactions are not gauge interactions. Additionally, both the Lieb-Liniger and Calogero models can be obtained from confining two-dimensional anyons to one-dimensional subspaces in two different ways [117].

Even if anyons obtained in this way are not described by gauge theories, they are intimately related to it. It is therefore not surprising that someone would try to find a many-body theory in one dimension that couples to a gauge field of zero curvature, to give a gauge theory model for such anyons. The main such proposal was given by Rabelo in [204], which tried to postulate a one-dimensional form of the Chern-Simmons potential and couple it to a Schrödinger field. However, as it was pointed out by Aglietti et al in [2], this procedure does not succeed on generating anyons, giving instead a special class of soliton solutions without fractional statistics.

The first theory of one-dimensional anyons that could be defined as a bona-fide theory described by statistical gauge fields, was given by Kundu in [155]. There, he showed that the version of the Lieb-Liniger model with the form given below

$$H_{Kundu} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial^2 x_i} + \sum_{<k,l>}\delta(x_k - x_l)\left[\eta + i\kappa\left(\frac{\partial}{\partial x_k}+\frac{\partial}{\partial x_l}\right)\right] + \lambda_2[\delta(x_k-x_l)]^2 + \\ + \sum_{<j,k,l>}\lambda_1\delta(x_j-x_k)\delta(x_l-x_k), \quad (2.69)$$

where η is the intensity of the Lieb-Liniger potential, and $< k, l >$ is the nearest-neighbours symbol, could be solved exactly using the techniques of the theory of integrable systems if $\lambda_1 = \lambda_2 = \kappa^2$.

He noticed that this many-body quantum Hamiltonian could be derived from a bosonic quantum field theory with Hamiltonian given by

$$\hat{H} = \int dx : \left(\frac{\partial\hat{\psi}^\dagger}{\partial x}\frac{\partial\hat{\psi}}{\partial x} + \eta\hat{\rho}^2 + i\kappa\hat{\rho}\left(\hat{\psi}^\dagger\frac{\partial\hat{\psi}}{\partial x} - \frac{\partial\hat{\psi}^\dagger}{\partial x}\hat{\psi}\right) + \kappa^2(\hat{\psi}^\dagger\hat{\rho}^2\hat{\psi})\right) :, \quad (2.70)$$

where $\hat{\rho} = \hat{\psi}^\dagger\hat{\psi}$ and the symbol :: is the normal ordering symbol. The operators $\hat{\psi}$ are bosonic in the sense that : $[\hat{\psi}(x); \hat{\psi}^\dagger(y)] := \delta(x-y)$.

Then, by executing a gauge transformation with the density $\hat{\rho}$ playing the role of a gauge field,

$$\tilde{\psi}(x) = e^{-i\kappa\int_{-\infty}^{x}\hat{\psi}^\dagger(x')\hat{\psi}(x')dx'}\hat{\psi}(x), \quad (2.71)$$

Kundu showed that the transformed Hamiltonian had the form

$$\tilde{H} = \int dx : \left(\frac{\partial\tilde{\psi}^\dagger}{\partial x}\frac{\partial\tilde{\psi}}{\partial x} + \eta(\tilde{\psi}^\dagger\tilde{\psi})^2\right) :_a, \quad (2.72)$$

where the normal ordering $::_a$ is now with given respect to the *anyonic commutation relations*

$$\tilde{\psi}^\dagger(x_1)\tilde{\psi}^\dagger(x_2) - e^{i\kappa sign(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}^\dagger(x_1) = 0, \quad (2.73a)$$

$$\tilde{\psi}(x_1)\tilde{\psi}^\dagger(x_2) - e^{-i\kappa sign(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}(x_1) = \delta(x_1-x_2). \quad (2.73b)$$

The multiparticle wave-functions $\Phi(x_{i_1}, ..., x_{i_N}) = \langle 0|\hat{\psi}^\dagger(x_{i_1})\cdots\hat{\psi}^\dagger(x_{i_N})|N\rangle$, given in terms of the gauge transformed operators with $|N\rangle$ being an arbitrary multiparticle state are then determined by the Lieb-Liniger Hamiltonian. Therefore, Kundu showed that a gauge transformation of the original field changes the Hamiltonian from a complicated interaction in terms of bosonic wave-functions, into an simpler Hamiltonian in terms of *anyonic wave-functions*, which have fractional statistics in the form

$$\Phi(x_1,...,x_j,...,x_i,...,x_N) = e^{i\kappa \sum_{k=i+1}^{j} sign(x_i-x_k) - \sum_{k=i+1}^{j-1} sign(x_j-x_k)} \Phi(x_1,...,x_i,...,x_j,...,x_N). \tag{2.74}$$

In this case the statistics is defined by the permutation of particle labels, as it is with other models of quantum statistics in one-dimension. However, since these wave-functions are not multi-valued, the particles are not transforming under a representation of the braid group, making them a completely different kind of anyon as the ones discussed in section 2.3.2

2.3.3.2 Statistics from deformed commutation relations

As we just saw with the Kundu model and with Green's ansatz for parastatistics in subsection 2.2.2, non-standard theories of quantum statistics also arise from creation operators obeying non-standard commutation relations. Such theories of quantum statistics can be roughly classified into three groups, which we discuss now. The first group is what can be called *quantum group particle statistics*, and are mostly related commutation relations for creation and annihilation operators that can represent quantum group symmetries, which were briefly discussed in subsection 2.3.2.

The best example of such particles are the q-deformed harmonic oscillator Biedenharn [26] and Macfarlane [174], given by the commutation relations

$$\hat{a}^*\hat{a} - q^{1/2}\hat{a}\hat{a}^* = q^{-\hat{N}/2}, \tag{2.75}$$

where $\hat{N} \neq \hat{a}^*\hat{a}$ acts as a number operator and $\hat{a}^*, \hat{a}$ are creation and annihilation operators respectively, but with $\hat{a}^* \neq \hat{a}^\dagger$. Using two uncoupled q-deformed oscillators $(\hat{a}_1^*, \hat{a}_1)$ and $(\hat{a}_2^*, \hat{a}_2)$, they have built the generators $\hat{J}_z, \hat{J}_+, \hat{J}_-$ of $SU_q(2)$ using the Schwinger representation

$$\hat{J}_z = \frac{1}{2}(\hat{a}_1\hat{a}_1^* - \hat{a}_2\hat{a}_2^*), \quad \hat{J}_+ = \hat{a}_1\hat{a}_2^*, \quad \hat{J}_- = \hat{a}_2\hat{a}_1^*. \tag{2.76}$$

that becomes the traditional Schwinger representation of angular momentum generators when $q = 1$. They have also discussed possible applications of these commutation relations as representing oscillators in a non-commutative phase space, or as variables in exactly solvable models.

The second group of statistical theories encompasses commutation relations broadly referred to as *quonic commutation relations*, due to its first example being Greenberg' quons [105], which were used to model the possibility for small violations of the Pauli principle. The existence of such violations was considered a possible solution to explain some problems in high energy physics at the time [104, 108–110, 134]. The commutation relations are

$$\hat{a}_k \hat{a}_l^\dagger - q \hat{a}_l^\dagger \hat{a}_k = \delta_{k,l}, \tag{2.77}$$

where $q \in \mathbb{R}$ with $-1 < q < 1$. This algebra is also called the *quon algebra.*

Greenberg then proceeds to show that a field theory quantized using these commutation relations would violate locality, but they would still uphold the CPT and clustering theorems, which meant that they could be used to build local non-relativistic field theories describing small violations of the Pauli principle. Other application of quons are as models for Brownian motion of particles in non-commutative spaces [34]. Some attempts at describing free quon gases were made, but they suffer from Gibbs' paradox [248].

The third are the commutation relations associated to fields describing anyons, both in one and two dimensions. This includes the commutation relations for field operators in the Kundu model, as well as other anyons arising from integrable models. However, two-dimensional anyons coming from the topological formalism of quantum statistics also give rise to non-trivial commutation relations for soliton and defect fields. The first example of these types of theories was provided [217], but to our purposes we study the one defined by Fradkin in [79–81].

Fradkin studied an abelian Chern-Simons field $\hat{A}_j(\mathbf{x})$ couple to a spinless fermion field $\hat{f}(\mathbf{x})$ on an arbitrary two-dimensional lattice L of points $\mathbf{x}$ with principal directions ϵ_i, with $i = 1, 2$. He showed that the Hamiltonian of these fields could be by mapped to the Hamiltonian of XY interactions between lattice sites by transforming the $\hat{f}(\mathbf{x})$ operators into the *anyonic operators*

$$\hat{a}(\mathbf{x}) = \exp\left\{ \frac{i}{\theta} \sum_{\mathbf{x}'} \Theta(\mathbf{x}, \mathbf{r}') \hat{f}^\dagger(\mathbf{x}') \hat{f}(\mathbf{x}') \right\} \hat{f}(\mathbf{x}), \tag{2.78a}$$

$$\hat{a}^\dagger(\mathbf{x}) = \exp\left\{ -\frac{i}{\theta} \sum_{\mathbf{x}'} \Theta(\mathbf{x}, \mathbf{r}') \hat{f}^\dagger(\mathbf{x}') \hat{f}(\mathbf{x}') \right\} \hat{f}^\dagger(\mathbf{x}), \tag{2.78b}$$

where $\Theta(\mathbf{x}, \mathbf{r}')$ is an angle variable between the vector $\mathbf{r}'$ living in the dual lattice to the vector $\mathbf{x}$ in the direct lattice, and θ is the statistical parameter for Chern-Simons.

This map from fermion to anyon operators is called a *Jordan-Wigner transformation*, and the anyon operators are bound to satisfy the commutation relations

$$\hat{a}(\mathbf{x}')\hat{a}^\dagger(\mathbf{x}) + e^{i\delta}\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{x}') = \delta_{\mathbf{x}',\mathbf{x}}, \tag{2.79}$$

where the phase δ is given by

$$\delta = \frac{1}{\theta}(\Theta(\mathbf{x}, \mathbf{r}') - \Theta(\mathbf{x}', \mathbf{r})) = \frac{1}{2\theta}. \tag{2.80}$$

These operators are multi-valued, and generate multi-valued anyonic wave-functions, typical of the topological formalism. This type of transformation was of the same type as the one used by Kundu to solve his model. Since statistical theories coming from the topological formalism have been thoroughly studied, there are many applications of this type of commutation relations, a few of which could be found in [70, 71, 125, 128, 203, 247]

The point, however, is that all of these three types of commutation relations can be studied without reference to any specific physical model or problem by studying the commutation relations themselves. As shown first by Van der Jeugt in [138], and later developed by Meljanac in [178], all of the previously mentioned commutation relations can be described by the family

$$\hat{a}_i\hat{a}_j - pR^{k,l}_{i,j}\hat{a}_k\hat{a}_l = 0 \tag{2.81a}$$

$$\hat{a}_i\hat{a}^*_j - p'R^{k,l}_{i,j}\hat{a}^*_k\hat{a}_l = \delta_{i,j} \tag{2.81b}$$

of commutation relations for N oscillators or modes, where the coefficients $R^{k.l}_{i,j}$ form the R-matrix $\mathbf{R}$ and, together with p, p', satisfy the equations

$$\sum_{u,v,w} R^{a,b}_{u,v}R^{v,w}_{c,d}R^{u,e}_{f,w} = \sum_{u,v,w} R^{b,e}_{u,v}R^{w,u}_{f,c}R^{a,v}_{w,d} \tag{2.81c}$$

$$(p\mathbf{PR} - 1)(p'\mathbf{PR} + 1) = 0 \tag{2.81d}$$

where $\mathbf{P}$ is the permutation operator with coefficients $P^{i,j}_{k,l} = \delta^i_l\delta^j_k$. This family is called the family of *braided commutation relations*, since they have a similar form to one of the defining equations of the braid group, represented in figure (9c).

However, not all possible Fock-space-representable commutation relations are contained in these braided families. The analysis of all possible relations with this property was made, first for single then for multi-mode oscillator modes, by Meljanac et al in [176, 177]. A multi-mode oscillator algebra generated by $\{\hat{a}^*_i, \hat{a}_i\}_{i=1,...,N}$ is Fock-space-representable if it satisfies

$$[\hat{n}_i;\hat{n}_j] = 0, \quad [\hat{n}_i;\hat{a}^*_j] = \delta_{i,j}\hat{a}^*_j, \quad [\hat{n}_i;\hat{a}_j] = -\delta_{i,j}\hat{a}_j, \quad [\hat{n}_i;\hat{a}^*_j\hat{a}_j] = [\hat{n}_i;\hat{a}_j\hat{a}^*_j] = 0 \tag{2.82}$$

where the set of operators $\{\hat{n}_i|i \in \{1,...,d\}\}$ are functions of the generators. They have showed that all algebras with this property, together with norm positivity conditions, have a normal form and can be described by a finite groups of parameters, where each group may contain infinite parameters. Therefore, in terms of a field theory of quantum statistics, all such algebras describe a sensible type of non-standard quantum statistics.

2.4 Conclusion and the definition of non-standard statistics used in this thesis

In conclusion, in the early period of non-standard quantum statistics, discussed in section 2.2, the statistical theories were not able to describe true forms of non-standard statistics. This means that either the models were in principle incompatible with the postulates of quantum mechanics, or were equivalent to either fermions or bosons. However in the modern period, which was discussed in section 2.3, true forms of non-standard statistics were indeed developed. Most of them can be directly related to the work of Leinaas and Myrheim, discussed in subsection 2.3.1. From it, the statistical theories get divided into those related to the quantization of classical identical particle systems in two-dimensions, giving origin to the topological trend, and those to the quantization of identical particles systems in one dimension, generating most of the other interesting statistical theories.

The theories of statistics in the topological trend, discussed in subsection 2.3.2, became inextricably related with gauge theory. Due to this fact, the concept of quantum statistic itself became a type of gauge interaction, that can confer statistics to all types of physical objects studied in quantum theory, most notably solitons and other extended objects. From these models two general classes of non-standard types of quantum statistics have flourished. The abelian and non-abelian anyons, which are particle or particle-like quantum objects that are described by multi-valued functions transformed by either abelian, or non-abelian irreducible representations of the braid groups when changed by a physical permutation. The non-abelian anyon case is peculiar, in the sense that the notion of particle identity depends on the local or non-local nature of the particle interactions.

For the theories in the non-topological trend, discussed in subsection 2.3.3, many can be seen as standard particles acquiring non-trivial exchange statistics via the action of a type of statistical interaction. However, they are anyons in the same sense as the ones related to the topological trend, either because the associated interactions are not gauge interactions, or because the multi-particle wave-functions exchange factors did not come from the braid group. However, all models considered here can still be defined in terms of multi-mode oscillator algebras, either discrete of continuous.

Having finished the general description of all relevant models of non-standard quantum statistics in the last section, I can proceed to discuss the type of statistics that will be studied in this thesis from an information and computation theory perspective. This theory is defined in terms of multi-mode oscillator algebras, and form two distinct families of statistical models.

We call the first family *fermionic anyons*, they are described by the algebra

$$\hat{\xi}_i\hat{\xi}_j^\dagger + e^{-i\varphi\epsilon_{i,j}}\hat{\xi}_j^\dagger\hat{\xi}_i = \delta_{ij}, \tag{2.83a}$$

$$\hat{\xi}_i\hat{\xi}_j + e^{i\varphi\epsilon_{i,j}}\hat{\xi}_j\hat{\xi}_i = 0, \tag{2.83b}$$

$$\hat{\xi}_i^\dagger\hat{\xi}_j^\dagger + e^{i\varphi\epsilon_{i,j}}\hat{\xi}_j^\dagger\hat{\xi}_i^\dagger = 0 \tag{2.83c}$$

where the i, j labels the lattice sites, $\varphi \in [0, 2\pi]$ is called the *statistical parameter*, and $\epsilon_{i,j}$

is the sign of $i-j$. When $\varphi=0$ the commutation relations become the one for standard fermions, given in eq.(2.29).

The operators for fermionic anyons are related to operators by standard fermions via the Jordan-Wigner transform

$$\hat{f}_i^\dagger \overset{J_\varphi}{\to} \hat{\xi}_i^\dagger = \exp\left\{-i\varphi\sum_{k=1}^{i-1}\hat{f}_k^\dagger\hat{f}_k\right\}\hat{f}_i^\dagger \tag{2.84a}$$

$$\hat{f}_i \overset{J_\varphi}{\to} \hat{\xi}^i = \exp\left\{i\varphi\sum_{k=1}^{i-1}\hat{f}_k^\dagger\hat{f}_k\right\}\hat{f}_i. \tag{2.84b}$$

Using the fact that J_φ is an algebra homomorphism, or in other words $J_\varphi(ab)=J_\varphi(a)J_\varphi(b)$, we can deduce that $J_\varphi(\hat{f}_i^\dagger\hat{f}_i)=\hat{\xi}_i^\dagger\hat{\xi}_i$ which implies that the number operators $\hat{n}_i=\hat{\xi}_i^\dagger\hat{\xi}_i$ for fermionic anyons have the same form as the ones for standard fermions.

The Fock space for fermionic anyons is built from a vacuum state $|0\rangle_\xi$ being acted upon by of creation operators, implying the Fock basis states have the form

$$|n_1,...,n_m\rangle_\xi = (\hat{\xi}_1^\dagger)^{n_1}...(\hat{\xi}_m^\dagger)^{n_m}|0\rangle_\xi\,. \tag{2.85}$$

Using fermionic anyon commutation relations, we can deduce that the action of creation and annihilation operators over the Fock basis is

$$\begin{cases}\hat{\xi}_i\,|n_1,...,0,...,n_m\rangle_\xi &= 0\\ \hat{\xi}_i\,|n_1,...,1,...,n_m\rangle_\xi &= \exp\left\{-i\varphi\sum_{k=1}^{i-1}n_k\right\}|n_1,...,0,...,n_m\rangle_\xi\end{cases} \tag{2.86a}$$

$$\begin{cases}\hat{\xi}_i^\dagger\,|n_1,...,0,...,n_m\rangle_\xi &= \exp\left\{i\varphi\sum_{k=1}^{i-1}n_k\right\}|n_1,...,1,...,n_m\rangle_\xi\\ \hat{\xi}_i^\dagger\,|n_1,...,1,...,n_m\rangle_\xi &= 0\end{cases} \tag{2.86b}$$

The model for particles in one-dimensional lattice was defined for the first time by Amico in [7], inspired by the discussion on one-dimensional fractional statistics done for the Calogero model in [117], and in the results of [100]. He showed that a Hubbard Hamiltonian for such particles was exactly solvable, and studied other deformed models in [190–193]. In the limit of $\varphi=\pi$, this Hamiltonian models a gas called a *hard-core boson* gas, which is equivalent to a Lieb-Lineger gas with $c\to\infty$ (see section 2.3.3.1).

The second family is called *bosonic anyons*, and they are described by the algebra

$$\hat{\beta}_i\hat{\beta}_j^\dagger - e^{-i\varphi\epsilon_{i,j}}\hat{\beta}_j^\dagger\hat{\beta}_i = \delta_{ij}, \tag{2.87a}$$

$$\hat{\beta}_i\hat{\beta}_j - e^{i\varphi\epsilon_{i,j}}\hat{\beta}_j\hat{\beta}_i = 0, \tag{2.87b}$$

$$\hat{\beta}_i^\dagger\hat{\beta}_j^\dagger - e^{i\varphi\epsilon_{i,j}}\hat{\beta}_j^\dagger\hat{\beta}_i^\dagger = 0, \tag{2.87c}$$

with φ and $\epsilon_{i,j}$ being the same as in the fermionic anyon case

The standard and anyonic boson operators are related by the Jordan-Wigner map

$$\hat{b}_i^\dagger \overset{J_\varphi}{\to} \hat{\beta}_i^\dagger = \exp\left\{-i\varphi \sum_{k=1}^{i-1} \hat{b}_k^\dagger \hat{b}_k\right\}\hat{b}_i^\dagger \tag{2.88a}$$

$$\hat{b}_i \overset{J_\varphi}{\to} \hat{\beta}_i = \exp\left\{i\varphi \sum_{k=1}^{i-1} \hat{b}_k^\dagger \hat{b}_k\right\}\hat{b}_i, \tag{2.88b}$$

as with fermionic anyons, J_φ is an algebra homomorphism, implying that $J_\varphi(\hat{b}_i^\dagger \hat{b}_i) = \hat{\beta}_i^\dagger \hat{\beta}_i$. Therefore, the number operators $\hat{n}_i = \hat{\beta}_i^\dagger \hat{\beta}_i$ have the same form as their standard boson counterpart.

This Fock space for bosonic anyons is built from a vacuum state $|0\rangle_\beta$, being acted upon by creation operators, implying that the Fock basis states have the form

$$|n_1, ..., n_m\rangle_\beta = \frac{(\hat{\beta}_1^\dagger)^{n_1} ... (\hat{\beta}_m^\dagger)^{n_m}}{\sqrt{n_1! ... n_m!}} |0\rangle_\beta\,, \tag{2.89}$$

Using bosonic anyon commutation relations, we can deduce that the action of creation and annihilation operators in the Fock basis states is given by

$$\hat{\beta}_i |n_1, ..., n_m\rangle_\beta = \exp\left\{i\varphi \sum_{k=1}^{i-1} n_k\right\}\sqrt{n_i}\, |n_1, ..., n_i - 1, ..., n_m\rangle_\beta\,, \tag{2.90a}$$

$$\hat{\beta}_i^\dagger |n_1, ..., n_m\rangle_\beta = \exp\left\{-i\varphi \sum_{k=1}^{i-1} n_k\right\}\sqrt{n_i + 1}\, |n_1, ..., n_i + 1, ..., n_m\rangle_\beta\,. \tag{2.90b}$$

This model is essentially a discrete version of the Kundu commutation relations for field operators in eq.(2.73). They were further studied by Batchelor et al in [20–22], as well as by Calabrese et al in [41, 42], after the first experimental synthesis of an strongly interacting ultracold gas with non-standard statistics was realized by Paredes [194] in an optical lattice setting.

Renewed interest in both fermionic and bosonic anyons developed after Keilmann [143] demonstrated that the variation of the anyonic statistical phase for bosonic anyons under a Hubbard interaction induced a phase transition. Lattice anyon models has been an active research topic by the ultracold gas/optical lattice community ever since [10, 28, 46, 66, 112–114, 119–121, 123, 126, 164, 171, 234].

From now on, unless otherwise stated, the terms "fermionic anyons" and "bosonic anyons" will refer specifically to these one-dimensional lattice anyon models. Anyons from the topological trend, or other non-standard theories of quantum statistics, will not be mentioned again. A detailed study of relevant Hamiltonian dynamics over these systems will be done in chapter 4, and are a part of this thesis' results.

3 Review: The computing power of standard quantum statistics

After discussing the concepts necessary for understanding the anyon models studied in this thesis, the last topic I need to address before proceeding is quantum computing and information theory. In this chapter I aim to introduce the basics of quantum information, and quantum computation with standard identical particles. In contrast to the last chapter, here I give an expository tone without aiming for historical continuity.

The organization of this chapter is as follows. In section 3.1, I introduce qubits as a quantum analog of classical bits. Next, I define the circuit model of quantum computing, studying quantum gates and their properties. I finish this section with a discussion on quantum entanglement and its role as a resource in the circuit model, showing how to define and measure the amount of entanglement produced by two-qubit gates.

In section 3.2, I describe the optical network formalism and its use in quantum computing with standard identical particles. First, I do a brief review of quantum interferometry of light to motivate the modeling of optical devices, such as beam-splitters and phase shifters, in terms of multi-mode, bosonic oscillator algebras. Then, I discuss optical networks and the ways they are used to represent quantum circuits. I finish this section with the generalization of optical networks to systems of identical fermions and their properties.

I finish this chapter in section 3.3, where I prove the equivalence between the quantum circuit model and the optical network model for hard-core bosons. First, I show how to describe qubits by an oscillator algebra, giving them a particle representation. Then, I generalize the optical network model to this algebra and provide an alternate description of quantum circuits. I finish this section by representing a family of quantum circuits called matchgates and discuss their relation with optical networks for fermionic oscillators.

3.1 Introduction to quantum computing in the circuit model

This goal of this section is to be a review on the basics of quantum information and computation theory. Almost every statement mentioned here can be found in reference books about these subjects, such as [148, 185, 254]. In subsection 3.1.1, I recall the basic concepts of classical information theory in order to introduce quantum information theory as its quantum analogue. Then, I use this analogy to motivate the definition of qubits and gates, comparing them to classical bits and their transformations. I finish by laying out an exposition of the most useful properties of single-qubit and two-qubit unitary gates.

In subsection 3.1.2, I define the circuit model of quantum computing proper. First, I give a definition of reversible classical circuits, and introduce the concept of universal gates. Then, I explain quantum circuits as a quantum analog of reversible classical circuits, and proceed to define the circuit model of quantum computing. To finish this discussion, I introduce the problem of approximating arbitrary single-qubit maps and the Solovay-Kitaev theorem.

Finally, in subsection 3.1.3, I review the density operator formalism, quantum entanglement, and their role in deciding the universality of gate sets. First, I describe the density operator as a model for classical ensembles of quantum states, and show the difference between pure and mixed states. Next, I use the definition of partial trace, and the description of subsystems to differentiate between product, separable and entangled states. Finally, I show how to measure the amount of entanglement in a two-qubit state and the entangling power of a two-qubit gate, which is related to the problem of deciding the universality of a gate set.

3.1.1 Qubits and gates

Here I argue, in an informal way, that quantum information theory is the quantum analogue of classical information theory. This is not a new idea, and it has been used as a way to highlight the differences and incompatibilities between them. In order to make sense of the analogy, we must first see classical information theory as a theory about physical processes, described by classical mechanics. Then, we need to see in what sense this theory can (or cannot) be quantized.

In order to use physical systems to process information, we must be able to describe them as reliably as possible, otherwise we can not manipulate information in a controllable way. This necessity for reliable description, contrary to intuition, does not create the need for restricting the set of computationally useful physical systems to only deterministic ones. In fact, probabilistic descriptions of a system's behavior can count as reliable, if the system outputs values according to a given probability distribution in a consistent way.

For us, the main prerequisite for a classical system to be considered *suitable for information processing* is that the system is closed and controllable. Controllable means that the system is characterized by measurable parameters with values that can be changed according to well defined processes. And closed means that these processes do not occur unintentionally. Information theory requires that we divide the phase space of such systems into some finite number of sub-spaces, and map them into a new, discrete phase-space. These sub-divisions may overlap into each other and, in the case they overlap, an approximation scheme is necessary for the map to be valid. This scheme is called the *digital approximation*, and is an integral component of the design of digital circuits, for example.

Discrete phase spaces, then, model classical systems suitable for information processing. For example, the space $\mathbb{B} = \{0, 1\}$, called a *classical bit*, represents a classical system whose phase space can be approximately divided into two representative sets, labeled by the symbols 0 and 1. This is the most fundamental of the discrete phase spaces, because it entails a mechanical representation of a *Boolean logical variable*.

Boolean variables are symbols standing for values in a *Boolean ring*. This ring is generated by two elements, 0 and 1, and has two operations called the *Boolean sum* (written using $\oplus$) and the *Boolean product* (written using $\cdot$)[1], represented in the tables:

A_1	A_2	$A_1 \oplus A_2$
0	0	0
0	1	1
1	0	1
1	1	0

A_1	A_2	$A_1 \cdot A_2$
0	0	0
0	1	0
1	0	0
1	1	1

Table 1 – Boolean operations

where the variables A_1 and A_2 are Boolean. Boolean variables and operations are an algebraic way to represent propositional logic, where logical sentences and deductions are translated as polynomials[2].

A physical system whose phase-space is given by the N-fold Cartesian product $\mathbb{B}^N$ is a physical representation of a system with N Boolean variables. This phase space is called an N-bit system, and its coordinates are the N-tuples $(A_1, \dots, A_N)$ called *bit strings of length N*, or N-bit strings. Bit strings are also represented as sequences of symbols without in-between spaces $A_1 \cdots A_N$. An N-bit system can hold the values used to evaluate Boolean expressions and, therefore, can be used to do propositional calculus. The conclusion is that, if we can automate the physical processes that maps N-bit strings into each other, we can automate formal reasoning, at least to the extent covered by propositional logic.

Logical operations on Boolean variables are induced by classical dynamical maps on the underlying physical system. Logical operations, then, are modelled by functions $f : \mathbb{B}^N \to \mathbb{B}^M$ from N-bit strings to M-bit strings. For example, when $N = M = 1$, there are four such functions, given by

$$Id(A) = A, \quad c_0(A) = 0, \quad c_1(A) = 1, \quad \mathrm{NOT}(A) = A \oplus 1, \tag{3.1}$$

1 The Boolean ring is isomorphic to the ring of integers modulo 2, where the Boolean sum and product are isomorphic to sum and multiplication modulo 2.

2 Boolean variables form a polynomial ring under Boolean sums and products.

where the first is the identity function, the next two are constant functions, and the last one is the transposition, or negation (NOT) function.

The identity and the negation have inverses, and for that reason are called *reversible functions*, while functions that do not have inverses, such as the constant functions, are called *irreversible*. The set R_N of reversible Boolean maps $f : \mathbb{B}^N \to \mathbb{B}^N$ on N-bit strings is isomorphic to the permutation group S^{2^N}, since the number of distinct N-bit strings is 2^N. The most important examples of reversible Boolean functions are the SWAP, CNOT and the TOFFOLI functions, given by the formulas

$$\mathrm{SWAP}(A_1, A_2) = (A_2, A_1) \tag{3.2a}$$

$$\mathrm{CNOT}(A_1, A_2) = (A_1, A_1 \oplus A_2), \tag{3.2b}$$

$$\mathrm{TOFFOLI}(A_1, A_2, A_3) = (A_1, A_2, [A_1 \cdot A_2] \oplus A_3). \tag{3.2c}$$

The physical model of Boolean algebras and reversible Boolean maps forms the basis of the physical analysis of information-processing tasks.

