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Some cobordism invariants for links

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Introduction. The purpose of this paper is to obtain some necessary conditions for a link in Euclidean 3-space to be spanned by a locally unknotted surface of given type in one half of 4-space. In particular necessary conditions for two links to be cobordant are proved.

In section 1 the problem is reduced to one concerning ribbon immersions in 3-space, in which form it is amenable to an algebraic approach. In section 2 some new algebraic invariants of link type are introduced. These generalize the signature and nullity, defined by Murasugi (8). Their relation to the geometric problem and the fact that they are cobordism invariants are shown. Sections 3 and 4 are devoted to computation of the invariants in certain cases and to corollaries arising from these computations. The most important corollaries concern the embeddability of 2-spheres in 4-manifolds to represent certain homology classes.

The results of section 2 can be improved and expanded in certain cases and this will be the object of a further paper.

1. *Geometry.* This section deals with aspects of link geometry. Definitions and certain proofs are given in some detail in an attempt to create a reference for future use. All maps and spaces are in the P.L. category and I have used the terminology introduced by Hudson and Zeeman (4, 5, 12).

Definition 1·1. If N is a manifold ∂N will denote its boundary and $\operatorname{int} N$ will denote $N - \partial N$.

Definition 1·2. If $g: M \to N$ is a map of manifolds, g is *proper* if $g(\partial M) \subset \partial N$ and $g(\operatorname{int} M) \subset \operatorname{int} N$.

Definition 1·3. If X is a topological space $\mu(X)$ will denote the number of its components.

Definition 1·4. A *link of n-components* is the oriented image in Euclidean 3-space, R^3, of an embedding $l: S_n \to R^3$ where $S_n = \bigcup_{i=1}^{n} S_i^1$, a disjoint union of n copies of the 1-sphere. For the purposes of this paper if $h: S_n \to S_n$ is an orientation preserving homeomorphism of S_n then $l(S_n)$ and $lh(S_n)$ are the same link. A *knot* is a link of one component.

Definition 1·5. Two links are said to be *equivalent*, $\equiv$, if they have ambient isotopic defining embeddings. Little distinction will be drawn between a link and its equivalence class.

Definition 1·6. If L is a link defined by $l: S_n \to R^3$ and $h: R^3 \to R^3$ is an orientation reversing homeomorphism denote by ρL the link defined by hl.

Definition 1·7. L is *amphiceiral* if $L \equiv \rho L$ (2).

For the purposes of this and the ensuing sections N denotes a compact, oriented 2-manifold such that every component of N has a non-empty boundary. The genus of N, $h(N)$ is the sum of the genera of its components. The genus of a connected surface N is $\frac{1}{2}(k(N)+1-\mu(\partial N))$ where $H_1(N)$ is free Abelian of rank $k(N)$.

If there exists an embedding $g: N \to R^3$ such that $g(\partial N) = L$ (∂N inherits an orientation from N) $g(N)$ is said to span L. Every link is spanned by some orientable surface (2).

Definition 1·8. The *genus* of a link, $h(L)$, is the minimum integer, m say, such that L is spanned by a surface of genus m.

Definition 1·9. The *degeneracy* of a link, $d(L)$, is the maximum integer, m say, such that L is spanned by a surface with m components.

Definition 1·10. If a link L is spanned by a surface, $g(N)$ say, such that $h(N) = 0$ and $\mu(L) = \mu(N) = n$ say then L is the *trivial link* and will be denoted by U_n.

Definition 1·11. Let L be a link and $b: I \times I \to R^3$ an embedding. b is said to be *compatible* with L if $b(I \times I) \cap L = b(I \times \partial I)$ and if the orientations inherited from L on $b(I \times \partial I)$ induce the same orientation on $b(I \times I)$. In this case the link

$$(L - b(I \times \partial I)) \cup b(\partial I \times I),$$

its orientations inherited from L, will be denoted by bL. It is clear that the map $b^T: I \times I \to R^3$ defined by $b^T(x,y) = b(y,x)$ is compatible with bL and that $b^T(bL) = L$. Compatibility implies that $|\mu(L) - \mu(bL)| = 1$.

Let L_1 and L_2 be links such that $L_1 \cap L_2 = \varnothing$. Put $L = L_1 \cup L_2$. It is possible to impose restrictions on L_1, L_2 and b as follows.

(A) $b(I \times 0) \subset L_1$ and $b(I \times 1) \subset L_2$.

(B) There exists a 2-plane $R^2 \subset R^3$ such that $R^2 \cap L = \varnothing$, L_1 and L_2 are separated by R^2 and $b(I \times I) \cap R^2$ is an arc of R^2.

Definition 1·12. If (A) holds define $L_1 +_b L_2$ to be bL.

If (B) holds define $L_1 \#_b L_2$ to be bL.

If K_1 and K_2 are knots then $K_1 \# K_2 = K_1 \#_b K_2$ is independent of the choice of b.

Definition 1·13. A *ribbon map* of N into R^3 is a map, g say, with no triple points satisfying: the doublepoint set consists of mutually disjoint arcs in N which may be paired (I_i, I_i') so that $g(I_i) = g(I_i')$, with I_i properly embedded in N and I_i' contained in int N, for all i in some finite indexing set. It is also assumed that the self-intersections of $g(N)$ at $g(I_i) = g(I_i')$ are transverse.

Definition 1·14. $g(N)$ will be called a *ribbon of type* N and $g(\partial N)$, denoted by $\partial(g(N))$, a *ribbon link of type* N. This generalizes the concept of ribbon disc and ribbon knot ((1), (2), p. 172, (3)).

In the following definition nB denotes the disjoint union of n copies of the 2-disc, i.e. $\bigcup_{i=1}^{n} B_i$.

Definition 1·15. $L \xrightarrow{r} L'$ if for some integer n there exists a ribbon map $g: nB \to R^3 - L$ such that $L' \equiv (\ldots((L +_{b_1} \partial B_1') +_{b_2} \partial B_2') \ldots) +_{b_n} \partial B_n'$ where $B_i' = g(B_i)$.

Definition 1·16. *L is ribbon equivalent to* L', denoted by $L \overset{r}{\equiv} L'$, if there exists a sequence of links, L_j $(j = 1, \ldots, m)$ say, such that $L_1 = L$, $L_m = L'$ and for $j = 1, \ldots, m-1$ either $L_j \xrightarrow{r} L_{j+1}$ or $L_{j+1} \xrightarrow{r} L_j$.

$\overset{r}{\equiv}$ is an equivalence relation on the set of links which preserves the number of components.

