GEOMETRIC STRING STRUCTURES

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Contents

1. introduction

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal m-category. An object $X \in \mathcal{C}$ is called invertible if there is another object $Y \in \mathcal{C}$, and equivalences $X \otimes Y \simeq \mathbb{1}$, and $Y \otimes X \simeq \mathbb{1}$. If the m-morphisms of $\mathcal{C}$ are equipped with an involution $a \mapsto a^*$, then we call an m-morphism a unitary if $aa^* = 1$ and $a^*a = 1$. We let $\mathcal{C}^\times$ denote the sub-category of invertible objects, invertible 1-morphisms, ... invertible $(m-1)$-morphisms, and unitary m-morphisms.

Let *Vect* be the groupoid of finite dimensional real Hilbert spaces, and suppose that we are given a symmetric monoidal functor

$$F : (\mathit{Vect}, \oplus) \to \mathcal{C}^\times.$$

Then for each $V \in \mathit{Vect}$, we can define a group $G(V)$ as follows: [GV]

$$G(V) := \left\{ (g, \beta) \,\middle|\, g \in O(V), \quad \begin{tikzcd} F(V) \arrow[r, bend left, "F(g)"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\beta"] & F(V) \end{tikzcd} \right\}. \tag{1}$$

It has an obvious map to $O(V)$, given by $(g, \beta) \mapsto g$. We also let $G(n) := G(\mathbb{R}^n)$. Strictly speaking, $G(V)$ is not a group, but rather a k-group, where $k = \max(1, m-1)$. For example, if $\mathcal{C}$ is a 3-category, then $G(V)$ also has arrows, given by

$$\hom_{G(V)}\Big((g, \beta), (h, \gamma)\Big) := \begin{cases} \emptyset & \text{if } g \neq h \\ \{\Xi : \beta \Rrightarrow \gamma\} & \text{if } g = h, \end{cases}$$

We have three situations in mind, where we can apply the above recipe.

In the first case, $\mathcal{C}$ is the category $\mathit{Hilb}_{\mathbb{R}}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded real Hilbert spaces. Its invertible objects are the one dimensional Hilbert spaces. We define $F(V)$ to be the top exterior power of V, put in degree $\dim(V)$. This yields $G(V) = SO(V)$, and is explained in Section 2.2.

Our second situation is the bicategory $\mathcal{C} = \mathit{VN2}_{\mathbb{R}}$ of $\mathbb{Z}/2$-graded von Neumann algebras over the reals. Its invertible objects are the type I factors i.e., algebras with minimal projections whose graded center is $\mathbb{R}$. The functor $F : \mathit{Vect} \to \mathcal{C}^{\times}$ is given by the Clifford algebra construction [cac]

$$C\ell(V) := \bigoplus_{i \geq 0} V^{\otimes i} \Big/ (v \otimes v - \|v\|^2), \qquad v \text{ is odd, } v^* = v. \tag{2}$$

Following our recipe, one finds $G(V) = \mathit{Spin}(V)$. This computation is the subject of Section 2.2.

Our last and most interesting example is the 3-category $\mathit{CN3}$ of $\mathbb{Z}/2$-graded conformal nets. The functor F is given by the free fermion construction $V \mapsto \mathit{Fer}(V)$, which is the subject of Section 4.1. The resulting 2-group $G(V)$ is the string group $\mathit{String}(V)$ i.e., the 3-connected cover of $O(V)$ (assuming $\dim V \geq 5$). Modulo a slight reinterpretation of (1), we also produce a model of $\mathit{String}(V)$ which is topological group.

We summarize the above discussion in the follwing table: [tab]

(3)

m	The m-category $\mathcal{C}$		$\mathcal{C}^{\times}$	$F : \mathit{Vect} \to \mathcal{C}^{\times}$	$G(n)$
1	$\mathit{Hilb}_{\mathbb{R}}^{\mathbb{Z}}$:	$\mathbb{Z}$-graded real Hilbert spaces	$\mathbb{Z}$-graded real lines	$\bigwedge^{\text{top}}$	$SO(n)$
2	$\mathit{VN2}_{\mathbb{R}}$:	$\mathbb{Z}/2$-graded von Neumann algebras over $\mathbb{R}$	$\mathbb{Z}/2$-graded type I factors over $\mathbb{R}$	Clifford algebra	$\mathit{Spin}(n)$
3	$\mathit{CN3}$:	$\mathbb{Z}/2$-graded conformal nets	$\mathbb{Z}/2$-graded conformal nets with μ-index equal to 1	The Free Fermion	$\mathit{String}(n)$

Perhaps more important than the groups $G(V)$ is the notion of a G-structure. An SO-structure on a vector space $V \in \mathit{Vect}$ is the same thing as an orientation. If V is n-dimensional, then an orientation can be described as a unitary isomoprhism between $\bigwedge^{\text{top}} V$ and $\mathbb{R}[n]$. More generally, a G-structure on V is a 1-morphism

$$f : F(V) \to F(\mathbb{R}^n)$$

in the groupoid $\mathcal{C}^{\times}$. Indeed, if V is equipped with such a 1-morphism f, then the automorphism group

$$\mathrm{Aut}(V, f) := \left\{ (g, \beta) \,\middle|\, g \in O(V), \begin{tikzcd} F(V) \arrow[rr, "F(g)"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & F(V) \arrow[dl, "f"] \\ & F(\mathbb{R}^n) & \end{tikzcd} \right\},$$

is isomophic to $G(V)$. We can thus extend table (3): [tab2]

(4)

Objects of $\mathcal{C}^{\times}$	Arrows of $\mathcal{C}^{\times}$	$F(\mathbb{R}^n)$	A G-structure $F(V) \xrightarrow{\sim} F(\mathbb{R}^n)$ is:	$G(V)$
$\mathbb{Z}$-graded real lines	Degree preserving unitary maps	$\mathbb{R}[n]$	An isometry $\bigwedge^{\text{top}} V \to \mathbb{R}[n]$	$SO(V)$
$\mathbb{Z}/2$-graded type I factors over $\mathbb{R}$	Morita equivalences	$C\ell(n)$	An invertible bimodule ${}_{C\ell(V)}S_{C\ell(n)}$	$\mathit{Spin}(V)$
$\mathbb{Z}/2$-gr. conformal nets with $\mu = 1$	Invertible defects	$\mathit{Fer}(n)$	An invertible defect ${}_{\mathit{Fer}(V)}D_{\mathit{Fer}(n)}$	$\mathit{String}(V)$

It is interesting to note that

$$\begin{aligned} \hom_{\mathit{VN2}_\mathbb{R}}(\mathbb{1},\mathbb{1}) &= \mathit{Hilb}_\mathbb{R}^{\mathbb{Z}/2} := \big\{\mathbb{Z}/2\text{-graded real Hilbert spaces}\big\},\\ \hom_{\mathit{CN3}}(\mathbb{1},\mathbb{1}) &= \mathit{VN2}_\mathbb{C} \;:= \big\{\mathbb{Z}/2\text{-graded von Neumann algebras over } \mathbb{C}\big\}, \end{aligned}$$

which establishes a connection between each row of (3), (4), and the previous one. The latter statement is our Theorem 3.19. We also have

$$\hom_{\mathit{VN2}_\mathbb{C}}(\mathbb{1},\mathbb{1}) = \mathit{Hilb}_\mathbb{C}^{\mathbb{Z}/2} := \big\{\mathbb{Z}/2\text{-graded complex Hilbert spaces}\big\}.$$

2. Statement of results, and sketch of the proofs

2.1. **The group** $SO(n)$ **[sec:SO].** In this section, we take $\mathcal{C} := \mathit{Hilb}_\mathbb{R}^\mathbb{Z}$ to be the category of $\mathbb{Z}$-graded real Hilbert spaces and F the top exterior power functor. Since $\mathit{Hilb}_\mathbb{R}^\mathbb{Z}$ is just a 1-category, the 2-cell β in (1) is simply stating the equality of $F(g)$ and $1_{F(V)}$. The group (1) then becomes

$$G_1(\mathbb{R}^n) = \Big\{(g,\beta)\,\Big|\, g\in O(n),\ \ \textstyle\bigwedge^n \mathbb{R}^n \underset{1}{\overset{\bigwedge^n g}{\underset{\Downarrow\beta}{\rightrightarrows}}} \textstyle\bigwedge^n \mathbb{R}^n\ \Big\} = \Big\{g\in O(n)\,\Big|\,\textstyle\bigwedge^n g = 1\Big\},$$

which is the usual definition of $SO(n)$. If V is n-dimensional, an orientation is then an isomorphism $\bigwedge^{\mathrm{top}} V \to \mathbb{R}[n]$ in the groupoid $\mathit{Hilb}_\mathbb{R}^{\mathbb{Z},\times}$.

2.2. **The group** $\mathit{Spin}(n)$ **[sec:Spin].** If A is a finite dimensional von Neumann algebra, then its identity arrow 1_A in $\mathit{VN2}_\mathbb{R}$ (a priori given by the Haagerup L^2-space) is a bimodule isomorphic to ${}_AA_A$. Any trace $tr : A \to \mathbb{R}$ provides such an isomorphism, and the induced inner product on A is given by

$$\langle a,b\rangle := tr(a^*b).$$

Let now ${}_AM_B$ and ${}_BN_C$ be 1-morphisms that are finite dimensional as vector spaces. Given a trace on B, one can identify their composition in $\mathit{VN2}_\mathbb{R}$ (a priori given by Connes fusion) with their usual tensor product. The inner product on ${}_AM \otimes_B N_C$ is then given by

$$\langle m\otimes n, m'\otimes n'\rangle := \sum_i \langle mb_i, m'\rangle\langle n, b_in'\rangle,$$

where $\{b_i\}$ is an orthonormal basis of B with respect to the inner product coming from the trace. More detail about Connes fusion, and about the 2-category $\mathit{VN2}_\mathbb{R}$ will be given in Section 3.1.

If G is a group and A a von Neumann algebra, then having an action of G on A means the following. For each element $g\in G$, we are given an A-A-bimodule M_g, and we have isomorphisms

$$a_{g,h} : M_g \boxtimes_A M_h \xrightarrow{\sim} M_{hg}, \qquad u : 1_A \xrightarrow{\sim} M_e$$

subject to associativity, left unit, and right unit axioms. A homomorphism $\rho : G \to \mathrm{Aut}(A)$ induces such an action: one takes M_g to be the bimodule 1_A equipped with its natural right A-action, and left A-action twisted by $\rho(g)$. The isomorphisms $a_{g,h}$ and u are the obvious ones.

We now wish to analyze the result of our recipe (1) in the case $\mathcal{C} := \mathit{VN2}_\mathbb{R}$ and $F(V) := C\ell(V)$. Given $g \in O(n)$, let M_g be the bimodule coming from the obvious action $\rho : O(n) \to \mathrm{Aut}(C\ell(n))$. It is given by $M_g := C\ell(n)$ with left action twisted by $\rho(g)$.

Proposition 2.1. *For $n \geq 2$, the group*

$$G_2(n) := \Big\{(g,\beta)\,\Big|\, g\in O(n),\ \ C\ell(n) \underset{1}{\overset{M_g}{\underset{\Downarrow\beta}{\rightrightarrows}}} C\ell(n),\ \ \beta \textit{ is unitary}\Big\}.$$

is a non-trivial double cover of $SO(n)$, and is therefore isomorphic to $Spin(n)$.

Proof. Let $\mathrm{A}_2 := \mathrm{Aut}_{\mathit{VN2}_{\mathbb{R}}}\big(C\ell(n)\big)$ denote the 2-group of endomorphisms of $C\ell(n)$ in the 2-groupoid $\mathit{VN2}_{\mathbb{R}}^{\times}$. The group $G_2(n)$ is the fiber of the homomorphism $O(n) \to \mathrm{A}_2$, induced by the functor $C\ell : \mathit{Vect}_{\mathbb{R}} \to \mathit{VN2}_{\mathbb{R}}$. The natural map

$$G_2(n) = \mathrm{fib}\big(O(n) \to \mathrm{A}_2\big) \longrightarrow O(n)$$

is then given by the projection $(g, \beta) \mapsto g$. Since $C\ell(n)$ is an invertible object, its automorphism 2-group is isomorphic to that of the identity object

$$\mathrm{A}_2 \;\simeq\; \mathrm{Aut}_{\mathit{VN2}_{\mathbb{R}}}(\mathbb{1}) = \big(\{\mathbb{Z}/2\text{-graded lines over } \mathbb{R}\}, \otimes\big) = \mathbb{Z}/2 \times B\mathbb{Z}/2.$$

We analyze $G_2(n)$ by computing the long exact sequence of homotopy groups [tsh]

$$\ldots \to \pi_1(G_2(n)) \to \pi_1(O(n)) \xrightarrow{(a)} \pi_1(\mathrm{A}_2) \to \pi_0(G_2(n)) \to \pi_0(O(n)) \xrightarrow{(b)} \pi_0(\mathrm{A}_2). \tag{5}$$

In order to prove the proposition, it is enough to show that the maps (5.a) and (5.b) are isomorphisms (except for $n = 2$, in which case (5.a) is only surjective).