We are now we in position to state what we mean by a quantum analogue. Given a discrete phase space, we can take its points as basis of an Hilbert space, and define a quantum state space. For example, we can take the classical bit $\mathbb{B} = \{0, 1\}$, and define the Hilbert space $\mathscr{B}$, generated by the basis elements $|0\rangle, |1\rangle$. This Hilbert space is called a *qubit*, or quantum bit, and a general state is given by the linear combination

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \tag{3.3}$$

where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$.

From a classical phase space of two points we arrive at a continuous Hilbert space with a particular geometry. To see this geometry, notice that the normalization condition and the global phase invariance of quantum states allow us to describe any single-qubit state using only two real parameters, called θ, φ, by the formula

$$|\theta, \varphi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\,. \tag{3.4}$$

The two angles give coordinates for a two-dimensional sphere, called the *Bloch sphere*.

The structure of the Bloch sphere is represented in figure Fig. 10. This parameterization for single-qubit states might suggest we interpret the pair of angles (θ, φ) as the analogues of Boolean variables. This seems intuitive, mostly because Boolean variables are, essentially, coordinate functions for a discrete phase space. However, Boolean variables form a ring structure, and this ring structure does not easily translate into an algebraic structure for quantum states, as we will see shortly.

Let us introduce the quantum state variable $|A\rangle$, where A is a Boolean variable. Given the Boolean sum $A_1 \oplus A_2$ and the Boolean product $A_1 A_2$, one can define the quantum state variables $|A_1 \oplus A_2\rangle$ and $|A_1 A_2\rangle$, without any problems. However, trying to create operations such as $|A_1\rangle \oplus |A_2\rangle \equiv |A_1 \oplus A_2\rangle$ and $|A_1\rangle \cdot |A_2\rangle \equiv |A_1 \cdot A_2\rangle$ fails, as they are always incompatible with the state normalization. Therefore, the quantum analogy between bits and qubits is not a formal quantization procedure.

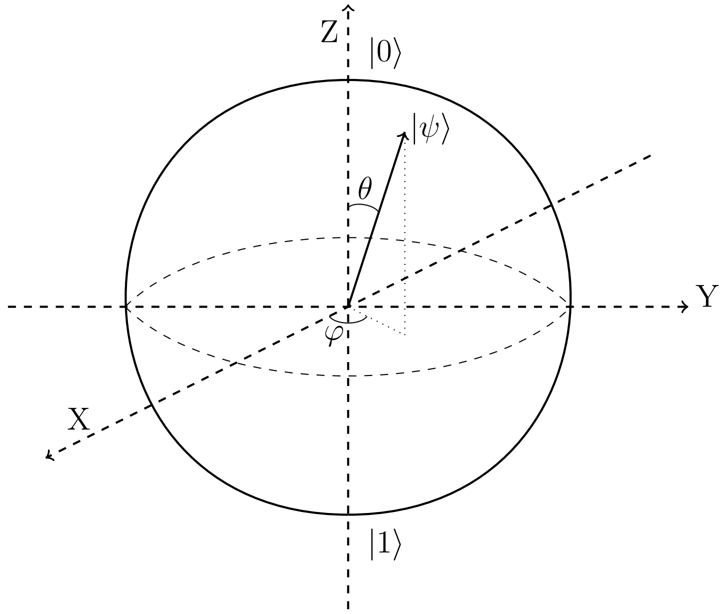


Figure 10 – Bloch sphere: The qubit basis states $|0\rangle$ and $|1\rangle$ are the north and south poles respectively, fixing the direction of the Z axis. The angles θ, φ are the typical angles of the spherical coordinate system in $\mathbb{R}^3$.

With this in mind, let us push the quantum analogy further, and define the Hilbert space $\mathscr{B}^{\otimes N}$, given by the N-fold tensor product of $\mathscr{B}$, as the analogue of the N-bit phase space $\mathbb{B}^N$. Since a general point in $\mathbb{B}^N$ has as its coordinates the string variable $A_1 \cdots A_N$, a general basis state of $\mathscr{B}^{\otimes N}$ is parametrized by

$$|(A_1, \ldots, A_N)\rangle \equiv |A_1 \cdots A_N\rangle \equiv \bigotimes_{i=1}^{N} |A_i\rangle \,. \tag{3.5}$$

This basis is called the *computational basis*, and plays an important role in the theory of quantum computing in general.

An interesting feature of the quantum analogy in the multi-qubit case is seen by defining the concatenation operation $\circ$, on Boolean strings $\mathbf{A}, \mathbf{B}$ of size N and M. This operation is such that $\mathbf{A} \circ \mathbf{B} = A_1 \cdots A_N B_1 \cdots B_M$ and it implies, using the definition of the computational basis, that

$$|\mathbf{A} \circ \mathbf{B}\rangle = |A_1 \cdots A_N B_1 \cdots B_M\rangle = |\mathbf{A}\rangle \otimes |\mathbf{B}\rangle \,. \tag{3.6}$$

Since the tensor product is compatible with a Hilbert space structure and is essentially unique[3], it is the most natural choice of structure to replace the concatenation $\circ$ in a

[3] The tensor product is universal [6] in the category of Hilbert spaces, and this makes it the unique natural analogue of concatenation.

quantization procedure. Therefore, the failure of quantization happens precisely at the single-bit level.

In other to obtain a better understanding on the lack of a natural quantization procedure from classical to quantum bits, we must discuss the structure of quantum bit maps, called *gates*. As we saw before, single-bit maps $f : \mathbb{B} \to \mathbb{B}$ can be either reversible or irreversible. The reversible subset has the structure of the permutation group S^2, which is generated by the NOT function. In the quantum case however the set of all maps $U : \mathscr{B} \to \mathscr{B}$ must preserve the Hilbert space structure of $\mathscr{B}$. This implies that any map U must be unitary, and the set of all such maps has a group structure, given by $SU(2)$. In other words, all qubit maps are necessarily reversible[4].

Now let us proceed to examples. The simplest single-qubit map is the quantum version of the NOT gate, which is called the X gate, and is given by

$$X |A\rangle = |A \oplus 1\rangle . \tag{3.7}$$

In general, any reversible Boolean map has a quantum version, obtained by exchanging their action over Boolean variables with an action over quantum state variables. This gives us, for example, the two-qubit and three-qubit gates

$$\mathrm{SWAP} |A_1 A_2\rangle = |A_2 A_1\rangle \tag{3.8a}$$

$$\mathrm{CNOT} |A_1 A_2\rangle = |A_1\rangle \otimes |A_1 \oplus A_2\rangle , \tag{3.8b}$$

$$\mathrm{TOFFOLI} |A_1 A_2 A_3\rangle = |A_1 A_2\rangle \otimes |[A_1 \cdot A_2] \oplus A_3\rangle . \tag{3.8c}$$

However, almost all single-qubit maps have no classical reversible Boolean function as an analogue. The simplest example of these non-classical maps is the Z operator, defined by

$$Z |A\rangle = (-1)^A |A\rangle . \tag{3.9}$$

The X and Z operators, together with the $Y = iXZ$, satisfy $X^2 = Y^2 = Z^2 = I$, where I is the identity operator over $\mathscr{B}$. They also satisfy the equations $[J_i; J_j] = \epsilon_{i,j,k} J_i$ with $J_1 = X$, $J_2 = Y$ and $J_3 = Z$, implying that they are a vector space basis of the Lie algebra of angular momentum, in the spin-1/2 representation. The X, Y and Z maps are called *Pauli operators*, since their matrix representations are the Pauli matrices given in eq. (1.1).

These three maps, the X, Y and Z operators, are associated to the X, Y and Z axes in the Bloch sphere of Fig. 10, in the sense that the intersection points of each axis with the sphere are the eigenvectors of the corresponding operator. If the eigenstate is obtained from the positive part of its respective axis, it will have eigenvalue 1, otherwise it will have eigenvalue -1. The positive and negative eigenstates of Z are the basis states $|0\rangle$

[4] For closed systems.

and $|1\rangle$ respectively, while for X and Y they are given in terms of Bloch sphere coordinates by

$$|+\rangle_x = \left|\frac{\pi}{2}, 0\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle_x = \left|\frac{\pi}{2}, \pi\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \text{ for } X \tag{3.10}$$

$$|+\rangle_y = \left|\frac{\pi}{2}, \frac{\pi}{2}\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \quad |-\rangle_y = \left|\frac{\pi}{2}, -\frac{\pi}{2}\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle), \text{ for } Y. \tag{3.11}$$

The identification of the Pauli operators with axes in Bloch sphere allows us to give a simple geometric representation of single-qubit operators. First, consider the operators

$$R_X(\alpha) = \exp\left\{-i\frac{\alpha}{2}X\right\} = \cos\left(\frac{\alpha}{2}\right)I - i\sin\left(\frac{\alpha}{2}\right)X, \tag{3.12a}$$

$$R_Y(\alpha) = \exp\left\{-i\frac{\alpha}{2}Y\right\} = \cos\left(\frac{\alpha}{2}\right)I - i\sin\left(\frac{\alpha}{2}\right)Y, \tag{3.12b}$$

$$R_Z(\alpha) = \exp\left\{-i\frac{\alpha}{2}Z\right\} = \cos\left(\frac{\alpha}{2}\right)I - i\sin\left(\frac{\alpha}{2}\right)Z. \tag{3.12c}$$

They act as rotations on the (θ, φ) angles, and are suitably called *rotation operators* with *rotation angle* α around the X, Y and Z axis respectively.

Since X, Y and Z are generators of rotations in the Bloch sphere, an arbitrary operator U can be written as

$$U = R_{\mathbf{n}}(\alpha) = \exp\left\{-i\frac{\alpha}{2}(n_x X + n_y Y + n_z Z)\right\}, \tag{3.13}$$

where the triple $\mathbf{n} = (n_x, n_y, n_z)$, satisfying $|n_x|^2 + |n_y|^2 + |n_z|^2 = 1$, is the *rotation axis*. However, it is more convenient to write these transformations as decompositions of rotations along the X, Y and Z axis. One of these decompositions is given by finding angles α, β and γ such that

$$U = R_{\mathbf{n}}(\alpha)R_{\mathbf{m}}(\beta)R_{\mathbf{n}}(\gamma), \tag{3.14}$$

where $\mathbf{n}, \mathbf{m}$ can be any pair of non-parallel axis in the Bloch sphere.

This characterization of single-qubit unitaries as rotations allows us to think about them as rotations in a more general sense, but in order to see this, we must first discuss more about Pauli operators. Pauli operators are also known to satisfy the anticommutation relations

$$\{\hat{\gamma}_i; \hat{\gamma}_j\} = 2\delta_{i,j}I, \tag{3.15}$$

if we take $\hat{\gamma}_0 = I$ $\hat{\gamma}_1 = X$, $\hat{\gamma}_2 = -iY$ and $\hat{\gamma}_3 = Z$. These relations define what is called a *Clifford algebra* [257], labelled by **Cliff**(2), where the number 2 refers to the number of algebraic generators (when we also consider products between vectors), given by X and Z.

We can use the Clifford algebra structure of Pauli operators to see single-qubit gates as special algebra elements K, acting as a group over the algebra itself by the *conjugate action*

$$K * \mathbf{v} \equiv K\mathbf{v}K^\dagger, \tag{3.16}$$

where **v** is any complex linear combination of Pauli operators. This group is called the *spin group*, or *Spin(3)* (because there are three Pauli operators), and it is isomorphic to $SU(2)$.

For us, the point of introducing Clifford algebras is that they are the quantization of a structure known as the *Grassmann algebra* [257]. This Grassmann algebra is used to define the classical mechanics of fermions, and it is not equivalent to a Boolean algebra. It is this fact that explains why qubits are not quantized classical bits, in the formal sense.

An important sub-group of *Spin(3)* is the *single-qubit Clifford group*, labelled by $Cl(1)$, given by the set of unitaries that preserve the Pauli operators under conjugate action. In other words, given the set of Pauli operators $\{X, Y, Z\}$, $K \in Cl(1)$ if and only if $K\{X, Y, Z\}K^\dagger = \{X, Y, Z\}$. The Clifford group is also the *normalizer*[5] of the *Pauli group* **P**, which is the group formed by all products of single-qubit Pauli operators.

The $Cl(1)$ group is generated the *Hadamard gate* H, and the *phase gate* S, which have matrix representations given in eq. (1.2), and is the symmetry group of the octahedron of eigenstates of Pauli operators in Fig. 11. This group has many applications in quantum computing and quantum error correction, as well as having an important role in measurement based quantum computing. The last important single-qubit gate example for us is the $\pi/8$ gate, also represented by T, with matrix given in eq. (1.2). The role of this gate is discussed in the next subsection.

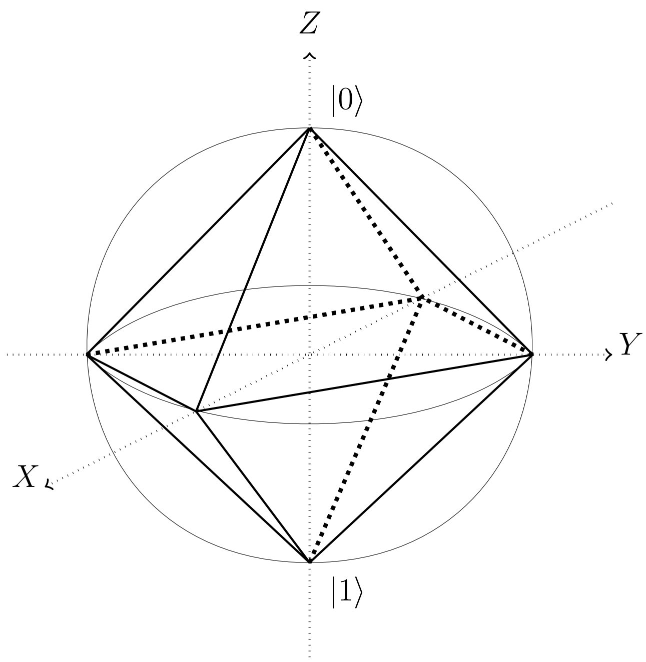


Figure 11 – Octahedron preserved by the single-qubit Clifford group. The vertices are eigenstates of Pauli operators.

Now we proceed to discuss multi-qubit gates. As we saw earlier, all reversible

[5] The normalizer of a subset S of group G is the set of elements $\{g \in G | gS = Sg\}$.

Boolean maps have direct quantum analogues. Just as in the single-qubit case, all N-qubit maps are reversible due to unitarity, and form the unitary group $SU(2^N)$. Before talking about the N-qubit gates, let us describe a special family of reversible Boolean maps.

Controlled gates $\Lambda^P(\mathrm{G})$, are N-bit reversible maps conditioned by value of the product of P bits, called *control bits*, which act non-trivially over *target bits* Q, according to the rule

$$\Lambda^P(\mathrm{G})(A_1,\ldots,A_N) = \begin{cases} (A_1,\ldots,A_N), \text{ if } \prod_{i=1}^P A_i = 0 \\ (A_1,\ldots,A_P, G(A_{P+1},\ldots,A_N)), \text{ if } \prod_{i=1}^P A_i = 1 \end{cases}, \tag{3.17}$$

where $G : \mathbb{B}^Q \to \mathbb{B}^Q$ is a reversible map on Q-bit strings, and $P+Q = N$. Both the CNOT and TOFFOLI are controlled gates, given by $\Lambda(\mathrm{NOT})$ and $\Lambda^2(\mathrm{NOT})$, respectively.

Controlled gates can be generalized to the quantum setting by exchanging Boolean variables for quantum states variables in the definition. In other words, a controlled quantum gate $\Lambda^P(G)$ is an N-qubit unitary of the form

$$\Lambda^P(\mathrm{G})\left|A_1\cdots A_N\right\rangle = \begin{cases} \left|A_1\cdots A_N\right\rangle, \text{ if } \prod_{i=1}^P A_i = 0 \\ \left|A_1\cdots A_P\right\rangle \otimes G(\left|A_{P+1}\cdots A_N\right\rangle), \text{ if } \prod_{i=1}^P A_i = 1 \end{cases}. \tag{3.18}$$

One of the simpler types of controlled quantum gates are the two-qubit *controlled rotation gates* $\Lambda(R_{\mathbf{n}}(\alpha))$. These are defined by

$$\Lambda(R_{\mathbf{n}}(\alpha))\left|A_1\right\rangle \otimes \left|A_2\right\rangle = \left|A_1\right\rangle \otimes (R_{\mathbf{n}}(\alpha))^{A_1}\left|A_2\right\rangle, \tag{3.19}$$

and have the $\Lambda(X)$(=CNOT) and $\Lambda(Z)$(=CZ) gates as the most important examples.

To sum up, here we established the basic concepts of quantum information theory as quantum analogues of classical concepts. We proved that this analogy cannot be extended to a formal quantization procedure due to the non-equivalence of Grassmann and the Boolean algebras. We analyzed the structure of single-qubit gates and presented some examples of two-qubit and three-qubit gates.

3.1.2 Classical and quantum circuits

Here, I present quantum circuits as the quantum analogue of classical reversible Boolean circuits. Instead of discussing the formal definitions of computational problems and tasks in the circuit models, my intent is to give an account of which *ingredients*, or *primitives* we need for circuits in order to express these problems in the first place. Therefore, I will not discuss classical, or quantum computability and complexity theory, and restrict myself to the subjects of circuit expressiveness, synthesis and cost analysis.

In order to define what is a classical circuit, first we need to extend the action of reversible maps over a small number of bits into a larger number of bits. This is done in the following way. Given an arbitrary N-bit string $A_1 \cdots A_N$, a reversible Boolean map f over M-bits, with $M < N$, is said to act as a map in $\mathbb{B}^N$ if it acts as f over an M-sized subset of the N bits, while acting as the identity over the remaining $N - M$ bits.

For example, the 2-bit CNOT map can act as the 7-bit map $\mathrm{CNOT}_{3,5}$, given by

$$\mathrm{CNOT}_{3,5}(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = (A_1, A_2, A_3, A_4, A_3 \oplus A_5, A_6, A_7), \qquad (3.20)$$

where bit 3 is the control bit and bit 5 is the target bit. Similarly, we can invert the control and target bits in $\mathrm{CNOT}_{3,5}$, obtaining the map $\mathrm{CNOT}_{5,3}$, given by

$$\mathrm{CNOT}_{5,3}(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = (A_1, A_2, A_5 \oplus A_3, A_4, A_5, A_6, A_7). \qquad (3.21)$$

As a last example, consider the following three 7-bit versions of the 3-bit TOFFOLI gate

$$\mathrm{TOFFOLI}_{2,4,5}(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = (A_1, A_2, A_3, A_4, [A_2 A_4] \oplus A_5, A_6, A_7) \quad (3.22)$$
$$\mathrm{TOFFOLI}_{3,1,2}(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = (A_1, [A_3 A_1] \oplus A_2, A_3, A_4, A_5, A_6, A_7) \quad (3.23)$$
$$\mathrm{TOFFOLI}_{5,7,1}(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = ([A_5 A_7] \oplus A_1, A_2, A_3, A_4, A_5, A_6, A_7) \quad (3.24)$$

In general $G_{i_1,\ldots,i_M}$ is an N-bit version of the the M-bit maps G, acting over the ordered subset of bits labelled by the i_j indices.

Using this construction we can define a *classical reversible circuit* W as the function composition

$$W \equiv G^L_{I_L} \circ \cdots \circ G^1_{I_1}, \qquad (3.25)$$

where $G^i_{I_i}$ are N-bit maps such that the sets I_i, with $i = 1, \ldots, L$, are ordered subsets of $|I_i| < N$ bits, and the G^i are $|I_i|$-bit maps. The number L is the called the *circuit size.* Any sequence of two maps, $G^i_{I_i}$ and $G^{i+1}_{I_{i+1}}$, of a circuit W is called *parallel* if and only if $I_i \cap I_{i+1} = \emptyset$. Sub-sequences of parallel gates form the *circuit layers* of W, and the number of layers is called its *depth* (d). The *width* (w) of a circuit is the number of bits over which W acts non-trivially. An example of circuit is given in fig. 12, it has size 9, depth 5 and width 5.

The main advantage of expressing reversible maps as circuits is to exhibit them as compositions of gates acting over fewer bits. This implies the possibility of implementing any N-bit reversible map by using a small, fixed set of M-bit maps, for some values of M, that can be executed reliably in the physical system of interest. An M-bit map G is called *universal* if and only if any N-bit map can be written as a circuit using only the N-bit versions of G, for every $N \in \mathbb{N}$. This is equivalent to saying that an M-bit map G is universal if and only if its N-bit versions generate the permutation group S^{2^N}, for every $N \in \mathbb{N}$.

In [244], there is a characterization theorem for reversible universal circuits. It says that any map that is not *affine-linear* is universal for reversible circuits. An M-bit map

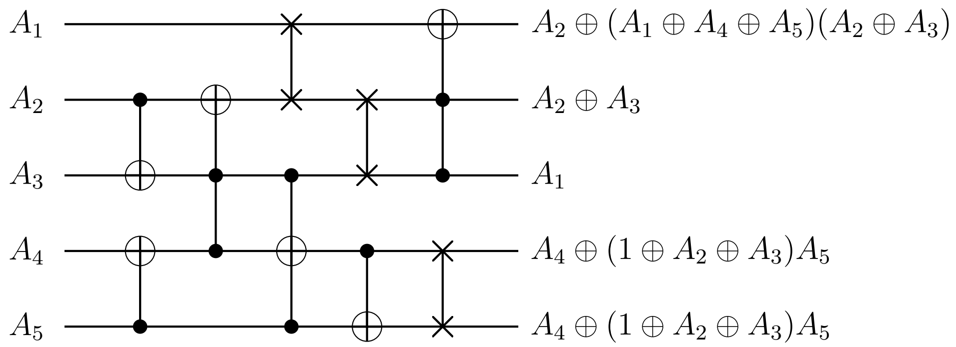


Figure 12 – Example of reversible classical circuit. Input Boolean variables are on the left end, and the output is on the right end. This circuit contains 3 CNOT gates, 3 SWAP gates and 3 TOFFOLI gates, with different arrangements of control and target bits.

is affine-linear if its action over M-bit strings can be put in the form

$$\begin{bmatrix} A'_1 \\ \vdots \\ A'_n \end{bmatrix} = \begin{bmatrix} c_0^1 \\ \vdots \\ c_0^n \end{bmatrix} \oplus \begin{bmatrix} c_1^1 & \cdots & c_1^1 \\ \vdots & \ddots & \vdots \\ c_1^n & \cdots & c_n^n \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, \tag{3.26}$$

where the c_j^i are Boolean constants. The set of all affine-linear M-bit maps form a group called the *affine-linear general group* $\mathbf{AGL}(M, \mathbb{B})$, which is a strictly smaller subgroup of S^{2^M} for every M. It happens that all 2-bit maps are affine-linear, and therefore the universal maps with smallest width are 3-bit non-linear maps, such as the TOFFOLI gate.

The idea of an N-bit version of an M-bit maps used up to this point only really applies to the case where $N > M$. In order for a universal M-bit reversible map to have an N-bit version with $N < M$, we need to designate some $M - N$ sized subsets of input and output bits as either *ancilla* or *garbage* bits. Ancillary bits are input bits designated to have a predefined value, while garbage bits are output bits whose value is irrelevant for the implementation of a particular function. An example of use is given in fig. 13, where we implement a version of the CNOT gate using the universal TOFFOLI gate, ancilla, and garbage bits.

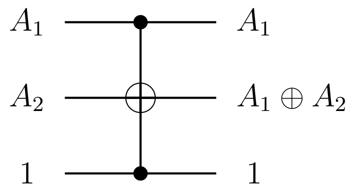


Figure 13 – Implementation of a $\mathrm{CNOT}_{1,2}$ gate using the $\mathrm{TOFFOLI}_{1,3,2}$ gate with input bit 3 designated as an ancillary bit and the output bit 3 as a garbage bit.

The existence of universal gates allow us to objectively measure the cost of implementing any particular N-bit map over an N-bit system. This measure is obtained by finding the circuit decompositions of this map in terms of the universal gate, and computing their circuit size, depth and width. However, the physical requirements for implementing any particular version of an M-bit universal gate may lead to different *figures of merit* that are only significant for the system in question.

For example, suppose that a physical system only allows the implementation of gates between bits that are nearest neighbours with respect to some spatial ordering. The implementation of versions of a universal gate G that connect non-nearest-neighbour bits will require the ability to perform sequences of SWAP gates, and this will invariably add to the circuit size and depth. If for some reason SWAP gates cannot be built using the available versions of G, then we may require *ancillas*, which increase the circuit depth, or we may just not be able to express all maps, which forfeits the universal property of G.

Now we are in position to discuss the quantum case. Just as with M-bit maps, M-qubit unitaries have many different N-qubit versions, for $N > M$. This is done defining the G action over a subset I of quantum state variables $\{\left|A_{i_j}\right\rangle\}_{j=1,\ldots,M}$ in the computational basis states of the N-qubit system. For example, the 3-qubit controlled rotation gate $\Lambda^2(R_{\mathbf{n}}(\alpha))$, given by

$$\Lambda^2(R_{\mathbf{n}}(\alpha))\left|A_1A_2A_3\right\rangle = \left|A_1A_2\right\rangle \otimes (R_{\mathbf{n}}(\alpha))^{A_1A_2}\left|A_3\right\rangle, \tag{3.27}$$

has as one of its 5-qubit versions the map $\Lambda^2(R_{\mathbf{n}}(\alpha))_{1,3,4}$, given by

$$\Lambda^2(R_{\mathbf{n}}(\alpha))_{1,3,4}\left|A_1A_2A_3A_4A_5\right\rangle = \left|A_1A_2A_3\right\rangle \otimes ((R_{\mathbf{n}}(\alpha))^{A_1A_3}\left|A_4\right\rangle) \otimes \left|A_5\right\rangle. \tag{3.28}$$

Therefore, the definition of a quantum circuit has exactly the same form as the definition of a reversible classical circuit, found in eq. (3.25), but with $G^i_{I_i}$ being N-qubit unitaries instead of N-bit reversible gates. However, in contrast to the classical case, it is impossible to build an exact circuit decomposition of an N-qubit unitary map using gates from a finite set, due to the continuous nature of the $SU(2^N)$ group.

Therefore, we instead speak of *strict universality* only when dealing with circuits built from N-qubit versions of gates that belong to a continuous *gate set*. As shown in [185], any N-qubit unitary W can be exactly decomposed as a circuit in terms of the gate set $G_{SU(2),\mathrm{CNOT}} = \{U, \mathrm{CNOT} | U \in SU(2)\}$. Notice that, while in the classical case, universal gates must be non-linear, in the quantum case, a universal gate set only needs single-qubit operations and the linear[6] CNOT gate.

This implies, in particular, that non-linear gates, such as TOFFOLI, must have a quantum circuit decomposition in terms of the linear CNOT gates (fig. 14), which is impossible for classical circuits. The decomposition of classical maps into quantum circuits leads to some very important *circuit identities* which are necessary for many types of exact circuit synthesis tasks. The most common of circuit identities are summarized in Fig.3.

6 In terms of its action over Boolean variables.

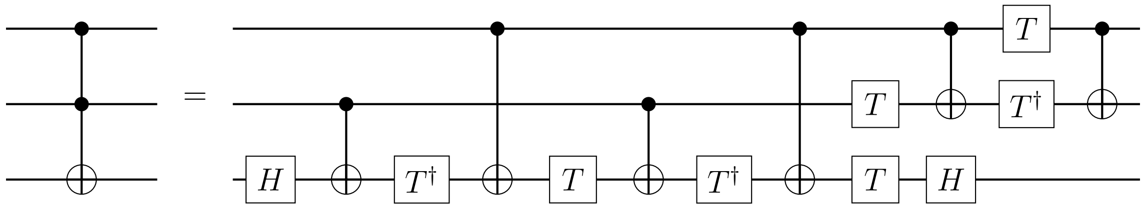


Figure 14 – Quantum circuit decomposition for the TOFFOLI gate.

Strict universality, although important to understand the structure of quantum circuits, is not the most practical notion of circuit universality. This is because it requires the capacity to implement any single-qubit map with arbitrary precision. A more useful notion is the concept of *computational universality*, where we only require that our circuit approximates the desired unitary map up to some non-zero precision $\epsilon > 0$. To measure the degree of precision we need to define what it means for two unitaries to be "close" to one another. This is done by the operator norm

$$\|U - V\| = \sup_{|\psi\rangle \in \mathscr{B}} \|(U - V) |\psi\rangle\|, \tag{3.29}$$

which measures the distance between two unitaries V and U.

The concept of computational universality also allows universal gates sets with a discrete number of elements. This is due to the existence of discrete gate sets that generate circuits that are dense over $SU(2^N)$, for every N. A set of N-qubit circuits is *dense* over $SU(2^N)$ if, for every $\epsilon > 0$ and every unitary $U \in SU(2^N)$, there exists a circuit W of size $l(\epsilon)$ built from this gate set, such that $\|U - W\| < \epsilon$.

However, we must also impose that the circuit W, written in terms of a dense gate set, can be found in a reasonable amount of time by some synthesis method, and have a reasonable size. By reasonable, I mean an execution time and output circuit size that do not grow exponentially fast as the approximation gets more precise. This problem can be solved for single-qubits, and by extension for N-qubits (with N fixed) [147], by employing the *Solovay-Kitaev algorithm*.

It shows how to synthesize a circuit W built from a computationally universal set, for any $U \in SU(2^N)$ with fixed N, such that the size of the circuit, as well as its synthesis run-time, is bounded by a function that grows no faster than an polynomial over $\ln(1/\epsilon)$. For example, in the single-qubit case, the circuit sizes grow as $\ln^{3.97}(1/\epsilon)$, and the synthesis run-time as $\ln^{2.71}(1/\epsilon)$ [56]. The three most common examples of computationally universal gates sets are $\{H, \Lambda^2(X)\}$ [3, 218], $\{G, \Lambda(X)\}$ where G is any single-qubit gate such that $[G^2; Z] \neq 0$ [218], and $\{H, T, \Lambda(X)\}$.