In what follows R^4 denotes Euclidean 4-space, a general point $\mathbf{w}$ having Cartesian coordinates (w_1, w_2, w_3, w_4). R^3 is the subspace $\{\mathbf{w} | w_1 = 0\}$ of R^4 and R^4_+ the subspace $\{\mathbf{w} | w_1 \geqslant 0\}$. The general surface, N, will have the same properties as before.

Let $g: N \to R^4_+$ be a proper simplicial embedding. Put $R^4_t = \{\mathbf{w} | w_1 = t\}$ and $R^4_{[t_1, t_2]} = \{\mathbf{w} | t_1 \leqslant w_1 \leqslant t_2\}$. Let $\gamma: R^4 \to R^3$ be the map defined by

$$\gamma(w_1, w_2, w_3, w_4) = (0, w_2, w_3, w_4)$$

and put $g(N)_t = R^4_t \cap g(N)$, $g(N)_{[t_1, t_2]} = R^4_{[t_1, t_2]} \cap g(N)$ and $L_t = \gamma(g(N)_t)$. It is now assumed (cf. (8)) that g has been modified to satisfy the following condition.

Condition 1·17. L_t is either a link, inheriting its orientation from $g(N)_{[t, \infty]}$ or a graph with a single vertex of order other than two. This order may only be zero or four.

Definition 1·18. Let X^g denote the subset of real numbers $\{x | L_x$ is not a link$\}$. X^g is a finite set. X^g will be called the *singular set* of $g(N)$ and $x \in X^g$ a *singular point*.

Let δ be a positive number such that δ is less than the difference between any two members of the set $\{x | x \in X^g \text{ or } x = 0\}$. It is easy to show that $L_{t_1} \equiv L_{t_2}$ if $[t_1, t_2] \cap X^g = \varnothing$ using the theorem of Zeeman and Hudson that l.u. isotopy implies ambient isotopy ((5), Theorem 2).

Put

$$X^g_m = \{x | x \in X^g \text{ and } L_x \text{ has a vertex of order zero}\}$$

and

$$X^g_c = \{x | x \in X^g \text{ and } L_x \text{ has a vertex of order four}\}.$$

It is clear that X^g_m and X^g_c partition X^g. The next step is to examine the relationship between $L_{x-\delta}$ and $L_{x+\delta}$ when $x \in X^g_m$ and $x \in X^g_c$.

Proposition 1·19. If $x \in X^g_m$ then the l.u. condition on $g(N)$ implies that

$$L_{x+\delta} \equiv L_{x-\delta} \cup k$$

where k has one component, is unknotted and is unlinked with $L_{x-\delta}$ (that is, k spans a disc in $R^3 - L_{x-\delta}$), or $L_{x-\delta} \equiv L_{x+\delta} \cup k$ with the same conditions. In the first case $g(N)_{[x-\delta, x+\delta]}$ is the union of $\mu(L_{x-\delta})$ annuli, each with a boundary in both $R^4_{x+\delta}$ and $R^4_{x-\delta}$, and a disc with boundary k. A similar situation (upside down) holds in the second case.

Proposition 1·20. The converse of Proposition 1·19 is true. That is, if L_a and L_b are links in R^4_a and R^4_b $(a > b)$ so that $\gamma(L_a) \equiv \gamma(L_b) \cup k$ where k spans a disc in $R^3 - \gamma(L_b)$, there is a surface in $R^4_{[b, a]}$ spanning them of the type described in Proposition 1·19.

Proposition 1·21. If $x \in X^g_c$ then, again using the l.u. condition on $g(N)$, it is possible to show that $L_{x+\delta} \equiv bL_{x-\delta}$ for some $b: I \times I \to R^3$, compatible with $L_{x-\delta}$. In this case

$g(N)_{[x-\delta, x+\delta]}$ is a union of annuli each with a boundary in $R^4_{x-\delta}$ and $R^4_{x+\delta}$ and a punctured disc with three boundaries meeting both $R^4_{x-\delta}$ and $R^4_{x+\delta}$.

Proposition 1·22. The converse of 1·21 is also true. In this case if L_c and L_d are links in R^4_c and R^4_d $(c < d)$ such that $\gamma(L_c) \equiv b\gamma(L_d)$ for some b then there is a surface in $R^4_{[c,d]}$ of the type described in 1·21 if $\mu(L_d) = \mu(L_c)+1$.

Proofs of these propositions are not given. Fox (2) and Murasugi (8, 9) state some of the results involved and it is easy to supply proofs.

It is necessary to have a further partition of X^g.

Definition 1·23. Define:

X^g_i to be $\{x | x \in X^g \text{ and } \mu(L_{x+\delta}) = \mu(L_{x-\delta})+1\}$,

X^g_d to be $\{x | x \in X^g \text{ and } \mu(L_{x+\delta}) = \mu(L_{x-\delta})-1\}$,

$X^g_{d_1}$ to be $\{x | x \in X^g_d \cap X^g_c \text{ and } h(g(N)_{[0,x-\delta]}) = h(g(N)_{[0,x+\delta]})\}$,

$X^g_{d_2}$ to be $\{x | x \in X^g_{d_1} \text{ and } \mu(g(N)_{[0,x-\delta]} \cup R_3) = \mu(g(N)_{[0,x+\delta]} \cup R_3)+1\}$,

and $\mathfrak{X}^g$ to be the set of subsets $X^g_m, X^g_c, X^g_i, X^g_d, X^g_{d_1}, X^g_{d_2}$ of X^g.

It follows from Propositions 1·19 and 1·21 that X^g_i and X^g_d partition X^g.

Definition 1·24. Define the *type* of $x \in X^g$ to be the subset of $\mathfrak{X}^g$ to whose members it belongs.

THEOREM 1·25. *Given a link L in R^3 there exists a l.u. proper embedding $g: N \to R^4_+$ such that $g(\partial N) = L$ if and only if $L \overset{r}{\equiv} L'$ where L' is a ribbon link of type N.*

The proof of Theorem 1·25 depends on lemmas to be stated. The vertices corresponding to points of X^g can be regarded as handles in a handle decomposition of $N \bmod \partial N$. In this light the points of $X^g_m \cap X^g_i$ correspond to 0-handles, the points of $X^g_m \cap X^g_d$ to 2-handles and those of X^g_c to 1-handles. It is possible to carry this analogy further by 'swapping' handles, the object of the next two lemmas.