The kernel $K := \ker(G_2(n) \to O(n))$ consists of the unitary automorphisms of ${}_{C\ell(n)}C\ell(n)_{C\ell(n)}$. Equivalently, it is the categorical loop space $\Omega\mathrm{A}_2 := \hom_{\mathrm{A}_2}(1, 1) = \mathrm{Aut}(\mathbb{R}[0]) = \{\pm 1\}$. The map (5.a) can then be identified with the boundary homomorphism

$$\partial : \pi_1\big(O(n)\big) \longrightarrow \pi_0(K).$$

To see that it is surjective, we first treat the case $n = 2$. In that case, the Clifford algebra

$$C\ell(2) = \big\langle e_1, e_2 \,\big|\, e_1^2 = e_2^2 = 1,\, e_1e_2 = -e_2e_1 \big\rangle$$

is isomorphic to the 2×2 matrix algebra $M_2(\mathbb{R})$ via

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The generator of π_1 is represented by the loop

$$[0, 2\pi] \to O(2) : \theta \mapsto r_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The action of r_θ on $C\ell(2)$ is given by $r_\theta \cdot e_1 = \cos(\theta)e_1 + \sin(\theta)e_2$, and $r_\theta \cdot e_2 = -\sin(\theta)e_1 + \cos(\theta)e_2$, and one can check that

$$r_\theta \cdot e_i = r_{\frac{\theta}{2}}\, e_i\, r_{\frac{\theta}{2}}^{-1}.$$

The loop $\theta \mapsto r_\theta$ lifts to a path $\gamma : [0, 2\pi] \to G_2(2) : \theta \mapsto (r_\theta, \beta_\theta)$ with β_θ given by

$$\beta_\theta(e_i) := r_{\frac{\theta}{2}} e_i.$$

Since γ begins at the identity and ends at the non-trivial element of K, the boundary homomorphism $\partial : \pi_1(O(2)) \to \pi_0(K)$ is surjective. For general n, the surjectivity of ∂ follows from the commutativity of the following diagram

$$\begin{tikzcd} \mathbb{Z} \,=\, \pi_1\big(O(2)\big) \arrow[r, two heads, "\partial"] \arrow[d] & \pi_0(K) \,=\, \mathbb{Z}/2 \arrow[d] \\ \mathbb{Z}/2 \,=\, \pi_1\big(O(n)\big) \arrow[r, "\partial"] & \pi_0(K) \,=\, \mathbb{Z}/2. \end{tikzcd}$$

To finish the proof, we need to show that (5.b) is an isomorphism for $n \geq 2$. For that purpose, we introduce spaces $G_2(\infty) := \mathrm{colim}\, G_2(n)$ and $O(\infty) := \mathrm{colim}\, O(n)$. They have actions of the linear isometries operad [EKMM], and are thus E_∞-spaces. The stack $\mathrm{A}_2 \simeq \mathrm{colim}\ \mathrm{Aut}_{\mathit{VN2}_{\mathbb{R}}}(C\ell(n))$ also has such an action, and therefore so does its geometric realization $|\mathrm{A}_2|$ (which is only well defined up to homotopy). The homotopy fiber sequence

$$G_2(\infty) \to O(\infty) \to |\mathrm{A}_2|$$

being compatible with the E_∞ structures, it deloops to a fiber sequence of spectra [goa]

$$g_2 \to o \to a_2. \tag{6}$$

The generators of $\pi_0(o)$ and $\pi_1(o)$ are related by mutliplication by the Hopf element $\eta \in \pi_1(\mathbb{S})$. Since $\pi_1(o) \to \pi_1(a_2)$ is an isomorphism, it follows that $\pi_0(o) \to \pi_0(a_2)$ is also an isomorphism. The long exact sequence associated to (6) therefore looks like this:

$$\begin{tikzcd}
\pi_4 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_3 & \mathbb{Z} \arrow[r, "\sim"'] & \mathbb{Z} \arrow[r] & 0 \arrow[dll] \\
\pi_2 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_1 & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\sim"] & \mathbb{Z}/2 \arrow[dll] \\
\pi_0 & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\sim"] \arrow[u, bend right, dash, "\eta"'] & \mathbb{Z}/2 \arrow[u, bend right, dash, "\eta"']
\end{tikzcd}.$$

The map $\pi_0(O(n)) \to \pi_0(o)$ being an isomoprhism, it follows that (5.b) is also an isomorphism. $\square$

2.3. **The group** *String*(n) [sec:String]**.** We now investigate the 2-group (1) in the case when $\mathcal{C}$ is the 3-category *CN3* of $\mathbb{Z}/2$-graded conformal nets, and F is the free fermion functor $\mathit{Fer} : \mathit{Vect} \to \mathit{CN3}^\times$. Conformal nets will be defined Section 3.2, and their 3-categorical structure will be explained in Section 3.3. The free fermion will be constructed in Section 4.1. Nevertheless, the only thing needed in order to understand this section is the knowledge that $\mathit{Fer}(n) := \mathit{Fer}(\mathbb{R}^n)$ are invertible nets (Theorem 4.4), and that $\hom_{\mathit{CN3}}(\mathbb{1}, \mathbb{1})$ is equivalent to the 2-category of $\mathbb{Z}/2$-graded von Neumann algebras over $\mathbb{C}$ (Theorem 3.19).

Let $\mathrm{A}_3 := \mathrm{Aut}_{\mathit{CN3}}\big(\mathit{Fer}(n)\big)$ denote the 3-group of endomorphisms of $\mathit{Fer}(n)$ in the 3-groupoid $\mathit{CN3}^\times$. The functor $\mathit{Fer} : \mathit{Vect} \to \mathit{CN3}^\times$ then induces a homomorphism $O(n) \to \mathrm{A}_3$ whose fiber is the 2-group

$$G_3(n) := \left\{(g,\beta)\,\middle|\, g \in O(n),\quad \begin{tikzcd} \mathit{Fer}(n) \arrow[r, bend left, "D_g"{name=U}] \arrow[r, bend right, "1"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\beta"] & \mathit{Fer}(n), \end{tikzcd}\ \beta \text{ is invertible}\right\}.$$

Here, D_g refers to the 1-morphism in *CN3* induced by the automorphism g of $\mathbb{R}^n$. The object $\mathit{Fer}(n)$ being invertible in *CN3*, we have

$$\mathrm{Aut}_{\mathit{CN3}}\big(\mathit{Fer}(n)\big) = \mathrm{Aut}_{\mathit{CN3}}(\mathbb{1}).$$

By Theorem 3.19, we also have $\hom_{\mathit{CN3}}(\mathbb{1}, \mathbb{1}) = \mathit{VN2}_{\mathbb{C}}$ which allows us to compute the homotopy groups of A_3 (or equivalently, of its geometric realization). Namely, we have

$$\begin{aligned}
&\mathrm{Aut}_{\mathit{CN3}}(\mathbb{1})/\mathrm{iso} = \mathit{VN2}_{\mathbb{C}}^\times/\mathrm{iso} = \big\{\mathit{C\ell}_{\mathbb{C}}(0), \mathit{C\ell}_{\mathbb{C}}(1)\big\},\\
&\mathrm{Aut}_{\mathrm{Aut}_{\mathit{CN3}}(\mathbb{1})}(1_{\mathbb{1}})/\mathrm{iso} = \mathrm{Aut}_{\mathit{VN2}_{\mathbb{C}}}(\mathbb{1})/\mathrm{iso} = \mathit{Hilb}_{\mathbb{C}}^{\mathbb{Z}/2,\times}/\mathrm{iso} = \big\{\mathbb{C}[0], \mathbb{C}[1]\big\},\\
&\mathrm{Aut}_{\mathrm{Aut}_{\mathrm{Aut}_{\mathit{CN3}}(\mathbb{1})}(1_{\mathbb{1}})}(1_{1_{\mathbb{1}}}) = \mathrm{Aut}_{\mathrm{Aut}_{\mathit{VN2}_{\mathbb{C}}}(\mathbb{1})}(1_{\mathbb{1}}) = \mathrm{Aut}_{\mathit{Hilb}_{\mathbb{C}}^{\mathbb{Z}/2}}(\mathbb{1}) = \mathrm{End}_{\mathit{Hilb}_{\mathbb{C}}^{\mathbb{Z}/2,\times}}(\mathbb{C}[0]) = S^1,
\end{aligned}$$

which gives us

$$\pi_0(\mathrm{A}_3) = \pi_1(\mathrm{A}_3) = \mathbb{Z}/2, \qquad \pi_3(\mathrm{A}_3) = \mathbb{Z}, \qquad \text{and} \qquad \pi_i(\mathrm{A}_3) = 0 \text{ for } i \notin \{0,1,3\}.$$

Proposition 2.2. *The map $\pi_i(O(n)) \to \pi_i(\mathrm{A}_3)$ is an isomorphism for $i = 0, 1, 2, 3$ and $n \geq 4$, except for $i = 3$, $n = 4$ in which case $\pi_3(O(4)) \to \pi_3(\mathrm{A}_3)$ is only surjective.*

Proof. Let $K := \mathrm{fib}\big(G_3(n) \to O(n)\big) = \Omega\mathrm{A}_3$ be the categorical loop space of A_3. It is given by

$$K \simeq \mathbb{Z}/2 \times BS^1.$$

The natural map $\pi_3(O(n)) \to \pi_3(\mathrm{A}_3)$ can then be identified with the boundary homomorphism [shj]

$$\partial : \pi_3\big(O(n)\big) \to \pi_2(K). \tag{7}$$

The hardest part of the proof is to show that $\partial : \pi_3(O(4)) \to \pi_2(K)$ is surjective, and will be carried out in section 4.4. Assuming that fact, it is easy to see from the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \;=\; \pi_3\big(O(4)\big) \arrow[r, two heads, "\partial"] \arrow[d] & \pi_2(K) \;=\; \mathbb{Z} \arrow[d] \\
\mathbb{Z} \;=\; \pi_3\big(O(n)\big) \arrow[r, "\partial"] & \pi_2(K) \;=\; \mathbb{Z}
\end{tikzcd}$$

that the other boundary maps (7) are isomorphisms.

Taking the colimit over n, the sequence $G_3(n) \to O(n) \to \mathrm{A}_3$ yields a fiber sequence of spectra [dgn]

$$g_3 \to o \to a_3. \tag{8}$$

Let $\eta \in \pi_1(\mathbb{S})$ be the Hopf element. The generators $x \in \pi_0(o)$, $y \in \pi_1(o)$, $z \in \pi_3(o)$ are related by $x\eta = y$, and by the Massey product $\langle y, \eta, 2\rangle = z$. The map $\pi_i(o) \to \pi_i(a_3)$ being an isomorphism for $i = 3$, it is therefore also an isomorphism for $i = 0, 1$. Thus, the long exact sequence of (8) looks as follows:

$$\begin{tikzcd}
\pi_7 & \mathbb{Z} \arrow[r, "\sim"'] & \mathbb{Z} \arrow[r] & 0 \arrow[dll] \\
\pi_6 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_5 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_4 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_3 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\sim"] \arrow[dd, bend left, dash, "\langle\ ,\eta,2\rangle"] & \mathbb{Z} \arrow[dll] \arrow[dd, bend left, dash, "\langle\ ,\eta,2\rangle"] \\
\pi_2 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_1 & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\sim"] \arrow[d, bend left, dash, "\eta"] & \mathbb{Z}/2 \arrow[dll] \arrow[d, bend left, dash, "\eta"] \\
\pi_0 & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\sim"] & \mathbb{Z}/2 \quad .
\end{tikzcd}$$

The map $\pi_i(O(n)) \to \pi_i(o)$ being an isomorphism for $i \leq 3$, $n \geq 4$, $(i, n) \neq (3, 4)$, it follows that $\pi_i(O(n)) \to \pi_i(\mathrm{A}_3)$ is also an isomorphism. □

As a corollary of the above proposition, we get our main result.

Theorem 2.3. *For $n \geq 5$, the geometric realization of $G_3(n) = \mathrm{fib}\big(O(n) \to \mathrm{A}_3\big)$ is the 3-connected cover of $O(n)$.* □

3. The 3-category of conformal nets

In this section, we describe the symmetric monoidal 3-category ($\mathit{CN3}, \otimes$). The objects of *CN3* are $\mathbb{Z}/2$-graded conformal nets with finite μ-index. The arrows between two nets $\mathcal{A}$ and $\mathcal{B}$ are called $\mathcal{A}$-$\mathcal{B}$-defects. The 2-morphisms between $\mathcal{A}$-$\mathcal{B}$-defects D and E are called D-E-sectors. Finally, the 3-morphisms of *CN3* are called homomorphisms of sectors. The main result of [BDH] is that *CN3* is indeed a symmetric monoidal 3-category.

3.1. **Von Neumann algebras** [sec:VNR]. Given a Hilbert space H over $\mathbb{R}$ or $\mathbb{C}$, we let $\mathrm{B}(H)$ denote its $*$-algebra of bounded linear operators. We equip it with the σ-weak topology, in other words, the topology of pointwise convergence with respect to the pairing with trace class operators. If H is $\mathbb{Z}/2$-graded, then $\mathrm{B}(H)$ inherits a corresponding $\mathbb{Z}/2$-grading. The $*$-operation on $\mathrm{B}(H)$ is the same as in the ungraded case.

Definition 3.1. A $\mathbb{Z}/2$-graded von Neumann algebra is a $\mathbb{Z}/2$-graded topological $*$-algebra that can be embedded as a closed subalgebra of some $\mathrm{B}(H)$. A module for a von Neumann algebra A consists of a $\mathbb{Z}/2$-graded Hilbert space H and a continuous homomorphism $A \to \mathrm{B}(H)$.