The last difference between classical reversible circuits and quantum circuits we need to discuss is the role of quantum measurements. Up to this point, whenever we talked about reversible circuits, we assumed that the input bits could be deterministically prepared in any desired state. Then, since reversible transformations were assumed to also be deterministic, the output is observed to be deterministic as well.

However, when we do quantum information processing with circuits, the end result is a probability distribution over the set of all N-bit strings, even if the input state preparation, the circuit itself and the output state are obtained deterministically. This could seem as a weakness of quantum information processing, but if we have the freedom to measure the output state of circuit with respect to a basis that is not the computational basis, we can exploit this as a resource [50, 139].

The essential concept used to exploit the nature of quantum measurement in information processing is that of *adaptivity.* An *adaptive quantum circuit* W_{ad} for an N-qubit unitary U, is defined in the following way. We start with an input state over a bigger qubit system, with the number of qubits being at most a polynomial on N. We designate all but N qubits as ancilla, and prepare the ancilla in some pre-defined initial state $|init\rangle$.

Given a choice of initial state, we implement a circuit V^1 over the $poly(N)$-qubit system and, after it, measure a set of d_1 qubits. This measurement give us a d_1-bit string $\mathbf{A}^1 = A_1^1 \cdots A_{d_1}^1$, which will be used to produce another quantum circuit $V^2(\mathbf{A}^1)$, acting on the $poly(N) - d_1$ qubits that were left unmeasured. After that, we measure d_2 qubits, obtaining a b_2-bit string $\mathbf{A}^2$ that is used, together with $\mathbf{A}^1$, to produce another circuit $V^3(\mathbf{A}^1, \mathbf{A}^2)$ on the remaining qubits. The process is repeated until all ancilla are measured, which will necessarily happen for some number k, called the number of *rounds* of W_{ad}.

Adaptivity does not allow us to create more quantum circuits than any universal set. However, if we are capable of implementing certain special initial states and special measurement basis, we are able to achieve computational universality using only single-qubit gates [102,235]. In this sense, being able to implement adaptive circuits is a resource in itself, and plays an important role in computational models based on the physics of standard identical particles. The special initial states and measurements that we just mentioned are obtained using another resource obtained from quantum states that does not exist for classical bit-states. This resource is called *quantum entanglement.*

3.1.3 Quantum entanglement and entanglement power

In order to understand quantum entanglement, we must introduce a generalization of the description of quantum systems known as the *density operator formalism.* This formalism allows us to deal with situations where a system is not described by a well defined quantum state, but by a statistical ensemble of quantum states. In classical information theory, a statistical ensemble $\{d(A), A\}$ of states on a bit is represented by a *random*

Boolean variable A, given by a probability distribution

$$d(A) = \begin{cases} d(0) = p_0 \\ d(1) = p_1 \end{cases}, \tag{3.30}$$

where p_0 is the probability of outcome $A = 0$ and p_1 the probability of $A = 1$. We also must have that $p_0, p_1 \in \mathbb{R}$, $p_0, p_1 \geq 0$ and $p_0 + p_1 = 1$.

In the quantum case, a statistical ensemble $\{p_i, |\psi_i\rangle\}$, where $|\psi\rangle_i$ are a collection of quantum states of size $K \in \mathbb{N}$, each occurring with probability p_i, is described by the *density operator*

$$\rho = \sum_{i=1}^{K} p_i \, |\psi_i\rangle\langle\psi_i| \,, \tag{3.31}$$

where we require that each $p_i \geq 0$, and $\sum_{i=1}^{K} p_i = 1$. These two conditions imply that any operator ρ, over any Hilbert space, can represent an ensemble of quantum states if they have unit trace ($\mathrm{Tr}\{\rho\} = 1$), and are *positive operators* ($\langle\psi|\rho|\psi\rangle \geq 0$, for all $|\psi\rangle$). The positivity condition ensures that ρ is Hermitian and has real, non-negative eigenvalues and, because of that, the eigensystem of any such ρ describes an ensemble of quantum states.

Given any observable O, its expected value in the density operator formalism is given by

$$\langle O \rangle = \mathrm{Tr}\{O\rho\}. \tag{3.32}$$

An ensemble that has a single quantum state $|\psi\rangle$ is called a *pure state*, and the corresponding density operator is $\rho_\psi = |\psi\rangle\langle\psi|$. So, in general, pure states are the ones that satisfy $\rho^2 = \rho$. Non-pure states are called *mixed*, and represent a classical "mixture" of more than one quantum state. The action of measurements over density operators is also described using traces, but we will not need it here.

Using density operators as descriptions of ensembles of quantum states entails a fundamental redundancy. Suppose that a quantum system is described by an ensemble $\{p_i, \rho_i\}$, where each ρ_i describes an ensemble $\{q_j^i, \rho_{\psi_j}^i\}$. It is easy to see that the quantum system must be described by both

$$\rho_{sys} = \sum_i p_i \rho_i, \quad \text{and} \quad \rho_{sys} = \sum_{i,j} p_i q_j^i \rho_{\psi_j}^i. \tag{3.33}$$

Notice that, in both cases, ρ_{sys} is a linear combination of density matrices, where the sum of all coefficients is always 1. We call such linear combinations *convex combinations*, and what this argument exemplifies is the fact that the same density operator can be described by many different convex combinations or, in other words, many ensembles are described by the same density operator. These ensembles are considered to be equivalent, and are characterized by an equivalence theorem which can be found in [185].

Density operators are used to describe settings in which the quantum state of a physical system is unknown, but are also useful for describing the states of *sub-systems*, as we see now. Let $\rho_{1,2}$ be the density operator representing a state of a two-qubit system. Suppose that we want to find the expectation value of an observable O_1 that depends only on the first qubit. We can use eq. (3.32) to write the general formula

$$\langle O_1 \rangle = \mathrm{Tr}_{1,2}\{O_1 \rho_{1,2}\}, \tag{3.34}$$

where the trace is taken over the computational basis state of the two-qubit system.

However, given any $\rho_{1,2}$, it is possible to find a density operator ρ_1 that only represents the state of the first qubit, in the sense that

$$\langle O_1 \rangle = \mathrm{Tr}_{1}\{O_1 \rho_{1}\}, \tag{3.35}$$

where the trace is taken over the basis of the first qubit only, which is called the *sub-system operator* for the first qubit. The operator ρ_1 itself is given by the partial trace

$$\rho_1 = \mathrm{Tr}_2\{\rho_{1,2}\}. \tag{3.36}$$

Similarly, one can define a $\rho_2 = \mathrm{Tr}_1\{\rho_{1,2}\}$ that gives description of the second qubit as a sub-system with respect to observables acting on the second qubit alone.

Having introduced the density operator formalism, we are now in position to define what is quantum entanglement. Let $|\psi\rangle$ be an arbitrary two-qubit (pure) state in $\mathscr{B}_2$. Using the canonical basis, such a state has the form

$$|\psi\rangle = c_{0,0}\,|00\rangle + c_{0,1}\,|01\rangle + c_{1,0}\,|10\rangle + c_{1,1}\,|11\rangle\,, \tag{3.37}$$

where $|xy\rangle = |x\rangle \otimes |y\rangle$. A two-qubit state is said to be a *product state* if there exists two single-qubit states, $|\zeta\rangle\,, |\eta\rangle \in \mathscr{B}$ such that,

$$|\psi\rangle = |\zeta\rangle \otimes |\eta\rangle\,, \tag{3.38}$$

otherwise the state is said to be *entangled*. Product states describe a physical situation where measurements made locally in each qubit are uncorrelated.

Any pure, product state of the form $|\zeta\rangle \otimes |\eta\rangle$ is described by a density operator

$$\rho_{1,2} = (|\zeta\rangle \otimes |\eta\rangle)(\langle\zeta| \otimes \langle\eta|) = |\zeta\rangle\langle\zeta| \otimes |\eta\rangle\langle\eta|\,. \tag{3.39}$$

It is easy to see that, for this $\rho_{1,2}$, one has $\rho_1 = |\zeta\rangle\langle\zeta|$ and $\rho_2 = |\eta\rangle\langle\eta|$. Notice that both ρ_1 and ρ_2 are pure states or, in other words, both satisfy $\rho_1^2 = \rho_1$ and $\rho_2^2 = \rho_2$.

This observation allows us to introduce the concept of *separable states*. A separable state of two-qubits is any $\rho_{1,2}$ that can be written as a convex combination

$$\rho_{1,2} = \sum_k p_k(\rho_1^k \otimes \rho_2^k), \tag{3.40}$$

where each ρ_1^k and ρ_2^k represent pure states. In other words, a separable state is a classical ensemble of product states $\{p_k, (|\zeta\rangle^k \otimes |\eta\rangle)^k\}$. Separable states describe a physical situation where measurements made locally on each qubit have classical probabilistic correlations.

An important set of examples of entangled states are the states of the *Bell basis*, given by

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{3.41}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{3.42}$$

Some applications of these states are on super-dense coding and quantum teleportation protocols, but their importance goes beyond them and encompasses the foundations of quantum theory itself. In fact, the special initial states needed for the single-qubit gate universality of adaptive quantum circuits (see subsection 3.1.2) need to be entangled states.

This characterization of product and separable states implies that two-qubit entangled states, either pure or mixed, cannot be described by classical ensembles of pure, product states. This fact is easy to see, for example, in the case of the Bell state $|\beta_{00}\rangle$, where the sub-system decomposition of $\rho_{\beta_{00}}$ leads to two mixed sub-system states. Therefore, entangled states represent a physical situation where measurements made locally on each qubit have non-classical probabilistic correlations. This implies that such correlations play some crucial role in the applications where entangled states are necessary, which means that they can also be treated as resources.

Among applications of entangled states, it was shown that entanglement is crucial for understanding the strict universality of gate sets of the form $G_{SU(2),V} = \{U, V | U \in SU(2)\}$, which contain all possible single-qubit gates and a specific two-qubit gate V. We saw earlier, in subsection 3.1.2, that when $V = \mathrm{CNOT}$ this gate set is universal, but this is not the case for all V. For example, if $V = \mathrm{SWAP}$, the gate set is not universal, because applying two single-qubit gates U_1 and U'_2 on qubits 1 and 2, respectively, and then applying the $\mathrm{SWAP}_{1,2}$ gate, is equivalent to just applying U'_1 and U_2 for any pair of unitaries.

Another way of explaining why this true is realizing that the SWAP gate always sends product states into product states. Therefore, if we initialize every circuit in the computational basis, we can never reach any entangled state. This means that not all $SU(2^N)$ are described by such a circuit, since many operations in $SU(2^N)$ create entangled states from product states. In fact, it was shown by Brylinski in [48], that any two-qubit gate V capable of creating an entangled state from some product state makes the gate set $G_{SU(2),V}$ universal for quantum computing. An example is the CNOT gate since, given the product state $|+\rangle_x \otimes |0\rangle$ we have

$$\text{CNOT}(|+\rangle_x \otimes |0\rangle) = |\beta_{00}\rangle . \tag{3.43}$$

To give a general description of the gates possessing this property, we must introduce an entanglement measure, a function that classifies the strength of the non-classical correlations that can be obtained from entangled states. Of all of the existing entanglement measures, the one which is commonly used for proving the universality of gates sets is the *linearized entanglement entropy*, defined by

$$E(|\psi\rangle) = 1 - \text{Tr}_1\{(\text{Tr}_2\{|\psi\rangle\langle\psi|\})^2\}. \tag{3.44}$$

The linearized entanglement entropy $E(|\psi\rangle)$ is 0 if $|\psi\rangle$ is separable, but non-zero if entangled.

The maximum of $E(|\psi\rangle)$ occurs when $Tr_2\{|\psi\rangle\langle\psi|\} = I/2$, which is the *maximally mixed state*. For a maximally mixed state the classical mixture contains all orthogonal quantum states with equal probability. In this case the original two-qubit state is said to be *maximally entangled*, and its linearized entropy is $E(|\psi\rangle_{max}) = 1/2$, with the Bell states $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ as examples. Therefore, we will scale $E(|\psi\rangle)$ by a factor of 2, such that it always stays between 0 and 1.

Using the (scaled) linearized entanglement entropy, Zanardi [263] proposed a way to measure if a particular two-qubit gate is capable of generating entanglement or not. The measure is called the *entangling power* $e_p(V)$ of a two-qubit gate V given by the expression

$$e_p(V) = \overline{E(V|\psi_1\rangle \otimes |\psi_2\rangle)}^{(\psi_1,\psi_2)}, \tag{3.45}$$

where the mean-value is taken over a classical probability distribution $p(\psi_1, \psi_2)$ over separable quantum states. It can be shown that, if the average is taken over the Haar distribution, the entangling power is both a local invariant and SWAP invariant (that is, it remains the same if U is conjugated by SWAP or by single-qubit gates).

This invariant can be easily calculated in terms of simpler invariants, which was done in [16]. Two-qubit gates have two local invariant quantities, given by

$$G_1(V) = \frac{\text{Tr}^2\, V_B^T V_B}{16 \det(V)}, \tag{3.46}$$

and

$$G_2(V) = \frac{\text{Tr}^2\, V_B^T V_B - \text{Tr}\left\{(V_B^T V_B)^2\right\}}{4 \det(V)}, \tag{3.47}$$

where V_B is the matrix representation of the gate V written in the Bell basis. With these invariants, the entangling power $e_p(V)$ of a two-qubit gate V over the uniform distribution is just given by

$$e_p(V) = 1 - |G_1(V)|. \tag{3.48}$$

This expression can be further simplified by finding a canonical form for arbitrary two-qubit unitaries which allows for easy calculation of G_1. Luckily, this was already done by Kraus and Cirac in [153], where they showed that any two-qubit gate can be put in the form

$$V = (U_1 \otimes U_2) \exp\{i(aX_1 \otimes X_2 + bY_1 \otimes Y_2 + cZ_1 \otimes Z_2)\}(W_1 \otimes W_2), \tag{3.49}$$

where the U and W operators are called the *local part*, and the exponential in the middle is called the *non-local part*, which is invariant up to permutation of a, b and c generated by local transformations.

Using this decomposition, one finds that the entangling power is given by

$$e_p(\{a, b, c\}) = 1 - \cos^2(a)\cos^2(b)\cos^2(c) - \sin^2(a)\sin^2(b)\sin^2(c), \tag{3.50}$$

where $\{a, b, c\}$ is the unique parametrization with $a > b > c$ of the non-local part of V. For the CNOT gate, we have $\{\pi/4, 0, 0\}$ and $e_p(\text{CNOT}) = 1$, while for the SWAP gate we have $\{\pi/4, \pi/4, \pi/4\}$ giving $e_p(\text{SWAP}) = 0$.

3.2 Introduction to the optical network model

Having introduced the quantum circuit model, we are now in position to discuss how it is implemented in systems of standard identical particles. In subsection 3.2.1, I introduce the quantum theory of optical devices, motivated by the classical description of a simple interferometer. Next, I present the Hamiltonian model of optical devices as quantum dynamical maps over a multi-mode system of bosonic oscillators. By the end of this subsection, I give explicit Hamiltonians representing the action of the most common optical devices, as well as their classification, and define the concept of optical networks.

In subsection 3.2.2, I calculate the action of optical devices over Fock-states, and use the results to obtain the general transmission amplitudes for linear multi-mode interferometers. I also discuss a special phenomena called the Hong-Ou-Mandel effect. Next, I make a brief introduction to the development of photonic computers, the definition of dual-rail qubits and logical qubit operations. I finish by pointing out that, up to this moment, the use of optical networks for implementing universal quantum gate sets requires the use of either non-linear devices or media, or off-line preparation of resource states via adaptive schemes, such as the KLM-protocol [150].

Then, in subsection 3.2.3, I finish by extending the concept of optical networks to systems of identical fermions, and studying their computing power. First, I give an isomorphism between the Hilbert space of an m-qubit system and the Fock-space of m fermionic modes. Next, I define fermionic optical devices, and networks, via the same Hamiltonian formalism applied to the bosonic case. Then, I explain why fermionic linear networks are easy to simulate in classical computers, and finish with a brief comparison between computation with bosonic and fermionic optical networks.

3.2.1 Quantum interferometry of light and optical networks

As put by Prasad, Scully and Martiessen in [201], interferometers have revolutionized technology of precision measurements, and their sensitivities are ultimately limited only by quantum mechanical rules. The development of optical technology led to the necessity for a fully-quantum description of optical devices. My purpose here in subsection 3.2.1 is to lay out this quantum description. Before doing so, let us do a review of the classical model of a general two-mode interferometer.

For the purpose of illustration, assume that we are dealing with a single-mode, travelling-wave electromagnetic field with wave-vector $\mathbf{k}$ and a given linear polarization labelled by λ. For both classical and quantum theories of interferometry, the relevant observables are written in terms of the electric field part of the wave, which is given in complex form by the expression

$$E(q,t,\omega) = E^{+}(q,t,\omega) + E^{-}(q,t,\omega) = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}} \left\{a e^{i(kq-\omega t+\pi/2)} + a^{*} e^{-i(kq-\omega t+\pi/2)}\right\}. \tag{3.51}$$

In this expression, q is the coordinate along an arbitrary direction of propagation $\mathbf{q}$ in three-dimensional space, and the E^{+} and E^{-} are conventionally called the positive and negative-frequency parts of E.

The classical, lossless, most general type of beam splitter can be understood in terms of the figure Fig. 15. A general beam splitter has four arms. Arms 1 and 2 host the *input fields* E_1 and E_2, which are assumed to have the form of eq. (3.51), but with propagation directions $\mathbf{x}$ and $\mathbf{y}$, respectively. Due to the linearity of the field, we must have the output fields E_3 and E_4 satisfying the relations

$$E_3 = R_{3,1}E_1 + T_{3,2}E_2 \quad E_4 = T_{4,1}E_1 + R_{4,2}E_2, \tag{3.52}$$

where $R_{3,1}, R_{4,2}$ and $T_{4,1}, T_{3,2}$ are called the reflection and transmission coefficients of the beam splitter, respectively.

Due to energy conservation, the matrix of coefficients below

$$\begin{bmatrix} R_{3,1} & T_{3,2} \\ T_{4,1} & R_{4,2} \end{bmatrix}, \tag{3.53}$$

must be a unitary matrix. This implies, given the polar decompositions $R_{i,j} = |R_{i,j}|e^{i\phi_{i,j}}$ and $T_{i,j} = |T_{i,j}|e^{i\phi_{i,j}}$, that $|R_{3,1}| = |R_{4,2}| = R$, $|T_{4,1}| = |T_{3,2}| = T$, $T^2 + R^2 = 1$ and $\phi_{3,1} + \phi_{4,2} - (\phi_{3,2} + \phi_{4,1}) = \pm\pi$. Therefore an arbitrary beam splitter can be written in terms of four real parameters: R, and three angles.

$$\begin{bmatrix} Re^{i\phi_{3,1}} & \sqrt{1-R^2}e^{i\phi_{3,2}} \\ \sqrt{1-R^2}e^{i\phi_{4,1}} & -Re^{i(\phi_{3,2}+\phi_{4,1}-\phi_{3,1})} \end{bmatrix} \tag{3.54}$$

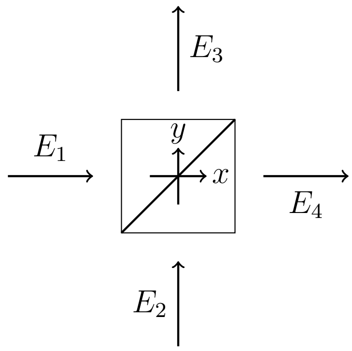


Figure 15 – A spatial depiction of a generic beam splitter. The arms labelled by the fields E_1 and E_2, are input arms, and the ones labelled by E_3 and E_4 are output arms.

Each E_i can be canonically quantized, which implies that each field operator $\hat{E}_i$ corresponds to a pair of creation and annihilation operators $\hat{a}_i^\dagger, \hat{a}_i$ that satisfy bosonic commutation relations. The action of the beam splitter, therefore, is relating output operators to input operators via the beam-splitter matrix in eq.(3.53), giving us

$$\begin{bmatrix} \hat{a}_3 \\ \hat{a}_4 \end{bmatrix} = \begin{bmatrix} R_{3,1} & T_{3,2} \\ T_{4,1} & R_{4,2} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}. \tag{3.55}$$

This result can be seem as a *spatial description* of the action of this device in the quantum regime, in the sense that the modes are associated to the input and output arms. For our purposes, the most useful description is in terms of the *scattering matrix* associated to the time evolution of field operators under a particular Hamiltonian. To show this, we generalize the previous setting in the following way.

First, instead of talking about a quantized electric field we talk about system of generalized *modes* described by bosonic oscillators $\hat{b}_i, \hat{b}_i^\dagger$ with commutation relations given in eq. (2.22). Then we consider only two modes instead of four, where the operators corresponding to the input and output arms are taken as operators before and after the time evolution. Calling the initial operators by $\hat{b}_1(0)$ and $\hat{b}_2(0)$ we have that the beam-splitter dynamics must correspond to

$$\begin{bmatrix} \hat{b}_1(t) \\ \hat{b}_2(t) \end{bmatrix} = \begin{bmatrix} R_{1,1} & T_{1,2} \\ T_{2,1} & R_{2,2} \end{bmatrix} \begin{bmatrix} \hat{b}_1(0) \\ \hat{b}_2(0) \end{bmatrix}, \tag{3.56}$$

where the reflection and transmission coefficients are functions of t. Since, after passing through the beam-splitter, the fields do not change, this model only makes sense if the interaction begins and ends in a finite time. Therefore, we presuppose that the dynamics generating the beam splitter matrix comes from adiabatically turning on and off an interaction Hamiltonian, in a controllable way.

Using this new formulation of the behavior of a general beam splitter, we see that its matrix must come from a time evolution operator $\hat{U}(t)$ such that

$$\hat{b}_i(t) = \hat{U}(t)\hat{b}_i(0)\hat{U}^\dagger(t). \tag{3.57}$$

Since the beam-splitter matrix is unitary, it must also be the case that $\hat{U}(t)$ is a unitary operator acting over the algebra of bosonic oscillators. Therefore, we can look for an *effective Hamiltonian*, written in terms of bosonic operators, whose action generates the matrix in eq. (3.56). The form of the Hamiltonian is given by two properties of eq. (3.56).

First, the time evolution affects only two modes. Second, since eq. (3.56) is linear in the operators, the commutators $[\hat{H}_{1,2}, \hat{b}_1]$ and $[\hat{H}_{1,2}, \hat{b}_2]$ must be linear combinations of $\hat{b}_1$ and $\hat{b}_2$. These properties imply that the Hamiltonian describing a general beam splitter must be a Hermitian linear combination of the operators $\hat{b}_1^\dagger\hat{b}_1$, $\hat{b}_2^\dagger\hat{b}_2$, $\hat{b}_1^\dagger\hat{b}_2$ and $\hat{b}_2^\dagger\hat{b}_1$. Any such linear combination can be rewritten as a real linear combination of the Hermitian basis of operators given by

$$\hat{T}^1_{1,2} = \frac{1}{2}(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1), \tag{3.58a}$$

$$\hat{T}^2_{1,2} = \frac{-i}{2}(\hat{b}_1^\dagger\hat{b}_2 - \hat{b}_2^\dagger\hat{b}_1), \tag{3.58b}$$

$$\hat{T}^3_{1,2} = \frac{1}{2}(\hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2), \tag{3.58c}$$

$$\hat{N}_{1,2} = \hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2. \tag{3.58d}$$

This basis has the very special property of being a representation of the angular momentum algebra or, in other words, we have that $[\hat{T}^i_{1,2}; \hat{T}^j_{1,2}] = \epsilon_{i,j,k}\hat{T}^k_{1,2}$ and $[\hat{N}_{1,2}; \hat{T}^i_{1,2}] = \hat{T}^i_{1,2}$ for all $i, j, k = 1, 2, 3$. This construction is famous, and is called the *Schwinger representation* of the angular momentum algebra [216]. Each of the four real parameters in the linear combination

$$\hat{H}_{1,2} = a\hat{N}_{1,2} + b_1\hat{J}^1_{1,2} + b_2\hat{J}^2_{1,2} + b_3\hat{J}^3_{1,2}, \tag{3.59}$$

are a function of the four real parameters that describe a general beam splitter. However, it is more convenient to represent a general beam splitter dynamics as being a composition of action of single-parameter devices that can be independently controlled.

Hereby, we define the single-parameter beam splitter, or just beam splitter for short, the unitary acting in modes i, j given by

$$BS_{1,2}(\theta) = \exp\left\{i2\theta\hat{J}^1_{1,2}\right\} = \exp\left\{i\theta(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1)\right\}, \tag{3.60}$$

where θ, is the *effective interaction time*, or the beam-splitter angle, with a pictorial representation given in Fig. 16. The action of this specific device over bosonic creation operators is

$$BS_{1,2}(\theta)\hat{b}_1^\dagger BS_{1,2}^\dagger(\theta) = \cos\theta\hat{b}_1^\dagger + i\sin\theta\hat{b}_2^\dagger, \tag{3.61a}$$
$$BS_{1,2}(\theta)\hat{b}_2^\dagger BS_{1,2}^\dagger(\theta) = \cos\theta\hat{b}_2^\dagger + i\sin\theta\hat{b}_1^\dagger. \tag{3.61b}$$

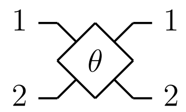


Figure 16 – Graphical representation of a beam-splitter acting on a pair of modes labelled by integers 1 and 2. The direction of time goes from left to right.

We also define a device *phase shifter* that, besides being a physical device, is also given by the operators describing the free propagation of a traveling-wave in a single mode. This operator is

$$PS_i(\tau) = \exp\left\{i\tau\hat{b}_i^\dagger\hat{b}_i\right\}, \tag{3.62}$$

and it corresponds to the Hamiltonian $\hat{N}_{1,2} \pm 2J_{1,2}^3$ in the Schwinger basis, where the sign depends on the mode the phase shifter acts on. This operator is represented graphically in Fig. 17. The action of this device over bosonic creation operators is

$$PS_i(\tau)\hat{b}_j^\dagger PS_i(-\tau) = e^{i\tau\delta_{ij}}\hat{b}_j^\dagger \tag{3.63}$$

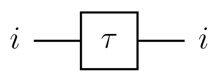


Figure 17 – Graphical representation of a phase-shifter, acting on a single mode labelled i. Time flows from left to right.

Using these two devices, a general beam-splitter can be written as the sequence of operators $PS_2(\alpha)BS_{1,2}(\beta)PS_2(\gamma)$, as in Fig. 18. This decomposition has the same form as the single-qubit unitary decomposition in eq. (3.14), but with $\mathbf{n} = \mathbf{z}$ and $\mathbf{m} = \mathbf{x}$. This fact is crucial for the quantum computing applications that we will see shortly.

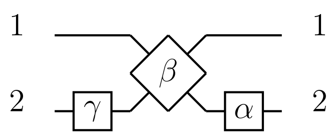


Figure 18 – Optical network decomposition of a general two-mode passive linear device. The order of elements in the network is the opposite of the order of operators in the operator decomposition.

Many other types of optical devices are described in terms of dynamical maps generated by second-quantized bosonic Hamiltonians. For a two-mode system, the most general Hamiltonian is a polynomial of arbitrary degree over $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_1^\dagger$ and $\hat{b}_2^\dagger$. In other words, the most general Hamiltonian is a Hermitian polynomial of the form

$$\hat{H}_{1,2} = \sum_{k_1,k_2,k_3,k_4} a_{k_1,k_2,k_3,k_4} (\hat{b}_1^\dagger)^{k_1} (\hat{b}_2^\dagger)^{k_2} (\hat{b}_1)^{k_3} (\hat{b}_2)^{k_4}. \tag{3.64}$$

From this general Hamiltonian we can obtain a description of many general classes of optical devices. For two-mode systems, we call *Gaussian devices* all which are described by Hamiltonians of the form

$$\hat{H} = \sum_{i_1,i_2=1}^{2} A_{i_1,i_2} \hat{b}_{i_1}^\dagger \hat{b}_{i_2} + \sum_{i_1,i_2=1}^{2} \left(B_{i_1,i_2} \hat{b}_{i_1} \hat{b}_{i_2} + C_{i_1,i_2} \hat{b}_{i_1}^\dagger \hat{b}_{i_2}^\dagger \right). \tag{3.65}$$

Among the Gaussian devices, the ones for which $A_{i_1,i_2} \neq 0$ and $B_{i_1,i_2} = C_{i_1,i_2} = 0$ are called *passive*, or *number-preserving*, since they commute with the total number operator $\hat{N}_{1,2}$. The ones for which $A_{i_1,i_2} = 0$, $B_{i_1,i_2} \neq 0$ and $C_{i_1,i_2} \neq 0$ are called *active*, or *parity-preserving*, since they commute with the total parity operator $(-1)^{\hat{N}_{1,2}}$. As we already saw, the set of passive Gaussian Hamiltonians forms a Lie algebra that is isomorphic to the direct sum $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, with the $\mathfrak{u}(1)$ generator being $\hat{N}_{1,2}$, and the generators of $\mathfrak{su}(2)$ being $\hat{T}^1_{1,2}$,$\hat{T}^2_{1,2}$ and $\hat{T}^3_{1,2}$.

The single-parameter beam splitter (BS) and phase shifter (PS) are the prime examples of passive Gaussian devices, while for active Gaussian devices we have the *squeezers* (SQ), and the *two-mode down-converters* (DC), given respectively by

$$SQ_i(\gamma) = \exp\left\{ i\gamma (\hat{b}_i^{\dagger 2} + \hat{b}_i^2) \right\}, \tag{3.66a}$$

$$DC_{1,2}(\kappa) = \exp\left\{ i\kappa (\hat{b}_1^\dagger \hat{b}_2^\dagger + \hat{b}_2 \hat{b}_1) \right\}. \tag{3.66b}$$

The complete set of Gaussian Hamiltonians also forms a Lie algebra, which is isomorphic to the real symplectic algebra $\mathfrak{sp}(4, \mathbb{R})$.