LEMMA 1·26. *If $x_1 \in X^g_m \cap X^g_i$ and $x_2 \in X^g$ are singular points of $g: N \to R^4_+$ such that $x_2 < x_1$ and $(x_2, x_1) \cap X^g = \varnothing$ then there exists $g': N \to R^4_+$ such that*:

(1) *$g(N)_{[0,x_2-\delta]} = g'(N)_{[0,x_2-\delta]}$ and $g(N)_{[x_1+\delta,\infty]} = g'(N)_{[x_1+\delta,\infty]}$.*

(2) *$X^{g'}$ has two points $y_1 < y_2$ in $[x_2-\delta, x_1+\delta]$ and y_i has the same type as x_i for $i = 1, 2$.*

Proof. If $x_2 \in X^g_c$ then $L_{x_1+\delta} \equiv (bL_{x_2-\delta}) \cup k$, where k spans a disc in $R^3 - bL_{x_2-\delta}$. Now $(bL_{x_2-\delta}) \cup k \equiv b(L_{x_2-\delta} \cup k')$ where k' spans a disc in $R^3 - (L_{x_2-\delta} \cup b(I \times I))$. So by propositions 1·20 and 1·22 $g(N)_{[x_2-\delta, x_1+\delta]}$ can be replaced by a surface with the required properties.

If $x_2 \in X_m$ a similar argument can be used.

LEMMA 1·27. *If X^g is the singular subset of $g: N \to R^4_+$ and $x_1 \in X^g_{d_1}$, $x_2 \in X^g - (X^g_m \cap X^g_i)$ are such that $x_1 > x_2$ and $(x_2, x_1) \cap X^g = \varnothing$ then there exists $g': N \to R^4_+$ such that g' satisfies conditions* (1) *and* (2) *of Lemma* 1·26.

Proof. If $x_2 \in X^g_c$ then $L_{x_1+\delta} \equiv b_2(b_1 L_{x_2-\delta})$. It is possible to show that there exists $b'_2: I \times I \to R^3$, isotopic to b_2 (as embeddings compatible with $b_1 L_{x_2-\delta}$), so that $b'_2(I \times I) \cap b_1(I \times I) = \varnothing$.

Then $b_2'(b_1 L_{x_2-\delta}) \equiv b_2(b_1 L_{x_2-\delta})$. The isotopy can be constructed by sliding $b_2(I \times \partial I)$ off $b_1(\partial I \times I)$ if necessary and pushing intersections of the bands off through $b_1(I \times 0)$. $L_{x_1+\delta}$ may now be regarded as $b_1(b_2' L_{x_2-\delta})$ and $g(N)_{[x_2-\delta, x_1+\delta]}$ can be replaced by a surface with the required properties, using two applications of Proposition 1·22. If $x_2 \in X_m^g \cap X_d^g$ then 2·13 follows from Lemma 1·26 (upside down).

It is possible to arrange in both Lemmas 1·26 and 1·27 that the embeddings g and g' are isotopic but this is unnecessary for the purposes of this paper.

LEMMA 1·28. *If L is a link in R^3 there exists a proper l.u. embedding $g: N \to R^4_+$, satisfying Condition* 1·17 *such that $X_m^g \cap X_i^g = \varnothing$ and $g(\partial N) = L \Leftrightarrow L$ is a ribbon link of type N.*

Proof. $\Rightarrow$: X^g is a finite set of points, x_i $(i = 1, \ldots, m)$ say, such that $x_i > x_{i+1}$ for all i. The proof is by induction on i. Assume that $L_{x_i+\delta}$ spans a ribbon, R_i say, of type $g(N)_{[x_i+\delta, \infty]}$ in R^3. If $x_i \in X_m^g$ then by the hypothesis of the lemma $x_i \in X_d^g$ and therefore $L_{x_i-\delta} \equiv L_{x_i+\delta} \cup k$ where k spans a disc in $R^3 - L_{x_i+\delta}$. This disc is ambient isotopic in $R^3 - L_{x_i+\delta}$ to a disc D with boundary k' say not meeting R_i. Now $L_{x_i-\delta} \equiv L_{x_i+\delta} \cup k'$ and $L_{x_i-\delta}$ spans $R_i \cup D$, which is a ribbon of type $g(N)_{[x_i-\delta, \infty]}$. The existence of $R_{i+1} = R_i \cup D$ spanning $L_{x_{i+1}+\delta} \equiv L_{x_i-\delta}$ proves the inductive step in this case.

If $x_i \in X_c^g$ then $L_{x_i-\delta} \equiv bL_{x_i+\delta}$ for some $b: I \times I \to R^3$. It is possible to isotop b so that $b(I \times I)$ meets R_i transversely and $b((\text{int}\, I) \times I) \cap (R_i^* \cup \partial R_i) = \varnothing$ where R_i^* is the doublepoint set of R_i. Then $b(I \times I) \cup R_i$ is a ribbon of type $g(N)_{[x_i-\delta, \infty]}$($b(I \times I)$ is a 1-handle of the same type as that of $g(N)$ at x_i). Putting $R_{i+1} = b(I \times I) \cup R_i$ in this case completes the induction because $L_{x_1+\delta} \equiv \varnothing$ and $L_{x_m-\delta} \equiv L$.

$\Leftarrow$: Let $g_1: L \times I \to R^4_{[0,1]}$ be the proper embedding such that $g_1(L \times 0) = L$ and $\gamma(g_1(L \times I)) \subset L$. $g_1(L \times I)$ spans a ribbon, R say, in R^4_1 defined by $g_2: N \to R^4_1$. Let N be subdivided so that the doublepoint set of g_2 is a subcomplex. Let D_i be the second derived neighbourhood of I_i' in N for all i (using the notation of Definition 1·13. Then $g_2|(N - \bigcup_i D_i)$ is an embedding as is $g_2|(\bigcup_i D_i)$. Let v_i be a point of R^4_+ above (with respect to w_1) $g_2(D_i)$. Then $g_1(I \times I) \cup g_2(N - \bigcup_i D_i) \cup (\bigcup_i D_i')$ where $D_i' = v_i g_2(\partial D_i)$ is a l.u. properly embedded copy of N in R^4_+. It is not difficult to modify this to meet the conditions of the lemma.

Remark 1·29. For every link L there is a surface N and a l.u. proper embedding $g: N \to R^4_+$ such that $g(\partial N) = L$. This well known result follows from the remark after Definition 1·7 and Lemma 1·28.

Definition 1·30. Two links L and L' are *cobordant* if for some N there exists a l.u. proper embedding $g: N \to R^4_+$ such that $L_0 = L$, $L_t = L'$ and $g(N)$ is a regular neighbourhood of $g(N)_{[t,\infty]}$ in $g(N)$.

It is easy to check that cobordism is an equivalence relation on the set of links. The next step is to show that this equivalence relation is identical to ribbon equivalence.