The graded commutant A' of a von Neumann algebra $A \subset \mathrm{B}(H)$ is defined by

$$A' := \{b \in \mathrm{B}(H)_0 : ab = ba,\ \forall a{\in}A\} \oplus \{b \in \mathrm{B}(H)_1 : ab = (-1)^{|a|}ba,\ \forall a{\in}A\}.$$

The bicommutant theorem then says that $A \mapsto A'$ is an involution on the poset of von Neumann subalgebras of $\mathrm{B}(H)$.

Given von Neumann algebras $A \subset \mathrm{B}(H)$ and $B \subset \mathrm{B}(K)$, their spacial tensor product $A \mathbin{\bar{\otimes}} B$ is defined as the closure of the algebraic tensor product $A \otimes_{alg} B$ inside $\mathrm{B}(H \otimes K)$. It is independent of the choice of Hilbert spaces H and K. The product and involution are given by the well known formulas

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd\,, \qquad (a \otimes b)^* = (-1)^{|a||b|} a^* \otimes b^*.$$

We also include the formulas for the graded opposite of a von Neumann algebra

$$a^{op}\, b^{op} = (-1)^{|a||b|}(ba)^{op}, \qquad (a^{op})^* = (-1)^{|a|}(a^*)^{op}.$$

Note that in the graded case, the map $* : A \to A$ is *not* an isomorphism between $\bar{A}$ and A^{op}. If A is an algebra over $\mathbb{C}$, then $\bar{A}$ and A^{op} are nevertheless isomorphic. Such natural isomorphisms are induced by

$$\#_j : A \to A, \qquad a^{\#_j} := \begin{cases} a^* & \text{if } a \text{ is even} \\ ja^* & \text{if } a \text{ is odd} \end{cases}$$

for $j = i$ and $j = -i$. They satisfy $a^{\#_j\#_j} = a$ and

$$(ab)^{\#_j} = (-1)^{|a||b|}\, b^{\#_j}\, a^{\#_j}.$$

Every von Neumann algebra A has a canonically associated A-A-bimodule $L^2(A)$. It is characterized (up to unique isomorphism) by the existence of an isometric antilinear involution $\mathbf{J} : L^2(A) \to L^2(A)$ and a cone $P \subset L^2(A)$, subject to the following conditions:

- the two actions of A on $L^2(A)$ are faithful, and are each other's commutants,
- the cone P is self-dual, meaning that $P = \{\xi \in L^2(A) : \langle p, \xi\rangle \geq 0,\ \forall p \in P\}$,
- $\mathbf{J}(a\xi b) = b^*\mathbf{J}(\xi)a^*$ for $a, b \in A$, and $\xi \in L^2(A)$,
- $a\xi a^* \in P$ for $a \in A$, $\xi \in P$,
- $\mathbf{J}(\xi) = \xi$ for $\xi \in P$,
- $c\xi = \xi c$ for c in the center of A, and $\xi \in L^2(A)$.

Any vector $\xi \in P$ induces a positive state $\phi : a \mapsto \langle a\xi, \xi\rangle$ and, under those circumstances, ξ may be identified with a formal square root of ϕ.

Even more, the space $L^2(A)$ is the completion of the vector space generated by formal square roots of positive states, with respect to the following inner product. To define $\langle\sqrt{\phi}, \sqrt{\psi}\,\rangle$, one considers the function $f : \mathbb{R} \to \mathbb{C}$, $t \mapsto \phi([D\phi : D\psi]_t)$, where $[D\phi : D\psi]_t \in A$ denotes the non-commutatie Radon-Nikodym derivative [Yam Alg asp.][1]. The function $f(t)$ has an analytic continuation to the strip $\mathrm{im}(t) \in [0, 1]$, and the inner product is given by $\langle\sqrt{\phi}, \sqrt{\psi}\,\rangle := f(i/2)$.

If A is $\mathbb{Z}/2$-graded, the grading involution $\gamma : A \to A$ induces an involution $\gamma : L^2(A) \to L^2(A)$ by functoriality, and thus a $\mathbb{Z}/2$-grading on $L^2(A)$.

[1]The Radon-Nikodym derivative formally satisfies $[D\phi : D\psi]_t = \phi^{it}\psi^{-it}$.

Given a right A-module H, and a left A-module K, their Connes fusion $H \boxtimes_A K$ is the completion of the vector space

$$\hom\big(L^2(A)_A, H_A\big) \otimes_{alg} L^2(A) \otimes_{alg} \hom\big({}_AL^2(A), {}_AK\big)$$

with respect to the inner product

$$\big\langle \phi_1 \otimes \xi_1 \otimes \psi_1,\, \phi_2 \otimes \xi_2 \otimes \psi_2 \big\rangle := \big\langle (\phi_2^*\phi_1)\xi_1(\psi_1\psi_2^*), \xi_2 \big\rangle.$$

In the above equation, we have written the action of ψ_i on the right, which means that $\psi_1\psi_2^*$ stands for the composite $L^2(A) \xrightarrow{\psi_1} K \xrightarrow{\psi_2^*} L^2(A)$. The functor

$$H \boxtimes_A - : A\text{-modules} \to \text{Hilbert spaces}$$

is characterized by the existence of an isomorphism $H \boxtimes_A L^2(A) \simeq H$ intertwining the two right A-actions. Connes fusion also satisfies $L^2(A) \boxtimes_A K \simeq K$.

The collection of $\mathbb{Z}/2$-graded von Neumann algebras forms the objets of a bicategory *VN2*. The arrows between A and B are the A-B-bimodules, and the composition

$$\hom_{\mathit{VN2}}(A, B) \times \hom_{\mathit{VN2}}(B, C) \to \hom_{\mathit{VN2}}(A, C)$$

is given by Connes fusion. We write $\mathit{VN2}_{\mathbb{R}}$ or $\mathit{VN2}_{\mathbb{C}}$ if we want to specify that the base field is $\mathbb{R}$ or $\mathbb{C}$, respectively.

3.2. **Conformal nets [sec:Nets].** Before describing the objects of our 3-category *CN3*, we need a few facts about pin structures on one dimensional manifolds. All manifolds will be of class C^1. By an *interval*, we shall always mean a manifold I of class C^1 that is diffeomorphic to $[0, 1]$.

Definition 3.2. A *pin interval* is an interval equipped with a complex line bundle $S \to I$, and an isomorphism $S^{\otimes 2} \xrightarrow{\sim} T^*_{\mathbb{C}} I$ between the square of S and the complexified cotangent bundle of I.

An embedding between pin intervals (I', S') and (I, S) consists of an embedding $f : I' \hookrightarrow I$, along with an isomorphism $\beta : f^*S \to S'$. We allow β to be either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. If β is linear, then the first one of the following two diagrams should commute. Otherwise, it is the second diagram that should commute: [2SQ]

$$\begin{tikzcd} f^*(S^{\otimes 2}) \arrow[r, "\sim"] \arrow[d, "\beta^{\otimes 2}"'] & f^*(T^*_{\mathbb{C}} I) \arrow[d, "T^*_{\mathbb{C}} f"] \\ S'^{\otimes 2} \arrow[r, "\sim"] & T^*_{\mathbb{C}} I' \end{tikzcd} \qquad \begin{tikzcd} f^*(S^{\otimes 2}) \arrow[r, "\sim"] \arrow[d, "\beta^{\otimes 2}"'] & f^*(T^*_{\mathbb{C}} I) = f^*(T^* I) \otimes \mathbb{C} \arrow[d, "T^*f \otimes (z \mapsto \bar{z})"] \\ S'^{\otimes 2} \arrow[r, "\sim"] & T^*_{\mathbb{C}} I' = T^* I' \otimes \mathbb{C} \end{tikzcd} \tag{9}$$

We say an embedding (f, β) is $\mathbb{C}$-linear, respectively $\mathbb{C}$-antilinear, if β is so.

Let (I, S) be a pin interval. Its *pin involution* is the map γ given by the identity on I and negation on S. There are two other non-trivial involutions c_i and c_{-i} that restricts to the identity on I. We call them the *conjugating involutions*.

In order to distinguish one from the other, we first introduce the notion of a *coorientation* of I. By this, we mean a coorientation of T^*I inside its complexification $T^*_{\mathbb{C}} I$. If the pin intervals (I, S), (I', S') are equipped with coorientations, we say that an embedding (f, β) preserves the coorientations if so does the right vertical arrow of the relevant diagram (9).

Remark 3.3. The data of a coorientation is equivalent to that of an orientation. However, it is not true that an orientation preserving embedding necessarily preserves the coorientation. The latter is only true for $\mathbb{C}$-linear embeddings.

So let (I, S) be a pin interval equipped with a coorientation, and let v be a section of $T^*_{\mathbb{C}} I$ representing the coorientation. Let $\sqrt{v}$ be the section of S (defined up to sign) determined by the equation $\sqrt{v} \otimes \sqrt{v} = v$. For $j = i$ or $-i$, the conjugating involution c_j of (I, S) acts by j on $\mathrm{span}_{\mathbb{R}}\{\sqrt{v}\}$ and by $-j$ on $\mathrm{span}_{\mathbb{R}}\{\sqrt{-v}\}$.

Definition 3.4. [diP] We let INT_{Pin} be the topological category whose objects are pin intervals equipped with a coorientation. The morphisms are pin embeddings (either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear) that do *not* need to preserve the coorientation.

Example 3.5. [ex:Sigma] Let Σ be a Riemann surface equipped with a chosen square root of $T^*\Sigma$. Then any embedded interval $I \subset \Sigma$ inherits a pin structure. As a special case of the above situation, consider an interval $I \subset \mathbb{C}$ in the complex plane. The trivial line bundle over $\mathbb{C}$ being a square root of the cotangent bundle, I acquires a pin structure.

If $I, J \subset \Sigma$ are two intervals and $h : \Sigma \times [0,1] \to \Sigma$ is an isotopy mapping J to I, then by the unique lifting property, h induces an isomorphism of pin intervals $J \to I$. Now consider two intervals $I, J \subset \mathbb{C}$ such that $\partial I = \partial J$, and let $f : J \to I$ be a diffeomorphism fixing ∂J (no compatibility between f and the pin structures on I and J). Then we may pick an isotopy h of $\mathbb{C}$ fixing ∂J and mapping J to I via f. Such an isotopy being unique up to homotopy, it induces a well defined pin isomorphism between J and I. Thus, we have enhanced f from a mere diffeomorphism to a pin diffeomorphism.

Convention 3.6. [C:Sigma] Given $I \subset \mathbb{C}$ we shall always equip it with the pin structure coming from its embedding. If $I, J \subset \mathbb{C}$ are intervals such that $\partial I = \partial J$, and if $f : J \to I$ is a diffeomorphism fixing ∂J, then we shall always upgrade f to a pin diffeomorphism, as explained in Example 3.5.

Let VN denote the category whose objects are complex $\mathbb{Z}/2$-graded separable von Neumann algebras, and whose morphisms are given by

$$\hom_{\mathsf{VN}}(A, B) := \hom(A, B) \cup \hom(A, \bar{B}) \cup \hom(A, B^{op}) \cup \hom(A, \bar{B}^{op}).$$

The hom-sets are given the topology of pointwise convergence.

Definition 3.7. [defCN] A $\mathbb{Z}/2$-graded conformal net is a continuous functor [aivn]

$$\mathcal{A} : \mathsf{INT}_{Pin} \to \mathsf{VN}. \tag{10}$$

To an embedding $f : J \hookrightarrow I$, it assigns a map $\mathcal{A}(f) : \mathcal{A}(J) \to \mathcal{A}(I)$ of the kind prescribed by the following table: [fus]

(11)

	f is $\mathbb{C}$-linear.	f is $\mathbb{C}$-antilinear.
f respects the coorientations of I and J.	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \mathcal{A}(I)\big)$	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \overline{\mathcal{A}(I)}\,\big)$
f does not respect the coorientations.	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \mathcal{A}(I)^{op}\big)$	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \overline{\mathcal{A}(I)}^{op}\big)$

Moreover, if γ is the pin involution of I, then $\mathcal{A}(\gamma)$ should be the grading involution of $\mathcal{A}(I)$, and if c_j is a conjugating involution, then $\mathcal{A}(c_j)$ should be the map $\#_j : \mathcal{A}(I) \to \overline{\mathcal{A}(I)}^{op}$. It is subject to the following axioms:

- ***Isotony***: The image of an embedding $J \hookrightarrow I$ is an injective map $\mathcal{A}(J) \hookrightarrow \mathcal{A}(I)$.
- ***Locality***: If $J \subset I$ and $K \subset I$ have disjoint interiors, then the images of $\mathcal{A}(J)$ and $\mathcal{A}(K)$ graded commute inside $\mathcal{A}(I)$.
- ***Strong additivity***: If $I = J \cup K$, then the images of $\mathcal{A}(J)$ and $\mathcal{A}(K)$ generate $\mathcal{A}(I)$.
- ***Haag duality***: If $I = J \cup K$ and $J \cap K$ is a point, then the image of $\mathcal{A}(J)$ is the graded commutant of $\mathcal{A}(K)$ inside $\mathcal{A}(I)$.
- ***Split property***: If J, K are disjoint subintervals of I and the inclusions are compatible with both orientations and coorientations, then the map from the algebraic tensor product $\mathcal{A}(J) \otimes_{alg} \mathcal{A}(K) \to \mathcal{A}(I)$ extends to the spacial tensor product

$$\mathcal{A}(J) \bar{\otimes} \mathcal{A}(K) \to \mathcal{A}(I).$$

- ***Diff covariance***: mIf $\varphi : I \to I$ is a diffeomorphism that restricts to the identity in a neighborhood of ∂I, then $\mathcal{A}(\varphi)$ is an inner automorphism of $\mathcal{A}(I)$.
- ***Vacuum***: Let $S^1 \subset \mathbb{C}$ denote the unit circle. Following Convention 3.6, every subinterval of S^1 aquires a pin structure from its embedding in $\mathbb{C}$. Let [IIp]

$$I := \exp\big([0, \pi i]\big), \qquad I' := \exp\big([\pi i, 2\pi i]\big). \tag{12}$$

Equip I and I' with the inward coorientation. Following Convention 3.6, the map $j : I' \to I$, $j(z) := z^{-1}$ can be upgraded to a pin isomorphism. Since j reverses the coorientation, it induces a homomorphism

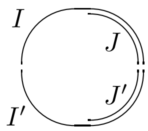
$$\mathcal{A}(j) : \mathcal{A}(I') \to \mathcal{A}(I)^{op}.$$

Let $H_0 := L^2(\mathcal{A}(I))$. We then have two left actions

$$\lambda : \mathcal{A}(I) \to \mathrm{B}(H_0), \qquad \rho : \mathcal{A}(I') \to \mathrm{B}(H_0),$$

given by the formulas $\lambda(a)(\xi) := a\,\xi$ and $\rho(b)(\xi) := (-1)^{|b||\xi|}\,\xi\,\mathcal{A}(j)(b)$.