Multi-mode interferometers can also be described by dynamical maps generated by Hamiltonians. In the case of m modes, the most general Hamiltonian is a polynomial of the form

$$\hat{H} = \sum_{\{k_i\},\{l_i\}} a_{\{k_i\},\{l_i\}} \prod_{i=1}^{m} \left(\hat{b}_i^{\dagger k_i} \hat{b}_i^{l_i} \right). \tag{3.67}$$

The set of all of these polynomials forms an algebra under operator multiplication and complex conjugation called the *Weyl algebra* $\mathbf{Weyl}(m)$, which plays and important role in bosonic quantization [257].

In a way similar to the two-mode case, Gaussian devices are described by Hamiltonians in eq. (3.65), but with the summation going through all the modes. The definition of active and passive devices is also the same as the one in the two-mode case, passive devices form a Lie algebra isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(m)$, while the full set of Gaussian devices forms one isomorphic to $\mathfrak{sp}(2m, \mathbb{R})$.

The action of passive Gaussian devices over bosonic creation operators is linear, and given by

$$\hat{b}_i^\dagger(t) = \sum_j U_{i,j}\hat{b}_j^\dagger(0). \tag{3.68}$$

For that reason, they are also called *linear optical devices*. The matrix U with coefficients $U_{i,j}$ is an element of the unitary group $\mathbf{U}(m)$. General Gaussian devices, on the other hand, act over bosonic creation operators by the equations

$$U\hat{b}_i^\dagger(t)U^\dagger = \sum_{j=1}^{m}(W_{i,j}\hat{b}_j^\dagger(0) + V_{i,j}\hat{b}_j(0)), \tag{3.69}$$

where the matrices of coefficients W and V are such that this transformation is canonical which implies that, taken together, they form a representation of the real symplectic group $\mathbf{Sp}(2m, \mathbb{R})$.

It is known [183] that any unitary matrix of size $m \times m$ can be decomposed as a product of $m(m-1)/2$, 2×2 unitaries, and one unitary diagonal matrix of size $m \times m$. Using this result, Reck showed in [207] that the action of any passive Gaussian multi-mode device can be decomposed in terms of sequences of single-parameter beam splitters and phase shifters. In general, an m mode passive Gaussian device can be decomposed into the successive applications of $2m^2 - m$ of beam spliters of the form $BS_{i,j}(\pi/4)$ (known as a balanced beam splitter), and $2m^2$ phase-shifters $PS_i(\tau)$.

The devices that act on arbitrary pairs of modes of an m-mode system are defined by the operators

$$PS_i(\tau) = \exp\Big\{i\tau(\hat{b}_i^\dagger\hat{b}_i)\Big\}, \tag{3.70a}$$

$$BS_{i,j}(\theta) = \exp\Big\{i\theta(\hat{b}_i^\dagger\hat{b}_j + \hat{b}_j^\dagger\hat{b}_i)\Big\}, \tag{3.70b}$$

and are analogous to the N-qubit versions of M-qubit maps discussed in subsection 3.1.2. Sequences of the optical devices defined above are called *passive linear optical networks*, and provide an alternate implementation of any m-mode passive Gaussian device U. When we allow other types of devices, we just call it an *optical network*. Optical networks are to general optical devices what quantum circuits are to general multi-qubit unitaries, and they can also be measured in terms of size, depth and width, as we will see shortly.

3.2.2 Quantum computing with bosonic optical networks

The purpose of this subsection is to show how optical networks are used in quantum computing and information theory. Up to this point, our description of optical devices was done in terms of their Hamiltonian, and classified according to their action over bosonic creation operators. In this section we discuss the action of optical devices over the bosonic Fock-space, with basis states given in eq. (2.25).

When considering the effects of devices over Fock-states, we will be primarily interested in devices whose action preserve the bosonic vacuum state. Given the general Hamiltonian of eq. (3.67), maps will only preserve the vacuum state if, for every monomial term, the sum of the degree of annihilation operators is the same as the sum of the degree of creation operators. This is equivalent to saying that the Hamiltonians describes a process that conserves the number of particles which, in the Gaussian case, corresponds to the set of passive devices. Therefore, all devices discussed in this subsection are passive.

Call the Fock-space for a set of m bosonic modes by the name $\mathscr{F}^b_m$. The action of an m-mode Gaussian device U over the single-particle subspace $N_1(\mathscr{F}^b_m)$ generated by the basis states $\{|i\rangle = \hat{b}_i^\dagger |0\rangle_b\}$, with $i = 1, \ldots, m$, can be obtained directly by applying eq. (3.68), which yields

$$|i\rangle_{out} = \sum_j U_{i,j} |j\rangle_{in} . \tag{3.71}$$

In fact, since the matrix of coefficients $U_{i,j}$ rules the transformation law over the creation operators themselves, the probability amplitudes for multi-particle states must be functions of these coefficients. This is why we call the matrix defined by the $U_{i,j}$ the *characteristic matrix* of the device.

In order to make the point about the role of $U_{i,j}$ more clear, consider the following example. Consider a balanced beam splitter ($\theta = \pi/4$) between modes 1 and 2. The characteristic matrix for this device is

$$[BS_{1,2}(\pi/4)] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} . \tag{3.72}$$

Now take the scenario where two particles are sent through this device. There are three different possible input states. In two of them, the two particles are sent into the same mode and, in the third, each particle goes into a different mode.

We can calculate the action of this device over all possible states using eq. (3.57), but let us restrict ourselves to the third of the aforementioned states. This third state has the form $|1,1\rangle_b = \hat{b}_1^\dagger \hat{b}_2^\dagger |0\rangle_b$, and its behavior under the balanced beam splitter exhibits an interesting effect. Doing the calculation we obtain

$$BS_{1,2}(\pi/4) |1,1\rangle_b = BS_{1,2}(\pi/4) \hat{b}_1^\dagger \hat{b}_2^\dagger |0\rangle_b = \frac{1}{2} (i(\hat{b}_1^\dagger)^2 + \hat{b}_1^\dagger \hat{b}_2^\dagger - \hat{b}_2^\dagger \hat{b}_1^\dagger + i(\hat{b}_2^\dagger)^2) |0\rangle_b . \tag{3.73}$$

The last result can be further simplified by applying eqs. (2.22), leading to

$$BS_{1,2}(\pi/4) |1,1\rangle_b = \frac{i}{\sqrt{2}} (|2,0\rangle_b + |0,2\rangle_b). \tag{3.74}$$

This result implies that two identical bosons, coming in trough different arms of a balanced beam splitter, always come out of together, as if there was an attractive force between the particles. This phenomena is called the *Hong-Ou-Mandel* [129], or *bunching effect.*

Now let us go to the general case. Given an arbitrary Fock state $|n_1, ..., n_m\rangle_b$ of $N = \sum_i n_i$ particles over m modes, and a Gaussian device U let $\Omega_n = (1^{n_1}, \ldots, m^{n_m})$ be the ordered set of N elements, such that the first n_1 elements are 1, the next n_2 elements are 2 and so on, up to the last n_m elements which are given by m. In [212] it was proven that

$$\langle k_1, \ldots, k_m|\hat{U}|l_1, \ldots, l_m\rangle_b = \frac{\text{perm}[\{U_{\Omega_k(i),\Omega_l(j)}\}_{i,j=1,\ldots,N}]}{\sqrt{\prod_i^m l_i! \times \prod_j^m k_j!}}. \tag{3.75}$$

where the $N \times N$ matrix $\{U_{\Omega_k(i),\Omega_l(j)}\}_{i,j=1,\ldots,N}$ is such that $\Omega_k(i)$ equals the i-th element in the list Ω_k and $\Omega_l(j)$ equals the j-th element in Ω_l.

The function perm(A) of an $N \times N$ matrix A is called the *permanent* of A, and is given by the expression

$$\text{perm}(A) = \sum_{\sigma \in S^N} \prod_i^N A_{i,\sigma(i)}, \tag{3.76}$$

where S^N is the permutation group over N elements. The permanent is very similar to the *determinant* of A, which, for the sake of completeness, is given by

$$\det(A) = \sum_{\sigma \in S^N} (-1)^{sgn(\sigma)} \prod_i^N A_{i,\sigma(i)}. \tag{3.77}$$

Later, we will see how the difference between permanents and determinants play a major role in the describing the computing power of standard identical particle systems.

Using the general formula for Fock-state amplitudes, we can see how the Hong-Ou-Mandel effect is determined by the permanent of the characteristic matrix in eq. (3.72). For the amplitude $\langle 1, 1|BS_{1,2}(\pi/4)|1, 1\rangle_b$ we have $\Omega_{1,1} = \{1, 2\}$, which implies that

$$\{BS_{1,2}(\pi/4)_{\Omega_{1,1}(i),\Omega_{1,1}(j)}\}_{i,j=1,2} = [BS_{1,2}(\pi/4)] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{3.78}$$

and therefore

$$\langle 1, 1|BS_{1,2}(\pi/4)|1, 1\rangle_b = \text{perm}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\right) = \frac{1}{2}(1 + i^2) = 0. \tag{3.79}$$

When the probability amplitude of a state with relation to a particular interferometer and input is zero, we have what we call a suppressed state. The suppression of the state $|1, 1\rangle_b$, signals the Hong-Ou-Mandel effect, and is a different way to state it.

As we saw in subsection 3.2.1, linear optical networks are an alternative implementation of a passive Gaussian device. Therefore, these networks also have a characteristic matrix, implying that the formula we presented for calculating transition amplitudes between N-particle states also applies the networks themselves. This fact was used to provide the first examples of application of bosonic systems in quantum computing [151].

Several works [47, 53, 72, 224, 227, 232, 240] used single photons and linear optical network over 2^m modes to simulate the action of of an m-qubit unitary. The idea is that every single particle state in $N_1(\mathscr{F}^b_{2^m})$ encodes a single computational basis state of an m-qubit system. Therefore, the characteristic matrix of an 2^m-mode linear optical network becomes the unitary matrix of a general m-qubit map. Since these networks generate all passive Gaussian maps, which in this case form the group $U(2^m)$, the set of all $BS_{i,j}(\theta)$ and $PS_i(\tau)$ is universal for quantum computing in this encoding scheme.

The single-photon encoding, discussed above, is extremely inefficient in terms of network resources, due to the exponential growth of the number of necessary modes. Therefore, realistic models of photonic computation use the so called *dual-rail encoding*. In this encoding, n qubits are mapped to the states of n particles in $2n$ modes, such that each logical qubit is supported in a pair of neighboring modes. The logical single-qubit states are defined by

$$|0\rangle \equiv |1,0\rangle_b\,, \tag{3.80a}$$

$$|1\rangle \equiv |0,1\rangle_b\,. \tag{3.80b}$$

while a two-qubit system needs four modes, with corresponding logical states

$$|0\rangle \otimes |0\rangle \equiv |1,0,1,0\rangle_b\,, \quad |0\rangle \otimes |1\rangle \equiv |1,0,0,1\rangle_b\,, \tag{3.81a}$$

$$|1\rangle \otimes |0\rangle \equiv |0,1,1,0\rangle_b\,, \quad |1\rangle \otimes |1\rangle \equiv |0,1,0,1\rangle_b\,, \tag{3.81b}$$

and so on.

It was shown that, with this encoding, it is possible to perform any logical single-qubit gate using only phase shifters and beam splitters. To prove this, consider a qubit encoded in modes 1 and 2. A phase shifter on mode 2 acts on the logical basis states as

$$PS_2(\tau)\,|1,0\rangle_b = |1,0\rangle_b\,, \tag{3.82a}$$

$$PS_2(\tau)\,|0,1\rangle_b = e^{i\tau}\,|0,1\rangle_b\,, \tag{3.82b}$$

which is a logical Z rotation on the Bloch sphere by τ. A beam splitter between modes 1 and 2 acts on the logical basis states as

$$BS_{12}(\theta)\,|1,0\rangle_b = \cos\theta\,|1,0\rangle_b + i\sin\theta\,|0,1\rangle_b\,, \tag{3.83a}$$

$$BS_{12}(\theta)\,|0,1\rangle_b = i\sin\theta\,|1,0\rangle_b + \cos\theta\,|0,1\rangle_b\,, \tag{3.83b}$$

which is a logical X rotation in the Bloch sphere by an angle θ. Therefore, in this encoding, the general beam splitter decomposition in Fig. 18 maps into the single-qubit unitary decomposition of eq. (3.14) over logical states.

To build a universal computer, we must also be able to synthesise an entangling two-qubit gate. But this task showed itself to be much harder than in the case of the single-photon encoding. This was mostly due to the Hong-Ou-Mandel effect not preserving the encoded subspaces used for computation. The works [51,52,67,132,133,180] circumvented the problem by introducing *non-Gaussian optical elements* into their optical networks.

One type of non-Gaussian device that was used in these works is the *cross-Kerr media*, described by the Hamiltonian

$$\hat{H} = \kappa \hat{b}_1^\dagger \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2, \tag{3.84}$$

which is a quartic operator. These devices leave the single-particle subspace invariant, but generate non-trivial transformations in the two-particle subspace. This fact allows them to implement the CZ gate in the dual-rail encoding. One of the problems with this approach, though, is that the κ parameters tends to be very small for most materials, making them hard to use in practice.

Another approach, which did not require the use of non-Gaussian devices, was introduced by Knill, LaFlamme and Milburn in [150]. They provided a protocol to generate an encoded CZ gate, which is an entangling gate, by executing the network in figure Fig. 19. This network only works if the *non-linear sign gate* NS in Fig. 20 is correctly implemented, which is not a deterministic process.

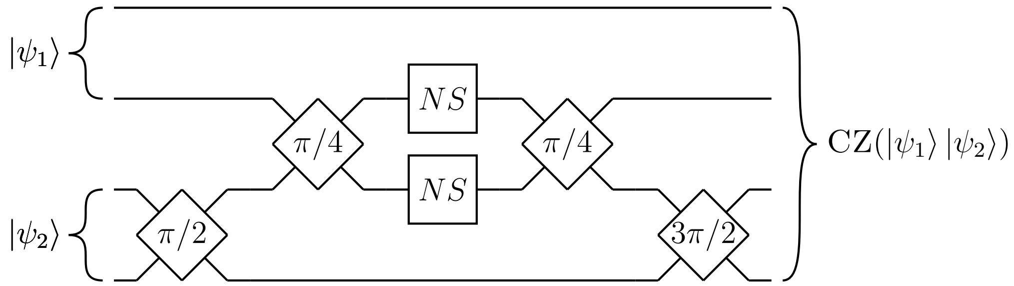


Figure 19 – Optical implementation of a CZ gate. The states $|\psi_1\rangle$ and $|\psi_2\rangle$ are arbitrary logical qubit states, and the NS gate is such that $NS(a\,|0\rangle + b\,|1\rangle + c\,|2\rangle) = a\,|0\rangle + b\,|1\rangle - c\,|2\rangle$. Therefore, if a single photon hits any of the two arms of the first $\pi/4$ beam-splitter the state is left unaltered, but if two photons hit each arm, the Hong-Ou-Mandel effect forces both to pass through the NS gates, in both output arms, which introduces the -1 characteristic of the CZ gate.

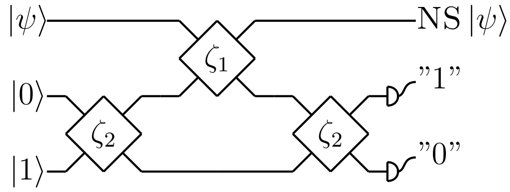


Figure 20 – Optical implementation of the NS gate. Of the three modes, the bottom two are ancillas prepared in a specific single-photon state. The three beam-splitters are such that $\cos(\zeta_1) = 1/(4 - 2\sqrt{2})$ and $\cos(\zeta_2) = 3 - 2\sqrt{2}$. To obtain the desired gate, one needs to measure the number of photons in the bottom modes using the photodetectors illustrated by the small semi-circles. The gate is applied correctly if one obtains the measurement indicated on the figure, which occurs with probability 1/4.

Since the NS gate is probabilistic, one cannot use it directly in the network without making the probability of implementing a general circuit exponentially small. Instead, the protocol is used to prepare an entangled resource state, that is used in a bigger network, together with adaptive measurements. Therefore, the KLM protocol has the advantage of not requiring non-Gaussian devices in its networks, but has the disadvantage of having to prepare states previously to any computation, and using it in an adaptive circuit scheme.

Even if we lack the capacity to reliably implement arbitrary quantum circuits using linear optical networks, the capacity of implementing linear networks themselves appears to offer some type of computational advantage over classical computation. This was discovered by Aaronson and Arkhipov in [1], where they showed that sending n bosons trough a random[7] multi-mode interferometer generates samples of a probability distribution that cannot be efficiently sampled by any classical algorithm. This is known as the *Boson Sampling problem*, and its difficulty is attributed to, among other factors, the fact that the amplitudes are proportional to permanents of large matrices [243].

3.2.3 Generalized quantum interferometry and Fermionic linear optics

We have just seen how systems of standard identical bosons can be used to represent quantum circuits. Interestingly, systems of identical fermions can also be used to do the same, as we see now [35, 62, 236]. For bosonic systems, quantum circuits were defined in terms of networks of optical devices, which are themselves described as dynamical maps generated by Hamiltonians of the form shown in eq. (3.67). From this representation, the definition of an optical device can be extended from bosonic systems to fermionic systems [149].

We define m-mode *fermionic optical devices* as dynamical maps generated by Hamiltonians of the form

$$\hat{H} = \sum_{\{k_i\},\{l_i\}} M_{\{k_i\},\{l_i\}} \prod_{i=1}^{m} \left(\hat{f}_i^{\dagger k_i} \hat{f}_i^{l_i} \right). \tag{3.85}$$

This set of polynomials forms an algebra called the *Clifford algebra* **Cliff**(m), which plays an important role in fermionic quantization [257]. By analogy to the bosonic case, we call *fermionic Gaussian devices* the maps generated by

$$\hat{H} = \sum_{i_1,i_2=1}^{2} A_{i_1,i_2} \hat{f}_{i_1}^{\dagger} \hat{f}_{i_2} + \sum_{i_1,i_2=1}^{2} \left(B_{i_1,i_2} \hat{f}_{i_1} \hat{f}_{i_2} + C_{i_1,i_2} \hat{f}_{i_1}^{\dagger} \hat{f}_{i_2}^{\dagger} \right), \tag{3.86}$$

which are further classified into passive ($A_{i_1,i_2} \neq 0$, and $B_{i_1,i_2} = C_{i_1,i_2} = 0$) and active ($A_{i_1,i_2} = 0$, and $B_{i_1,i_2} = C_{i_1,i_2} \neq 0$) devices.

[7] In the Haar measure [130, 131].

In two-mode systems, Hamiltonians of passive Gaussian devices a real vector space with basis $\{N_{1,2}, T^1_{1,2}, T^2_{1,2}, T^3_{1,2}\}$, given by

$$\hat{T}^1_{1,2} = \frac{1}{2}(\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_1), \tag{3.87a}$$

$$\hat{T}^2_{1,2} = \frac{-i}{2}(\hat{f}_1^\dagger \hat{f}_2 - \hat{f}_2^\dagger \hat{f}_1), \tag{3.87b}$$

$$\hat{T}^3_{1,2} = \frac{1}{2}(\hat{f}_1^\dagger \hat{f}_1 - \hat{f}_2^\dagger \hat{f}_2), \tag{3.87c}$$

$$\hat{N}_{1,2} = \hat{f}_1^\dagger \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_2. \tag{3.87d}$$

These basis elements are, as in the bosonic case, generators of the Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$.

Interestingly, the real vector space of Hamiltonians given by the basis elements

$$\hat{R}^1_{1,2} = \frac{1}{2}(\hat{f}_1^\dagger \hat{f}_2^\dagger + \hat{f}_2 \hat{f}_1), \tag{3.88a}$$

$$\hat{R}^2_{1,2} = \frac{-i}{2}(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_2 \hat{f}_1), \tag{3.88b}$$

$$\hat{R}^3_{1,2} = \frac{1}{2}(\hat{f}_1^\dagger \hat{f}_1 + \hat{f}_2^\dagger \hat{f}_2 - I), \tag{3.88c}$$

is also isomorphic to $\mathfrak{su}(2)$ as a Lie algebra. The set of all two-mode, fermionic Gaussian Hamiltonians is a real vector space with basis $\{N_{1,2}, T^1_{1,2}, T^2_{1,2}, T^3_{1,2}, R^1_{1,2}, R^2_{1,2}.R^3_{1,2}\}$, and is isomorphic to $\mathfrak{u}(1) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)$ as a Lie algebra. This contrasts with the bosonic case, where the corresponding algebra is $\mathfrak{sp}(4, \mathbb{R})$.

The action of an m-mode fermionic passive Gaussian device over creation operators has the form

$$\hat{f}_i^\dagger(t) = \sum_j U_{i,j} \hat{f}_j^\dagger(0), \tag{3.89}$$

where the matrix U of coefficients $U_{i,j}$ belongs to the unitary group $\mathbf{U}(m)$, which is identical to the bosonic counterpart. On the other hand, general fermionic Gaussian devices act as

$$U \hat{f}_i^\dagger(t) U^\dagger = \sum_{j=1}^{m} (W_{i,j} \hat{f}_j^\dagger(0) + V_{i,j} \hat{f}_j(0)), \tag{3.90}$$

where the matrices W and V, with coefficients $W_{i,j}$ and $V_{i,j}$, are such that this map is canonical, which implies that, taken together, the pair gives a representation of the $\mathbf{SO}(2m)$ group.

As pointed out in [258], while the group of passive Gaussian devices for fermions is isomorphic to the one for bosons, the group of all Gaussian devices is not. This is true because the dimensions of the Lie groups $\mathbf{SO}(2m)$ and $\mathbf{Sp}(2m, \mathbb{R})$ are different, being given respectively by $2m^2 - m$ and $8m^2 - 2m$. The difference in the size is due to the differences between the fermionic and bosonic oscillator algebras, given that Gaussian devices act as linear canonical transformations for both bosons and fermions. These maps are also called *Bogoliubov transformations*, and play an important role in characterizing entanglement for identical particle systems [68].

Fermionic, passive linear optical networks are built from the fermionic beam splitters and phase shifters defined bellow

$$PS_i(\tau) = \exp\left(i\tau \hat{f}_i^\dagger \hat{f}_i\right),$$
$$BS_{i,j}(\theta) = \exp\left(i\theta(\hat{f}_i^\dagger \hat{f}_j + \hat{f}_j^\dagger \hat{f}_i)\right).$$

Their actions over fermionic creation operators are given by

$$PS_i(\tau) f_j^\dagger PS_i^\dagger(\tau) = e^{i\tau\delta_{ij}} f_j^\dagger \tag{3.91}$$
$$BS_{i,j}(\theta)\hat{f}_i^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta \hat{x}_i^\dagger + i\sin\theta \hat{f}_j^\dagger, \tag{3.92}$$
$$BS_{i,j}(\theta)\hat{f}_j^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta \hat{f}_j^\dagger + i\sin\theta \hat{f}_i^\dagger. \tag{3.93}$$

which are identical to their bosonic counterparts. Again, just as in the bosonic case, the group structure of passive Gaussian devices implies that every linear network over m modes has an $m \times m$ unitary characteristic matrix U, defined in terms of the network action over the single-particle subspace $N_1(\mathscr{F}_m^f)$ with basis $\{\hat{f}_i^\dagger |0\rangle_f\}$.

To understand the action of a general network over an arbitrary fermionic Fock state, we begin with an example. As explored in the bosonic case (see 3.2.2), consider a scenario where two fermions are sent through a balanced fermionic beam-splitter. The only two-particle fermionic state allowed by the exclusion principle is given by $|1,1\rangle_f = \hat{f}_1^\dagger \hat{f}_2^\dagger |0\rangle_f$. The action of the beam splitter over this state is given by

$$BS_{1,2}(\pi/4)|1,1\rangle_f = BS_{1,2}(\pi/4)\hat{f}_1^\dagger \hat{f}_2^\dagger |0\rangle_f = \frac{1}{2}(i(\hat{f}_1^\dagger)^2 + \hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_2^\dagger \hat{f}_1^\dagger + i(\hat{f}_2^\dagger)^2)|0\rangle_f, \tag{3.94}$$

which is equal to $|1,1\rangle_f$. In fact, for any θ, the fermionic two-particle state is unaltered under propagation through a fermionic beam splitter, an effect called *anti-bunching*.

In the general case, using the notation in eq. (3.75), we have that the action of a general m-mode fermionic linear network, with characteristic matrix U, over the states $|\mathbf{k}\rangle$ and $|\mathbf{l}\rangle$, each of N particles, is given by

$$\langle k_1,\ldots,k_m|\hat{U}|l_1,\ldots,l_m\rangle_b = \frac{\det\left[\{U_{\Omega_k(i),\Omega_l(j)}\}_{i,j=1,\ldots,N}\right]}{\sqrt{\prod_i^m l_i! \times \prod_j^m k_j!}}. \tag{3.95}$$

The denominator is often omitted, due to restriction on the values of occupation numbers imposed by the exclusion principle. But here I chose to maintain it, to emphasize the structural similarity with eq. (3.75). Using this general result, we can show that the previous calculation of the anti-bunching effect is given by

$$\langle 1,1|BS_{1,2}(\pi/4)|1,1\rangle_f = \det\left[\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\right] = 1. \tag{3.96}$$

Having the general behavior of fermionic linear networks, we are now in position to discuss how they can simulate quantum circuits. First, in contrast to the bosonic case, for fermions there exists natural correspondence between the m-fermion Fock space $\mathscr{F}_m^f$, and the m-qubit Hilbert space $\mathscr{B}^m$. This is true because the Pauli exclusion principle limits the maximum occupation number per mode to be one, which allows fermionic Fock states to represent computational basis states using the correspondence

$$|n_1, \ldots, n_m\rangle_f \equiv |n_1\rangle \otimes \cdots \otimes |n_m\rangle \,. \tag{3.97}$$

We hereby call this map the *fermion-qubit correspondence*, and use it to define a natural encoding of m-qubit systems, called the *single-rail encoding.*

In the single-rail encoding, all fermionic Gaussian devices yield encoded circuits for m-qubits. To take this fact into consideration, we extend the definition of fermionic linear networks to include networks that are also built using the fermionic two-mode down converter

$$D_{i,j}(\kappa) = \exp\Big\{i\kappa(\hat{f}_i^\dagger \hat{f}_j^\dagger + \hat{f}_j \hat{f}_i)\Big\}. \tag{3.98}$$

This element, together with fermionic phase shifters and beam splitters, generate networks capable of implementing any general fermionic Gaussian map.

The most general, two-mode Gaussian device $U_{1,2}$ has the form

$$U_{1,2} = \exp\Big\{i\phi \hat{N}_{1,2}\Big\} \exp\{i\mathbf{a} \cdot \mathbf{T}_{1,2}\} \exp\{i\mathbf{b} \cdot \mathbf{R}_{1,2}\}, \tag{3.99}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{a} = (b_1, b_2, b_3)$ and $\mathbf{T}_{1,2} = (T_{1,2}^1, T_{1,2}^2, T_{1,2}^3)$, $\mathbf{R}_{1,2} = (R_{1,2}^1, R_{1,2}^2, R_{1,2}^3)$. The action of such device over the two-particle subspace $N_2(\mathscr{F}_2^f)$ of basis $(|0,0\rangle_f$, $|0,1\rangle_f$, $|1,0\rangle_f$, $|1,1\rangle_f)$ has the form

$$U = e^{i\phi} \begin{bmatrix} U_{1,1}^R(\mathbf{b}) & 0 & 0 & U_{1,2}^R(\mathbf{b}) \\ 0 & U_{1,1}^T(\mathbf{a}) & U_{1,2}^T(\mathbf{a}) & 0 \\ 0 & U_{2,1}^T(\mathbf{a}) & U_{2,2}^T(\mathbf{a}) & 0 \\ U_{2,1}^R(\mathbf{b}) & 0 & 0 & U_{2,2}^R(\mathbf{b}) \end{bmatrix}, \tag{3.100}$$

where the matrix functions $U^T(\mathbf{a})$ and $U^R(\mathbf{b})$ both belong to $\mathbf{SU}(2)$, for all coefficients $\mathbf{a}$ and $\mathbf{b}$. The corresponding two-qubit gate is called a *matchgate*, and we discuss their properties in the next section.

Since the group of all fermionic Gaussian operators in m-modes is $\mathbf{SO}(2m)$, whose dimension is exponentially smaller than $\mathbf{SU}(2^m)$, the set of fermionic linear optical networks is not universal for quantum computing. Universality is achieved using non-Gaussian devices ($\hat{H} = \kappa \hat{n}_1 \hat{n}_2$) [35] but it is not achieved using adaptivity. This was shown in [149] and [236] by noticing that any post-measurement[8] state of a fermionic linear network is a

[8] Measurements of individual occupation numbers.

computational state of a fermionic system with less modes, up to linear optical transformations [62]. Therefore, the use of fermionic number measurements does not change the group structure of the transformations, and it stays in a group with dimension exponentially smaller than $\mathbf{SU}(2^m)$.

To compare this result with the computing power of bosonic linear optics, we must restrict ourselves to passive networks, and qubits in dual-rail encoding. For $2m$ mode systems of fermions and bosons in the dual-rail encoding (which has m particles), passive networks forms a high dimensional representation a subgroup of the $\mathbf{U}(m)$, but the group dimension (which is related to the number of different group elements) is exponentially smaller than $\mathbf{SU}(2^m)$. On the other hand, adjoining the capacity of implementing occupation number measurements during computation leads bosonic linear networks into computational universality, while it does not do the same for the fermionic case. It appears that, even with the higher dimensionality of the associated representations, the restriction to networks that preserve the encoding spoils their universality in the bosonic case.

Another way to compare both systems is to ask if there exists an efficient classical algorithm that samples the distribution of obtaining particular occupation number outcomes for particular subsets of modes. For bosonic systems, this is equivalent to the Boson Sampling problem, for which there is strong computational evidence of the non-existence of an efficient classical sampling algorithm. As mentioned in subsection 3.2.2, this is mostly due to this distribution being given by the permanents of large matrices. On the other hand, it was shown in [236] that, for the fermionic case, such algorithms do exist, since the relevant distribution involves functions of the determinant, for which efficient algorithms are known [13].