LEMMA 1·31. *If L and L' are links such that $L \xrightarrow{r} L'$ then L is cobordant to L'.*

Proof. Using the notation adopted in Lemma 1·15 $L' = (L \cup \bigcup_j \partial B_j') - \bigcup_j J_j$ where

$B'_j = g(B_j)$ and g is a ribbon map of $\bigcup_j B_j s$ being disjoint 2-discs. J_j is $L \cap B'_j$, an arc of B'_j.

Let g_1 and g_2 be proper embeddings,

$$g_1: L' \times I \to R^4_{[-1,0]}, \quad g_2: L \times I \to R^4_{[0,1]}$$

such that $\gamma g_1(L' \times s) = L'$ and $\gamma g_2(L \times s) = L$ for all $s \in I$. Let D_i, v_i and $D'_i (v_i \in \operatorname{int} R^4_{[0,1]})$ be defined as in the proof of 1·28 for the ribbon B'_j in R^3. By Remark 1·29 $g_2(L \times I) \cap R^4_1$ spans a l.u. properly embedded surface, N' say, in $R^4_{[1,\infty]}$. Then the embedded surface

$$H(g_1(L' \times I) \cup (\bigcup_j B'_j - \bigcup_j D_i) \cup (\bigcup_j D'_i) \cup g_2(L \times I) \cup N')$$

where $H: R^4 \to R^4$ is defined by

$$H(w_1, w_2, w_3, w_4) = (w_1 + 1, w_2, w_3, w_4)$$

satisfies the criteria ($t = 2$) of Definition 1·30 and L is cobordant to L'.

Lemma 1·32. *If two links L and L' are cobordant then $L \overset{r}{\equiv} L'$.*

Proof. Using the notation of Definition 1·30 let N' be $g^{-1}(R^4_{[0,t]})$, a union of disjoint annuli. Put $g' = g|N'$, and modify g' to satisfy Condition 1·17.

By Lemmas 1·26 and 1·27 g' may be modified further so that

$$x_1 \in X^{g'}_m \cap X^{g'}_i, \quad x_2 \in X^{g'}_c \cap X^{g'}_d, \quad x_3 \in X^{g'}_c \cap X^{g'}_i \quad \text{and} \quad x_4 \in X^{g'}_m \cap X^{g'}_d$$

implies that $x_1 < x_2 < \frac{1}{2}t < x_3 < x_4$. (Note that for g', $X^{g'}_c \cap X^{g'}_d = X^{g'}_{d_1} = X^{g'}_{d_2}$.) It is now possible to show, using a similar induction to that in the proof of Lemma 1·28, that $L_t \overset{r}{\to} L_{\frac{1}{2}t}$ and $L_0 \overset{r}{\to} L_{\frac{1}{2}t}$. Thus $L \overset{r}{\equiv} L'$. From Lemmas 1·31 and 1·32 the following corollary can be deduced.

Corollary 1·33. *$L \overset{r}{\equiv} L' \Leftrightarrow L$ is cobordant to L'.*

Proof of Theorem 1·25. If $g: N \to R^4_+$ is a l.u. proper embedding it may be modified, keeping $g(\partial N)$ fixed, to satisfy Condition 1·17. Again using Lemmas 1·26 and 1·27 g may be modified, keeping $g(\partial N)$ fixed so that

$$x_1 \in X^g_m \cap X^g_i, \quad x_2 \in X^g_{d_2}, \quad x_3 \in X^g - ((X^g_m \cap X^g_i) \cup X^g_{d_2})$$

implies that $x_1 < x_2 < 1 < x_3$. Then L_0 is cobordant to L_1 and $g(N)_{[1,\infty]}$ satisfies the hypothesis of Lemma 1·28. So $L = L_0 \overset{r}{\equiv} L_1$ and L_1 is a ribbon link of type N.

This proves one implication of Theorem 1·25. The converse follows directly from Lemmas 1·28 and 1·33.

2. *Algebra.* In what follows Z^n with any subscript will denote a free Abelian group of rank n. Z will denote the integers, C the complex numbers and P the ring of integral polynomials of a single variable t.

Definition 2·1. A *bilinear form* is a triple (l, Z^n, R) where R is a commutative ring with unit and l is a bilinear mapping $l: Z^n \times Z^n \to R$. Where no ambiguity arises the form will be denoted by l.

Two forms (l_1, Z^n_1, R) and (l_2, Z^n_2, R) are equal, denoted by $l_1 = l_2$, if there is an isomorphism $i: Z^n_1 \to Z^n_2$ such that $l_2(i(x), i(y)) = l_1(x, y)$ for all x, y in Z^n_1.

If Z^m is a subgroup of Z^n and l a bilinear form on Z^n the induced form on Z^m will be denoted by $l|Z^m$.

Definition 2·2. A bilinear form (l, Z^n, R) has a well-defined *determinant*, denoted by $\det(l)$, the determinant of any matrix of l computed from some basis of Z^n. The *rank* of l is the maximum integer, m say, such that there exists $Z^m \subset Z^n$ with $\det(l|Z^m) \neq 0$. The *nullity* of l is defined to be $n-m$ and is denoted by $n(l)$.

Definition 2·3. Let (l, Z^n, Z) be a bilinear form and w a complex number such that $|w| = 1$ and $w \neq 1$. Then $({}^w l, Z^n, C)$ denotes the form given by

$$ {}^w l(x,y) = \tfrac{1}{2}(1-\overline{w})\{l(x,y) - wl(y,x)\} $$

and is Hermitian. (l^t, Z^n, P) is the form given by $l^t(x,y) = l(x,y) - tl(y,x)$.

Let $w_p = \exp(2m\pi i/2m+1)$ for all odd prime p where $p = 2m+1$ and let $w_2 = -1$. Then $\sigma_p(l)$ will denote the signature of ${}^{w_p}l$ for all prime p and $n_p(l)$ its nullity. These are integral invariants of l.

Definition 2·4. (l, Z^n, Z) is *proper* if the polynomial $\det(l^t)$ evaluated at $t = 1$ has modulus 1.

LEMMA 2·5. *If (l, Z^n, Z) is proper then $n_p(l) = 0$ for all prime p and $\lim_{i\to\infty} \sigma_{p_i}(l) = \sigma_2(l)$, where p_i is the sequence of primes in ascending order.*

Proof. Let $f(t) = \det(l^t)$. Then $|f(1)| = 1$. If, for some prime p, $n_p(l) > 0$ then $\det({}^{w_p}l) = 0$ for that p. This implies that w_p is a root of $f(t) = 0$. Put

$$ f_p(t) = f(t).f(t^2)\dots f(t^{p-1}). $$

Then $|f_p(1)| = 1$ (*). $(w_p)^r$ is a primitive root of unity for all $r = 1, \dots, p-1$. Thus $(1+t+t^2\dots+t^{p-1})$ is a factor of $f_p(t)$ which contradicts (*). The proof of the second statement of the lemma requires only simple analysis.