Let $J \subset I$ be a subinterval such that $J \cap I' = \{1\}$ or $\{-1\}$, and let $J' := j(J)$.



Then the action of the algebraic tensor product

$$\lambda \otimes \rho \;:\; \mathcal{A}(J) \otimes_{alg} \mathcal{A}(J') \longrightarrow \mathrm{B}(H_0)$$

extends (uniquely) to an action of $\mathcal{A}(J \cup J')$.

Remark 3.8. Our definition of conformal net is slightly different from the classical one [K's survey?]. In the classical definition, one only assigns von Neumann algebras to subintervals of S^1, and instead of (10), one has a map of posets [mopo]

$$\mathcal{A} : \big\{\text{subintervals of } S^1\big\} \to \big\{\text{subalgebras of } \mathrm{B}(H_0)\big\}. \tag{13}$$

Such a net is called *Diff*-covariant if there is a projective action of (the coorientation preserving part of) $\mathit{Diff}_{Pin}(S^1)$ on H_0, and the map (13) is equivariant with respect to that group. Under that condition, one can recover a conformal net in our sense. One first uses the *Diff*-covariance to extends (13) to a functor

$$\left\{\begin{matrix}\text{Full subcategory of } \mathsf{INT}_{Pin}\\ \text{consisting of subintervals of } S^1\end{matrix}\right\} \to \left\{\begin{matrix}\text{Full subcategory of } \mathsf{VN} \text{ con-}\\ \text{sisting of subalgebras of } \mathrm{B}(H_0)\end{matrix}\right\};$$

the left hand side being equivalent to INT_{Pin}, it is then easy to extend it to a functor $\mathsf{INT}_{Pin} \to \mathsf{VN}$. We refer to (29) for a concrete model.

Like with von Neumann algebras, every conformal net $\mathcal{A}$ has a complex conjuate $\bar{\mathcal{A}}$, and an opposite $\mathcal{A}^{op}$. They are given by [bno]

$$\begin{aligned}
&\bar{\mathcal{A}}(I) := \overline{\mathcal{A}(I)}, &&\bar{\mathcal{A}}(f) := \begin{cases}\overline{\mathcal{A}(f)} & \text{if } f \text{ is } \mathbb{C}\text{-linear,}\\ \gamma \circ \overline{\mathcal{A}(f)} & \text{if } f \text{ is } \mathbb{C}\text{-antilinear,}\end{cases}\\
&\mathcal{A}^{op}(I) := \mathcal{A}(I)^{op}, &&\mathcal{A}^{op}(f) := \begin{cases}\mathcal{A}(f)^{op} & \text{if } f \text{ respects the coorientations,}\\ \gamma \circ \mathcal{A}(f)^{op} & \text{if it doesn't,}\end{cases}
\end{aligned} \tag{14}$$

where γ is the grading involution. There is an isomorphism between $\bar{\mathcal{A}}$ and $\mathcal{A}^{op}$ given by applying $\#_i$ objectwise.

3.3. **Defects** [sec:ds]**.** The arrows of our 3-category *CN3* are called defects: given two conformal nets $\mathcal{A}$ and $\mathcal{B}$ there is a notion of $\mathcal{A}$-$\mathcal{B}$-defect. One has an identity defect $1_{\mathcal{A}}$ associated to any conformal net $\mathcal{A}$, and one can compose an $\mathcal{A}$-$\mathcal{B}$-defect with a $\mathcal{B}$-$\mathcal{C}$-defect to form an $\mathcal{A}$-$\mathcal{C}$-defect.

Remark 3.9. Our defects probably correspond to the *conformal* defects of [Runkel], as opposed to the more restrictive topological defects.

Definition 3.10. Equip $\mathbb{R}$ with the pin structure coming from its embedding in $\mathbb{C}$. A *bicoloring* of a pin interval I is an equivalence class $[\imath]$ of pin embeddings $\imath : I \to \mathbb{R}$, $0 \notin \imath(\partial I)$, where $\imath \sim \imath'$ if $\imath(x) > 0 \Leftrightarrow \imath'(x) > 0$ and $\imath|_N = \imath'|_N$ for some neighborhood N of $\imath^{-1}(0)$ (the last condition is empty if $0 \notin \imath(I)$ since we may then take $N = \emptyset$).

Given a bicoloring of I, we let $I_\bullet := \imath^{-1}(\mathbb{R}_{\geqslant 0})$ and $I_\circ := \imath^{-1}(\mathbb{R}_{\leqslant 0})$, where $\imath$ is any representative of the equivalence class. We think of $I_\bullet$ as being painted in black, and $I_\circ$ as being painted in white. A bicolored interval can be either entirely black (case $I = I_\bullet$), entirely white (case $I = I_\circ$), or it can have two halves $I_\circ \neq I \neq I_\bullet$. In the last case, I is also equipped with a local coordinate around the point $\imath^{-1}(0)$. If $(I, [\imath])$ and $(J, [\jmath])$ are bicolored intervals, then we say that an embedding $f : J \hookrightarrow I$ is compatible with the bicoloring if $[\jmath] = [\imath \circ f]$. This is equivalent to the statement that $f(J_\bullet) \subset I_\bullet$, $f(J_\circ) \subset I_\circ$, and that f respects the local coordinate. Similarly to Definition 3.4, we have:

Definition 3.11. Let $\mathsf{INT}_{Pin}^{\bullet\circ}$ be the topological category whose objects are pin bicolored intervals equipped with a coorientation, and whose morphisms are pin embeddings compatible with the bicolorings, but not necessarily with the coorientation.

Let $\mathsf{INT}_{Pin}^{\bullet}$, $\mathsf{INT}_{Pin}^{\circ}$, and $\mathsf{INT}_{Pin}^{\LEFTcircle}$ denote the full subcategories of $\mathsf{INT}_{Pin}^{\bullet\circ}$ consisting of intervals I such that $I = I_\bullet$, $I = I_\circ$, and $I_\circ \neq I \neq I_\bullet$, respectively. When we compose the obvious equivalences $\mathsf{INT}_{Pin} \xrightarrow{\sim} \mathsf{INT}_{Pin}^{\bullet}$, $\mathsf{INT}_{Pin} \xrightarrow{\sim} \mathsf{INT}_{Pin}^{\circ}$ with the inclusions $\mathsf{INT}_{Pin}^{\bullet} \hookrightarrow \mathsf{INT}_{Pin}^{\bullet\circ}$ and $\mathsf{INT}_{Pin}^{\circ} \hookrightarrow \mathsf{INT}_{Pin}^{\bullet\circ}$, we obtain fully faithful functors

$$\iota_\bullet : \mathsf{INT}_{Pin} \to \mathsf{INT}_{Pin}^{\bullet\circ} \qquad \text{and} \qquad \iota_\circ : \mathsf{INT}_{Pin} \to \mathsf{INT}_{Pin}^{\bullet\circ}.$$

Informally speaking, $\iota_\bullet$ paints in black, and $\iota_\circ$ paints in white.

Definition 3.12. Let $\mathcal{A}$ and $\mathcal{B}$ be conformal nets. An $\mathcal{A}$-$\mathcal{B}$-defect is a continuous functor [dan]

$$D : \mathsf{INT}_{Pin}^{\bullet\circ} \to \mathsf{VN} \tag{15}$$

equipped with natural isomorphisms $D \circ \iota_\bullet \simeq \mathcal{A}$ and $D \circ \iota_\circ \simeq \mathcal{B}$, and subject to the requirements of Table (11). It satisfies the following five axioms:

- ***Isotony***: If $I, J \in \mathsf{INT}_{Pin}^{\LEFTcircle}$ and $f : J \hookrightarrow I$ is an embedding, then $D(f) : D(J) \to D(I)$ is injective.
- ***Locality***: If $J \subset I$ and $K \subset I$ have disjoint interiors, then the images of $D(J)$ and $D(K)$ are graded-commuting subalgebras of $D(I)$.
- ***Strong additivity***: If $I = J \cup K$, then the images of $D(J)$ and $D(K)$ generate $D(I)$.
- ***Haag duality***: If $I = J \cup K$, $J \cap K$ is a point, and $J \in \mathsf{INT}_{Pin}^{\LEFTcircle}$, then the image of $D(J)$ is the graded commutant of $D(K)$ inside $D(I)$.
- ***Vacuum***: Let $I, I' \subset \mathbb{C}$ be as in (12). Pick a bicoloring $[\imath]$ of I such that $\imath(z) = \mathrm{Re}(z)$ in a neighborhood of $i \in I$. Let $j : I' \to I$ be the map given by $z \mapsto \bar{z}$ (note the use of Convention 3.6), and let $[\imath \circ j]$ be the induced bicoloring of I'. Letting $H_0 := L^2(D(I))$, we have two actions
$$\lambda : D(I) \to \mathrm{B}(H_0), \qquad \rho : D(I') \to \mathrm{B}(H_0),$$
given by $\lambda(a)(\xi) := a\,\xi$ and $\rho(b)(\xi) := (-1)^{|b||\xi|}\,\xi\,D(j)(b)$. Let $J \in \mathsf{INT}_{Pin}^{\bullet} \cup \mathsf{INT}_{Pin}^{\circ}$ be a subinterval of I such that $J \cap I'$ consists of a single point. Finally, let $J' := j(J)$. Then the action of the algebraic tensor product
$$\lambda \otimes \rho \,:\, D(J) \otimes_{alg} D(J') \longrightarrow \mathrm{B}(H_0)$$
extends to an action of $D(J \cup J')$.

Remark 3.13. The split property and diff covariance for defects are consequences of the corresponding axioms for conformal nets.

Remark 3.14. As with (10) versus (13), there is a "big" versus "small" description of defects. In the latter description, an $\mathcal{A}$-$\mathcal{B}$-defect is a poset map

$$D : \big\{ \text{subintervals } I \subset \mathbb{R},\ 0 \notin \partial I \big\} \to \big\{ \text{subalgebras of } \mathrm{B}(H) \big\}$$

equipped with natural homomorphisms $\mathcal{A}(I) \to D(I)$ for $I \subset \mathbb{R}_{<0}$ and $\mathcal{B}(I) \to D(I)$ for $I \subset \mathbb{R}_{>0}$.