3.3 The optical equivalent of the quantum circuit model

After discussing quantum circuits and optical networks, now I discuss under what conditions they are equivalent. In subsection 3.3.1, I introduce a multi-mode oscillator algebra of qubits and show that they describe a system of hard-core bosons. Next, I define Gaussian optical devices and optical networks for hard-core bosons and show that their generators do not form a closed Lie algebras.

In subsection 3.3.2, I prove that the action of linear optical devices over creation and annihilation operators for hard-core bosons is not linear due to the particle statistics. I demonstrate that arbitrary quantum gates are equivalent to dynamical maps for hard-core bosons, giving single-qubit gates as an example. Finally, I exhibit the hard-core boson algebra as a fermionic anyon algebra via a Jordan-Wigner transform and explain the nature of their relationship with matchgate circuits.

3.3.1 Qubits and particles

As we saw in section 3.1, the Hilbert space $\mathscr{B}^m$ of an m-qubit system has as its canonical basis the states

$$|A_1\rangle \otimes \cdots \otimes |A_m\rangle\,, \tag{3.101}$$

where the A_i are Boolean variables. We also saw, this time in section 3.2, that the fermionic Fock space $\mathscr{F}_m^f$ is isomorphic to $\mathscr{B}^m$, via the fermion-qubit correspondence. Since the Fock space is the representation space of a multi-mode oscillator algebra, it makes sense to ask if the multi-qubit space $\mathscr{B}^m$ can also be seen in this way. In other words, one might ask if there exists a multi-mode oscillator algebra which has $\mathscr{B}^m$ as its representation space.

The existence of the Fock space representation for an oscillator algebra entails a natural particle interpretation, as discussed in subsection 2.3.3.2. Therefore, if such an algebra exists, we must interpret the Boolean variables A_i in the last equation, as the i-th occupation number n_i for a type of particle system. This new interpretation entails, for example, that the vacuum state of this particle system must be

$$|0\rangle_q \equiv \underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{m}, \tag{3.102}$$

where q means qubit. Similarly, single-particle states must be defined as

$$\hat{q}_i^\dagger\,|0\rangle_q \equiv \underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{i-1} \otimes |1\rangle \otimes \underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{m-(i+1)}, \tag{3.103}$$

where $\hat{q}_i^\dagger$ is the creation operator for the qubit oscillator algebra.

By using the properties of computational basis states, we can obtain the commutation relations for the qubit oscillator algebra. These were first stated in [258], and are

$$[\hat{q}_i; \hat{q}_j^\dagger] = [\hat{q}_i; \hat{q}_j] = [\hat{q}_i^\dagger; \hat{q}_j^\dagger] = 0, \tag{3.104}$$

for all modes i, j with $i \neq j$, [compare with eq. (2.22)] and

$$\hat{q}_i\hat{q}_i^\dagger + \hat{q}_i^\dagger\hat{q}_i = 1, \tag{3.105a}$$
$$(\hat{q}_i^\dagger)^2 = (\hat{q}_i)^2 = 0, \tag{3.105b}$$

for each mode i [compare with eq.(2.29)].

The particles described by these commutation relations satisfy the Pauli exclusion principle, just as standard fermions do, but do not acquire any exchange phase under particle permutation, just as standard bosons do. Therefore, these particles are called *hard-core bosons*, and their Fock-space $\mathscr{F}_m^q$ has basis given by

$$|n_1, ..., n_m\rangle = (\hat{q}_1^\dagger)^{n_1} ... (\hat{q}_m^\dagger)^{n_m}\,|0\rangle_q\,, \tag{3.106}$$

which is also isomorphic to $\mathscr{B}^m$, as expected.

Given this construction for the hard-core boson algebra, we can treat it as an abstract particle system and study the structure of quantum circuits in terms of the operator algebra alone, as done in [258]. In order to accomplish this, we define optical devices for hard-core bosons as unitary operators generated by the Hamiltonians of the form

$$\hat{H} = \sum_{\{k_i\},\{l_i\}} M_{\{k_i\},\{l_i\}} \prod_{i=1}^{m} \left(\hat{q}_i^{\dagger k_i} \hat{q}_i^{l_i}\right). \tag{3.107}$$

Just as before, Gaussian devices are generated by

$$\hat{H} = \sum_{i_1,i_2=1}^{2} A_{i_1,i_2} \hat{q}_{i_1}^\dagger \hat{q}_{i_2} + \sum_{i_1,i_2=1}^{2} \left(B_{i_1,i_2} \hat{q}_{i_1} \hat{q}_{i_2} + C_{i_1,i_2} \hat{q}_{i_1}^\dagger \hat{q}_{i_2}^\dagger\right), \tag{3.108}$$

which are further classified into passive ($A_{i_1,i_2} \neq 0$, and $B_{i_1,i_2} = C_{i_1,i_2} = 0$) and active ($A_{i_1,i_2} = 0$, and $B_{i_1,i_2} = C_{i_1,i_2} \neq 0$) devices.

The set of two-mode Gaussian Hamiltonians is a real vector space with basis $\{N_{1,2}, T_{1,2}^1, T_{1,2}^2, T_{1,2}^3, R_{1,2}^1, R_{1,2}^2, R_{1,2}^3\}$, where

$$\hat{T}_{1,2}^1 = \frac{1}{2}(\hat{q}_1^\dagger \hat{q}_2 + \hat{q}_2^\dagger \hat{q}_1), \quad \hat{R}_{1,2}^1 = \frac{1}{2}(\hat{q}_1^\dagger \hat{q}_2^\dagger + \hat{q}_2 \hat{q}_1), \tag{3.109a}$$

$$N_{1,2} = \hat{q}_1^\dagger \hat{q}_1 + \hat{q}_2^\dagger \hat{q}_2 \quad \hat{T}_{1,2}^2 = \frac{-i}{2}(\hat{q}_1^\dagger \hat{q}_2 - \hat{q}_2^\dagger \hat{q}_1), \quad \hat{R}_{1,2}^2 = \frac{-i}{2}(\hat{q}_1^\dagger \hat{q}_2^\dagger - \hat{q}_2 \hat{q}_1), \tag{3.109b}$$

$$\hat{T}_{1,2}^3 = \frac{1}{2}(\hat{q}_1^\dagger \hat{q}_1 - \hat{q}_2^\dagger \hat{q}_2), \quad \hat{R}_{1,2}^3 = \frac{1}{2}(\hat{q}_1^\dagger \hat{q}_1 + \hat{q}_2^\dagger \hat{q}_2 - I). \tag{3.109c}$$

Using the commutation relations for hard-core bosons, is not hard to see that the these basis elements are generators of the Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

We saw, in sections 3.2.1 and 3.2.3, that m-mode bosonic and fermionic passive Gaussian devices are described by Hamiltonians that belong to the $\mathfrak{u}(1) \oplus \mathfrak{su}(m)$ Lie algebra. Nonetheless, m-mode passive Gaussian Hamiltonians for hard-core bosons, written as Hermitian linear combinations of the operator basis $\{\hat{q}_i^\dagger \hat{q}_j\}_{i,j=1,\ldots,m}$, do not form a closed Lie algebra. To see this, notice that the commutators of this basis have quartic operators in them, as shown below

$$[\hat{q}_i^\dagger \hat{q}_j; \hat{q}_k^\dagger \hat{q}_l] = \delta_{j,k} \hat{q}_i^\dagger \hat{q}_l - \delta_{i,l} \hat{q}_k^\dagger \hat{q}_j + \Delta_{i,j,k,l}^q \hat{q}_i^\dagger \hat{q}_j^\dagger \hat{q}_k \hat{q}_l, \tag{3.110}$$

where

$$\Delta_{i,j,k,l}^q = \begin{cases} 2 & \text{, if } i = l \neq k \neq j \\ -2 & \text{, if } i \neq j = k \neq l \\ 0 & \text{, otherwise} \end{cases}. \tag{3.111}$$

Interestingly, when we look at general, Gaussian Hamiltonians for hard-core bosons acting on a specific pair of modes i and j, in place of modes 1 and 2, the real vector space with basis $\{N_{i,j}, T^1_{i,j}, T^2_{i,j}, T^3_{i,j}, R^1_{i,j}, R^2_{i,j}, R^3_{i,j}\}$ forms the Lie algebra $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Therefore, we can still define Gaussian optical networks as sequences of phase shifters, beam splitters and two-mode down converters, given respectively by

$$PS_i(\tau) = \exp\Big\{i\tau\hat{q}_i^\dagger\hat{q}_i\Big\}, \tag{3.112a}$$

$$BS_{i,j}(\theta) = \exp\Big\{i\theta(\hat{q}_i^\dagger\hat{q}_j + \hat{q}_j^\dagger\hat{q}_i)\Big\}, \tag{3.112b}$$

$$D_{i,j}(\kappa) = \exp\Big\{i\kappa(\hat{q}_i^\dagger\hat{q}_j^\dagger + \hat{q}_j\hat{q}_i)\Big\}. \tag{3.112c}$$

Using this definition, we are able to reinterpret the quantum circuit model in terms of the optical networks model, which we explore next.

3.3.2 Quantum circuits and Gaussian networks for hard-core bosons

Having understood the oscillator algebra of hard-core bosons in their own terms, we can now see how they can be written in terms of qubit unitary maps. We can see, from the matrix representations of creation and annihilation operators for single qubits, that these operators can be written as linear combinations of the Pauli matrices below

$$\hat{q}_i^\dagger = \frac{1}{2}(X_i + iY_i), \tag{3.113a}$$

$$\hat{q}_i = \frac{1}{2}(X_i - iY_i), \tag{3.113b}$$

$$\hat{n}_i = \hat{q}_i^\dagger\hat{q}_i = 2(Z_i - I), \tag{3.113c}$$

where $\hat{n}_i$ is the number operator for qubit oscillators.

These identities allow us to write any m-qubit map in terms of qubit oscillators. For example, the most general single-qubit gate, proportional to the one given in eq. (3.13), can be written as the the unitary

$$U = \exp\Big\{i(a + b\hat{q} + c\hat{q}^\dagger + d\hat{q}^\dagger\hat{q})\Big\}, \tag{3.114}$$

for some set of coefficients a, b, c, d.

The map in eq. (3.113a) also allows us to do the opposite, write hard-core boson Hamiltonians in terms of Pauli operators. For example, the beam splitter, has the form

$$BS_{i,j}(\theta) = \exp\Big\{i\theta(\hat{q}_i^\dagger\hat{q}_j + \hat{q}_j^\dagger\hat{q}_i)\Big\} = \exp\left\{i\frac{\theta}{2}(X_iX_j + Y_iY_j)\right\}. \tag{3.115}$$

Using it, we can calculate the action of the beam splitter over linear combinations of Pauli operators, and then rewrite this in terms of hard-core bosons operators. This procedure allows us to obtain the action of passive Gaussian devices over creation and annihilation operators, which are given by

$$PS_i(\tau)\hat{q}_j^\dagger PS_i^\dagger(\tau) = e^{i\delta_{i,j}\tau}\hat{q}_j^\dagger, \tag{3.116a}$$

$$BS_{i,j}(\theta)\hat{q}_i^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta\hat{q}_i^\dagger + i\sin\theta\hat{q}_j^\dagger(1 - 2\hat{q}_i^\dagger\hat{q}_i), \tag{3.116b}$$

$$BS_{i,j}(\theta)\hat{q}_j^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta\hat{q}_j^\dagger + i\sin\theta\hat{q}_i^\dagger(1 - 2\hat{q}_j^\dagger\hat{q}_j). \tag{3.116c}$$

Therefore, in contrast to both bosonic and fermionic devices, the action of hard-core bosonic beam splitters is non-linear over creation operators. Using these equations for beam splitters, and their analogues for two-mode down converters, we deduce that two-quit unitary associated to a general two-mode Gaussian device $U_{1,2}$ has exactly the same form as eq. (3.100).

The reason behind the isomorphism between these two algebras is the fermion-qubit correspondence, which in terms of oscillators can be put as the map

$$\hat{f}_i^\dagger \overset{J_\pi}{\to} \hat{q}_i^\dagger = \exp\left\{-i\pi \sum_{k=1}^{i-1} \hat{f}_k^\dagger \hat{f}_k\right\} \hat{f}_i^\dagger, \tag{3.117a}$$

$$\hat{f}_i \overset{J_\pi}{\to} \hat{q}_i = \exp\left\{i\pi \sum_{k=1}^{i-1} \hat{f}_k^\dagger \hat{f}_k\right\} \hat{f}_i. \tag{3.117b}$$

which is the Jordan-Wigner map J_φ, defined in section 2.4, with $\varphi = \pi$. This characterizes hard-core bosons in one-dimensions as anyonic particles, in the sense of section 2.4, but their specific anyonic properties will be described only in the next chapter.

Nevertheless, the fact that all two-mode Gaussian devices for hard-core bosons are matchgates poses an interesting paradox. This is because we known that fermionic linear networks are not universal for quantum computing, since the group they generate has dimension exponentially smaller than $\mathbf{SU}(2^m)$. But it can be proven that matchgate circuits acting over qubits (hard-core bosons) are universal for quantum computing, if we allow them to act between pairs of modes that are not nearest-neighbours [38, 39, 140, 241, 242].

This paradox is only apparent, though, as proven in [38, 140]. The reason is that, although the Fock-space representation of Gaussian devices acting over a particular pair of modes is the same in both fermionic and hard-core bosonic systems, the action over the remaining modes is not. To show this, consider the states $|1,1,0\rangle_q$, and $|1,1,0\rangle_f$. The action of a beam splitter between modes 1 and 3, for each type of particle, has the form

$$BS_{1,3}(\theta)\,|1,1,0\rangle_q = \cos\theta\,|1,1,0\rangle_q + i\sin\theta\,|0,1,1\rangle_q\,, \tag{3.118a}$$

$$BS_{1,3}(\theta)\,|1,1,0\rangle_f = \cos\theta\,|1,1,0\rangle_f - i\sin\theta\,|0,1,1\rangle_f\,. \tag{3.118b}$$

In the last equation, the commutation relations for standard fermions force the appearance of an exchange phase, which changes the way a beam-splitter normally acts. In fact, in the absence of a fermion, both actions would be identical.

This last example shows that the exchange statistics of the particles play an important role in determining the behavior of networks containing Gaussian devices between non-nearest-neighbour modes. However, when acting on nearest-neighbour modes, the action of Gaussian devices for both types of particles are identical, and this is due to the

Jordan-Wigner map, that implies the identities

$$\hat{q}_i^\dagger \hat{q}_{i+1} = J_\pi(\hat{f}_i^\dagger \hat{f}_{i+1}) = \hat{f}_i^\dagger \hat{f}_{i+1}, \tag{3.119a}$$
$$\hat{q}_i^\dagger \hat{q}_{i+1}^\dagger = J_\pi(\hat{f}_i^\dagger \hat{f}_{i+1}^\dagger) = \hat{f}_i^\dagger \hat{f}_{i+1}^\dagger. \tag{3.119b}$$

The difference between the standard fermions and hard-core bosons, when non-nearest-neighbour devices are used in networks, comes from the way exchange of creation operators happen. This was first noticed in [35], where it was shown that a generalized swap transformation, which here we call $\mathrm{SWAP}_a^{k,k+1}$, could be defined in terms of their action over creation and annihilation operators. This action is given by defined by

$$(\mathrm{SWAP}_a^{k,k+1})\hat{a}_k^\dagger(\mathrm{SWAP}_a^{k,k+1}) = \hat{a}_{k+1}^\dagger, \tag{3.120}$$
$$(\mathrm{SWAP}_a^{k,k+1})\hat{a}_k(\mathrm{SWAP}_a^{k,k+1}) = \hat{a}_{k+1}, \tag{3.121}$$
$$(\mathrm{SWAP}_a^{k,k+1})\hat{a}_{k+1}^\dagger(\mathrm{SWAP}_a^{k,k+1}) = \hat{a}_k^\dagger, \tag{3.122}$$
$$(\mathrm{SWAP}_a^{k,k+1})\hat{a}_{k+1}(\mathrm{SWAP}_a^{k,k+1}) = \hat{a}_k, \tag{3.123}$$

where the a symbol stands for b, f or q.

With this definition, we can find the Hamiltonian that generates the dynamical map responsible for the swap transformation. For both standard bosons and fermions, this Hamiltonian is Gaussian, and given by the network $PS_k(-\pi/2)BS_{k,k+1}(\pi/2)PS_{k+1}(-\pi/2)$. For hard-core bosons however, the dynamical map is

$$\mathrm{SWAP}_q^{k,k+1} = \exp\left\{i\frac{\pi}{2}\left(I + (\hat{q}_k^\dagger \hat{q}_{k+1} + \hat{q}_{k+1}^\dagger \hat{q}_k) + \frac{1}{2}(\hat{n}_k + \hat{n}_{k+1}) + \frac{1}{4}\hat{n}_k\hat{n}_{k+1}\right)\right\}, \tag{3.124}$$

which contains a quartic term $\hat{n}_k\hat{n}_{k+1}$ in the Hamiltonian, and therefore is not Gaussian.

For fermions, the fact that the swap transformation is Gaussian,means that one cannot leave the group of fermionic Gaussian transformations by introducing non-nearest-neighbour devices. This is the reason why matchgates obtained from fermionic devices are not universal, while the ones defined in terms of standard quantum circuits (or hard-core boson devices) are. In fact, by the Jordan-Wigner mapping, the swap transformation for hard-core bosons is mapped into a quartic Hamiltonian over fermions, which is known to make fermionic optical networks universal.

4 Results: The computing power of non-standard quantum statistics

I have introduced the theories of non-standard quantum statistics in chapter 2, and the models of quantum computing based on the dynamics of particles with standard quantum statistics in chapter 3. In this chapter, I introduce a generalization of such quantum computing models to two particular families of particles having non-standard quantum statistics. In 4.1, I recapitulate the definitions in section 2.4, and do a brief review of some mathematical properties necessary for expressing this thesis' results. In particular, I show how to use these properties to calculate the action of the unitary dynamics generated by two-mode, quadratic Hamiltonians for fermionic and bosonic anyons.

In 4.2, I use the unitaries generated by quadratic Hamiltonians to define abstract anyonic optical devices and an optical network model. Next, I give an interpretation of the action of these devices in terms of anyonic phenomena. I finish the section showing the action of anyonic optical devices in encoded qubits, and prove that using only anyonic beam-splitters and phase-shifters allows for universal quantum computing, which is our main result.

In 4.3, I study bosonic anyon coherent states and their behavior under optical devices. First, I give a review of the quantum theory of optical coherence, introducing the concept of generalized coherent states. Next, I explain that single-mode coherent states for both standard and anyonic bosons are equivalent, but that there is no natural physical definition of multi-mode coherent states for anyons. I finish this chapter by discussing the action of anyonic optical devices on single-mode coherent states.

4.1 Mathematical properties of anyonic operators

In subsection 4.1.1, I review the definition of anyonic oscillator algebras and compare them with standard quantum oscillators (bosons and fermions), as well as with hard-core boson oscillators. I end this subsection showing that a sub-algebra of quadratic operators acting on pairs of modes is isomorphic to a direct sum of two $\mathfrak{su}(2)$ sub-algebras, and use them to define anyonic optical devices.

In subsection 4.1.2, I explore the structure of quadratic fermionic operators to find explicit expressions for anyonic beam-spliters, phase shifters and two-mode down converters. I also show how to use these expressions to calculate the action of these devices over creation and annihilation operators for fermionic anyons.

In subsection 4.1.3, I extend the algebraic analysis done for quadratic operators over fermionic anyons to the bosonic anyon case. First, I show remark that the absence of an exclusion principle for bosonic anyons makes it impossible to write explicit expressions for operators that define optical devices. Next, I show that the action of optical devices over creation operators can still be obtained by recursion relations for commutations between them and optical operations. I end this section with a comment on the exact solubility of non-linear dynamics of one-dimensional anyons.

4.1.1 Quadratic Hamiltonian algebras over anyonic oscillators

In order to compare anyonic, standard and qubit multi-mode oscillator algebras, I introduced a special notation for their generators. As explained in section 1.2, standard fermionic and bosonic oscillators can be represented by the *symbolic variable* "x", which can stand for both the letter "f" and "b" in expressions involving fermionic and bosonic operators. For example, the commutation relations for N oscillators in eqs. (2.29,2.22) can be expressed jointly by the system of equations

$$\hat{x}_i\hat{x}_j^\dagger - (-1)^{sgn(x)}\hat{x}_j^\dagger\hat{x}_i = \delta_{i,j}, \tag{4.1a}$$

$$\hat{x}_i^\dagger\hat{x}_j^\dagger - (-1)^{sgn(x)}\hat{x}_j^\dagger\hat{x}_i^\dagger = 0, \tag{4.1b}$$

$$\hat{x}_i\hat{x}_j - (-1)^{sgn(x)}\hat{x}_j\hat{x}_i = 0 \tag{4.1c}$$

where $sgn(f) = 1$ and $sgn(b) = 0$.

Similarly, fermionic and bosonic anyon oscillators, as well as qubit oscillators (see subsection 3.3.1), are represented by the symbolic variable "χ_φ", which is shortened to χ, and is a stand-in for all possible types of bosonic ("β") and fermionic ("ξ") anyons, parametrized by the statistical angle φ. The anyonic commutation relations in eqs. (2.83,2.87) are expressed by

$$\hat{\chi}_i\hat{\chi}_j^\dagger - (-1)^{sgn(\chi)}e^{-i\varphi\epsilon_{i,j}}\hat{\chi}_j^\dagger\hat{\chi}_i = \delta_{i,j}, \tag{4.2a}$$

$$\hat{\chi}_i^\dagger\hat{\chi}_j^\dagger - (-1)^{sgn(\chi)}e^{i\varphi\epsilon_{i,j}}\hat{\chi}_j^\dagger\hat{\chi}_i^\dagger = 0, \tag{4.2b}$$

$$\hat{\chi}_i\hat{\chi}_j - (-1)^{sgn(\chi)}e^{i\varphi\epsilon_{i,j}}\hat{\chi}_j\hat{\chi}_i = 0, \tag{4.2c}$$

where $sgn(\xi) = sgn(q) = 1$ and $sgn(\beta) = 0$. In the case $\chi = q$ we have $\varphi = \pi$, which reproduces the commutation relations in eqs. (3.104,3.105a).

The Jordan-Wigner mappings defined in eqs. (2.84,2.88) can be compactly described by

$$\hat{x}_i^\dagger \xrightarrow{J_\varphi} \hat{\chi}_i^\dagger = \exp\left\{-i\varphi\sum_{k=1}^{i-1}\hat{x}_k^\dagger\hat{x}_k\right\}\hat{x}_i^\dagger \tag{4.3a}$$

$$\hat{x}_i \xrightarrow{J_\varphi} \hat{\chi}_i = \exp\left\{i\varphi\sum_{k=1}^{i-1}\hat{x}_k^\dagger\hat{x}_k\right\}\hat{x}_i, \tag{4.3b}$$

whenever the condition $sgn(\chi) = sgn(x)$ is valid.

Due to the existence of these Jordan-Wigner maps, which are algebra homomorphisms, the polynomial algebra

$$\hat{H} = \sum_{\{k_i\},\{l_i\}} M_{\{k_i\},\{l_i\}} \prod_{i=1}^{m} \left(\hat{\chi}_i^{\dagger k_i} \hat{\chi}_i^{l_i} \right), \tag{4.4}$$

is isomorphic to their standard counterpart. In other words, the polynomial algebra of general fermionic-anyon and bosonic-anyon interactions over m modes are, respectively, the Clifford algebra $\mathbf{Cliff}(m)$, and the Weyl algebra $\mathbf{Weyl}(m)$. Therefore, essentially, anyonic operators are an alternative choice of generators for these algebras.

However, the specification of dynamical maps over particles are made by finding representations of the Lie algebras of physical observables over sub-algebras of both Weyl and Clifford algebras, and this process depends explicitly on the choice between standard and anyonic generators. To illustrate this point, we must first discuss two sub-algebras of importance, the sub-algebra of *number-preserving* operators and the sub-algebra of *parity-preserving* operators, both of which were defined in section 3.2.1.

The sub-algebra of number-preserving operators can be expressed in terms of standard oscillators by the generators $\{\hat{x}_i^\dagger \hat{x}_j\}_{i,j=1,\ldots,m}$. This sub-algebra carries a representation of a Lie algebra, given by the commutation relations

$$[\hat{x}_i^\dagger \hat{x}_j ; \hat{x}_k^\dagger \hat{x}_l] = \delta_{j,k} \hat{x}_i^\dagger \hat{x}_l - \delta_{i,l} \hat{x}_k^\dagger \hat{x}_j. \tag{4.5}$$

The elements of the number-preserving sub-algebra that belongs to this Lie algebra are always quadratic. Therefore, its Hermitian elements describe passive, Gaussian devices, for both bosonic and fermionic oscillators. The sub-algebra of parity-preserving operators on the other hand, is generated by $\{\hat{x}_i^\dagger \hat{x}_j, \hat{x}_i \hat{x}_j, \hat{x}_i^\dagger \hat{x}_j^\dagger\}_{i,j=1,\ldots,m}$, and also entails a Lie algebra representation given by linear combinations of quadratic polynomials, whose Hermitian elements are Hamiltonians for general Gaussian devices.

Both the number-preserving and parity-preserving sub-algebras contain monomials of degree $2n$ over creation and annihilation operators, with $n \in \mathbb{N}$ for bosons, and $n \in \{1, \ldots, m\}$ for fermions. But the sub-algebra of Gaussian devices, which is also a Lie algebra in both cases, contains only real linear combinations of monomials of degree 2. This implies we can never obtain non-Gaussian dynamical maps from composing Gaussian devices alone, a point that was made in subsections 3.2.2 and 3.2.3.

However, in terms of anyonic generators, the situation is quite different. First, it is worth noticing that the Jordan-Wigner maps are such that $J(\hat{x}_i^\dagger \hat{x}_i) = \hat{\chi}_i^\dagger \hat{\chi}_i = \hat{x}_i^\dagger \hat{x}_i$ for all i. And not just that, but the $\hat{\chi}_i^\dagger \hat{\chi}_i$ also behave as number operators for anyonic oscillators. Therefore, the sub-algebra of number-preserving and parity-preserving operators for anyonic oscillators are also isomorphic to their standard counterparts and generated, respectively, by $\{\hat{\chi}_i^\dagger \hat{\chi}_j\}_{i,j=1,\ldots,m}$ and $\{\hat{\chi}_i^\dagger \hat{\chi}_j, \hat{\chi}_i \hat{\chi}_j, \hat{\chi}_i^\dagger \hat{\chi}_j^\dagger\}_{i,j=1,\ldots,m}$.

The differences begin when we consider the commutation relations of those sets of generators. In the case of generators for the number-preserving sub-algebra, we have that

$$[\hat{\chi}_i^\dagger\hat{\chi}_j;\hat{\chi}_k^\dagger\hat{\chi}_l]=\delta_{j,k}\hat{\chi}_i^\dagger\hat{\chi}_l-\delta_{i,l}\hat{\chi}_k^\dagger\hat{\chi}_j+\Delta^{\chi}_{i,j,k,l}\hat{\chi}_i^\dagger\hat{\chi}_j^\dagger\hat{\chi}_k\hat{\chi}_l, \tag{4.6}$$

where $\Delta^{\chi}_{i,j,k,l}=(-1)^{sgn(\chi)}(e^{-i\varphi\epsilon_{j,k}}-e^{-i\varphi(\epsilon_{l,i}-\epsilon_{k,i}-\epsilon_{l,j})})$. Therefore, the set of Hamiltonians given by the Hermitian linear combinations of quadratic anyonic generators does not close into a Lie algebra by itself since, by taking commutators, we can generate monomials of higher degree. This means that the equations of motion for creation and annihilation operators under Gaussian Hamiltonians are non-linear, and this fact makes obtaining exact solutions of these equations intractable.

In spite of the non-linearity of equations of motion determined by general Gaussian Hamiltonians for anyons, two-mode Gaussian Hamiltonians satisfy algebraic properties that enables us to find exact solutions in those cases. We saw in sections 3.2.1 and 3.2.3, that the set of Gaussian Hamiltonians has the Hermitian basis given by

$$\hat{T}^1_{i,j}=\frac{1}{2}(\hat{x}_i^\dagger\hat{x}_j+\hat{x}_j^\dagger\hat{x}_i),\ \hat{R}^1_{i,j}=\frac{1}{2}(\hat{x}_i^\dagger\hat{x}_j^\dagger+\hat{x}_j\hat{x}_i),\ \hat{P}^1_a=\frac{1}{2}\left((\hat{x}_a^\dagger)^2+(\hat{x}_a)^2\right), \tag{4.7a}$$

$$\hat{T}^2_{i,j}=\frac{-i}{2}(\hat{x}_i^\dagger\hat{x}_j-\hat{x}_j^\dagger\hat{x}_i),\ \hat{R}^2_{i,j}=\frac{-i}{2}(\hat{x}_i^\dagger\hat{x}_j^\dagger-\hat{x}_j\hat{x}_i),\ \hat{P}^2_a=\frac{-i}{2}\left((\hat{x}_a^\dagger)^2-(\hat{x}_a)^2\right), \tag{4.7b}$$

$$\hat{T}^3_{i,j}=\frac{1}{2}(\hat{x}_i^\dagger\hat{x}_i-\hat{x}_j^\dagger\hat{x}_j),\ \hat{R}^3_{i,j}=\frac{1}{2}(\hat{x}_i^\dagger\hat{x}_i+\hat{x}_j^\dagger\hat{x}_j+(-1)^{sgn(x)}I),\ \hat{N}_{i,j}=\hat{x}_i^\dagger\hat{x}_i+\hat{x}_j^\dagger\hat{x}_j, \tag{4.7c}$$

where i and j are an arbitrary pair of modes, a is an index that can take either i or j as values, and the $\hat{P}^k_a$ generators only exist for bosonic particles.