There now follows a definition of an equivalence relation between bilinear forms which is important in the context of later sections.

Definition 2·6. Given two bilinear forms (l_1, Z^n, Z) and (l_2, Z^m, Z), with $m > n$ and $m = n + 2r$ for some integer r, write $l_1 \to l_2$ if there exists a decomposition

$$ Z^m \cong Z_1^n \oplus Z_1^r \oplus Z_2^r $$

such that:

(*a*) $l_2|Z_1^n = l_1$,

(*b*) $l_2|Z_1^r \oplus Z_2^r$ is proper,

(*c*) $l_2|Z_2^r$ is the zero form, and

(*d*) $l_2(x,y) = l_2(y,x) = 0$ if $x \in Z_1^n$ and $y \in Z_2^r$.

Two forms (l, Z^n, Z) and (l', Z^m, Z) are equivalent, denoted by $l \equiv l'$, if there exists a sequence of forms $l = l_0, l_1, \dots, l_t = l'$ such that for all $i = 1, \dots, t$ either $l_{i-1} \to l_i$ or $l_i \to l_{i-1}$. This is clearly an equivalence relation and it is easy to prove the following lemma.

LEMMA 2·7. *If $l \equiv l'$ then $\sigma_p(l) = \sigma_p(l')$ and $n_p(l) = n_p(l')$ for all prime p.*

Definition 2·8. (l, Z^{m+1}, Z) is an *extension* of (l', Z^m, Z) if there is a decomposition $Z^{m+1} \cong Z_1^m \oplus Z^1$ such that $l|Z_1^m = l'$.

LEMMA 2·9. *If l is an extension of l' then for all prime p either*

$$|\sigma_p(l) - \sigma_p(l')| = 1 \quad \text{and} \quad n_p(l) = n_p(l'),$$

or

$$|n_p(l) - n_p(l')| = 1 \quad \text{and} \quad \sigma_p(l) = \sigma_p(l').$$

Definition 2·10. If (l_1, Z^m, Z) and (l_2, Z^n, Z) are bilinear forms then $(l_1 \oplus l_2, Z^m \oplus Z^n, Z)$ is defined by:

(a) $l_1 \oplus l_2 | Z^m = l_1$ and $l_1 \oplus l_2 | Z^n = l_2$,

(b) $l_1 \oplus l_2(x, y) = l_1 \oplus l_2(y, x) = 0$ if $x \in Z^m$ and $y \in Z^n$.

LEMMA 2·11. *If $l = l_1 \oplus l_2$ then*

$$\left.\begin{aligned} \sigma_p(l) &= \sigma_p(l_1) + \sigma_p(l_2) \\ \text{and} \quad n_p(l) &= n_p(l_1) + n_p(l_2) \end{aligned}\right\} \text{ for all prime } p.$$

The proofs of Lemmas 2·9 and 2·11 are straightforward, requiring only simple manipulation.

The rest of this section is devoted to definitions and discussion of algebraic invariants of link type. These invariants generalize the signature and nullity of links defined by Murasugi (8).

Let $g: N \to R^3$ be an embedding of a connected surface spanning a link L. N is orientable and can be identified with the subset $N \times 0$ of $N \times I$ with $N \times I$ oriented suitably. There exists an embedding $G: N \times I \to R^3$ such that $G(x, 0) = g(x)$ for all $x \in N$. G is unique up to ambient isotopy (rel $N \times 0$). This may be proved by the collaring techniques introduced in chapter 5 of (12).

Let $i: g(N) \to R^3 - g(N)$ be the embedding defined by $i(g(x)) = G(x, 1)$.

Definition 2·12. Let $\bar{x}, \bar{y} \in H_1(g(N))$ be represented by cycles x and y respectively on $g(N)$. Then define $(l_g, H_1(g(N)), Z)$ by $l_g(\bar{x}, \bar{y}) = \langle x, i(y) \rangle$, where if a and b are cycles in R^3 $\langle a, b \rangle$ denotes the algebraic linking number of a and b. lg is a well-defined bilinear form on $H_1(g(N))$ and will be referred to as a form for the link L.

LEMMA 2·13. *If l_g and $l_{g'}$ are forms for equivalent links L and L' corresponding to embeddings of connected surfaces spanning L and L' then $l_g \equiv l_{g'}$.*

No proof of this will be given. A proof using projections of links is summarised by Murasugi in (8). A proof using the above definitions is given in (11).

Definition 2·14. If L is a link spanned by a connected surface $g(N)$ define

$$\left.\begin{aligned} &\sigma_p(L) \text{ to be } \sigma_p(l_g) \\ \text{and} \quad &n_p(L) \text{ to be } n_p(l_g) + 1 \end{aligned}\right\} \text{ for all prime } p.$$

LEMMA 2·15. *$\sigma_p(L)$ and $n_p(L)$ are well defined. This is a consequence of Lemmas* 2·7 *and* 2·13.

LEMMA 2·16. *If K is a knot then $\sigma_p(K)$ is even and $n_p(K) = 1$ for all prime p. Furthermore, $\lim_{i \to \infty} (\sigma_{p_i}(K)) \to \sigma_2(K)$.*

Proof. If $g(N)$ spans K then $H_1(g(N))$ has even rank. The form $l_g(\bar{x},\bar{y})-l_g(\bar{y},\bar{x})$ is the intersection form for $H_1(g(N))$ and so is unimodular. Thus l_g is proper and the results follow from Lemma 2·5.

LEMMA 2·17. *$n_p(L) \geqslant d(L)$ for any link L and all prime p.*

Proof. Let $d(L) = m+1$. Then L is spanned by $g(N)$ where $g(N)$ has $m+1$ components. A connected surface $g'(N')$ spanning L can be constructed from $g(N)$ by piping components of $g(N)$ together using m pipes. $H_1(g'(N'))$ has a direct summand Z^m generated by simple closed curves, one on each pipe. Then $l_{g'}(\bar{x},\bar{y}) = 0$ if $\bar{x}\in Z^m$ or $\bar{y}\in Z^m$. The result follows immediately.

The following three lemmas are proved by Murasugi in (8), using projections. They may be proved directly from the above definitions by manipulation of the relevant spanning surfaces.