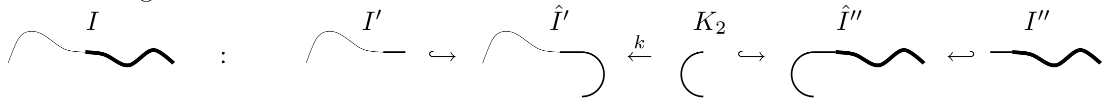
Given a conformal net $\mathcal{A}$, its identity arrow in *CN3* is the defect $1_\mathcal{A}$ given by $1_\mathcal{A}(I) := \mathcal{A}(I)$. Let us use the notation ${}_\mathcal{A}D_\mathcal{B}$ to indicate that D is an $\mathcal{A}$-$\mathcal{B}$-defect. The composition in *CN3* of defects ${}_\mathcal{A}D_\mathcal{B}$ and ${}_\mathcal{B}E_\mathcal{C}$ is denoted ${}_\mathcal{A}D \circledast_\mathcal{B} E_\mathcal{C}$ and is called the fusion of the two defects. We now explain how it is defined. Let $J_1, K_1, \hat{J}_1, J_2, K_2, \hat{J}_2 \subset \mathbb{C}$ be the pin intervals given by

$$J_1 := [-1, 0], \qquad K_1 := \exp([-\tfrac{\pi i}{2}, \tfrac{\pi i}{2}]) - i, \qquad \hat{J}_1 := J_1 \cup K_1$$
$$J_2 := [0, 1], \qquad K_2 := \exp([\tfrac{\pi i}{2}, \tfrac{3\pi i}{2}]) - i, \qquad \hat{J}_2 := J_2 \cup K_2$$

If $I \in \mathsf{INT}_{Pin}^{\circ\!\bullet}$, we can use its local coordinate to form new pin intervals

$$I' := I_\circ \cup J_1, \qquad \hat{I}' := I_\circ \cup \hat{J}_1, \qquad I'' := J_2 \cup I_\bullet, \qquad \hat{I}'' := \hat{J}_2 \cup I_\bullet,$$

where the glueing is performed at the points $-1 \in J_1$ and $1 \in J_2$ respectively. These intervals have an obvious bicoloring satisfying $I'_\circ = \hat{I}'_\circ = I_\circ$ and $I''_\bullet = \hat{I}''_\bullet = I_\bullet$. Equip I', $\hat{I}'$, I'', $\hat{I}''$ with the coorientation induced from $I_\circ$ and $I_\bullet$. Similarly, equip K_1 and K_2 with the coorientation inherited from $\hat{I}'$ and $\hat{I}''$. Following Convention 3.6, we let the formula $k(z) := -\bar{z}$ define a map $K_2 \to K_1$. Note that k reverses both the orientation and the coorientation. Given our choices, the pin embeddings



induce homomorphisms $D(I')^{op} \to D(\hat{I}')^{op} \leftarrow \mathcal{B}(K_2) \to E(\hat{I}'') \leftarrow E(I'')$. Pick faithful left modules M of $D(\hat{I}')$ and N of $E(\hat{I}'')$. We define $(D \circledast_\mathcal{B} E)(I)$ to be the von Neumann algebras generated by $D(I')$ and $E(I'')$ in their action on the Hilbert space $M \boxtimes_{\mathcal{B}(K_2)} N$, where the right action of $\mathcal{B}(K_2)$ on M is induced by $D(k)$. That algebra is independent of the choices of M and N. More generally, the fusion of D and E is defined by [MN]

$$(16) \qquad \big(D \circledast_\mathcal{B} E\big)(I) := \begin{cases} \mathcal{A}(I) & I \in \mathsf{INT}_{Pin}^{\circ} \\ \mathcal{C}(I) & I \in \mathsf{INT}_{Pin}^{\bullet} \\ \overline{D(I') \otimes E(I'')}^{\mathrm{B}\left(M \underset{\mathcal{B}(K_2)}{\boxtimes} N\right)} & I \in \mathsf{INT}_{Pin}^{\circ\!\bullet}, \end{cases}$$

where the bar indicates the closure of the algebraic tensor product with the respect to the σ-weak topology on $\mathrm{B}(M \boxtimes_{\mathcal{B}(K_2)} N)$. Fusion of defects is associative and unital, modulo the appropriate 2-morphisms of *CN3*. For more details, we refer the reader to our future work [BDH].

3.4. **Sectors** [sec:ds]**.** Given conformal nets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}$-$\mathcal{B}$-defects D and E, we now introduce the notion of D-E-sector. These are the 2-morphisms of *CN3* between the 1-morphisms D and E. Given two D-E-sectors, there is also a notion of map between them. These are the 3-morphisms of *CN3*.

Let $J_1 := \exp([0, \pi i])$ and $J_2 := \exp([\pi i, 2\pi i])$. Following Convention 3.6, we let $j : J_2 \to J_1$ be the pin isomorphism given by $z \mapsto \bar{z}$. Pick a bicoloring $[\imath]$ of J_1 such that $\imath(z) = \mathrm{re}(z)$ in a neighborhood of i, and let $[\imath \circ j]$ be the induced bicoloring of J_2. For every interval $I \subset S^1$, $i \notin \partial I$, $-i \notin \partial I$, such that $i \notin I$ or $-i \notin I$, there is a unique bicoloring that is compatible with the chosen bicolorings of J_1 and J_2. It satisfies

$$I_\circ = \big\{ z \in I \,\big|\, \mathrm{re}(z) \leqslant 0 \big\} \qquad \text{and} \qquad I_\bullet = \big\{ z \in I \,\big|\, \mathrm{re}(z) \geqslant 0 \big\}.$$

We equip all those intervals with the inward coorientation.

Definition 3.15. Given nets $\mathcal{A}$, $\mathcal{B}$, and defects ${}_{\mathcal{A}}D_{\mathcal{B}}$, ${}_{\mathcal{A}}E_{\mathcal{B}}$, a D-E-sector is a Hilbert space H equipped with homomorphisms [4rho]

$$\rho_I : \mathcal{A}(I) \to \mathrm{B}(H), \quad I = I_\circ, \qquad \rho_I : \mathcal{B}(I) \to \mathrm{B}(H), \quad I = I_\bullet,$$
$$\rho_I : D(I) \to \mathrm{B}(H), \quad i \in I, \qquad \rho_I : E(I) \to \mathrm{B}(H), \quad -i \in I \tag{17}$$

for every $I \subset S^1$, $i \notin \partial I$, $-i \notin \partial I$, such that $i \notin I$ or $-i \notin I$. It is subject to the condition $\rho_I|_J = \rho_J$, whenever $J \subset I$. Moreover, if $I \ni i$ and $J \ni -i$ are intervals with disjoint interiors, then $\rho_I(D(I))$ and $\rho_J(E(J))$ should commute. Here, $\rho_I|_J$ is an abbreviation for either $\rho_I \circ \mathcal{A}(J \hookrightarrow I)$, $\rho_I \circ \mathcal{B}(J \hookrightarrow I)$, $\rho_I \circ D(J \hookrightarrow I)$, or $\rho_I \circ E(J \hookrightarrow I)$, depending on the position of I in S^1.

If H and K are D-E-sectors, a homomorphism of sectors is a bounded linear map $H \to K$ that is equivariant with respect to the actions (17).

We now shortly explain the identity 2-morphisms of an arrow of *CN3*, and the vertical composition of 2-morphisms in *CN3*. Let $J_1 = \exp([0, \pi i]) \in \mathsf{INT}^{\bullet}_{Pin}$ be as above. The identity sector of a defect D is given by

$$1_D := L^2\big(D(J_1)\big),$$

and the fact that this is indeed a D-D-sector is a consequence of the vacuum axiom of D. Given $\mathcal{A}$-$\mathcal{B}$-defects D, E, F, the composition

$$H \boxtimes_E K \ \in \ \hom_{\hom(\mathcal{A},\mathcal{B})}(D, F)$$

of a D-E-sector H with an E-F-sector K is given by $H \boxtimes_{E(J_1)} K$. Here, the right action of $E(J_1)$ on H is given by $E(j)$, where $j(z) := \bar{z}$ maps J_1 to its complement $J_2 := \exp([\pi i, 2\pi i])$.

Remark 3.16. [uptoI] The composition of sectors is defined using specific choices of intervals. However, using diff covariance, one can show that homotopic choices yield isomorphic answers.

We refer to [BDH] for further operations and properties of sectors.

Definition 3.17. We let $\mathrm{Rep}(\mathcal{A}) := \mathrm{End}_{\mathrm{End}_{CN3}(\mathcal{A})}(1_{\mathcal{A}})$ denote the 1-category of sectors of $\mathcal{A}$, also called representations of $\mathcal{A}$.

Remark 3.18. If $X \in \mathcal{C}$ is an object in a 3-category, then $\mathrm{End}_{\mathrm{End}_{\mathcal{C}}(X)}(1_X)$ is always braided monoidal. As a special case of the above remark, we get the following result: let $\mathcal{A}$ be a $\mathbb{Z}/2$-graded conformal net, then $\mathrm{Rep}(\mathcal{A})$ is a braided tensor category.

The 3-category *CN3* is symmetric monoidal, with product and unit objects given by

$$(\mathcal{A} \otimes \mathcal{B})(I) := \mathcal{A}(I) \mathbin{\bar{\otimes}} \mathcal{B}(I), \qquad \text{and} \qquad \mathbb{1}(I) := \mathbb{C}.$$

The following is an important property of *CN3*.

Theorem 3.19. *[ThDe] Let $\mathbb{1} \in CN3$ denote the unit object, then $\hom_{CN3}(\mathbb{1}, \mathbb{1})$ is equivalent to the 2-category $VN2_{\mathbb{C}}$ of $\mathbb{Z}/2$-graded von Neumann algebras over $\mathbb{C}$.*

Proof. We construct a functor $F : VN2_{\mathbb{C}} \to \hom_{CN3}(\mathbb{1}, \mathbb{1})$. Equip $\mathbb{R} \subset \mathbb{C}$ with the downward coorientation. If a bicolored interval $(I, [\imath])$ is in $\mathsf{INT}^{\bullet}_{Pin}$, then it makes sense to ask whether $\imath$ is $\mathbb{C}$-linear, and whether it preserves the coorientation. We record that information in two variables [CO]

$$c := \begin{cases} \mathrm{T} & \text{if } \imath \text{ is } \mathbb{C}\text{-linear} \\ \mathrm{F} & \text{if it is } \mathbb{C}\text{-antilinear,} \end{cases} \qquad o := \begin{cases} \mathrm{T} & \text{if } \imath \text{ respects the coorientations} \\ \mathrm{F} & \text{if it doesn't.} \end{cases} \tag{18}$$

Given an algebra A, the defect $F(A)$ is then given by

$$F(A)(I) := \begin{cases} \mathbb{C} & I \in \mathsf{INT}^{\bullet}_{Pin} \cup \mathsf{INT}^{\circ}_{Pin} \\ \begin{cases} A & (c,o) = (\mathrm{T},\mathrm{T}) \\ \bar{A} & (c,o) = (\mathrm{F},\mathrm{T}) \\ A^{op} & (c,o) = (\mathrm{T},\mathrm{F}) \\ \bar{A}^{op} & (c,o) = (\mathrm{F},\mathrm{F}) \end{cases} & I \in \mathsf{INT}^{\bullet\circ}_{Pin}\,. \end{cases}$$

The functor $F(A)$ is constant on $\mathsf{INT}^{\bullet\circ}_{Pin}$ in the sense that it assigns the identity map to every embedding $J \hookrightarrow I$. If H is an A-B-bimodule, then $F(H)$ is the $F(A)$-$F(B)$-sector given by the same Hilbert space H. Thus, F is fully faithful almost by definition. To see that F is an equivalence of categories, we must show that it is also essentially surjective.

Equip $\overline{\mathbb{R}} := [-\infty, +\infty]$ with C^1 and pin structures extending the corresponding structures on $\mathbb{R}$. It is bicolored by letting $\overline{\mathbb{R}}_{\bullet} := \overline{\mathbb{R}}_{\geqslant 0}$, $\overline{\mathbb{R}}_{\circ} := \overline{\mathbb{R}}_{\leqslant 0}$, and by letting the local coordinate be the identity map. Given $D \in \hom_{CN3}(1,1)$, let $A := D(\overline{\mathbb{R}})$. We claim that D is isomorphic to $F(A)$. If $I \in \mathsf{INT}^{\bullet}_{Pin} \cup \mathsf{INT}^{\circ}_{Pin}$, then $D(I) = \mathbb{C}$ and there is nothing to show. Otherwise, if $I \in \mathsf{INT}^{\bullet\circ}_{Pin}$, then $[\imath]$ provides a map

$$D(\imath) \,:\, D(I) \to F(A)(I).$$

That map is an isomorphism because as soon as $[-\varepsilon, \varepsilon] \subset \imath(I)$, the algebra $D(\overline{\mathbb{R}})$ is generated by $D([\varepsilon, +\infty]) = \mathbb{C}$, $D([-\infty, -\varepsilon]) = \mathbb{C}$, and $D(I)$. To see that $D(\imath)$ is independent of the choice of $\imath$, consider another $\jmath \in [\imath]$. Since $\jmath \sim \imath$, there is an inclusion $f : J \hookrightarrow I$, $J \in \mathsf{INT}^{\bullet\circ}_{Pin}$, such that $\jmath \circ f = \imath \circ f$. $D(f)$ being an isomorphism, it follows that $D(\jmath) = D(\imath)$. We have thus canonically identified $D(I)$ with $F(A)(I)$ in all cases. $\square$

3.5. **$\boldsymbol{\mu = 1}$ and invertibility.** Let $\mathcal{A}$ be a conformal net. Its vacuum sector

$$H_0 := L^2\big(\mathcal{A}(\exp([0, \pi i]))\big)$$

is the identity on the identity defect of $\mathcal{A}$. It is an $\mathcal{A}(I)$-module for every $I \subset S^1$, where S^1 is endowed with the inward coorientation. We shall say that $\mathcal{A}$ is irreducible if its vacuum sector is irreducible. In other words, $\mathcal{A}$ is irreducible if $\mathcal{A} \neq 0$, and if

$$\bigvee_{I \subset S^1} \mathcal{A}(I) = \mathrm{B}(H_0).$$

This is equivalent to the algebras $\mathcal{A}(I)$ being factors i.e., having trivial center.