The generators $\hat{T}^k_{i,j}$ and $\hat{R}^k_{i,j}$ satisfy the commutation relations $[\hat{T}^k_{i,j};\hat{T}^l_{i,j}]=i\epsilon_{k,l,m}\hat{T}^m_{i,j}$, and $[\hat{R}^k_{i,j};\hat{R}^l_{i,j}]=i\epsilon_{k,l,m}\eta^m_n(x)\hat{R}^n_{i,j}$, with $\eta^1_1(x)=\eta^2_2(x)=1$, $\eta^3_3(x)=-(-1)^{sgn(x)}$ and all other elements equal to zero. This tells us that for both fermionic and bosonic oscillators, the equations of motion where the generators $\hat{T}^k_{i,j}$ and $\hat{R}^k_{i,j}$ are the dynamical variables are linear and exactly solvable. The point is that the two-mode Lie algebra structure of Gaussian maps for anyonic particles are isomorphic to their standard counterparts. To see this, notice that the generators

$$\hat{T}^1_{i,j}=\frac{1}{2}(\hat{\chi}_i^\dagger\hat{\chi}_j+\hat{\chi}_j^\dagger\hat{\chi}_i),\ \hat{R}^1_{i,j}=\frac{1}{2}(\hat{\chi}_i^\dagger\hat{\chi}_j^\dagger+\hat{\chi}_j\hat{\chi}_i),\ \hat{P}^1_a=\frac{1}{2}\left((\hat{\chi}_a^\dagger)^2+(\hat{\chi}_a)^2\right), \tag{4.8a}$$

$$\hat{T}^2_{i,j}=\frac{-i}{2}(\hat{\chi}_i^\dagger\hat{\chi}_j-\hat{\chi}_j^\dagger\hat{\chi}_i),\ \hat{R}^2_{i,j}=\frac{-i}{2}(\hat{\chi}_i^\dagger\hat{\chi}_j^\dagger-\hat{\chi}_j\hat{\chi}_i),\ \hat{P}^2_a=\frac{-i}{2}\left((\hat{\chi}_a^\dagger)^2-(\hat{\chi}_a)^2\right), \tag{4.8b}$$

$$\hat{T}^3_{i,j}=\frac{1}{2}(\hat{\chi}_i^\dagger\hat{\chi}_i-\hat{\chi}_j^\dagger\hat{\chi}_j),\ \hat{R}^3_{i,j}=\frac{1}{2}(\hat{\chi}_i^\dagger\hat{\chi}_i+\hat{\chi}_j^\dagger\hat{\chi}_j+(-1)^{sgn(\chi)}I),\ \hat{N}_{i,j}=\hat{\chi}_i^\dagger\hat{\chi}_i+\hat{\chi}_j^\dagger\hat{\chi}_j, \tag{4.8c}$$

satisfy the relations $[\hat{T}^k_{i,j};\hat{T}^l_{i,j}]=i\epsilon_{k,l,m}\hat{T}^m_{i,j}$, $[\hat{R}^k_{i,j};\hat{R}^l_{i,j}]=i\epsilon_{k,l,m}\eta^m_n(\chi)\hat{R}^n_{i,j}$ and, when $sgn(\chi)=1$, $[\hat{R}^k_{i,j};\hat{T}^l_{i,j}]=0$.

These properties are proven by using the non-trivial commutation relations

$$[\hat{\chi}_i^\dagger\hat{\chi}_j;\hat{\chi}_j^\dagger\hat{\chi}_i]=\hat{\chi}_i^\dagger\hat{\chi}_i-\hat{\chi}_j^\dagger\hat{\chi}_j,\quad [\hat{\chi}_i^\dagger\hat{\chi}_j^\dagger;\hat{\chi}_j\hat{\chi}_i]=-(1+(-1)^{sgn(\chi)}(\hat{\chi}_i^\dagger\hat{\chi}_i+\hat{\chi}_j^\dagger\hat{\chi}_j)), \tag{4.9}$$

together with identities involving the number operators $\hat{\chi}_i^\dagger \hat{\chi}_i$, for all $i = 1, \ldots, m$. However, these are the only commutation relations that have the same form as their standard counterparts. The commutation relations between the generators $\hat{T}^k_{i,j}$, $\hat{R}^k_{i,j}$ and $\hat{P}^k_{i,j}$ for bosonic anyons do indeed depend on the statistical parameter. Nevertheless, there is an isomorphism between the Lie algebra of Gaussian Hamiltonians for fermions and fermionic anyons, and between the Lie algebra of passive Gaussian Hamiltonians for bosons and bosonic anyons. These isomorphisms enable us to give an exact solution for the dynamical map used to define the optical network models for such particles, which we use in section 4.2.

4.1.2 Two-mode sub-algebras of fermionic anyons

Let i and j be two specific modes of an m mode fermionic anyon system. The fermionic nature of the same-site commutation relations imposes $(\hat{\xi}_i^\dagger)^2 = (\hat{\xi}_j^\dagger)^2 = 0$. This implies that the dimension of the sub-algebra of parity-preserving, two-mode operators, as a vector space, is finite. A general element A of this sub-algebra has the form

$$A = a_0 I + a_1 \hat{n}_i + a_2 \hat{n}_j + a_3 \hat{t}_{i,j} + a_4 \hat{t}^\dagger_{i,j} + a_5 \hat{r}_{i,j} + a_6 \hat{r}^\dagger_{i,j} + a_7 \hat{n}_i \hat{n}_j, \tag{4.10}$$

where $\hat{t}_{i,j} = \hat{\xi}_i^\dagger \hat{\xi}_j$ and $\hat{r}_{i,j} = \hat{\xi}_i^\dagger \hat{\xi}_j^\dagger$. Therefore, this sub-algebra has dimension 8 over $\mathbb{C}$.

From the isomorphism described in section 4.1.1, we infer that the generators $\hat{T}^k_{i,j}, \hat{R}^k_{i,j}$ for fermionic anyons form two commuting copies of the angular momentum algebra $\mathfrak{su}(2)$. Then, we can write the Hermitian elements H of this sub-algebra as the sum

$$H = (a_0 I + a_1 \hat{n}_i \hat{n}_j) + (b_1 \hat{T}^1_{i,j} + b_2 \hat{T}^2_{i,j} + b_3 \hat{T}^3_{i,j}) + (c_1 \hat{R}^1_{i,j} + c_2 \hat{R}^2_{i,j} + c_3 \hat{R}^3_{i,j}), \tag{4.11}$$

where all terms between parentheses commute with each other. In particular both I and $\hat{n}_i \hat{n}_j$ commute with all operators, and are the center of the sub-algebra.

The most general dynamical map over two fermionic anyons modes is given by $U = \exp\{i\theta H\}$, and therefore, we can write any two-mode map as the decomposition

$$U = \exp\{i\theta a_0\} \exp\{i\theta(a_1 \hat{n}_i \hat{n}_j)\} \exp\Big\{i\theta(\mathbf{b} \cdot \hat{\mathbf{T}}_{i,j})\Big\} \exp\Big\{i\theta(\mathbf{c} \cdot \hat{\mathbf{R}}_{i,j})\Big\}, \tag{4.12}$$

where $\hat{\mathbf{T}}_{i,j} = (\hat{T}^1_{i,j}, \hat{T}^2_{i,j}, \hat{T}^3_{i,j})$ and $\hat{\mathbf{R}}_{i,j} = (\hat{R}^1_{i,j}, \hat{R}^2_{i,j}, \hat{R}^3_{i,j})$ are vector operators. Since $(\hat{n}_i \hat{n}_j)^2 = \hat{n}_i \hat{n}_j$, we have that $\exp\{i\theta(a_1 \hat{n}_i \hat{n}_j)\} = 1 - (e^{i\theta a_0} - 1)\hat{n}_i \hat{n}_j$. Therefore, to understand all two-mode dynamical maps, we only need to describe the maps generated by Gaussian Hamiltonians.

In the case of passive Gaussian Hamiltonians, the dynamical maps have the form $\exp\Big\{i\theta(\mathbf{b} \cdot \hat{\mathbf{T}}_{i,j})\Big\}$ and, without loss of generality, we can impose that $|\mathbf{b}| = 1$. Due to the angular momentum algebra of the $\hat{T}^k_{i,j}$ operators, this unitary is a rotation map, and one can find a decomposition of any such unitary into a sequence of three rotation maps given by eq. (3.14), as discussed in sections 3.1.1 and 3.2.1. Therefore, to characterize all such maps we need only to study the rotations around $T^1_{i,j}$ and rotations around $T^3_{i,j}$.

Consider rotations around $T^1_{i,j}$ first. The rotation operator has the form

$$Q_1(\theta) = \exp\left\{i\frac{\theta}{2}(\hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i)\right\}. \tag{4.13}$$

To find the explicit expression of $Q_1(\theta)$ as a vector in the parity-preserving sub-algebra, we must first find the powers of the term $\hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i$. Using the commutation relations for quadratic operators we find that

$$(\hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i)^2 = \hat{\xi}_i^\dagger\hat{\xi}_i + \hat{\xi}_j^\dagger\hat{\xi}_j - 2\hat{\xi}_i^\dagger\hat{\xi}_i\hat{\xi}_j^\dagger\hat{\xi}_j, \tag{4.14a}$$
$$(\hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i)^3 = \hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i. \tag{4.14b}$$

Therefore, the rotation operator $Q_1(\theta)$ has the decomposition

$$Q_1(\theta) = I + i\sin\frac{\theta}{2}(\hat{\xi}_i^\dagger\hat{\xi}_j + \hat{\xi}_j^\dagger\hat{\xi}_i) + (\cos\frac{\theta}{2} - 1)(\hat{\xi}_i^\dagger\hat{\xi}_i + \hat{\xi}_j^\dagger\hat{\xi}_j - 2\hat{\xi}_i^\dagger\hat{\xi}_i\hat{\xi}_j^\dagger\hat{\xi}_j). \tag{4.15}$$

Using this decomposition, we can obtain how this dynamical map evolves the anyonic oscillators. Using the commutation relations for fermionic anyons we obtain

$$Q_1(\theta)\hat{\xi}_i^\dagger Q_1(-\theta) = \cos\frac{\theta}{2}\hat{\xi}_i^\dagger + i\sin\frac{\theta}{2}\hat{\xi}_j^\dagger e^{i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i}, \tag{4.16a}$$
$$Q_1(\theta)\hat{\xi}_j^\dagger Q_1(-\theta) = \cos\frac{\theta}{2}\hat{\xi}_j^\dagger + i\sin\frac{\theta}{2}\hat{\xi}_i^\dagger e^{-i\varphi\hat{\xi}_j^\dagger\hat{\xi}_j}. \tag{4.16b}$$

while the respective identities for annihilation operators are obtained by taking the conjugate of these equations.

A few comments are in order. First, these formulas are the solutions of the non-linear Heisenberg equations of motion

$$i\frac{d\hat{\xi}_i^\dagger}{d\theta} = [\hat{T}^1_{ij}, \hat{\xi}_i^\dagger] = \hat{\xi}_j^\dagger\{1 + (e^{i\varphi} - 1)\hat{\xi}_i^\dagger\hat{\xi}_i\} = \hat{\xi}_j^\dagger e^{i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i}, \tag{4.17a}$$
$$i\frac{d\hat{\xi}_j^\dagger}{d\theta} = [\hat{T}^1_{ij}, \hat{\xi}_j^\dagger] = \hat{\xi}_i^\dagger\{1 + (e^{-i\varphi} - 1)\hat{\xi}_j^\dagger\hat{\xi}_j\} = \hat{\xi}_i^\dagger e^{-i\varphi\hat{\xi}_j^\dagger\hat{\xi}_j}. \tag{4.17b}$$

Second, the appearance of the non-linear factors $\exp\left\{\pm i\varphi\hat{\xi}_j^\dagger\hat{\xi}_j\right\}$ can also be understood by noticing that simple linear combinations of $\hat{\xi}_i^\dagger$ and $\hat{\xi}_j^\dagger$ do not preserve the anyonic commutation relations, and the non-linear factors appear to solve this problem.

Similarly, the rotation operator associated to $\hat{T}^3_{i,j}$ is given by the formula

$$Q_3(\theta) = I + (e^{i\theta/2} - 1)\hat{\xi}_i^\dagger\hat{\xi}_i + (e^{-i\theta/2} - 1)\hat{\xi}_j^\dagger\hat{\xi}_j + (1 - \cos\theta)2\hat{\xi}_i^\dagger\hat{\xi}_i\hat{\xi}_j^\dagger\hat{\xi}_j, \tag{4.18}$$

which gives us the action

$$Q_3(\theta)\hat{\xi}_i^\dagger Q_3(-\theta) = \exp\left\{i\frac{\theta}{2}\right\}\hat{\xi}_i^\dagger, \tag{4.19a}$$
$$Q_3(\theta)\hat{\xi}_j^\dagger Q_3(-\theta) = \exp\left\{-i\frac{\theta}{2}\right\}\hat{\xi}_j^\dagger. \tag{4.19b}$$

For completeness' sake, the general rotation around an axis $\mathbf{b}$ has the form

$$\exp\Big\{i\theta(\mathbf{b}\cdot\hat{\mathbf{T}}_{i,j})\Big\} = I + i2\sin\frac{\theta}{2}(\mathbf{b}\cdot\hat{\mathbf{T}}_{i,j}) + 4(\cos\frac{\theta}{2}-1)(\mathbf{b}\cdot\hat{\mathbf{T}}_{i,j})^2, \tag{4.20}$$

since it can also be proven that $(2\mathbf{b}\cdot\hat{\mathbf{T}}_{i,j})^3 = (2\mathbf{b}\cdot\hat{\mathbf{T}}_{i,j})$. Similarly, for the case of number-destroying Hamiltonians we find that

$$\exp\Big\{i\theta(\mathbf{c}\cdot\hat{\mathbf{R}}_{i,j})\Big\} = I + i2\sin\frac{\theta}{2}(\mathbf{c}\cdot\hat{\mathbf{R}}_{i,j}) + 4(\cos\frac{\theta}{2}-1)(\mathbf{c}\cdot\hat{\mathbf{R}}_{i,j})^2. \tag{4.21}$$

These are the preliminary results of my work with fermionic anyons, and they are used when we discuss how to perform quantum computing with such systems.

4.1.3 Two-mode sub-algebras of bosonic anyons

Let i and j be two specific modes of an m-mode bosonic anyon system. The bosonic nature of the same-site commutation relations implies that both the number-preserving and parity-preserving sub-algebras are infinite dimensional as vector spaces. Therefore, the analysis carried out for the fermionic anyon case is not feasible, and a new method for obtaining the action of dynamical maps over creation and annihilation operators is required.

Let us restrict to the case of passive, Gaussian Hamiltonians for bosonic anyons, generated by $\hat{T}^k_{i,j}$. As with the fermionic anyon case, we only need to describe the maps generated by $T^1_{i,j}$ and $T^3_{i,j}$ to obtain the action of the general case. From the anyonic commutation relations we obtain

$$(2\hat{T}^1_{i,j})\hat{\beta}^\dagger_i = \hat{\beta}^\dagger_i\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j})\right] + \hat{\beta}^\dagger_j, \tag{4.22a}$$

$$(2\hat{T}^1_{i,j})\hat{\beta}^\dagger_j = \hat{\beta}^\dagger_j\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j})\right] + \hat{\beta}^\dagger_i. \tag{4.22b}$$

Next, we sum and subtract the two equations to get

$$(2\hat{T}^1_{i,j})(\hat{\beta}^\dagger_i + \hat{\beta}^\dagger_j) = (\hat{\beta}^\dagger_i + \hat{\beta}^\dagger_j)\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}) + 1\right], \tag{4.23a}$$

$$(2\hat{T}^1_{i,j})(\hat{\beta}^\dagger_i - \hat{\beta}^\dagger_j) = (\hat{\beta}^\dagger_i - \hat{\beta}^\dagger_j)\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}) - 1\right]. \tag{4.23b}$$

We can use the equations above to commute any power of $(2\hat{T}^1_{i,j})$ with the sum and difference of creation operators. This allows us to commute any function of $(2\hat{T}^1_{i,j})$ by commuting each term of its Taylor series expansion. Then, by using this property for $\exp\Big\{i\theta(2\hat{T}^1_{i,j})\Big\}$ we obtain

$$\exp\Big\{i\theta(2\hat{T}^1_{i,j})\Big\}(\hat{\beta}^\dagger_i + \hat{\beta}^\dagger_j) = (\hat{\beta}^\dagger_i + \hat{\beta}^\dagger_j)\exp\Big\{i\theta\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}) + 1\right]\Big\}, \tag{4.24a}$$

$$\exp\Big\{i\theta(2\hat{T}^1_{i,j})\Big\}(\hat{\beta}^\dagger_i - \hat{\beta}^\dagger_j) = (\hat{\beta}^\dagger_i - \hat{\beta}^\dagger_j)\exp\Big\{i\theta\left[2(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}) - 1\right]\Big\}. \tag{4.24b}$$

Finally, after taking the sum and subtraction of the last two equations, and after a re-scaling of θ, we obtain

$$\exp\Big\{i\theta\hat{T}^1_{i,j}\Big\}\hat{\beta}^\dagger_i = (\cos\frac{\theta}{2}\hat{\beta}^\dagger_i + i\sin\frac{\theta}{2}\hat{\beta}^\dagger_j)\exp\Big\{i\theta(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j})\Big\}, \tag{4.25a}$$

$$\exp\Big\{i\theta\hat{T}^1_{i,j}\Big\}\hat{\beta}^\dagger_j = (i\sin\frac{\theta}{2}\hat{\beta}^\dagger_i + \cos\frac{\theta}{2}\hat{\beta}^\dagger_j)\exp\Big\{i\theta(\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j})\Big\}. \tag{4.25b}$$

The linear combination $\cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}$ can be seen as the result of applying a $\hat{T}^3_{i,j}$ rotation by an angle φ over $T^1_{i,j}$, since

$$\exp\Big\{i\varphi\hat{T}^3_{i,j}\Big\}\hat{T}^1_{i,j}\exp\Big\{-i\varphi\hat{T}^3_{i,j}\Big\} = \cos\varphi\hat{T}^1_{i,j} - \sin\varphi\hat{T}^2_{i,j}. \tag{4.26}$$

Therefore, defining the compound rotation operator

$$G^{(\alpha)}_{i,j|1}(\theta) = \exp\Big\{i\alpha\hat{T}^3_{i,j}\Big\}\exp\Big\{i\theta\hat{T}^1_{i,j}\Big\}\exp\Big\{-i\alpha\hat{T}^3_{i,j}\Big\}, \tag{4.27}$$

allows us to rewrite eq. (4.26) in the form

$$G^{(0)}_{i,j|1}(\theta)\hat{\beta}^\dagger_i = (\cos\frac{\theta}{2}\hat{\beta}^\dagger_i + i\sin\frac{\theta}{2}\hat{\beta}^\dagger_j)G^{(\varphi)}_{i,j|1}(\theta), \tag{4.28a}$$

$$G^{(0)}_{i,j|1}(\theta)\hat{\beta}^\dagger_j = (i\sin\frac{\theta}{2}\hat{\beta}^\dagger_i + \cos\frac{\theta}{2}\hat{\beta}^\dagger_j)G^{(\varphi)}_{i,j|1}(\theta). \tag{4.28b}$$

These identities allow us to write the formal solutions

$$G^{(0)}_{i,j|1}(\theta)\hat{\beta}^\dagger_i G^{(0)}_{i,j|1}(-\theta) = (\cos\frac{\theta}{2}\hat{\beta}^\dagger_i + i\sin\frac{\theta}{2}\hat{\beta}^\dagger_j)G^{(\varphi)}_{i,j|1}(\theta)G^{(0)}_{i,j|1}(-\theta), \tag{4.29a}$$

$$G^{(0)}_{i,j|1}(\theta)\hat{\beta}^\dagger_j G^{(0)}_{i,j|1}(-\theta) = (i\sin\frac{\theta}{2}\hat{\beta}^\dagger_i + \cos\frac{\theta}{2}\hat{\beta}^\dagger_j)G^{(\varphi)}_{i,j|1}(\theta)G^{(0)}_{i,j|1}(-\theta). \tag{4.29b}$$

However, these are not useful to do more complex calculations. The action of a rotation by $\hat{T}^3_{i,j}$ over bosonic anyon oscillators is trivially given by

$$\exp\Big\{i\varphi\hat{T}^3_{i,j}\Big\}\hat{\xi}^\dagger_i\exp\Big\{-i\varphi\hat{T}^3_{i,j}\Big\} = \exp\Big\{i\frac{\theta}{2}\Big\}\hat{\xi}^\dagger_i, \tag{4.30a}$$

$$\exp\Big\{i\varphi\hat{T}^3_{i,j}\Big\}\hat{\xi}^\dagger_j\exp\Big\{-i\varphi\hat{T}^3_{i,j}\Big\} = \exp\Big\{-i\frac{\theta}{2}\Big\}\hat{\xi}^\dagger_j. \tag{4.30b}$$

Then, using the above formula is easy to see that, by induction on n,

$$G^{(n\varphi)}_{i,j|1}(\theta)\hat{\beta}^\dagger_i = (\cos\frac{\theta}{2}\hat{\beta}^\dagger_i + ie^{-in\varphi}\sin\frac{\theta}{2}\hat{\beta}^\dagger_j)G^{((n+1)\varphi)}_{i,j|1}(\theta), \tag{4.31a}$$

$$G^{(n\varphi)}_{i,j|1}(\theta)\hat{\beta}^\dagger_j = (\cos\frac{\theta}{2}\hat{\beta}^\dagger_j + ie^{in\varphi}\sin\frac{\theta}{2}\hat{\beta}^\dagger_i)G^{((n+1)\varphi)}_{i,j|1}(\theta). \tag{4.31b}$$

These identities are called the *propagation identities*, and I use them to propagate the action of a dynamical map through any polynomial of creation particle operators. This is another of my preliminary results, and it enables us to calculate the dynamical evolution of any two-mode bosonic-anyon Fock-state.

To sum up, the effects of two-mode, passive, Gaussian dynamical maps, and also of active Gaussian maps in the case of fermionic anyons, are exactly solvable, and their solutions resemble the action of their standard counterparts to some degree. However, in contrast to standard oscillators, these actions have an explicit non-linear nature, leading to unexpected consequences that are treated in the next section.

4.2 Anyonic optics and quantum computation

In this section we apply the algebraic facts demonstrated earlier in the context of optical networks. In subsection 4.2.1, I give the definition of anyonic optical devices, and analyze their action on anyonic states. I prove that we can describe these results using the language of optical and of anyonic effects, such as bunching and Aharonov-Bohm phases.

In subsection 4.2.2, I describe the properties of anyonic optical networks in contrast to networks for standard particles. First, I argue that networks of anyonic beam-splitters can be crafted to act trivially in the presence of one particle, but non-trivially in the presence of two or more. I also provide an example of such network, and show that it gives a one-dimensional representation of the braid group.

In subsection 4.2.3, I define the computational model of anyonic optical circuits. First, I give a general description of single-qubit operators for logical qubits in the dual-rail encoding for both fermionic and bosonic anyon modes. Next, I provide two networks using only anyonic beam-splitters and phase-shifters that create entangling two-qubit gates. The first one applies only to the fermionic anyon case, while the second applies to both fermionic and bosonic anyons. I finish this section by highlighting the differences between standard and anyonic optical computing models.

4.2.1 Phenomenology of anyonic optical devices

Since the algebras of two-mode, passive and active Gaussian Hamiltonians are the same for standard and anyonic particles, it makes sense to define the optical elements for anyons analogously to those of fermions and bosons. Thus we have the anyonic phase shifters and beam splitters described respectively by

$$PS_i(\tau) = \exp\left(i\tau\hat{\chi}_i^\dagger\hat{\chi}_i\right), \tag{4.32a}$$

$$BS_{i,j}(\theta) = \exp\left[i\theta(\hat{\chi}_i^\dagger\hat{\chi}_j + \hat{\chi}_j^\dagger\hat{\chi}_i)\right]. \tag{4.32b}$$

The graphical notation for these devices is the same as the one in figs. 16 and 17. Similarly, we can define squeezers and two-mode down converters by

$$\hat{S}_i(\gamma) = \exp\left\{i\gamma((\hat{b}_i^\dagger)^2 + (\hat{b}_i)^2)\right\}, \tag{4.33a}$$

$$\hat{D}_{i,j}(\kappa) = \exp\left\{i\kappa(\hat{x}_i^\dagger\hat{x}_j^\dagger + \hat{x}_j\hat{x}_i)\right\}, \tag{4.33b}$$

but we do not include them in our analysis directly.

With the results of section 4.1, we can calculate the effect of these devices over anyonic Fock-space basis states. First, let us restrict ourselves to single-particle, two-mode systems, with the modes labelled 1 and 2. In this subspace, generated by $\{|1,0\rangle_\chi, |0,1\rangle_\chi\}$ with χ indicating the anyon type, the action of general passive devices is simple, and is given by

$$PS_1(\tau)|1,0\rangle_\chi = e^{i\tau}|1,0\rangle_\chi, \quad PS_2(\tau)|1,0\rangle_\chi = |1,0\rangle_\chi, \tag{4.34a}$$

$$PS_1(\tau)|0,1\rangle_\chi = |0,1\rangle_\chi, \quad PS_2(\tau)|0,1\rangle_\chi = e^{i\tau}|0,1\rangle_\chi, \tag{4.34b}$$

and

$$BS_{1,2}(\theta)|1,0\rangle_\chi = \cos\theta|1,0\rangle_\chi + i\sin\theta|0,1\rangle_\chi \tag{4.34c}$$

$$BS_{1,2}(\theta)|0,1\rangle_\chi = i\sin\theta|1,0\rangle_\chi + \cos\theta|0,1\rangle_\chi. \tag{4.34d}$$

To prove the last six identities for fermionic anyons ($\chi = \xi$), we need to rewrite eqs. (4.16a,4.19a) in terms of the definitions for passive devices, which gives us

$$PS_i(\tau)\xi_j^\dagger PS_i(-\tau) = e^{i\tau\delta_{ij}}\xi_j^\dagger, \tag{4.35a}$$

$$BS_{i,j}(\theta)\xi_i^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta\xi_i^\dagger + i\sin\theta\xi_j^\dagger e^{i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i}, \tag{4.35b}$$

$$BS_{i,j}(\theta)\xi_j^\dagger BS_{i,j}^\dagger(\theta) = \cos\theta\xi_j^\dagger + i\sin\theta\xi_i^\dagger e^{-i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i}. \tag{4.35c}$$

However, it is more convenient to rewrite the last three equations as propagation relations, obtaining

$$PS_i(\tau)\xi_j^\dagger = (e^{i\tau\delta_{ij}}\xi_j^\dagger)PS_i(\tau), \tag{4.36a}$$

$$BS_{i,j}(\theta)\xi_i^\dagger = (\cos\theta\xi_i^\dagger + i\sin\theta\xi_j^\dagger e^{i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i})BS_{i,j}(\theta), \tag{4.36b}$$

$$BS_{i,j}(\theta)\xi_j^\dagger = (\cos\theta\xi_j^\dagger + i\sin\theta\xi_i^\dagger e^{-i\varphi\hat{\xi}_i^\dagger\hat{\xi}_i})BS_{i,j}(\theta). \tag{4.36c}$$

Then, since $|1,0\rangle_\xi = \hat{\xi}_1^\dagger|0\rangle_\xi$ and $|0,1\rangle_\xi = \hat{\xi}_1^\dagger|0\rangle_\xi$, we just need to propagate the device operators through $\hat{\xi}_1^\dagger, \hat{\xi}_2^\dagger$, and use the fact that $PS_i|0\rangle_\chi = BS_{i,j}|0\rangle_\chi = |0\rangle_\chi$ for all i and j.

Putting the equations for the action of device operators as propagation relations allows us to use the same calculation method for both fermionic and bosonic anyons. To see this, notice that for bosonic-anyon beam-splitters we can define the operator

$$BS_{i,j}^{(\alpha)}(\theta) = PS_i(\alpha)PS_j(\alpha)BS_{i,j}(\theta)PS_i(-\alpha)PS_j(-\alpha), \tag{4.37}$$

such that the propagation relations become

$$BS_{i,j}^{(n\varphi)}(\theta)\hat{\beta}_i^\dagger = (\cos\theta\hat{\beta}_i^\dagger + ie^{-in\varphi}\sin\theta\hat{\beta}_j^\dagger)BS_{i,j}^{((n+1)\varphi)}(\theta), \tag{4.38a}$$

$$BS_{i,j}^{(n\varphi)}(\theta)\hat{\beta}_j^\dagger = (\cos\theta\hat{\beta}_j^\dagger + ie^{in\varphi}\sin\theta\hat{\beta}_i^\dagger)BS_{i,j}^{((n+1)\varphi)}(\theta). \tag{4.38b}$$

$$BS_{i,j}^{(n\varphi)}(\theta)\hat{\beta}_k^\dagger = \hat{\beta}_k^\dagger BS_{i,j}^{(n\varphi)}(\theta), \text{ if } k < i \text{ or } k > j \tag{4.38c}$$

$$BS_{i,j}^{(n\varphi)}(\theta)\hat{\beta}_k^\dagger = \hat{\beta}_k^\dagger BS_{i,j}^{((n+2)\varphi)}(\theta), \text{ if } i < k < j \tag{4.38d}$$

Since for bosonic anyons we can also write

$$PS_i(\tau)\beta_j^\dagger = (e^{i\tau\delta_{ij}}\beta_j^\dagger)PS_i(\tau), \tag{4.38e}$$

the propagation relations for these device operators determine the action over single-particle states in the same way as for fermionic anyons.