LEMMA 2·18. *If $L' \equiv \rho L$ then there exist forms l_g and $l_{g'}$ for L and L' respectively such that $l_{g'} = -l_g$.*

COROLLARY 2·19.
$$\left.\begin{aligned}\sigma_p(L) &= -\sigma_p(\rho L)\\ \text{and}\quad n_p(L) &= n_p(\rho L)\end{aligned}\right\}\ \textit{for all links } L \textit{ and all prime } p.$$

LEMMA 2·20. *If $L = L_1 \#_b L_2$ there exist forms l_g, l_{g_1} and l_{g_2} for L, L_1 and L_2 respectively such that $l_g = l_{g_1}\oplus l_{g_2}$.*

COROLLARY 2·21. *For all prime p, any two links L_1 and L_2 and any allowable b*
$$\sigma_p(L_1 \#_b L_2) = \sigma_p(L_1)+\sigma_p(L_2)$$
and
$$n_p(L_1 \#_b L_2) = n_p(L_1)+n_p(L_2)-1.$$

LEMMA 2·22. *If $L' = bL$ for some link L and compatible b then there exist forms l_g and $l_{g'}$ for L and L' respectively such that $l_{g'}$ is an extension of l_g.*

COROLLARY 2·33. *With L and L' as in Lemma 2·22 it follows from 2·9 that for all prime p*

either $\quad |\sigma_p(L)-\sigma_p(L')| = 1 \quad$ *and* $\quad n_p(L) = n_p(L')$,

or $\quad |n_p(L)-n_p(L')| = 1 \quad$ *and* $\quad \sigma_p(L) = \sigma_p(L')$.

COROLLARY 2·24. *It can be deduced at once from Corollary 2·23 and Lemma 2·16 that for any link L and prime p, $n_p(L) \leqslant \mu(L)$.*

The following lemma can be deduced from a result of Murasugi (8). It is also proved in (11).

LEMMA 2·25. *If L and L' are links such that $L \overset{*}{\to} L'$ then there exists form l_g and $l_{g'}$ for L and L' respectively such that $l_g \to l_{g'}$.*

COROLLARY 2·26. *It follows from Lemma 2·25 and Corollary 1·33 that if two links L and L' are corbordant then for all prime p $\sigma_p(L) = \sigma_p(L')$ and $n_p(L) = n_p(L')$.*

The main result of this section is the following generalization of a theorem of Murasugi (8).

Theorem 2·27. *If $g\colon N \to R^4_+$ is a proper locally unknotted embedding and $L = g(\partial N)$ then for all prime p.*

$$|\sigma_p(L)| + |n_p(L) - \mu(N)| \leqslant \mu(L) - \mu(N) + 2h(N) \quad (*).$$

This is a consequence of Theorem 1·25 and the following lemma.

Lemma 2·28. *If L spans a ribbon of type N then the inequality (*) holds.*

Proof. Let $g\colon N \to R^3$ be the ribbon map such that $\partial(g(N)) = L$. It is possible to embed properly in N arcs J_j $(j = 1, \dots, t)$ where $t = \mu(L) - \mu(N) + 2h(N)$ with open regular neighbourhoods $N(J_j)$ so that $N - \bigcup_j N(J_j)$ is the disjoint union of $\mu(N)$ discs. This is proved by induction on t. It is possible to ensure at the same time that $\bigcup_j J_j \cap \bigcup_i I'_i = \varnothing$ (Definition 1·13) and that the intersections of $\bigcup_j J_j$ with $\bigcup_i I_i$ are transverse. This condition implies that $g|(N - \bigcup_j N(J_j))$ is a ribbon map. Furthermore, the closure of $g(N(J_j))$ is the image of a map $b_j\colon I \times I \to R^3$ compatible with

$$b_{j-1}(b_{j-2}(\dots(b_1 L)\dots)) \quad \text{for each} \quad j = 1, \dots, t.$$

Now $\partial(g(N) - \bigcup_j N(J_j))$ is $b_t(b_{t-1}(\dots(b_1 L)\dots))$, L' say. By Corollary 2·23

$$|\sigma_p(L) - \sigma_p(L')| + |n_p(L') - n_p(L')| \leqslant t \quad \text{for all} \quad p.$$

L' is, by definition, ribbon equivalent to $U_{\mu(N)}$ for which

$$\sigma_p(U_{\mu(N)}) = 0 \quad \text{and} \quad n_p(U_{\mu(N)}) = \mu(N).$$

By Corollary 2·26 $\sigma_p(L') = 0$ and $n_p(L') = \mu(N)$. Therefore

$$|\sigma_p(L)| + |n_p(L) - \mu(N)| \leqslant t = \mu(L) - \mu(N) + 2h(N).$$

3. *Computation.* This section concerns the computation of σ_p and n_p for certain types of link. The first result is useful in the study of certain properties of 4-manifolds and its use is illustrated in section 4.

Definition 3·1. Let L be a link, K one of its components and $h\colon S^1 \times I \to R^3$ an embedding such that $h(S^1 \times 0) = K = h(S^1 \times I) \cap L$. Let $K_1 = h(S^1 \times \frac{1}{2})$ and $K_2 = h(S^1 \times 1)$ oriented in 'opposite directions' so that K_2 and K induce different orientations on $h(S^1 \times I)$. Then the link $L \cup K_1 \cup K_2$ is a (n, L, K)-pair where $n = \langle K_2, L\rangle$ (2·12).

Theorem 3·2. *With the notation of* 3·1 *and $n \neq 0$ put $L' = L \cup K_1 \cup K_2$; then if $p|n$ $\sigma_p(L') = \sigma_p(L)$ and $n_p(L') = n_p(L) + 2$.*

Proof. Let $g(N)$ be a connected surface spanning L and let $A = h(S^1 \times [\frac{1}{2}, I])$. Then it can be assumed without loss of generality that A intersects $g(N)$ in m ribbon intersections where $m = |n|$ (Figure 1). It is possible to use A and $g(N)$ to create a connected surface $g'(N')$ spanning L' (Figure 2). It can be seen that $k(N') = k(N) + 2m$. It is possible to choose a regular neighbourhood T of K containing A such that $g'(N')$ and $g(N)$ coincide outside T and

1. $H_1(g(N))$ is generated by elements $\bar{c}_j$ $(j = 1, \dots, k(N))$ represented by cycles c_j on N in the complement of T;