If ${}_AH_B$ is a bimodule between factors, then its statistical dimension $\dim({}_AH_B)$ is an invariant that lives in $\{0\} \cup [1, \infty]$, see [Longo: A theory of dimension] for a definition. The statistical dimension is additive under direct sums, and multiplicative under tensor product and Connes fusion:

$$\begin{aligned} \dim{}_A\big(H \oplus K\big)_B &= \dim\big({}_AH_B\big) + \dim\big({}_AK_B\big), \\ \dim{}_{A \bar{\otimes} B}\big(H \otimes K\big)_{C \bar{\otimes} D} &= \dim\big({}_AH_C\big) \cdot \dim\big({}_BK_D\big), \\ \dim{}_A\big(H \boxtimes_B K\big)_C &= \dim\big({}_AH_B\big) \cdot \dim\big({}_BK_C\big). \end{aligned}$$

Furthermore, it satisfies

$$\dim\big({}_AH_B\big) = 0 \;\Leftrightarrow\; H = 0 \qquad\text{and}\qquad \dim\big({}_AH_B\big) = 1 \;\Leftrightarrow\; B^{op} = A',$$

where A' denotes the graded commutant of A in $\mathrm{B}(H)$. In the latter case, the bimodule ${}_B\bar{H}_A$ with actions [actD]

$$(19) \qquad b\,\bar{\xi}\,a := (-1)^{|a||\xi| + |a||b| + |\xi||b|}\;\overline{a^{\#_i}\xi\, b^{\#_i}}$$

is an inverse of ${}_AH_B$ with respect to Connes fusion.

Recall that if $I, J \subset S^1$ don't intersect, then $\mathcal{A}(J)\bar{\otimes}\mathcal{A}(I)$ acts on H_0 by the split property. Define intervals [4i]

$$(20) \qquad I_1 := \exp\big([0, \tfrac{\pi i}{2}]\big), \quad I_2 := \exp\big([\tfrac{\pi i}{2}, \pi i]\big), \quad I_3 := \exp\big([\pi i, \tfrac{3\pi i}{2}]\big), \quad I_4 := \exp\big([\tfrac{3\pi i}{2}, 2\pi i]\big).$$

Definition 3.20. The μ-index $\mu(\mathcal{A})$ of an irreducible conformal net $\mathcal{A}$ is the square of the statistical dimension of the bimodule [4I]

$$ {}_{\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)}\, H_0 \, {}_{\mathcal{A}(I_2)^{op}\bar{\otimes}\mathcal{A}(I_4)^{op}} \,. \tag{21}$$

An important result of [KLM] says that for even nets, [klm]

$$ \mu(\mathcal{A}) \;=\; \mathrm{Dim}\big(\mathrm{Rep}(\mathcal{A})\big) \;:=\; \frac{1}{2}\sum_{\lambda} \dim(H_\lambda)^2, \tag{22}$$

where the sum runs over all isomorphism classes of irreducible representations of $\mathcal{A}$, and $\dim(H_\lambda)$ is the categorical dimension

$$\dim(H_\lambda) = \lim_{n\to\infty} \sqrt[n]{\#\text{ of irreducible summands in } H_\lambda^{\otimes n}}.$$

The latter is also equal to the statistical dimension of H_λ, viewed as an $\mathcal{A}(\frown)$-$\mathcal{A}(\smile)$-bimodule. The factor $1/2$ in (22) comes from the fact that we consider $\mathbb{Z}/2$-graded representations, and that for each even sector H_λ there is a corresponding odd sector $\Pi H_\lambda := H_\lambda \otimes \mathbb{R}^{0|1}$.

Remark 3.21. The (super)conformal nets considered in [KLM], [CKL] are somewhat different than ours. Thus, even if Kawahigashi-Longo-Mueger had proved the $\mathbb{Z}/2$-graded version of (22), we would not have been able to quote their result directly.

We expect formula (22) to also hold for our notion of conformal nets, and also in the $\mathbb{Z}/2$-graded case. The definition of Dim(Rep($\mathcal{A}$)) being purely categorical, it would imply the invariance of the μ-index:

$$\mathcal{A} \;\simeq_{CN3} \mathcal{B} \quad\Rightarrow\quad \mathrm{Rep}(\mathcal{A}) \simeq \mathrm{Rep}(\mathcal{B}) \quad\Rightarrow\quad \mu(\mathcal{A}) = \mu(\mathcal{B}).$$

In this paper, we shall only need the following weaker result.

Proposition 3.22. *[IRm1] Let $\mathcal{A}$ and $\mathcal{B}$ be two irreducible nets that are isomorphic in CN3. Then $\mu(\mathcal{A}) = 1$ if and only if $\mu(\mathcal{B}) = 1$.*

Proof. Clearly, we have

$$\mathcal{A} \;\simeq_{CN3} \mathcal{B} \qquad\Rightarrow\qquad \mathrm{Rep}(\mathcal{A}) \simeq \mathrm{Rep}(\mathcal{B}).$$

Therefore, it is enough to show that

$$\mu(\mathcal{A}) = 1 \qquad\Longleftrightarrow\qquad \mathrm{Rep}(\mathcal{A}) \simeq \mathit{Hilb}_{\mathbb{C}}^{\mathbb{Z}/2}.$$

We shall abusively denote the latter condition by $\mathrm{Rep}(\mathcal{A}) = 1$. By Proposition 56 of [KLM], any representation of $\mathcal{A}$ decomposes as a direct integral of irreducible representations (this is a property shared by separable C^*-algebras). Therefore, $\mathrm{Rep}(\mathcal{A}) = 1$ if and only if the only irreducible representations are H_0 and $\Pi H_0 := H_0 \otimes \mathbb{R}^{0|1}$.

Let $\mathcal{A}$ have μ-index equal to 1, and let H_λ be an irreducible representation. The isomorphisms

$$j_1 \,:\, I_1 \to I_2\cup I_3\cup I_4, \quad z \mapsto \frac{iz+(1-i)}{(1+i)z-i}, \qquad j_2 \,:\, I_3 \to I_4\cup I_1\cup I_2, \quad z \mapsto \frac{-iz+(1-i)}{(1+i)z+i}.$$

can be used to form

$$H_\lambda \boxtimes_{\mathcal{A}(I_1)} H_0 \qquad\text{and}\qquad H_0 \boxtimes_{\mathcal{A}(I_3)^{op}} H_\lambda = H_\lambda \boxtimes_{\mathcal{A}(I_3)} H_0,$$

where the right actions of $\mathcal{A}(I_1)$ and $\mathcal{A}(I_3)$ on H_λ are given by $\mathcal{A}(j_1)$ and $\mathcal{A}(j_2)$ respectively. The remaining left actions of $\mathcal{A}(I_1)$, $\mathcal{A}(I_2)$, $\mathcal{A}(I_3)$, and $\mathcal{A}(I_4)$ equip these Hilbert spaces with the structure of representations of $\mathcal{A}$. By Remark 3.16, those are isomorphic to $H_\lambda\boxtimes_{1_\mathcal{A}}H_0$ and $H_0\boxtimes_{1_\mathcal{A}}H_\lambda$ respectively. Since H_0 is an identity 2-morphism, it follows that

$$H_\lambda \boxtimes_{\mathcal{A}(I_1)} H_0 \;\simeq\; H_0 \boxtimes_{\mathcal{A}(I_3)^{op}} H_\lambda$$

in $\mathrm{Rep}(\mathcal{A})$. Fusing with the inverse of (21), the left hand side becomes

$$H_\lambda \underset{\mathcal{A}(I_1)}{\boxtimes} L^2\Big(\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)\Big) \;\simeq\; H_\lambda \underset{\mathcal{A}(I_1)}{\boxtimes} \Big(L^2\big(\mathcal{A}(I_1)\big)\otimes L^2\big(\mathcal{A}(I_3)\big)\Big) \;\simeq\; H_\lambda \otimes L^2\big(\mathcal{A}(I_3)\big)$$

while the right hand side becomes

$$H_\lambda \underset{\mathcal{A}(I_3)}{\boxtimes} L^2\Big(\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)\Big) \simeq H_\lambda \underset{\mathcal{A}(I_3)}{\boxtimes} \Big(L^2\big(\mathcal{A}(I_1)\big) \otimes L^2\big(\mathcal{A}(I_3)\big)\Big) \simeq L^2\big(\mathcal{A}(I_1)\big) \otimes H_\lambda.$$

It follows that $H_\lambda \otimes H_0$ and $H_0 \otimes H_\lambda$ are isomorphic as representations of $\mathcal{A}\otimes\mathcal{A}$. This can only happen if $H_\lambda = H_0$ or if $H_\lambda = \Pi H_0$. Therefore $\mathrm{Rep}(\mathcal{A}) = 1$.

Now let us assume that $\mu(\mathcal{A}) \neq 1$. The bimodule [4J]

$$(23) \qquad {}_{\mathcal{A}(I_2)\bar{\otimes}\mathcal{A}(I_4)}\,\bar{H}_0\,{}_{\mathcal{A}(I_1)^{op}\bar{\otimes}\mathcal{A}(I_3)^{op}}$$

with actions as in (19) is no longer the inverse of (21). But the statistical dimension being finite, it is still its adjoint [Longo Index, statistic of q fields, II, page 289] [Nets of subf]. By strong additivity and by the irreducibility of $\mathcal{A}$, the bimodules (21) and (23) are irreducible. Among other things, this implies that

$$\hom\Big(L^2\big(\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)\big), H_0 \boxtimes_{\mathcal{A}(I_2)\bar{\otimes}\mathcal{A}(I_4)} \bar{H}_0\Big) = \mathbb{C},$$

where the hom is taken in the category of $\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)$-$\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)$-bimodules. Upon identifying the appropriate intervals, we can view $L^2\big(\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)\big)$ and $H_0 \boxtimes_{\mathcal{A}(I_2)\bar{\otimes}\mathcal{A}(I_4)} \bar{H}_0$ as representations of $\mathcal{A}\otimes\mathcal{A}$. The former corresponds to the vacuum, while the latter has statistical dimension $\mu(\mathcal{A}) > 1$. Hence, $H_0 \boxtimes_{\mathcal{A}(I_2)\bar{\otimes}\mathcal{A}(I_4)} \bar{H}_0$ contains at least one irreducible summand not isomorphic to $H_0 \otimes H_0$ or $\Pi H_0 \otimes H_0$. It follows that $\mathrm{Rep}(\mathcal{A}\otimes\mathcal{A}) \neq 1$ and hence that $\mathrm{Rep}(\mathcal{A}) \neq 1$ (see Lemma 27 of [KLM] for the details of this very last step). $\square$

Theorem 3.23. *[inet] A conformal net $\mathcal{A}$ is invertible in CN3 if and only if it is irreducible and $\mu(\mathcal{A}) = 1$.*

Proof. Let $\mathcal{A}$ be an invertible net. Tensoring with $\mathcal{A}^{-1}$ provides an equivalence $\mathrm{End}_{CN3}(\mathcal{A}) \simeq \mathrm{End}_{CN3}(\mathbb{1})$ and therefore, by Theorem 3.19, an equivalence

$$\mathrm{End}_{CN3}(\mathcal{A}) \simeq VN2_{\mathbb{C}}.$$

If $\mathcal{A}$ were not irreducible, then $\mathbb{C}\oplus\mathbb{C}$ would be a subalgebra of the endomorphism algebra

$$\mathrm{End}_{\,\mathrm{Rep}(\mathcal{A})}(H_0) = \mathrm{End}_{\mathrm{End}_{\mathrm{End}_{CN3}(\mathcal{A})}(1_{\mathcal{A}})}(H_0),$$

which is impossible since the latter is isomorphic to $\mathrm{End}_{\mathrm{End}_{VN2_{\mathbb{C}}}(\mathbb{1})}(1_{\mathbb{1}}) = \mathbb{C}$. Thus $\mathcal{A}$ is irreducible. By Proposition 3.22, we see that

$$\mu(\mathcal{A})\,\mu(\mathcal{A}^{-1}) = \mu(\mathcal{A}\otimes\mathcal{A}^{-1}) = \mu(\mathbb{1}) = 1,$$

which implies $\mu(\mathcal{A}) = 1$ since the μ-index is always at least 1.