For single-particle states, the action of anyonic optical devices is identical to their standard counterparts. However, this changes drastically when we deal with states of two or more particles. Let us discuss fermionic anyons first. Since the only two-particle states of two-mode fermionic anyon system is $|1,1\rangle_\xi$, it is not hard to see that

$$BS_{12}(\theta)\hat{\xi}_1^\dagger\hat{\xi}_2^\dagger|0\rangle_\xi = \left(\cos^2\theta\hat{\xi}_1^\dagger\hat{\xi}_2^\dagger - \sin^2\theta\hat{\xi}_2^\dagger e^{i\varphi\hat{\xi}_1^\dagger\hat{\xi}_1}\hat{\xi}_1^\dagger e^{-i\varphi\hat{\xi}_2^\dagger\hat{\xi}_2}\right)|0\rangle_\xi \tag{4.39a}$$

$$= \left(\cos^2\theta\hat{\xi}_1^\dagger\hat{\xi}_2^\dagger - \sin^2\theta e^{i\varphi}\hat{\xi}_2^\dagger\hat{\xi}_1^\dagger\right)|0\rangle_\xi \tag{4.39b}$$

$$= \hat{\xi}_1^\dagger\hat{\xi}_2^\dagger|0\rangle_\xi\,. \tag{4.39c}$$

This result shows that the anti-bunching phenomenon persists in fermionic anyon optics.

The new behaviors begin to appear for fermionic anyons when we consider the case of two particles in three modes. The Hilbert space in such case is generated by $\{|0,1,1\rangle_\xi, |1,0,1\rangle_\xi, |1,1,0\rangle_\xi\}$. For beam-splitters acting in nearest-neighbour modes the action is identical to the standard fermion case. On the other hand, the action of the beam-splitter $BS_{1,3}(\theta)$ is this subspace is given by

$$BS_{1,3}(\theta)|1,1,0\rangle_\xi = \cos\theta|1,1,0\rangle_\xi - ie^{-i\varphi}\sin\theta|0,1,1\rangle_\xi\,, \tag{4.40}$$

$$BS_{1,3}(\theta)|1,0,1\rangle_\xi = |1,0,1\rangle_\xi\,, \tag{4.41}$$

$$BS_{1,3}(\theta)|0,1,1\rangle_\xi = \cos\theta|0,1,1\rangle_\xi - ie^{i\varphi}\sin\theta|1,1,0\rangle_\xi\,. \tag{4.42}$$

In the last result, the statistical angle of the anyon type appears explicitly on the coefficients of the states. Therefore, these extra phases must be somehow related to the anyonic character of the particles. To see how this anyonic character appear, consider the previous example in the case of $\theta = \pi/2$, which behaves as a mirror up to a global phase. We have in this that the states transforming non-trivially under the mirror are such that

$$BS_{1,3}(\pi/2)|1,1,0\rangle_\xi = -ie^{-i\varphi}|0,1,1\rangle_\xi\,, \tag{4.43}$$

$$BS_{1,3}(\pi/2)|0,1,1\rangle_\xi = -ie^{i\varphi}|1,1,0\rangle_\xi\,. \tag{4.44}$$

So, in the first equation, when the particle in mode 1 hits the mirror it gets reflected upward, in terms of the mode ordering, tunneling trough mode 2, which is occupied by a single particle. If we think of this process as one-dimensional generalization of particle exchange, the $-e^{-i\varphi}$ phase can be seem as an exchange phase. Similarly, in the second equation, the particle in mode 3 gets reflected downward, and tunnels trough the particle occupying mode 2, acquiring an exchange phase $-e^{i\varphi}$.

Our claim is that the phase $-e^{-i\varphi}$ can be seen as an Aharonov-Bohm phase that arises from the anyonic character of the particles. In other to prove this, we need to show that the presence of more particles in intermediary modes adds up to the total relative phase. Consider the set of l-mode states

$$|1, k_{2,l-1}, 0\rangle_\xi = \hat{\xi}_1^\dagger \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi \, . \tag{4.45}$$

where k is the number of particles occupying k modes i_j lying between 2 and $l-1$. We can calculate the action of an beam-splitter $BS_{1,l}(\theta)$ in the following way

$$\begin{aligned}
BS_{1,l}(\theta)\, |1, k_{2,l-1}, 0\rangle_\xi &= BS_{1,l}(\theta) \hat{\xi}_1^\dagger \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi = \\
&= (\cos\theta \hat{\xi}_1^\dagger + i \sin\theta \hat{\xi}_l^\dagger e^{-i\varphi \hat{\xi}_1^\dagger \xi_1}) \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi = \\
&= \cos\theta \hat{\xi}_1^\dagger \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi + i \sin\theta \hat{\xi}_l^\dagger \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi = \\
&= \cos\theta \hat{\xi}_1^\dagger \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) |0\rangle_\xi + i \sin\theta e^{-ik(\varphi+\pi)} \left(\prod_{j=1}^{k} \hat{\xi}_{i_j}^\dagger \right) \hat{\xi}_l^\dagger \, |0\rangle_\xi = \\
&= \cos\theta \, |1, k_{2,l-1}, 0\rangle_\xi + i \sin\theta e^{-ik(\varphi+\pi)} \, |0, k_{2,l-1}, 1\rangle_\xi \, .
\end{aligned}$$

The previous calculation proves that the action of a beam-splitter over two distant modes 1 and l, where mode 1 is occupied, under the presence of k particles induces a relative phase $e^{ik(\varphi+\pi)}$. Therefore, it makes sense to consider fermionic anyons as particles with a effective magnetic flux[1] of φ, that undergo an Aharonov-Bohm effect under the action of beam-splitters between distant modes. This interpretation has a deep implication in terms of quantum computing, not just for fermionic anyons, but also for fermions and qubits, as we will see in subsection 4.2.3.

Now let us analyze the action of beam-splitters over two-particle states for bosonic anyons. As we saw in section 3.2, the action of a balanced beam-splitter over the two-particle state $|1, 1\rangle_b$ exhibits the Hong-Ou-Mandel effect. For bosonic anyons, the action

1 The terminology is a bit misleading, since there is no "magnetic field" in one-dimensional theories, however, we maintain the use due to the fact that the particle-number dependent phases appearing in one dimensional theories of interacting anyons are called "Aharonov-Bohm" phases [21].

of a balanced beam-splitter over $|1,1\rangle_\beta$ is given by

$$\begin{aligned}
BS_{1,2}(\pi/4)\,|1,1\rangle_\beta &= BS_{1,2}(\pi/4)\hat{\beta}_1^\dagger\hat{\beta}_2^\dagger\,|0\rangle_\beta = \\
&= \frac{1}{\sqrt{2}}\left(\hat{\beta}_1^\dagger + i\hat{\beta}_2^\dagger\right) BS_{1,2}^{(\varphi)}(\pi/4)\hat{\beta}_2^\dagger\,|0\rangle_\beta = \\
&= \frac{1}{2}\left(\hat{\beta}_1^\dagger + i\hat{\beta}_2^\dagger\right)\left(ie^{i\varphi}\hat{\beta}_1^\dagger + \hat{\beta}_2^\dagger\right) BS_{1,2}^{(2\varphi)}(\pi/4)\,|0\rangle_\beta = \\
&= \frac{i}{2}\left(e^{i\varphi}(\hat{\beta}_1^\dagger)^2 + (\hat{\beta}_2^\dagger)^2\right)|0\rangle_\beta = \\
&= \frac{i}{\sqrt{2}}(e^{i\varphi}\,|2,0\rangle_\beta + |0,2\rangle_\beta).
\end{aligned}$$

where the last equality follows from the commutation relations and the fact that $BS_{1,2}^{(2\varphi)}(\theta)$ acts trivially on the vacuum state.

Note that this recovers the original bosonic Hong-Ou-Mandel effect when $\varphi = 0$, as expected. Interestingly, however, the $|1,1\rangle_\beta$ state is still suppressed for any value of the exchange phase. The only difference to the bosonic case is a relative phase between states $|2,0\rangle_\beta$ and $|0,2\rangle_\beta$. This phase has, up to this moment, not been given an interpretation in terms of the anyonic character of the particles. Nevertheless, we can establish the existence of a one-dimensional Aharonov-Bohm effect for bosonic anyons in the following way. Consider the three mode state

$$|1,k,0\rangle_\beta = \hat{\beta}_1^\dagger\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)|0\rangle_\beta\,, \tag{4.48}$$

and the beam-splitter $BS_{1,3}(\theta)$.

We have that,

$$\begin{aligned}
BS_{1,3}(\theta)\,|1,k,0\rangle_\beta &= BS_{1,3}(\theta)\hat{\beta}_1^\dagger\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)|0\rangle_\beta = \\
&= (\cos\theta\hat{\beta}_1^\dagger + i\sin\theta\hat{\beta}_3^\dagger)BS_{1,3}^{(\varphi)}(\theta)\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)|0\rangle_\beta = \\
&= (\cos\theta\hat{\beta}_1^\dagger + i\sin\theta\hat{\beta}_3^\dagger)\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)BS_{1,3}^{((2k+1)\varphi)}(\theta)\,|0\rangle_\beta = \\
&= \cos\theta\hat{\beta}_1^\dagger\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)|0\rangle_\beta + i\sin\theta e^{-ik\varphi}\left(\frac{(\hat{\beta}_2^\dagger)^k}{\sqrt{k!}}\right)\hat{\beta}_3^\dagger\,|0\rangle_\beta = \\
&= \cos\theta\,|1,k,0\rangle_\beta + i\sin\theta e^{-ik\varphi}\,|0,k,1\rangle_\beta\,.
\end{aligned}$$

This proves that bosonic anyons undergo the one-dimensional Aharonov-Bohm effect under the same conditions as fermionic anyons. In particular, bosonic anyons can be understood to have effective magnetic flux φ. As in the fermionic anyon case, this also has striking implications for quantum computing with anyonic optical networks.

4.2.2 Anyonic optical networks and interferometry

Now we proceed to describe the action of anyonic optical networks. As already discussed in section 4.1, the algebra of m-mode, passive, Gaussian Hamiltonians does not form a Lie algebra. For standard particles this algebra does indeed form a Lie algebra, and this fact enables us to characterize a general m-mode passive, Gaussian devices by their matrix of single-particle amplitudes, defined in subsection 3.2.2. This happens exactly because the commutation relations for m-mode anyonic Gaussian generators do not close, which means that the action of general Gaussian devices over creation operators is non-linear, and the coefficients of non-linear factors may appear in multi-particle amplitudes without appearing in single-particle amplitudes.

To see this consider, for example, the network in fig. 21. Its interferometer matrix in the single-particle ordered basis $(|1,0,0\rangle_{x,\chi}, |0,1,0\rangle_{x,\chi}, |0,0,1\rangle_{x,\chi})$ is the identity matrix, and therefore this network describes a trivial interferometer for all single-particle subspaces for both standard and anyonic particles. This implies, for standard particles, that its action over all multi-particle subspaces is also the identity. However, for fermionic and bosonic anyons, the action of this network is not the identity, for example, in the two-particle subspace.

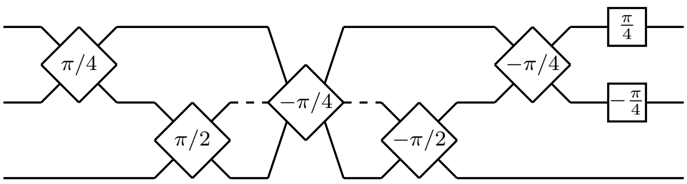


Figure 21 – Decomposition of the trivial interferometer. This is equivalent to the network found in [238], up to phase-shifters

In the fermionic anyon case, the two-particle subspace is written in the ordered basis $(|1,1,0\rangle_\xi, |1,0,1\rangle_\xi, |0,1,1\rangle_\xi)$, and the interferometer matrix has the form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}}\frac{1+i\cos\varphi}{\sqrt{2}} & e^{-i\frac{\pi}{4}}\left(\frac{i\sin\varphi}{\sqrt{2}}\right) \\ 0 & e^{i\frac{\pi}{4}}\left(\frac{i\sin\varphi}{\sqrt{2}}\right) & e^{i\frac{\pi}{4}}\frac{1-i\cos\varphi}{\sqrt{2}} \end{bmatrix}, \tag{4.50}$$

where φ is the statistical parameter for the anyon. Notice that the bottom 2×2 block is by itself a unitary matrix, that can be written in the form $R_{(0,0,1)}(\pi/2)R_{(\sin\varphi,0,\cos\varphi)}(-\pi/2)$ in terms of products of rotations. Given this form, it is easy to see that for $\varphi = 0$, we recover the identity matrix.

For bosonic anyons, the two-particle interferometer matrix is a 6×6 matrix, that can be written in block diagonal form with an appropriate ordering of the two-particle basis vectors $|2,0,0\rangle_\beta$,$|1,1,0\rangle_\beta$,$|1,0,1\rangle_\beta$,$|0,2,0\rangle_\beta$,$|0,1,1\rangle_\beta$,$|0,0,2\rangle_\beta$. The first block is written in terms of the subspace generated by the ordered basis $(|2,0,0\rangle_\beta$,$|1,1,0\rangle_\beta$,$|0,2,0\rangle_\beta)$, and has the form

$$\begin{bmatrix} \left(\frac{1}{2}-\frac{i}{4}\right)+\left(\frac{1}{2}+\frac{i}{4}\right)\cos\varphi & \frac{1}{\sqrt{2}}\left(\frac{1-e^{2i\varphi}}{4}-i\frac{1-e^{i\varphi}}{2}\right) & -\left(\frac{1-e^{2i\varphi}}{4}+i\frac{(1-e^{i\varphi})^2}{8}\right) \\ \frac{1}{\sqrt{2}}\left(\frac{e^{-i\varphi}-1}{2}+i\frac{(e^{-i2\varphi}-1)}{4}\right) & \frac{1+\cos\varphi}{2} & \frac{1}{\sqrt{2}}\left(\frac{1-e^{i\varphi}}{2}-\frac{\sin\varphi}{2}\right) \\ \left(\frac{1-e^{-2i\varphi}}{4}+i\frac{(1-e^{-i\varphi})^2}{8}\right) & \frac{-i}{\sqrt{2}}\left(\frac{1-e^{i\varphi}}{2}+\frac{\sin\varphi}{2}\right) & \left(\frac{1}{2}+\frac{i}{4}\right)+\left(\frac{1}{2}-\frac{i}{4}\right)\cos\varphi \end{bmatrix}.$$

The second block is written in the subspace generated by the remaining basis vectors in the order ($|1,0,1\rangle_\beta$,$|0,1,1\rangle_\beta$,$|0,0,2\rangle_\beta$), and is given by

$$\begin{bmatrix} e^{-i\frac{\pi}{4}}\frac{1+i\cos\varphi}{\sqrt{2}} & e^{-i\frac{\pi}{4}}\left(\frac{i\sin\varphi}{\sqrt{2}}\right) & 0 \\ e^{i\frac{\pi}{4}}\left(\frac{i\sin\varphi}{\sqrt{2}}\right) & e^{i\frac{\pi}{4}}\frac{1-i\cos\varphi}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.51}$$

As in the fermionic anyon case, it is simple to verify that for $\varphi = 0$, both blocks are identity matrices.

This example proves the point about the existence of non-trivial decompositions of the trivial interferometer, which can be adjoined to any non-trivial network. Therefore, we can always find alternate decompositions of a multi-mode interferometer defined by a single-particle matrix, where each of them act differently in the multi-particle anyonic subspaces. However, this example does not help us to understand this behavior in terms of the anyonic properties of the particles. To this end, we can consider the interferometer in fig. (22).

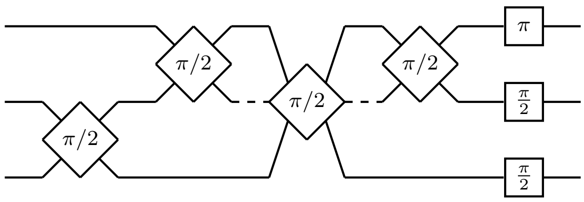


Figure 22 – Another decomposition of the trivial interferometer. This is the same network found in [239]

This network has the very convenient property of being composed only of mirrors and phase-shifters. This means that each component acts as a permutation over basis states of the Fock-space, up to a multiplication by complex phases. This particular network acts as a trivial interferometer in the single-particle subspace, as in the last example. The matrix of this network for all other multi-particle bosonic anyon and fermionic anyon subspaces is diagonal. Most importantly, the matrix for the two-particle subspace spanned by the basis vectors ($|1,1,0\rangle_\chi$,$|1,0,1\rangle_\chi$,$|0,1,1\rangle_\chi$) is the same matrix for both bosonic and fermionic anyons, given by

$$\begin{bmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.52}$$

The explanation for this effect can be obtained by using the physical interpretation of the action of a mirror, given in subsection 4.2.1. Consider the state $|1,1,0\rangle_\chi$. The first mirror sends the particle from mode 2 to mode 3, while leaving the particle in mode 1 alone. The second mirror sends the particle in mode 1 to mode 2, leaving the one at 3 alone. The non-local mirror in the middle leads to a downward exchange, taking the particle from mode 3 to mode 1, tunneling through the particle at mode 2, which leads to the state acquiring the exchange phase $e^{i\varphi}$. A similar reasoning also applies for the state $|1,0,1\rangle_\chi$, where what occurs is a upward exchange, leading to a phase of $e^{-i\varphi}$.

In both examples, the presence of a non-local gate was fundamental to achieve the results. For fermionic anyons this is mandatory, since fermionic particles are unable to exchange meaningfully under beam-splitters due to the exclusion principle. However, for the bosonic anyon case, it may be the case that local beam-splitters are enough to generate these types of non-local effects. This possibility is still open.

4.2.3 Anyonic optical networks and quantum computation

Given our understanding of the distinct properties possessed by anyonic optical networks, and the differences from their standard counterparts, we can now see how to use them to build quantum computing models. Since the single-particle behavior of all types of networks are equivalent, it makes sense to define logical qubits with a dual-rail encoding

$$|0\rangle \equiv |1,0\rangle_\chi\,, \tag{4.53a}$$

$$|1\rangle \equiv |0,1\rangle_\chi\,. \tag{4.53b}$$

The encoding of more qubits is straightforward. A two-qubit system needs four modes, with corresponding logical states

$$|0\rangle\otimes|0\rangle \equiv |1,0,1,0\rangle_\chi\,,\quad |0\rangle\otimes|1\rangle \equiv |1,0,0,1\rangle_\chi\,, \tag{4.54a}$$

$$|1\rangle\otimes|0\rangle \equiv |0,1,1,0\rangle_\chi\,,\quad |1\rangle\otimes|1\rangle \equiv |0,1,0,1\rangle_\chi\,. \tag{4.54b}$$

Three qubits need six modes, and so on, as exemplified in subsections 3.2.2 and 3.3.2.

Following subsection 3.2.2, single-qubit gates are generated by networks of phase shifters and beam splitters in mode pairs that encode a logical qubit. Phase-shifters lead to Z rotations in the Bloch sphere, as indicated by

$$PS_1(\theta)\,|1,0\rangle_\chi = e^{i\theta}\,|1,0\rangle_\chi\,,\quad PS_2(\theta)\,|1,0\rangle_\chi = |1,0\rangle_\chi\,, \tag{4.55a}$$

$$PS_1(\theta)\,|0,1\rangle_\chi = |0,1\rangle_\chi\,,\quad PS_2(\theta)\,|0,1\rangle_\chi = e^{i\theta}\,|0,1\rangle_\chi\,, \tag{4.55b}$$

while a beam splitter between modes 1 and 2 acts in the logical basis states as

$$BS_{12}(\theta)\,|1,0\rangle = \cos\theta\,|1,0\rangle + i\sin\theta\,|0,1\rangle\,,$$

$$BS_{12}(\theta)\,|0,1\rangle = i\sin\theta\,|1,0\rangle + \cos\theta\,|0,1\rangle\,,$$

which is a logical X rotation in the Bloch sphere by an angle θ.

To build a universal quantum computer, we must also have an entangling two-qubit gate [169], as explained in subsection 3.1.3. We also saw in subsections 3.2.2 and 3.3.2 that, for standard particles, deterministic entangling gates require non-linear interactions. However, for anyonic particles we can generate these entangling gates using only passive, Gaussian anyonic networks.

To see this, first consider the network in fig. 23. This network acts as the one found in fig. 21 over the first three modes. The Q_1 indicates the modes encoding the first logical qubit and Q_2 indicates the second. Given the results in subsection 4.2.2 for this specific network, it is easy to see that its action over the encoded subspace is given by the gate

$$[C(\varphi)] = \begin{bmatrix} e^{-\frac{i\pi}{4}}\frac{(1+i\cos\varphi)}{\sqrt{2}} & 0 & e^{-\frac{i\pi}{4}}\frac{i\sin\varphi}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ e^{\frac{i\pi}{4}}\frac{i\sin\varphi}{\sqrt{2}} & 0 & e^{\frac{i\pi}{4}}\frac{(1+i\cos\varphi)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.57}$$

which can be written as the quantum circuit decomposition

$$C(\varphi) = X_2(\Lambda(R_{0,0,1}(\pi/2)))_{2,1}(\Lambda(R_{\sin\varphi,0,\cos\varphi}(-\pi/2)))_{2,1}X_2. \tag{4.58}$$

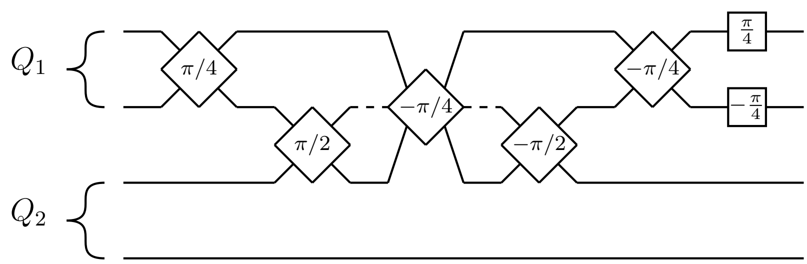


Figure 23 – Optical network acting over two logical qubits Q_1 and Q_2. This network the same as the one found in [238], up to phase-shifters.

This circuit decomposition helps us to build intuition about the gates properties. For example, its easy to see that when $\varphi = 0$, the gate is trivial, and when $\varphi = \pi$ it is locally equivalent to the $\Lambda(S)_{2,1}$ gate (see subsection 3.1.1 for notation). To determine the entangling power of this gate, we need to calculate the $G_1(C(\varphi))$ invariant. First, in order to use the formula in eq. (3.46), we need to write the matrix of $C(\varphi)$ in the Bell basis (given in eq. (3.41)). We have that

$$[C(\varphi)]_B = \begin{bmatrix} e^{i\frac{\varphi}{2}}\cos\varphi & 0 & 0 & -e^{i\frac{\varphi}{2}}\sin\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{-i\frac{\varphi}{2}}\sin\varphi & 0 & 0 & e^{-i\frac{\varphi}{2}}\cos\varphi \end{bmatrix}. \tag{4.59}$$

Then, since $\det\{[C(\varphi)]\} = 1$, we obtain from eq. (3.46) the entanglement power formula

$$e(C(\varphi)) = 1 - \cos^4 \frac{\varphi}{2} \tag{4.60}$$

Therefore, passive, Gaussian optical networks for fermionic anyons is universal for quantum circuits with respect to the dual-rail encoding, for all values $\varphi \neq 0$, even if very small. This fact has a nice interpretation in terms of the anyonic character of the particles. First, notice that for fermionic anyons specifically, we have that

$$J_\varphi(\hat{f}_i^\dagger \hat{f}_{i+1} + \hat{f}_{i+1}^\dagger \hat{f}_i) = \hat{\xi}_i^\dagger \hat{\xi}_{i+1} + \hat{\xi}_{i+1}^\dagger \hat{\xi}_i = \hat{f}_i^\dagger \hat{f}_{i+1} + \hat{f}_{i+1}^\dagger \hat{f}_i, \tag{4.61}$$
$$J_\varphi(\hat{f}_i^\dagger \hat{f}_{i+1}^\dagger + \hat{f}_{i+1} \hat{f}_i) = \hat{\xi}_i^\dagger \hat{\xi}_{i+1}^\dagger + \hat{\xi}_{i+1} \hat{\xi}_i = \hat{f}_i^\dagger \hat{f}_{i+1}^\dagger + \hat{f}_{i+1} \hat{f}_i. \tag{4.62}$$

for all modes i. This means that nearest-neighbour fermionic devices act in the same way on all Fock-spaces, regardless of them being anyonic or standard.

In particular, as was argued in subsection 3.3.2, optical networks built from Gaussian devices acting on nearest-neighbour modes generate a special, classically simulable family of circuits called matchgates. It was also shown in subsection 3.3.2 that extending the class of devices to include next-to-nearest-neighbour mode pairs allows the synthesis of gates outside the group of matchgates in the case of hard-core bosons, but not in the case of standard fermions. Therefore, our result suggests that one way of interpreting why non-local beam-splitters create different encoded gates for different particles is to see that next-to-nearest-neighbour Gaussian devices induce the one-dimensional Aharonov-Bohm effect, which is only present for non-standard particles.

However, the network of Fig. 23 only works to prove the encoded universality of fermionic anyons. In order to prove that this is also true for bosonic anyons, we need a network that preserves the dual-rail encoding for both bosonic and fermionic cases. Luckily, this is achieved by the network given in fig. 22. In fig. 24, it is shown how the network of fig. 22 is applied to the three modes in the middle, leaving the outermost two untouched. The first and last pairs of modes encodes the first and second logical qubits, respectively. Therefore, if we initialize the auxiliary mode A with one particle, the results of section 4.2.2 show that it generates the two-qubit gate

$$CP(\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{bmatrix}$$

This gate is a controlled phase gate, which is an entangling gate for all $\varphi \neq 0$.

The protocol explained in fig. 24 proves quantum universality for both fermionic and bosonic anyons for any $\varphi \neq 0$, using only optical networks and one auxiliary mode, with one particle that never leaves the circuit. Besides also holding for bosonic anyons,

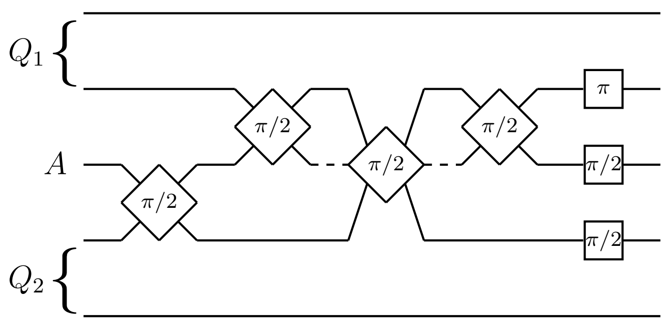


Figure 24 – Optical network acting over two logical qubits Q_1 and Q_2. This network the same as the one found in [239].

it improves the protocol given in fig. 23 in terms of simplicity, with a minor resource addition given by the presence of an auxiliary mode.

4.3 Bosonic anyons and coherent states

In this section we depart from the previous discussion on the action of optical devices over Fock states, and move onto the theory of coherent states for standard bosons and for bosonic anyons. In subsection 4.3.1, I review the theory of optical coherence for standard bosons, with a particular focus on the difference between eigenstates of annihilation operators and generalized coherent states. The actual definition of coherent states lie in the concept of n-th order coherence coefficients.

In subsection 4.3.2, I discuss the meaning of coherence for bosonic anyons. First, I point out that the eigenstate of a single annihilation operator for bosonic anyons is a coherent state. Next, I show that two-mode coherent states are harder to define, since annihilation operators for different modes do not commute. Nevertheless, by the use of anyonic displacement operators, we can define multi-mode anyonic coherent states.

Finally, in subsection 4.3.3, I discuss the action of optical devices on anyonic coherent states. First, I review the action of beam-splitters and phase-shifters on annihilation operator eigenstates for standard bosons, and calculate the action of anyonic beam-splitters over single-mode coherent states. Next, I discuss how to transform between the different classes of generalized coherent states. I finish by showing an applications where cat-states are obtained from passive Gaussian devices acting over single-mode anyonic coherent states.

4.3.1 Review of the theory of coherence

Coherent states are usually defined as eigenstates of annihilation operators. For example, for standard bosons, a coherent state satisfies

$$\hat{b}\,|g\rangle_b = g\,|g\rangle_b\,. \tag{4.63}$$

Here g, known as the amplitude of the coherent state, can be any complex number (due to the non-Hermiticity of $\hat{b}$). In the Fock basis, the coherent state can be written as

$$|g\rangle_b = e^{-\frac{1}{2}|g|^2} \sum_n \frac{(g\hat{b}^\dagger)^n}{n!} |0\rangle_b \,, \tag{4.64}$$

The states defined by eq. (4.63), however, are only a particular kind of coherent state. The theory of optical quantum coherence arose from the task of discriminating experimentally between different states of the electromagnetic field by the amplitude of n-photon absorption events [96, 97]. Given a single-mode input state $|input\rangle_b$ coming from some field source, the probability of detecting n photons is given by the n-th order correlation function

$$C_b(n) = \langle input| \, (\hat{b}^\dagger)^n (\hat{b})^n \, |input\rangle_b \,.$$

These correlators can be used, for example, as a measure to attest the quality of single-photon sources [55, 223, 225], since for such sources we should have $C_b(1)$ as high as possible. It is more common, however, to use the so-called n-th order single-mode coherence functions

$$c_b(n) = \frac{\left\langle (\hat{b}^\dagger)^n (\hat{b})^n \right\rangle_b}{\langle n \rangle_b^n}, \tag{4.65}$$

where $c_b(n)$ is calculated relative to some specific state. We say that a state $|\psi\rangle_b$ is n-order coherent if $c_b(m) = 1$ for all $m \leq n$.

In the general theory of coherence, a *coherent state* is one for which $c_b(n) = 1$ for all $n \in \mathbb{N}$. In other words, this state has full coherence in the sense that it is n-order coherent for all n. The most general single-mode coherent state is of the form

$$|g|\{\rho_n\}\rangle_b = e^{-\frac{1}{2}|g|^2} \sum_n e^{i\rho_n} \frac{(g\hat{b}^\dagger)^n}{n!} |0\rangle_b \,, \tag{4.66}$$

where the notation $g|\{\rho_n\}$, indicated that g is the amplitude of the coherent states, and ρ_n is an arbitrary sequence of real numbers [25, 237]. Note that we recover the states defined in eq. (4.63) in the particular case of $\rho_n = 0$ for all n.