2. The cycles a_i $(i = 1, \ldots, m)$ and b_i $(i = 1, \ldots, m)$ $(b_{m+1} = b_2)$ of $g'(N')$ indicated in Figure 2 are contained in T. Then the totality of elements $\bar{c}_j, \bar{a}_i, \bar{b}_i \in H_1(g'(N'))$ represented by c_j, a_i and b_i respectively, each with an orientation form a basis for $H_1(g'(N'))$. Let $Z^{k(N)}$ be the direct summand of $H_1(g'(N'))$ generated by $\bar{c}_j$ $(j = 1, \ldots, k(N))$ and Z^{2m} the direct summand generated by $\bar{a}_i$ $(i = 1, \ldots, m)$ and $\bar{b}_i (i = 1, \ldots, m)$. Then clearly $l_{g'}|Z^{k(N)} = l_g$ and $l'_g(x, y) = l_{g'}(y, x) = 0$ if $x \in Z^{k(N)}$ and $y \in Z^{2m}$. It can be checked from Figure 2 that (with appropriate conventions for orientation and linking number and with $b_0 = b_m$)

$$\left.\begin{array}{ll} l_{g'}(\bar{a}_i, \bar{b}_j) = \delta_{ij}. & l_{g'}(\bar{b}_{i-1}, \bar{a}_j) = -\delta_{ij}, \\ l_{g'}(\bar{a}_i, \bar{a}_j) = 0, & l(\bar{b}_i, \bar{b}_j) = 0. \end{array}\right\} \quad \text{for all} \quad i, j = 1, \ldots, m.$$

These formulae define $l_{g'}|Z^{2m}$ and it is easy to check that $n_p(l_{g'}|Z^{2m}) \geqslant 2$ if $p|n$. This proves that $n_p(L') \geqslant n_p(L) + 2$.

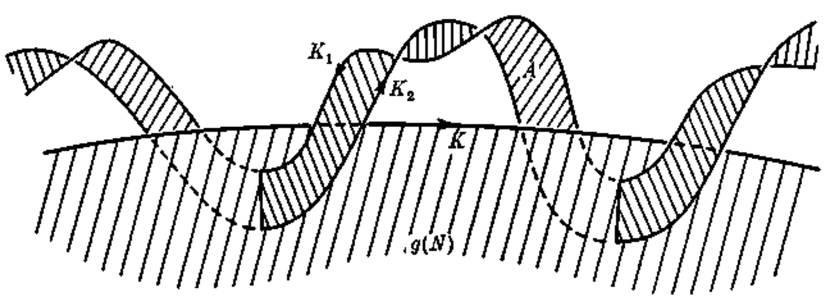

Fig. 1

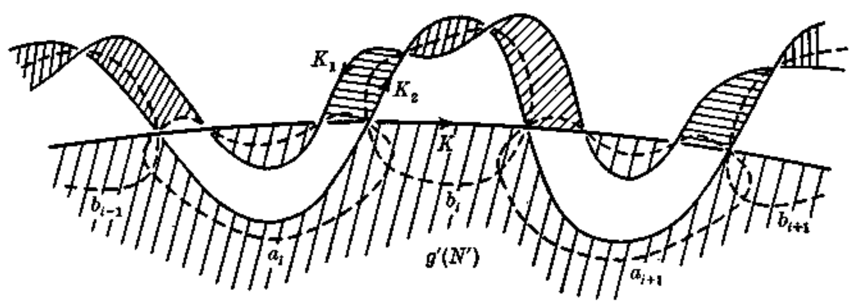

Fig. 2

If $L^* = L \cup U_1$ where U_1 is spanned by a disc in $R^3 - L$ it is clear that there is an embedding $b: I \times I \to R^3$ compatible with L^* (with ends on U_1) such that $b(L^*) = L'$. $n_p(L^*) = n_p(L) + 1$ and $\sigma_p(L^*) = \sigma_p(L)$. By Corollary 2·23 $n_p(L') \leqslant n_p(L) + 2$. Therefore $n_p(L') = n_p(L) + 2$ and, again by 2·23,

$$\sigma_p(L') = \sigma_p(L^*) = \sigma_p(L).$$

Remark. It is possible to show that if $n = 0$ then for all p, $n_p(L') = n_p(L) + 2$ and $\sigma_p(L) = \sigma_p(L')$. The proof is similar to that given above.

Definition 3·3. Let $L_{q,r}$ be the link with the projection shown in Figure 3. The shaded regions of the projection indicate a Seifert surface (2), $g(N)$ say, spanning $L_{q,r}$.

LEMMA 3·4. $\sigma_p(L_{q,r}) \geqslant q+r-1$ *for all prime* p

and $n_p(L_{q,r}) \geqslant 2$ *if* $p|q$ *or* $p|r$.

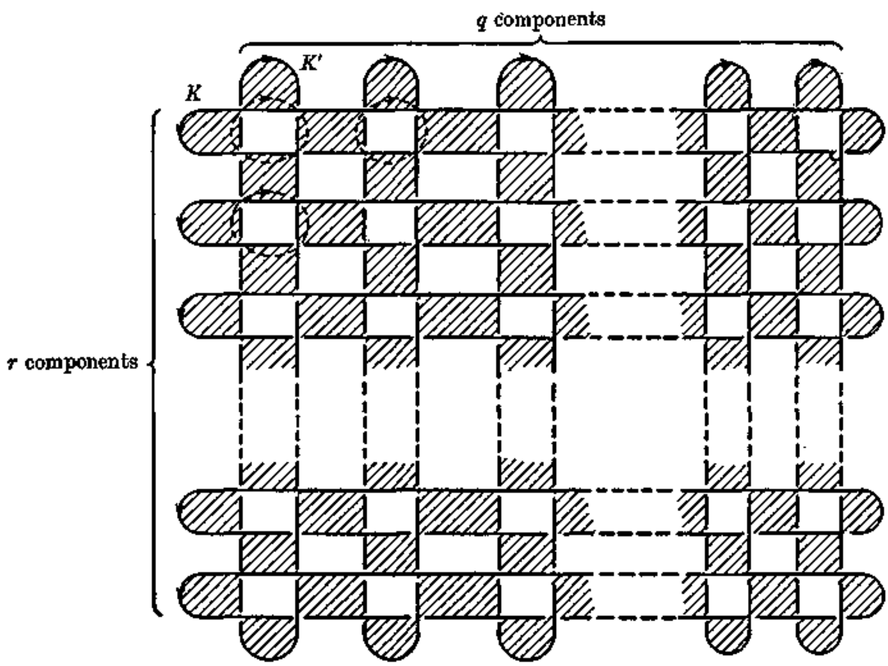

Fig. 3

Proof. Let $\bar{a}_i$ $(i = 1, \ldots, qr) \in H_1(g(N))$ be those elements represented by the dotted curves in Figure 3. Then $l_g(\bar{a}_i, \bar{a}_j) = \delta_{ij}$ (with an appropriate convention for linking number). Also $\bar{a}_i$ $(i = 1, \ldots, qr)$ generate a direct summand Z^{qr} of $H_1(g(N))$ and $k(N) = 2qr-q-r+1$. For all p, $\sigma_p(l_g|Z^{qr}) = qr$. This implies that $\sigma_p(L) \geqslant q+r-1$. It can be shown using this projection of $L_{q,r}$ and an induction argument that det (l_g^t) contains a factor of the form $(t^q-1)(t^r-1)$. This is sufficient to show that $n_p(L_{q,r}) \geqslant 2$ if $p|q$ or $p|r$.