Now we assume $\mu(\mathcal{A}) = 1$, and show that $\mathcal{A}$ is invertible in *CN3*. The inverse net is given by

$$\mathcal{A}^{-1} := \mathcal{A}^{op}$$

as in (14). To show that it is indeed an inverse of $\mathcal{A}$, we construct an invertible defect D from $\mathcal{A}\otimes\mathcal{A}^{-1}$ to $\mathbb{1}$. Given $I \in \mathsf{INT}^{◐}_{Pin}$, we may use its local coordinate to construct a doubled interval

$$I_{\circ\circ} := I_\circ \underset{i}{\cup} \exp([-\tfrac{\pi i}{2}, \tfrac{\pi i}{2}]) \underset{-i}{\cup} I_\circ.$$

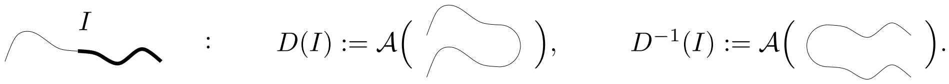
We equip $I_{\circ\circ}$ with the coorientation inherited from the first copy of $I_\circ$. The two inclusions $I_\circ \hookrightarrow I_{\circ\circ}$ induce maps $\mathcal{A}(I_\circ) \to \mathcal{A}(I_{\circ\circ})$, $\mathcal{A}(I_\circ)^{op} \to \mathcal{A}(I_{\circ\circ})$, and so it makes sense to define

$$D(I) := \begin{cases} A(I)\,\bar{\otimes}\,\mathcal{A}(I)^{op} & I \in \mathsf{INT}^{\circ}_{Pin} \\ \mathcal{A}(I_{\circ\circ}) & I \in \mathsf{INT}^{◐}_{Pin} \\ \mathbb{C} & I \in \mathsf{INT}^{\bullet}_{Pin}. \end{cases}$$

The inverse defect is given by

$$D^{-1}(I) := \begin{cases} \mathbb{C} & I \in \mathsf{INT}^{\circ}_{Pin} \\ \mathcal{A}(I_{\bullet\bullet}) & I \in \mathsf{INT}^{◐}_{Pin} \\ A(I)\,\bar{\otimes}\,\mathcal{A}(I)^{op} & I \in \mathsf{INT}^{\bullet}_{Pin}, \end{cases}$$

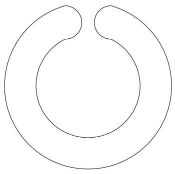
where $I_{\bullet\bullet} := I_\bullet \cup \exp([\frac{\pi i}{2}, \frac{3\pi i}{2}]) \cup I_\bullet$. Here is a pictorial description of D and D^{-1}, where the little arrows indicate the coorientations:



We must now show that $D \circledast D^{-1}$ and $D^{-1} \circledast D$ are equivalent to the identity defects of $\mathcal{A} \otimes \mathcal{A}^{-1}$ and $\mathbb{1}$ respectively. The begin with the first one:

$$\big(D \circledast_{\mathbb{1}} D^{-1}\big)(I) = \begin{cases} \mathcal{A}(I) \,\bar{\otimes}\, \mathcal{A}(I)^{op} & I \in \mathsf{INT}^{\circ}_{Pin} \\ \mathcal{A}(I_{\circ\circ}) \,\bar{\otimes}\, \mathcal{A}(I_{\bullet\bullet}) & I \in \mathsf{INT}^{\bullet}_{Pin} \\ \mathcal{A}(I) \,\bar{\otimes}\, \mathcal{A}(I)^{op} & I \in \mathsf{INT}^{\bullet}_{Pin}. \end{cases}$$

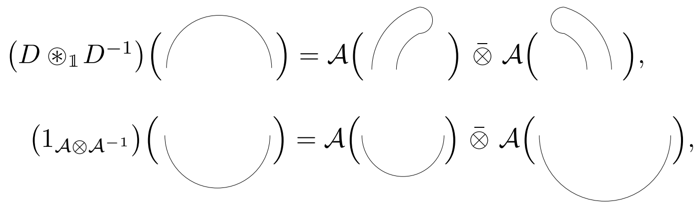
To show $D \circledast_{\mathbb{1}} D^{-1} \simeq 1_{\mathcal{A}\otimes\mathcal{A}^{-1}}$, we construct an invertible sector K. To construct the actions (17) on K, it is enough to have actions of $\mathcal{A}(I)$ for every

$I \in$  .

Here, the last picture is a distorted image of the closed manifold

$$\Big(\exp([\tfrac{\pi i}{2}, \tfrac{3\pi i}{2}]) \underset{i,-i}{\cup} \exp([-\tfrac{\pi i}{2}, \tfrac{\pi i}{2}]) \underset{i,i}{\cup} \exp([\tfrac{\pi i}{2}, \tfrac{3\pi i}{2}])\Big) \cup \Big(\exp([-\tfrac{\pi i}{2}, \tfrac{\pi i}{2}]) \underset{i,-i}{\cup} \exp([\tfrac{\pi i}{2}, \tfrac{3\pi i}{2}]) \underset{i,i}{\cup} \exp([-\tfrac{\pi i}{2}, \tfrac{\pi i}{2}])\Big),$$

and the arrow indicates the chosen coorientation. We identify that manifold with S^1 and define K to be the vacuum representation of $\mathcal{A}$. The invertibility of K as a $(D \circledast_{\mathbb{1}} D^{-1})$-$(1_{\mathcal{A}\otimes\mathcal{A}^{-1}})$-sector is equivalent to its invertibility as a $(D \circledast_{\mathbb{1}} D^{-1})(\;)$-$(1_{\mathcal{A}\otimes\mathcal{A}^{-1}})(\;)$-bimodule. The latter follows from $\mu(\mathcal{A}) = 1$ since

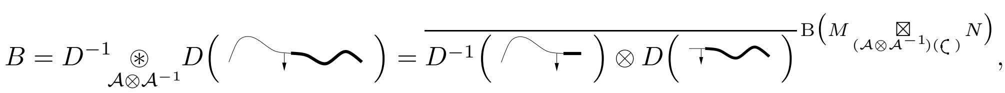


and the four intervals on the right hand side can be identified with I_1, I_3, I_2, I_4 of (21).

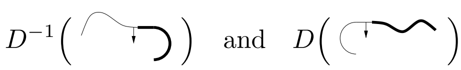
We now show that $D^{-1} \circledast_{\mathcal{A}\otimes\mathcal{A}^{-1}} D$ is equivalent to the identity defect of $\mathbb{1}$. Given the equivalence between $\hom_{CN3}(\mathbb{1}, \mathbb{1})$ and $VN2_{\mathbb{C}}$, it is enough to show that the von Neumann algebra corresponding to $D^{-1} \circledast_{\mathcal{A}\otimes\mathcal{A}^{-1}} D$ is Morita equivalent to $\mathbb{C}$. Pick $I \in \mathsf{INT}^{\bullet}_{Pin}$ with $(c, o) = (\mathrm{T}, \mathrm{T})$, as defined in (18). By the proof of Theorem 3.19, the algebra B corresponding to $D^{-1} \circledast D$ is given by

$$B := \Big(D^{-1} \underset{\mathcal{A}\otimes\mathcal{A}^{-1}}{\circledast} D\Big)(I),$$

and is independent of I. We now show that B is isomorphic to $\mathrm{B}(H)$, and hence Morita equivalent to $\mathbb{C}$. By the definition of fusion of defects, that algebra is given by



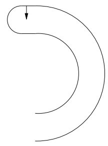
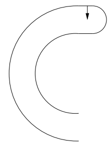
where M and N are faithful representations of



respectively. Filling in the definition of D and D^{-1} in the above formula, we get

$$(24)\qquad B \;=\; \overline{\mathcal{A}\big(\quad\big)\otimes\mathcal{A}\big(\quad\big)}^{\mathrm{B}\left(M \underset{\mathcal{A}(\ )\bar{\otimes}\mathcal{A}(\ )}{\boxtimes} N\right)},$$

where M and N are faithful representations of

$\mathcal{A}\Big($  $\Big)$ and $\mathcal{A}\Big($  $\Big)$.

Here, these two last pictures represent the intervals

$$(25)\qquad \exp([-\tfrac{\pi i}{2},\tfrac{\pi i}{2}]) \underset{i,0}{\cup} [-1,0] \underset{-1,i}{\cup} \exp([\tfrac{\pi i}{2},\tfrac{3\pi i}{2}]) \underset{-i,-1}{\cup} [-1,0] \underset{0,i}{\cup} \exp([-\tfrac{\pi i}{2},\tfrac{\pi i}{2}])$$

$$(26)\qquad \exp([\tfrac{\pi i}{2},\tfrac{3\pi i}{2}]) \underset{i,0}{\cup} [0,1] \underset{1,i}{\cup} \exp([-\tfrac{\pi i}{2},\tfrac{\pi i}{2}]) \underset{-i,1}{\cup} [0,1] \underset{0,i}{\cup} \exp([\tfrac{\pi i}{2},\tfrac{3\pi i}{2}])$$

and the arrows indiciate the coorientations. Let I_1, I_2, I_3, I_4 be as in (20), and let us identify (25) and (26) with $I_4\cup I_1\cup I_2$ in such a way that $[-1,0]\cup\exp([\frac{\pi i}{2},\frac{3\pi i}{2}])\cup[-1,0]$ and $[0,1]\cup\exp([-\frac{\pi i}{2},\frac{\pi i}{2}])\cup[0,1]$ map to I_1.

After those identifications, we may pick M to be H_0 with actions as in (21), and N to be its inverse $\bar{H}_0$. The expression (24) then becomes

$$\begin{aligned} B &= \overline{\mathcal{A}(I_1)\otimes\mathcal{A}(I_1)^{op}}^{\mathrm{B}\left(H_0 \underset{\mathcal{A}(I_2)^{op}\bar{\otimes}\mathcal{A}(I_4)^{op}}{\boxtimes} \bar{H}_0\right)} \\ &= \overline{\mathcal{A}(I_1)\otimes\mathcal{A}(I_1)^{op}}^{\mathrm{B}\left(L^2\left(\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)\right)\right)} \\ &= \overline{\mathcal{A}(I_1)\otimes\mathcal{A}(I_1)^{op}}^{\mathrm{B}\left(L^2\left(\mathcal{A}(I_1)\right)\otimes L^2\left(\mathcal{A}(I_3)\right)\right)} \;=\; \mathrm{B}\big(L^2(\mathcal{A}(I_1))\big) \end{aligned}$$

since $L^2(\mathcal{A}(I_1))$ is an irreducible $\mathcal{A}(I_1)$-$\mathcal{A}(I_1)$-bimodule. □

4. Fermionic model for the string group [sec:F]

4.1. The free fermions [sec:ff]. In this section, we construct the *free fermion* functor

$$Fer : (Vect_{\mathbb{R}}, \oplus) \to (CN3^{\times}, \otimes).$$

Its input is a finite dimensional real Hilbert space V, and its output is an invertible conformal net Fer_V.

Let $I = (I, S)$ be a pin interval. The space of continuous sections $\Gamma(I, S)$ has a canonical sesquilinear pairing

$$\Gamma(I,S)\otimes\overline{\Gamma(I,S)} \longrightarrow \Gamma\big(I, S\otimes\overline{S}\big) \simeq \Gamma\big(I,\text{densities}\big) \xrightarrow{\ \int\ } \mathbb{C},$$

and so we may form the complex Hilbert space $L^2(I,S)$. The von Neumann algebra $Fer_V(I)$ is a completion of a Clifford algebra on $L^2(I, S\otimes V)$. To make sense of that Clifford algebra, we first need to equip $L^2(I, S\otimes V)$ with a $\mathbb{C}$-bilinear form. For that, we use the coorientation of I.

Define real subbundles $S_+, S_- \subset S$ in the following way. Pick a section $v \in \Gamma(I, T^*I\otimes\mathbb{C})$ representing the coorientation and set [scd]

$$(27)\qquad \begin{aligned} S_+ &:= \big\{s\in S \,\big|\, s\otimes s \in \mathbb{R}_{\geq 0} v\big\}, \\ S_- &:= \big\{s\in S \,\big|\, s\otimes s \in \mathbb{R}_{\leq 0} v\big\}. \end{aligned}$$

The line bundles S_+ and S_- are orthogonal to each other and satisfy $S_+\oplus S_- = S$. Being a real Hilbert space, $L^2(I, S_+)$ comes with an $\mathbb{R}$-bilinear pairing $\langle\ ,\ \rangle_+$. Its complexification

$$(\ ,\ )_+ := \langle\ ,\ \rangle_+ \otimes \mathbb{C}$$

is then a $\mathbb{C}$-bilinear pairing on $L^2(I,S)$. We define $CAR(I)$ to be the Clifford C^*-algebra

$$C\ell\big(L^2(I,S);(\ ,\ )_+\big) := \overline{\bigoplus_{i\geq 0}\big(L^2(I,S)\big)^{\otimes i}\Big/\xi\otimes\xi-(\xi,\xi)_+},\quad \xi \text{ is odd},\ \ \xi^* := \begin{cases}\xi, & \xi\in L^2(I,S_+)\\ -\xi, & \xi\in L^2(I,S_-),\end{cases}$$

where the bar refers to the norm closure in the action on some universal Hilbert space. This can also be described as the complexified Clifford C^*-algebra $C\ell_{\mathbb{C}}(L^2(I,S_+))$ of the real Hilbert space $L^2(I,S_+)$. Similarly, we define

$$\begin{aligned} CAR_V(I) &:= C\ell\big(L^2(I,S\otimes V);(\ ,\ )_+\big),\\ &= C\ell_{\mathbb{C}}\big(L^2(I,S_+\otimes V);\langle\ ,\ \rangle_+\big).\end{aligned}$$

The fermion algebra $Fer_V(I)$ is a completion of $CAR_V(I)$. Before describing it, we show how the construction $I\mapsto CAR_V(I)$ extends to a functor

$$CAR : \mathsf{INT}_{Pin}\to\big\{C^*\text{-algebras}\big\}$$

satisfying the requirements of table (11). Let $f:J\hookrightarrow I$ be a morphism of INT_{Pin}.

• If f preserves the coorientation then it preserves the subbundles S_+ and S_-.

∘ If it is $\mathbb{C}$-linear, then the corresponding isometry $L^2(J,S\otimes V)\to L^2(I,S\otimes V)$ preserves the bilinear forms $(\ ,\ )_+$, and so it induces a map of Clifford algebras $CAR_V(J)\to CAR_V(I)$.

∘ If f is $\mathbb{C}$-antilinear, then the linear map $L^2(J,S\otimes V)\to\overline{L^2(I,S\otimes V)}$ pulls back the complex conjugate $\overline{(\ ,\ )_+}$ to the form $(\ ,\ )_+$. We therefore get a map $CAR_V(J)\to\overline{CAR_V(I)}$.

• If f doesn't preserve the coorientation, then it exchanges S_+ and S_-.