Coherent states that are also eigenstates of annihilation operators satisfy a minimum uncertainty relation with respect to the quadrature operators

$$\hat{q} = \frac{1}{2}(\hat{b}^\dagger + \hat{b}) \tag{4.67}$$

$$\hat{p} = \frac{1}{2i}(\hat{b}^\dagger - \hat{b}). \tag{4.68}$$

The quadrature operators satisfy the commutation relation $[\hat{q}, \hat{p}] = 1$, and form a representation of the quantum harmonic oscillator. Only eigenstates of $\hat{b}$ can be simultaneously single-mode coherent states and minimum uncertainty states [25], which is a property that sets them apart from more general coherent states.

Eigenstates of the annihilation operator can be created from the vacuum by the action of the displacement operator

$$\hat{D}_b(g) = \exp\Big\{g\hat{b}^\dagger - g^*\hat{b}\Big\}, \tag{4.69}$$

which can be written in the equivalent forms

$$\hat{D}_b(g) = e^{-\frac{1}{2}|g|^2} \exp\Big\{g\hat{b}^\dagger\Big\} \exp\Big\{-g^*\hat{b}\Big\}, \tag{4.70a}$$

$$\hat{D}_b(g) = e^{\frac{1}{2}|g|^2} \exp\Big\{-g^*\hat{b}\Big\} \exp\Big\{g\hat{b}^\dagger\Big\}. \tag{4.70b}$$

Note that $\hat{D}_b(g)$ is a unitary operator, with $\hat{D}_b^\dagger(g) = \hat{D}_b(-g)$.

Several properties of displacement operators can be derived from those identities. The most important is that these operators "displace" the vacuum state. This follows from the equations

$$\hat{D}_b(-g)\hat{b}\hat{D}_b(g) = \hat{b} + g \tag{4.71a}$$

$$\hat{D}_b(-g)\hat{b}^\dagger\hat{D}_b(g) = \hat{b}^\dagger + g^*, \tag{4.71b}$$

called the displacement identities, from which we can show that the state

$$|g\rangle_b = \hat{D}_b(g)\,|0\rangle_b = e^{-\frac{1}{2}|g|^2} \sum_n \frac{(g\hat{b}^\dagger)^n}{n!}\,|0\rangle_b \tag{4.72}$$

is, in fact, an eigenstate of $\hat{b}$.

Displacement operators form an algebra, given by the relation

$$\hat{D}_b(g)\hat{D}_b(h) = e^{gh^*-hg^*} D(g+h). \tag{4.73}$$

From this algebra, one can calculate the overlap function

$$\langle h|g\rangle_b = e^{-\frac{1}{2}(|g|^2+|h|^2-2gh^*)}, \tag{4.74}$$

which shows that coherent states are not orthogonal in general. Nonetheless they still satisfy the relation

$$\int_{\mathbf{C}} \frac{d^2g}{\pi}\,|g\rangle\langle g|_b = \mathbb{I}, \tag{4.75}$$

with $\mathbb{I}$ as the identity operator, which makes them an over-complete basis for the single-mode state space.

As pointed out in [229], generalized coherent states can be written in terms of the action of an operator over annihilation operator eigenstates. For every sequence of real numbers $\{\rho_n\}_{n\in\mathbb{N}}$, define the operator

$$T_{\{\rho_n\}_{n\in\mathbb{N}}} = \sum_{n\in\mathbb{N}} e^{i\rho_n}\,|n\rangle\langle n|_b\,. \tag{4.76}$$

It is not hard to see that this operator has the aforementioned property, and by using it, we are able to describe different classes of generalized coherent states. We will be mainly interested in generalized coherent states for which the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is periodic, with period $p \in \mathbb{Z}$.

Due to the over-completeness of annihilation eigenstates, it was proven in [25] that any generalized coherent state can be written as a superposition

$$|g|\{\rho_n\}\rangle_b = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\gamma f(\gamma) \left||g|e^{i\gamma}\right\rangle_b , \tag{4.77}$$

where $f(\gamma)$ is the Fourier series

$$f(\gamma) = \sum_{n\in\mathbb{N}} e^{i\rho_n} e^{-in\gamma}, \tag{4.78}$$

and the state $||g|e^{i\gamma}\rangle_b$ is an eigenstate of $\hat{b}$ with eigenvalue $e^{i\gamma}|g|$.

When the state is periodic with period $p \in \mathbb{Z}$, the Fourier series can be summed and described as a linear combination of p Dirac delta functions of the form $\delta(\gamma - k2\pi/p)$ with $k = 0, \ldots, p-1$, each with a different weight. This fact implies that the general superposition formula can be simplified to

$$|g|\{\rho_n\}, p\rangle_b = \sum_{k=0}^{p-1} w_k \left||g|e^{ik\frac{2\pi}{p}}\right\rangle_b , \tag{4.79}$$

where the notation $g|\{\rho_n\}, p$ implies that the sequence $\{\rho_n\}$ has period $p \in \mathbb{Z}$, and w_k are the respective weights. Such states also have the property of being eigenstates of $\hat{b}^p$. In fact, by doing cyclic permutations of the weights in the previous equations, we obtain a set of p orthonormal eigenstates of $\hat{b}^p$. This fact will be very important later on.

Now let us go to the multi-mode case. In this case, the definition of coherent states is given in terms of the multi-mode n-th order coherence coefficients

$$c_{b_i}(n) = \frac{\left\langle (\hat{b}_i^\dagger)^n (\hat{b}_i)^n \right\rangle_b}{\langle n_i \rangle_b^n}. \tag{4.80}$$

A multi-mode coherent state must satisfy the condition $c_{b_i}(n) = 1$ for all n and for all modes i. Since, for standard bosons, annihilation operators commute with each other, we can define simultaneous eigenstates of annihilation operators, and they happen to be multi-mode coherent states of this kind.

The most general type of multi-mode coherent states over an m mode system is given by the expression

$$|g_1, \ldots, g_m|\{\rho_\mathbf{n}\}\rangle_b = T_{\{\rho_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}^m}} \prod_{i=1}^m \hat{D}_{b_i}(g_i) \left|0\right\rangle_b , \tag{4.81}$$

where the sequence $\{\rho_{\mathbf{n}}\}$ is a real number sequence indexed by vectors of natural number coefficients, $\hat{D}_{b_i}(g_i)$ are displacement operators over the modes i, which also commute with each other, and $T_{\{\rho_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^m}}$ is an operator having the form

$$T_{\{\rho_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^m}} = \sum_{n_1,\ldots,n_m} e^{i\rho_{n_1,\ldots,n_m}} \left|n_1,\ldots,n_m\middle\rangle\!\middle\langle n_1,\ldots,n_m\right|_b . \tag{4.82}$$

4.3.2 Anyonic coherent states

After describing the theory of coherent states, let us now consider its generalization for bosonic anyons. For single-mode systems, we may define eigenstates of annihilation operators:

$$\hat{\beta}\left|g\right\rangle_\beta = g\left|g_\beta\right\rangle , \tag{4.83}$$

where again g is any complex number. These states have full coherence if we define the n-th order coherence coefficients $c_\beta(n)$ in the analogous form

$$c_\beta(n) = \frac{\left\langle (\hat{\beta}^\dagger)^n(\hat{\beta})^n \right\rangle_\beta}{\langle n\rangle_\beta^n}. \tag{4.84}$$

We can take these states as the first examples of single-mode bosonic anyons coherent states.

In fact, since single-mode commutation relations for bosonic anyons are the same as their standard bosonic counterparts, the algebra of anyonic, single-mode displacement operators $\hat{D}_\beta(g)$ is isomorphic to the standard version. Therefore, as long as we only consider a single mode, coherent states for anyonic bosons have exactly the same properties as those of standard bosons. In particular, we can also define the anyonic version of $T_{\{\rho_n\}_{n\in\mathbb{N}}}$ (see eq. (4.76)) by the same expression as the standard case. This implies that single-mode generalized coherent states for anyons are also analogous to the standard case.

Since, in the single-mode case, the theory is exactly the same, to look for meaningful differences we must proceed to the multi-mode case. Consider first the case of two-mode bosonic anyons. Using the form of displacement operators for anyonic modes, we can define the set of states

$$\left|u;v\right\rangle_\beta^{(1,2)} = \hat{D}_{\beta_1}(u)\hat{D}_{\beta_2}(v)\left|0\right\rangle_\beta , \tag{4.85}$$

for any $u, v \in \mathbb{C}$. By defining multi-mode n-th order coherent coefficients for bosonic anyons, it is not hard to see that this state is two-mode coherent.

However, in contrast to the standard case, this state is not a simultaneous eigenstate of $\hat{\beta}_1$ and $\hat{\beta}_2$. In particular, we have that

$$\hat{\beta}_1\left|u;v\right\rangle_\beta^{(1,2)} = u\left|u;v\right\rangle_\beta^{(1,2)} , \tag{4.86}$$

$$\hat{\beta}_2\left|u;v\right\rangle_\beta^{(1,2)} = v\left|ue^{-i\varphi};v\right\rangle_\beta^{(1,2)} . \tag{4.87}$$

These expressions are obtained from the propagation relation

$$\hat{\beta}_2 \hat{D}_{\beta_1}(u) = \hat{D}_{\beta_1}(ue^{i\varphi})\hat{\beta}_2. \tag{4.88}$$

In this example, the system is an eigenstate of the annihilation operators in the first mode, but not of the second. In particular, this means that such state is a minimum uncertainty state for the quadrature variables of the first mode, but not for the second.

The lack of commutativity between $\hat{D}_{\beta_1}(u)$ and $\hat{\beta}_2$ translates into a lack of commutativity between $\hat{D}_{\beta_1}(u)$ and $\hat{D}_{\beta_2}(v)$. This prompts us to define the state

$$|u;v\rangle_\beta^{(2,1)} = \hat{D}_{\beta_2}(v)\hat{D}_{\beta_1}(u)\,|0\rangle_\beta\,, \tag{4.89}$$

which ends up being different from $|u;v\rangle_\beta^{(1,2)}$. This seems to imply that multi-mode coherent states for anyonic bosons are, in one way or another, generalized coherent states.

Similarly to first example, the state $|u;v\rangle_\beta^{(2,1)}$ is only an eigenstate of $\hat{\beta}_2$, and not of $\hat{\beta}_1$. In fact, the action of these operators is given by

$$\hat{\beta}_1\,|u;v\rangle_\beta^{(2,1)} = u\left|u;ve^{i\varphi}\right\rangle_\beta^{(2,1)}, \tag{4.90}$$

$$\hat{\beta}_2\,|u;v\rangle_\beta^{(2,1)} = v\,|u;v\rangle_\beta^{(2,1)}\,. \tag{4.91}$$

Whatever choice we make, both of these states are coherent, but none is a simultaneous eigenstate of both of the annihilation operators.

The lack of a common eigenstate changes the way we treat generalized eigenstates in the case of anyonic system, since in order to define a T operator, characterizing the relative phases of each power in the coherent state expansion, we used the eigenstates as reference states. However, since we still have the notion of normal ordering for bosonic anyon operators, we can choose $|u;v\rangle_\beta^{(1,2)}$ as a reference state for two-mode coherent states. With this choice in mind, consider the identities

$$|u;v\rangle_\beta^{(1,2)} = \left[\sum_{l,k} N\frac{(u\hat{\beta}_1^\dagger)^l}{l!}\frac{(v\hat{\beta}_2^\dagger)^k}{k!}\right]|0\rangle_\beta\,, \tag{4.92a}$$

$$|u;v\rangle_\beta^{(2,1)} = \left[\sum_{l,k} Ne^{-i\varphi kl}\frac{(u\hat{\beta}_1^\dagger)^l}{l!}\frac{(v\hat{\beta}_2^\dagger)^k}{k!}\right]|0\rangle_\beta\,. \tag{4.92b}$$

where $N = \exp\left\{-1/2(|u|^2+|v|^2)\right\}$.

It is easy to see that the $|u;v\rangle_\beta^{(2,1)}$ can be written as

$$|u;v\rangle_\beta^{(2,1)} = T_{-\varphi n_1 n_2}\,|u;v\rangle_\beta^{(1,2)}\,, \tag{4.93}$$

where

$$T_{-\varphi n_1 n_2} = \sum_{n_1,n_2} e^{-i\varphi n_1 n_2}\,|n_1,n_2\rangle\!\langle n_1,n_2|_\beta\,, \tag{4.94}$$

is the operator associated to the sequence of relative phases $\{-\varphi n_1 n_2\}_{(n_1,n_2)\in\mathbb{N}^2}$. This operator has exactly the same representation as the time evolution of the state $|u;v\rangle_\beta^{(1,2)}$ under the cross-Kerr interaction $\hat{H}_{c-Kerr} = -\hat{n}_1\hat{n}_2$ with time parameter given by the statistical angle φ.

This shows us that, in some way, the anyonic character of the particles is introducing an effective quartic interaction that depends on the order of the modes. However, to make this statement more precise, we need to discuss the effects of optical devices on anyonic coherent states.

4.3.3 Anyonic coherent states under optical networks

First, let us do a quick review on the action of beam-splitters and phase-shifters on standard bosonic coherent states. Take a system initialized in either of the coherent states

$$|g,0\rangle_b = \hat{D}_{b_1}(g)\,|0\rangle_b\,,$$
$$|0,h\rangle_b = \hat{D}_{b_2}(h)\,|0\rangle_b\,,$$

The action of a phase shifter $PS_1(\tau)$ on $|g,0\rangle_b$ is simply given by

$$PS_1(\tau)\,|g,0\rangle_b = \left|ge^{i\tau},0\right\rangle_b,$$

and the action of $PS_2(\tau)$ on $|0,h\rangle_b$ is, similarly, given by $|0,he^{i\tau}\rangle$.

More interesting is the action of the beam splitter $BS_{12}(\theta)$, which can be obtained using eq. (3.56), resulting in

$$BS_{12}(\theta)\,|g,0\rangle_b = |\cos(\theta)g, i\sin(\theta)g\rangle_b\,,$$
$$BS_{12}(\theta)\,|0,h\rangle_b = |i\sin(\theta)h, \cos(\theta)h\rangle_b\,.$$

It is easy to see that this state is two-mode coherent, in the sense that $c_b^1(n) = 1$ and $c_b^2(n) = 1$ for all n. We now depart from standard naming conventions and call any state of the form

$$|g_1,\ldots,g_m\rangle_b = \prod_{i=1}^{m} \hat{D}_i(g_i)\,|0\rangle_b\,, \tag{4.97}$$

an exact multi-mode coherent state, or *exact coherent state* for short.

In general, the action of an arbitrary two-mode linear map $\hat{A}$ over an arbitrary two-mode coherent state is then

$$\hat{A}\,|u,v\rangle_b = |A_{1,1}u + A_{1,2}v, A_{2,1}u + A_{2,2}v\rangle_b\,, \tag{4.98}$$

where $\hat{A}$ is determined by the coefficient matrix $A = [A_{i,j}]$. It is a simple fact of linear algebra that, for any nonzero complex vector, there is a unitary matrix which rotates it into a vector with a single nonzero component. Therefore, given any $|u,v\rangle_b$, one can find two linear maps $\hat{A}^1(u,v)$ and $\hat{A}^2(u,v)$, such that

$$\hat{A}^1(u,v)\,|u,v\rangle_b = \left|\sqrt{|u|^2+|v|^2},0\right\rangle_b \tag{4.99a}$$

$$\hat{A}^2(u,v)\,|u,v\rangle_b = \left|0,\sqrt{|u|^2+|v|^2}\right\rangle_b . \tag{4.99b}$$

This observation motivates the following definition. When, for a bosonic state $|\psi\rangle_b$, a linear dynamic $\hat{A}(\psi)$ can be found such that $\hat{A}(\psi)\,|\psi\rangle_b$ is a single-mode coherent state, we say that $|\psi\rangle_b$ is a *dynamically coherent* state. It is not hard to see that all dynamically coherent states are exactly coherent. These two notions of coherent state—dynamic and exact—are not standard in the quantum optics literature, since there they coincide. However, as we already saw, annihilation operators do not commute for bosonic anyons. This suggests that these definitions might not agree, as we now show.

Naturally, exact single-mode coherent states for bosonic anyons should be defined in the same way as they were defined for standard bosonic ones

$$|g,0\rangle_\beta = \hat{D}_{\beta_1}(g)\,|0\rangle_\beta\,, \tag{4.100a}$$

$$|0,h\rangle_\beta = \hat{D}_{\beta_2}(h)\,|0\rangle_\beta\,. \tag{4.100b}$$

The action of a phase shifter on these states is exactly the same as in the standard case, due to their diagonal action over creation and annihilation states.

The action of a beam splitter operator can be calculated from the propagation identities of eq. (2.87), leading to the expressions

$$BS_{1,2}(\theta)\,|g,0\rangle_\beta = N_g \sum_n \frac{1}{n!}\prod_{k=0}^{n-1}(g\cos(\theta)\hat{\beta}_1^\dagger + ie^{-ik\varphi}g\sin(\theta)\hat{\beta}_2^\dagger)\,|0\rangle_\beta\,, \tag{4.101a}$$

$$BS_{1,2}(\theta)\,|0,h\rangle_\beta = N_h \sum_n \frac{1}{n!}\prod_{k=0}^{n-1}(ie^{ik\varphi}h\sin(\theta)\hat{\beta}_1^\dagger + h\cos(\theta)\hat{\beta}_2^\dagger)\,|0\rangle_\beta\,, \tag{4.101b}$$

where $N_x = \exp\left\{-1/2|x|^2\right\}$ for any x.

In order to get rid of the binomial product, we need to prove the deformed binomial identity below. Let a, b be arbitrary complex numbers and $i<j$, then

$$\prod_{k=0}^{n-1}(a\hat{\beta}_i^\dagger + e^{-ik\varphi}b\hat{\beta}_j^\dagger) = e^{-i\varphi\frac{n(n-1)}{2}}\prod_{k=0}^{n-1}(e^{ik\varphi}a\hat{\beta}_i^\dagger + b\hat{\beta}_j^\dagger) \tag{4.102a}$$

where

$$\prod_{k=0}^{n-1}(e^{ik\varphi}a\hat{\beta}_i^\dagger + b\hat{\beta}_j^\dagger) = \sum_{l=0}^{n}\binom{n}{l}e^{i\varphi\frac{l(l-1)}{2}}(a\hat{\beta}_i^\dagger)^l(b\hat{\beta}_j^\dagger)^{n-l}. \tag{4.102b}$$

This is proven in the following way. For the first equality, is easy to prove see that

$$a\hat{\beta}_i^\dagger + e^{-ik\varphi}b\hat{\beta}_j^\dagger = e^{-ik\varphi}(e^{ik\varphi}a\hat{\beta}_i^\dagger + b\hat{\beta}_j^\dagger), \tag{4.103}$$

which implies

$$\prod_{k=0}^{n-1}(a\hat{\beta}_i^\dagger + e^{-ik\varphi}b\hat{\beta}_j^\dagger) = \left[\prod_{k=0}^{n-1} e^{-i\varphi k}\right]\prod_{k=0}^{n-1}(e^{ik\varphi}a\hat{\beta}_i^\dagger + b\hat{\beta}_j^\dagger). \tag{4.104}$$

Now just notice that

$$\prod_{k=0}^{n-1} e^{-i\varphi k} = e^{i\varphi\sum_{n-1}^{k=0} k} = e^{i\varphi\frac{n(n-1)}{2}}. \tag{4.105}$$

The next identity though is a bit more challenging, and requires the use of induction over n. Is easy to see that the identity is trivial for $n = 1$, now suppose it is valid for the n case. The $n + 1$ can be written as

$$\sum_{l=0}^{n}\binom{n}{l}e^{i\varphi\frac{l(l-1)}{2}}(a\hat{\beta}_i^\dagger)^l(b\hat{\beta}_j^\dagger)^{n-l}(e^{in\varphi}a\hat{\beta}_i^\dagger + b\hat{\beta}_j^\dagger), \tag{4.106}$$

where we used the induction hypothesis for n. Expanding the factors and rearranging the operators such that they are normally ordered we obtain

$$\sum_{l=0}^{n}\binom{n}{l}e^{i\varphi\frac{l(l+1)}{2}}(a\hat{\beta}_i^\dagger)^{l+1}(b\hat{\beta}_j^\dagger)^{n-l} + \sum_{l=0}^{n}\binom{n}{l}e^{i\varphi\frac{l(l-1)}{2}}(a\hat{\beta}_i^\dagger)^{l}(b\hat{\beta}_j^\dagger)^{n-l+1}. \tag{4.107}$$

To proceed we take the terms $l = n$ from the first summation and $l = 0$ from the second summation, writing then explicitly. After that we make the substitution $k = l+1$ in the first sum and $k = l$ in the second sum, obtaining

$$e^{i\varphi\frac{n(n+1)}{2}}(a\hat{\beta}_i^\dagger)^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]e^{i\varphi\frac{k(k-1)}{2}}(a\hat{\beta}_i^\dagger)^{k}(b\hat{\beta}_j^\dagger)^{(n+1)-k} + (b\hat{\beta}_j^\dagger)^{n+1}, \tag{4.108}$$

then, using the identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, \tag{4.109}$$

it easy to see that the case $n + 1$ is also valid, thus proving the theorem.

Direct application of the deformed binomial identities lead to the results

$$BS_{1,2}(\theta)\,|g,0\rangle_\beta = \left[\sum_{l,k} N_g c^1_{l,k}\frac{(\cos(\theta)g\hat{\beta}_1^\dagger)^l}{l!}\frac{(i\sin(\theta)g\hat{\beta}_2^\dagger)^k}{k!}\right]|0\rangle_\beta\,, \tag{4.110a}$$

$$BS_{1,2}(\theta)\,|0,h\rangle_\beta = \left[\sum_{l,k} N_h c^2_{l,k}\frac{(i\sin(\theta)h\hat{\beta}_1^\dagger)^l}{l!}\frac{(\cos(\theta)h\hat{\beta}_2^\dagger)^k}{k!}\right]|0\rangle_\beta\,, \tag{4.110b}$$

where we have

$$c^1_{l,k} = e^{-i\varphi(lk+\frac{k(k-1)}{2})},$$
$$c^2_{l,k} = e^{i\varphi\frac{l(l-1)}{2}}.$$

Due to the group property of beam splitters, the form of the states in eqs. (4.110) stay the same, even after successive applications of beam splitters with different angles.

This observation allows us to find at least two different types of states with that have dynamical coherence, represented by the equations

$$|u,v\rangle^1_\beta = N\sum_n \frac{1}{n!}\prod_{k=0}^{n-1}(u\hat{\beta}_1^\dagger + e^{-ik\varphi}b\hat{\beta}_2^\dagger)\,|0\rangle_\beta\,,$$
$$|u,v\rangle^2_\beta = N\sum_n \frac{1}{n!}\prod_{k=0}^{n-1}(e^{ik\varphi}u\hat{\beta}_1^\dagger + v\hat{\beta}_2^\dagger)\,|0\rangle_\beta\,,$$

where $N = \exp\Big\{-(|u|^2 + |v|^2)\Big\}$. We refer to these states as type 1 and type 2 dynamically coherent states, respectively. The action of a general two-mode interferometer $\hat{A}$ is then

$$\hat{A}\,|u,v\rangle^i_\beta = |(A_{11}u + A_{12}v),(A_{21}u + A_{22}v)\rangle^i_\beta\,,$$

for $i = 1, 2$.

By the definition, its easy to see that the unitary $\hat{T} = \exp\Big\{i\varphi\hat{K}\Big\}$ with K given by the anyonic Kerr Hamiltonian

$$\hat{K}_{12} = \frac{(\hat{n}_1 + \hat{n}_2)(\hat{n}_1 + \hat{n}_2 - 1)}{2},$$

is such that

$$|u,v\rangle^2_\beta = \hat{T}\,|u,v\rangle^1_\beta\,.$$

Therefore, there is no passive, Gaussian Hamiltonian that can convert a type 1 dynamically coherent state into a type 2 dynamically coherent state. By inspection, we see that no exactly coherent state can be mapped into a dynamically coherent state using passive, Gaussian anyonic Hamiltonians.

This fact seems to indicate that introducing the capacity of "displacing the anyonic vacuum" in passive, Gaussian optical networks for bosonic anyons is in some sense equivalent to introducing quartic interactions. However, the details of such construction are not yet clear, and will be left for future research. On a last note, however, notice that both dynamically coherent states can also be mapped into the reference exact coherent state $|u,v\rangle^{(1,2)}_\beta$ by quartic two-mode, number-preserving interactions.

We believe that this incursion into the different kinds of two-mode coherent states of bosonic anyons is enough to illustrate the drastic effects the anyonic exchange phase has on coherent state dynamics. As a last example, let us study the effect of a mirror, i.e. the network given by $PS_1(\pi/2)BS_{12}(\pi/2)PS_2(\pi/2)$, on single-mode coherent states:

$$|u,0\rangle_\beta = \hat{D}_{\beta_1}(u)\,|0\rangle_\beta\,,$$
$$|0,v\rangle_\beta = \hat{D}_{\beta_2}(v)\,|0\rangle_\beta\,.$$

From our previous discussion it follows that the output states are given by

$$|0,u\rangle_\beta^1 = N_u \sum_k e^{-i\varphi\frac{k(k-1)}{2}} \frac{(u\hat{\beta}_2^\dagger)^k}{k!} |0\rangle_\beta ,$$
$$|v,0\rangle_\beta^2 = N_v \sum_k e^{i\varphi\frac{k(k-1)}{2}} \frac{(v\hat{\beta}_1^\dagger)^k}{k!} |0\rangle_\beta ,$$

where the first state is a reflection of $|u,0\rangle_\beta$ and the second a reflection of $|0,v\rangle_\beta$.

For all values of φ, we can use the Fourier representation of eq.(4.77) with $f_i(\gamma)$ given by

$$f_i(\gamma) = \sum_{n\in\mathbb{N}} e^{\pm\varphi\frac{k(k-1)}{2}} e^{-in\gamma}, \tag{4.115}$$

where the value of $\pm$ depends on whether $i = 1$ or $i = 2$. However, this representation can be further simplified in case of $\varphi = 2p\pi/q$ for integers p, q. In this case, the generalized single-mode coherent states are periodic with period q, and must have the form in eq. (4.79). Interestingly, for this specific example, the associated weights w_k can be obtained by finding the solutions to the *quadratic congruence*

$$k(k-1) \equiv 0 \mod 2q. \tag{4.116}$$

The full description of the solutions would involve the use of theory of quadratic residues, and is outside the scope of this work.

For the sake of illustration, let us take the simplest case $\varphi = \pi$. We see that the mirror acts as

$$|0,u\rangle_\beta^1 = \frac{1}{N\sqrt{2}} \left[(-1)^{\frac{1}{4}} |0,-iu\rangle_\beta - (-1)^{\frac{3}{4}} |0,iu\rangle_\beta\right],$$
$$|v,0\rangle_\beta^2 = \frac{1}{N\sqrt{2}} \left[(-1)^{\frac{1}{4}} |-iv,0\rangle_\beta - (-1)^{\frac{3}{4}} |iv,0\rangle_\beta\right],$$

up to a normalization factor N. Such states are called cat states, and they have multiple applications in quantum information theory, such as encoding logical qubits or as a resource for teleportation protocols [94, 137, 144, 158, 173, 175, 182, 206].

To sum up, the theory of coherent states for bosonic anyons has many interesting and surprising features, due to the intrinsic non-linearities imposed by the anyonic character of the particles. The extent to which this framework can be used to explain the computing power of quantum information processing with continuous variables, as it was used for discrete ones in subsection 4.3.2 remains largely unknown.

5 Conclusion, open questions, and final remarks

In the end, our research suggests a physical way of understanding the computing power of both ballistic (billiard-ball-like) quantum computing modes, and the quantum circuit model itself. I believe to have successfully demonstrated that: (i) the optical network model can be consistently generalized for systems of identical particles with one-dimensional, anyonic exchange statistics, as argued in section 4.1, (ii) the action of Gaussian optical devices over such systems can be interpreted using the physics of one-dimensional anyons, and (iii) that it is this physics that determines the expressive power of optical network models, as argued in section 4.2. Our results also suggest that this understanding can be used as another bridge between the physics of one-dimensional quantum systems and quantum computer science. We have began to cross this bridge, showing some results about the effects of anyonic exchange statistics over the dynamics of coherent states for bosonic anyons in section 4.3.

But such a bridge can be walked down much further. For example, we might ask whether further extensions of the optical network formalism are possible. As described in subsection 2.3.3.2, a simple, general class of particle systems with non-standard statistics is given by the braided commutation relations of eq. (2.81a). There is no a priori reason to believe that the formalism developed here could not be generalized to these classes of particles, especially if they can be obtained via more general Jordan-Wigner maps from commutation relations for standard bosons and fermions. Another interesting type of identical particles for which an extension might be interesting are the *Fock parafermions* defined in [54]. These particles have both anyonic exchange statistics, and fractional exclusion statistics, obeying a generalized exclusion principle of the type proposed by Gentile in [91].

We can also take the bridge backwards, and instead look for results in quantum computer science and translate them into results about the physics of exotic particle systems. We can, for example, ask if one can use the optical network model to provide quantum algorithms for simulating the behavior of one-dimensional, anyonic-Hubbard models [10, 264]. Or, we could also ask if results about random quantum circuits [27] can tell us anything about disorder in gases of hard-core bosons, due to their equivalence with the quantum circuit model. Finally, we could also take this bridge to another directions, and see if our results can help elucidate the debates around the quantum entanglement and the physics of identical particles. We could ask, for example, what the presence of the one-dimensional Aharanov-Bohm effect in fermionic anyons tells us about the one-particle-reduced entropy [68, 82, 172, 263] of particles as we vary the statistical parameter from standard fermions to hard-core bosons. The list of potential connections is limited only by the imagination and mathematical tractability of the non-commutative algebras involved.

As a final thought, I point out that as quantum technologies progress, the possibilities of simulating exotic particle systems grow larger by the day, and with it, the possibility of exploring bridges, such as the one opened by this work, grow too. As we have not yet found the ways to build reliable, fault-tolerant computers, bridges between the physics of condensed matter systems and the structure of quantum computers will become increasingly important in overcoming the challenges posed to us in the NISQ era.

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