This section is concluded with one further result which is easily computed.

LEMMA 3·5. *Let* K_n *be the torus knot determined by the pair of integers* $(2, 2n+1)$, *that is* $(2n+1)_1$ *in Reidemeister's table* (10). *Then for odd primes* $|\sigma_p(K_n)| = 2(n-[2n+1/2p])$ (*where* [ ] *means 'integral part of'*) *and* $|\sigma_2(K_n)| = 2n$. *The sign depends on conventions.*

Remark. By 3·5 $|p_i\sigma_{p_j}(K_1) - \sigma_{p_j}(K_{p_i})| = 2$ if $i = j$, 0 if $i < j$.

4. *Corollaries.* This section is concerned with two applications of the results of section 2.

Definition 4·1. Let $\mathfrak{K}$ be the set of cobordism classes of knots. Then it is well known (2) that $\mathfrak{K}$ forms a group under the addition operation.

Definition 4·2. Let Z^∞ be the abelian group of infinite sequences of integers, $\{n_i\}$ say, such that $\lim_{i\to\infty} n_i$ exists and is finite. Addition is defined by $\{n_i\}+\{m_i\} = \{n_i+m_i\}$.

THEOREM 4·3. *There exists an epimorphism $h_\sigma: \mathfrak{K}\to Z^\infty$.*

Proof. Let $h_\sigma(\{K\}) = \{\frac{1}{2}\sigma_{p_i}(K)\}$ where p_i is as before the sequence of primes in ascending order. This is well defined by 2·5 and 2·6 and is a homomorphism by Corollary 2·21. The remark at the end of section 3 shows that $\{\sigma_{p_i}(K_n)\}$ for $n = 1, \ldots, \infty$ form a generating set for Z^∞ and so h is an epimorphism.

The inequality of Theorem 2·27 has several simplifications in particular circumstances. The one of use in the second corollary follows.

Definition 4·4. L is a *weakly slice link* (8) if there exists a locally unknotted proper embedding $g: N\to R^4_+$ such that $g(\partial N) = L$ and $h(N) = 0$. Then if L is weakly slice by 2·27 it follows that for all prime p $|\sigma_p(L)|+n_p(L) \leqslant \mu(L)$.

The problems to which the following results give partial answers were posed by Wall and published in ((7), problem 8). Theorem 4·15 is an improvement of a result of Kervaire and Milnor's (6).

Let Q denote the 4-manifold $S^2\times S^2$ and let x be a point of S^2. $\bar{a}$ and $\bar{b}$ will denote the generators of $H_2(Q)$ represented by $S^2\times x$ and $x\times S^2$ respectively.

THEOREM 4·5. *Given integers q and r there does not exist a locally unknotted embedding of S^2 in Q whose image represents $q\bar{a}+r\bar{b}$ in $H_2(Q)$ if* g.c.d. $(q, r) > 1$.

Proof. It can be assumed without loss of generality that $q > 0$ and $r > 0$. Let D be a disc in S^2 such that $x\in \operatorname{int} D$. Then $(S^2\times D)\cup(D\times S^2)$, Q' say, is a regular neighbourhood of $(S^2\times x)\cup(x\times S^2)$ in Q and $(S^2\times x)\cup(x\times S^2)$ is a spine of Q. Let $g: S^2\to Q$ be a locally unknotted embedding such that $g(S^2)$ represents $q\bar{a}+r\bar{b}$ in $H_2(Q)$. g may be modified isotopically so that $g(S^2)\cap(D\times D) = \varnothing$ and the intersections of $g(S^2)$ with $(S^2-D)\times x$ and $x\times(S^2-D)$ are transverse (with respect to the product structure on Q'). Let $r+2n_1$ and $q+2n_2$ be the number of points of

$$g(S^2)\cap((S^2-D)\times x), \quad g(S)\cap(x\times(S^2-D))$$

respectively. $L_{q,r}$ again denotes the link described in Figure 1 with named components K and K'. Let $L_{q,r}$ be the link obtained by adding $n_1(q, L_{q,r}, K)$-pairs and n_2 $(r, L_{q,r}, K')$-pairs to $L_{q,r}$ in such a way that the embedded annuli used to define each pair intersect mutually in K, K' or not at all.

$Q-(\operatorname{int} Q')$ is a 4-ball and $\partial Q'$ a 3-sphere. It is easily shown that the link $\partial Q'\cap g(S^2)$, with orientations inherited from $g(S^2)\cap(Q-\operatorname{int} Q')$ is equivalent to $L'_{q,r}$ or $\rho L'_{q,r}$. Without loss of generality it will be assumed to be $L'_{q,r}$. The genus of $g(S^2)\cap(Q-\operatorname{int} Q')$ is zero. This is equivalent to saying that $L'_{q,r}$ is weakly slice.

Choose a prime p such that p divides both q and r. By Theorem 3·2 and Lemma 3·4

$$\sigma_p(L'_{q,r}) = \sigma_p(L_{q,r}) \geqslant q+r-1,$$
$$n_p(L'_{q,r}) = n_p(L_{q,r})+2n_1+2n_2 \geqslant 2(n_1+n_2+1)$$
and
$$\mu(L'_{q,r}) = q+r+2n_1+2n_2.$$

Thus the invariants of $L'_{q,r}$ do not satisfy the inequality of 4·4 and this contradicts the existence of the embedding g.

The drawback to using the above technique if g.c.d. $(q, r) = 1$ is that 3·2 cannot be applied. It is known that locally unknotted embeddings do exist when q or $r = 0$ or 1. Using the same technique it is possible to prove the final theorem, also solved by Kervaire and Milnor (6) in a number of cases.

Note. If $\bar{c}$ is a generator of $H_2(CP^2)$ it is not possible to embed a locally unknotted copy of S^2 in CP^2 to represent $r\bar{c}$ if $|r| \geqslant 3$.

To prove this it is necessary to use σ_p and n_p where p divides r. It is easy to show that embeddings do exist when $|r| < 3$ and this therefore provides a complete solution to the problem concerning CP^2.

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