∘ If it is $\mathbb{C}$-linear, then the from that pulls back to $(\ ,\ )_+$ is the opposite $-(\ ,\ )_+$. So we get a map $CAR_V(J)=C\ell\big(L^2(J,S\otimes V),(\ ,\ )_+\big)\to C\ell\big(L^2(I,S\otimes V),-(\ ,\ )_+\big)=CAR_V(I)^{op}$.

∘ If f is $\mathbb{C}$-antilinear, $-\overline{(\ ,\ )_+}$ pulls back to $(\ ,\ )_+$ and so we get a map $CAR_V(J)\to\overline{CAR_V(I)}^{op}$.

As an example of the first situation, the pin involution of I acts by -1 on $L^2(I,S\otimes V)$ and hence gives rise to the grading involution on $CAR_V(I)$; as an example of the last situation, the conjugating involution $c_i: I\to I$ acts by i on S_+ and thus by $\#_i$ on $CAR_V(I)$.

We now describe $Fer_V(I)$ in two steps. We first construct it for subintervals of S^1, and then extend it to the general case. Equip S^1 with the inward coorientation, and with the pin structure induced from its embedding in $\mathbb{C}$. Let S_+ denote the line bundle over S^1 given by (27). Its topology being that of a Möbius band, we rename it *Möb*. Explicitly, its fiber over a point $z\in S^1$ is given by

$$M\ddot{o}b_z=\big\{w\in\mathbb{C}\,\big|\,\tfrac{w^2}{z}\in\mathbb{R}_{\leq 0}\big\}.$$

Consider the Hilbert transform $J: L^2(S^1,M\ddot{o}b)\to L^2(S^1,M\ddot{o}b)$ given by

$$Jf(z) := \tfrac{1}{\pi}\,\mathrm{P.V.}\!\int_{S^1} f(w)\,\frac{zw^{-1}}{z-w}\,dw,$$

where P.V. stand for the Cauchy principal value of the singular integral[2]. The map J is unitary and satisfies $J^2=-1$. It can also be described it in terms of the Fourrier transform

$$\begin{array}{ccc} J\ :\ L^2\big(I,M\ddot{o}b\big) \xrightarrow{\ \mathcal{F}\ } & \big\{f\in\ell^2\big(\mathbb{Z}+\tfrac12\big)\,\big|\,f(-n)=\bar f(n)\big\} & \\ & \Big\downarrow \ \longleftarrow \begin{array}{l}\text{times } i \text{ on } (\mathbb{Z}+1/2)_{>0}, \text{ and}\\ \text{times } -i \text{ on } (\mathbb{Z}+1/2)_{<0}.\end{array} & \\ & \big\{f\in\ell^2\big(\mathbb{Z}+\tfrac12\big)\,\big|\,f(-n)=\bar f(n)\big\} & \xrightarrow{\ \mathcal{F}^{-1}\ } L^2\big(I,M\ddot{o}b\big),\end{array}$$

where $\mathcal{F}: f\mapsto\hat f$ is given by $\hat f(n)=\frac{1}{\sqrt{2\pi}}\int_{S^1} f(z)z^{n-\frac32}dz$.

[2] I computed: $J:(z-1)\mapsto -i(z+1)$

Let us expand the definition of the CAR algebra to include $CAR_V(S^1) := C\ell_{\mathbb{C}}\big(L^2(S^1, \textit{Möb}\otimes V)\big)$. Given a vector $\xi \in L^2(S^1, \textit{Möb} \otimes V)$, we denote the corresponding element of $CAR_V(S^1)$ by $c(\xi)$. Consider the fermionic Fock space of $L^2(S^1, \textit{Möb}\otimes V)$ with respect to the complex structure $J \otimes 1_V$

$$H_0^{(V)} := \bigoplus_{i=0}^{\infty} \bigwedge\nolimits^{i}_{J\otimes 1_V}\big(L^2(S^1, \textit{Möb}\otimes V)\big).$$

The algebra $CAR_V(S^1)$ acts on $H_0^{(V)}$ by

$$\begin{aligned} \pi_0 \,:\, CAR_V(S^1) \quad &\longrightarrow \quad \mathrm{B}\big(H_0^{(V)}\big)\\ c(\xi) \quad &\mapsto \quad (\xi\wedge -) + (\xi \wedge -)^*. \end{aligned}$$

The von Neumann algebra $Fer_V(I)$ is then defined as the closure of $CAR_V(I) \subset CAR_V(S^1)$ with respect to the σ-weak topology on $\mathrm{B}(H_0^{(V)})$

$$Fer_V(I) := \overline{C\ell_{\mathbb{C}}\big(L^2(I, \textit{Möb}\otimes V)\big)}^{\mathrm{B}(H_0^{(V)})}.$$

Let $\mathsf{INT}^{S^1}_{Pin}$ be the full subcategory of INT_{Pin} whose objects are the subintervals of S^1. Given an arrow $f : J \to I$ of $\mathsf{INT}^{S^1}_{Pin}$, we now show that $CAR(f)$ extends to a continuous map $Fer(J) \to Fer(I)$. Clearly, it is enough to do this for a set of generators of $\mathsf{INT}^{S^1}_{Pin}$.

i. If f is $\mathbb{C}$-linear and preserves the coorientation, pick a diffeomorphism $\varphi \in \mathit{Diff}_{Pin}(S^1)$ extending f. Let φ_* denote its action on $L^2(S^1, \textit{Möb} \otimes V)$ and let $CAR_V(\varphi)$ denote its action on $CAR_V(S^1)$. For $CAR_V(f)$ to induce a map $Fer_V(J) \to Fer_V(I)$, we must show that the representations π_0 and $\pi_\varphi := \pi_0 \circ CAR_V(\varphi)$ are equivalent. By Segal's quantization criterion, this holds if and only if $J \otimes 1_V - \varphi_*^{-1} \circ (J \otimes 1_V) \circ \varphi_*$ is a Hilbert-Schmidt operator. That operator is indeed Hilbert-Schmidt because V is finite dimensional and because its integral kernel

$$K(z,w) \;=\; \frac{zw^{-1}}{z-w} \;-\; \frac{\varphi(z)\varphi(w)^{-1}}{\varphi(z)-\varphi(w)}\sqrt{\varphi'(z)}\sqrt{\varphi'(w)}$$

is continuous on $S^1 \times S^1$.

ii. Let $g \in \mathit{Diff}_{Pin}(S^1)$ be the map given by complex congugation, both on S^1 and on the spinor bundle. The induced involution on $L^2(S^1, \textit{Möb}\otimes V)$ anticommutes with $J \otimes 1_V$, and thus extends to a map [mnk]

$$H_0^{(V)} \longrightarrow \overline{H_0^{(V)}} \tag{28}$$

that intertwines π_0 and $\pi_0 \circ CAR_V(g)$. If $f : I' \to I$ is the restriction of g to some interval $I' \subset S^1$, then by the same argument as above, we see that $CAR_V(f)$ extends to a map $Fer_V(f) : Fer_V(I') \to \overline{Fer_V(I)}$.

iii. Finally, if c_i the conjugating involution of $I \in \mathsf{INT}^{S^1}_{Pin}$, then $CAR_V(c_i) = \#_i$, which clearly extends to a map on $Fer_V(I)$.

Every arrow f of $\mathsf{INT}^{S^1}_{Pin}$ is a composite of arrows of the form *i*, *ii*, and *iii*. It follows that for every f, the map $CAR_V(f)$ extends to a map $Fer_V(f)$, unique by density.

The identification of $H_0^{(V)}$ with the vacuum sector of Fer_V is a well known result.

Proposition 4.1. *Let $I := \exp([0,\pi i])$, $j(z) := \bar{z}$, and consider the right action of $Fer_V(I)$ on $H_0^{(V)}$ given by $Fer_V(j)$. Equipped with this action, the Hilbert space $H_0^{(V)}$ is isomorphic to $L^2(Fer_V(I))$ as a $Fer_V(I)$-$Fer_V(I)$-bimodule.*

Proof. The Fock space $H_0^{(V)}$ has a vacuum vector $\Omega \in \bigwedge^0(L^2(S^1, \textit{Möb} \otimes V))$, and an antilinear involution $\mathbf{J}$ given by (28). The positive cone P is the closure of $\{a\Omega a^* \,|\, a \in Fer_V(I)\}$. By [Takesaki: *Tomita's...*] and [Connes: *Caracterization...*], it is enough to show that Ω is cyclic for the left action of $Fer_V(I)$ and that $\mathbf{J}$ is the modular conjugation with respect to Ω.

The first property is known as the Reeh-Schlieder theorem, and is true under very general assumptions [Corollary 2.8 of G&F] [Thm 1 of Borchers: *On the converse of the Reeh Schlieder theorem*].

To see that $\mathbf{J}$ agrees with the modular conjugation, we need to

$\square$

Remark 4.2. For the nets $Fer_{\mathbb{C}^n} = Fer_{\mathbb{R}^{2n}}$, the fact that Ω is cyclic for $Fer_V(I)$, and that $\mathbf{J}$ is the corresponding modular conjugation are proved in [Was, Sec 15]

- the two actions of A on $L^2(A)$ are faithful, and are each other's commutants,
- the cone P is self-dual, meaning that $P = \{\xi \in L^2(A) : \langle p, \xi\rangle \geq 0, \forall p \in P\}$,
- $\mathbf{J}(a\xi b) = b^*\mathbf{J}(\xi)a^*$ for $a, b \in A$, and $\xi \in L^2(A)$,
- $a\xi a^* \in P$ for $a \in A$, $\xi \in P$,
- $\mathbf{J}(\xi) = \xi$ for $\xi \in P$,
- $c\xi = \xi c$ for c in the center of A, and $\xi \in L^2(A)$.

To finish the construction of Fer_V, we must extend it from $\mathsf{INT}^{S^1}_{Pin}$ to INT_{Pin}. Given an object $I \in \mathsf{INT}_{Pin}$, consider the set of all $\mathbb{C}$-linear coorientation preserving embeddings of $I \to S^1$. Given two elements φ, ψ of that set, the map $Fer_V(\psi\circ\varphi^{-1})$ determines an isomorphism between $Fer_V(\varphi(I))$ and $Fer_V(\psi(I))$. We then define [fcf]

$$Fer_V(I) := \left\{ (a_\varphi) \in \prod_{\varphi: I \hookrightarrow S^1} Fer_V\big(\varphi(I)\big) \;\middle|\; Fer_V(\psi\circ\varphi^{-1})(a_\varphi) = a_\psi \quad \forall \varphi, \psi \right\}. \tag{29}$$

Alternatively, $Fer_V(I)$ can be defined as the completion of $CAR_V(I)$ inside $Fer_V(S^1) := \mathrm{B}(H_0^{(V)})$, where the map $CAR_V(I) \to Fer_V(S^1)$ is induced by an arbitrary embedding $\varphi : I \hookrightarrow S^1$, as above.

Remark 4.3. Our construction makes it obvious that the orthogonal group $O(V)$ acts on Fer_V. But more is true: the path group

$$P_I O(V) := \{\gamma : I \to O(V), \text{ of class } C^1\}$$

acts on each algebra $Fer_V(I)$. Indeed, that group acts on $L^2(I, S_+ \otimes V)$ and hence on $CAR_V(I)$. Finally, this action extends to $Fer_V(I)$ by an application of Segal's quantization criterion. Equivalent to the existence of an action

$$\rho : P_I O(n) \to \mathrm{Aut}\big(Fer_{\mathbb{R}^n}(I)\big)$$

is the fact that one can extend the free fermion construction to intervals with vector bundles [fbu]

$$\begin{array}{rcl} Fer : \left\{ \begin{array}{l} \text{Intervals } I \in \mathsf{INT}_{Pin} \text{ equipped} \\ \text{with a bundle } V \to I \text{ of finite} \\ \text{dimensional real Hilbert spaces.} \end{array} \right\} & \longrightarrow & \mathsf{VN} \\ \big(V \to I\big) & \mapsto & Fer_V(I) \end{array} \tag{30}$$

Indeed, if $V \to I$ is a vector bundle one defines

$$Fer_V(I) := \left\{ (a_\varphi) \in \prod_{\varphi: V \,\simeq\, \mathbb{R}^n \times I} Fer_{\mathbb{R}^n}(I) \;\middle|\; \rho\big(\psi\circ\varphi^{-1}\big)(a_\varphi) = a_\psi \quad \forall \varphi, \psi \right\},$$

where the product is indexed by all the possible trivializations φ of the vector bundle V.

We finish this section by stating an important property of Fer_V.

Theorem 4.4. *The free fermion conformal net Fer_V is an invertible object of CN3.*

Proof. Let $m := \dim(V)$. By Theorem 3.23, it is enough to show that $\mu(Fer_V) = \mu(Fer_{\mathbb{R}^m}) = 1$. By the multiplicativity of the μ-index, it is enough to treat the case $m = 2n$. The net constructed in [Was, Sec 15] is isomorphic to $Fer_{\mathbb{C}^n} = Fer_{\mathbb{R}^{2n}}$, and the fact that $\mu(Fer_{\mathbb{C}^n}) = 1$ follows directly from [Was, Sec 13-15]. □

4.2. **The string group as a 2-group [sec:2g].** Recall the definition of $String(n)$.
Explain the meaning of the defect D_g induced by an element $g \in O(n)$.
The groupoid underlying $String(n)$.
$String(n)$ as a topological stack.

4.3. **The string group as a topological group [sec:tg].** Replace D_g by its weak version.

4.4. **Computation of the boundary homomorphism [sec:bh].**