A note on strong dinaturality, initial algebras and uniform parameterized fixpoint operators

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Abstract

I show a basic Yoneda-like lemma relating strongly dinatural transformations and initial algebras. Further, I apply it to reprove known results about unique existence of uniform parameterized fixpoint operators.

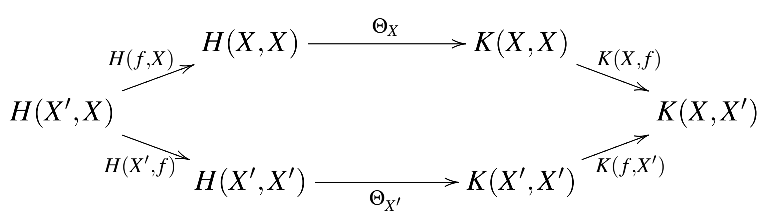
1 Introduction

I present a Yoneda-like lemma relating strongly dinatural (a.k.a. Barr dinatural) transformations and initial functor-algebras. It is very basic, but I do not know whether it has appeared in the literature. I have found it quite useful: it can be used, for example, to prove the validity of some Mendler-style structured recursion schemes for initial algebras or recursive coalgebras [12, 14] and to prove properties of Church representations of inductive types [9, 5]. Here, I use it to reprove some known results [7, 10] about existence and unique existence of uniform parameterized fixpoint operators, exploiting that uniformity is a strong dinaturality condition. I would not dare to claim that the proofs become simpler, but they obtain a structure that nicely localizes the invocations of the various initial and bifree algebra existence assumptions made.

2 Strong dinaturality and a Yoneda lemma for initial algebras

Dinatural transformations [2] and strongly dinatural (a.k.a. Barr dinatural) transformations [7, 8] are two generalizations of natural transformations from (covariant) functors to mixed-variant functors that have components only defined for the diagonal of the domain. We recall the definitions, starting with dinaturality.

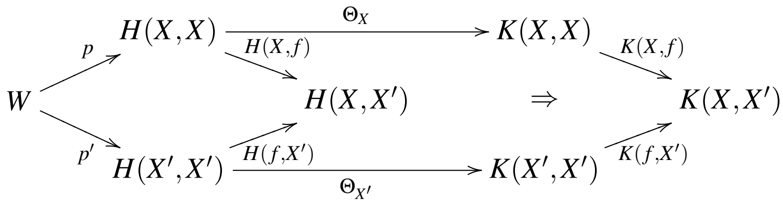
Definition 1 (Dinaturality)**.** *A* dinatural transformation *between $H, K \in \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{E}$ is given by, for any $X \in |\mathbb{C}|$, a map $\Theta_X \in \mathbb{E}(H(X,X), K(X,X))$ such that, for any $f \in \mathbb{C}(X, X')$, the following hexagon commutes in $\mathbb{E}$:*



Dinatural transformations are used, for example, in the definitions of coend and end. A coend is an initial cowedge, where a cowedge is given by an object and an accompanying dinatural transformation (just as a colimit is defined as an initial cocone, a cocone being an object with a natural transformation).

Dinatural transformations do not generally compose and so do not give a category. Strongly dinatural transformations do not suffer from this shortcoming.

Definition 2 (Strong dinaturality)**.** *A* strongly dinatural transformation *between $H, K \in \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{E}$ is given by, for any $X \in |\mathbb{C}|$, a map $\Theta_X \in \mathbb{E}(H(X,X), K(X,X))$ such that, for any map $f \in \mathbb{C}(X,X')$ and any span (W, p, p') on (X,X'), if the square in the following diagram commutes in $\mathbb{E}$, then so does the hexagon:*



If $\mathbb{E}$ is a category with pullbacks such as, e.g., **Set**, one can equivalently require that, for every map $f \in \mathbb{C}(X,X')$, the outer hexagon of the above diagram commutes for (W, p, p') the chosen pullback of the cospan $(H(X,X'), H(X,f), H(f,X'))$.

We write $[\mathbb{C},\mathbb{E}]_{\mathrm{sd}}$ for the category of mixed-variant functors from $\mathbb{C}$ to $\mathbb{E}$ and strongly dinatural transformations.

Any strongly dinatural transformation is also dinatural, but the converse does not hold in general.

This note is centered around the following observation, which I have not noticed published (I have mentioned it in an unpublished talk abstract [11] ten years ago, and also in a paper on the recursion scheme from the cofree recursive comonad [14]). Please be so kind and tell me, if you know of a reference where it might appear. It is a kind of a Yoneda lemma for strongly dinatural transformations and initial algebras.

Proposition 1 (Yoneda lemma for initial algebras)**.** *Let $\mathbb{C}$ be a locally small category, $F \in \mathbb{C} \to \mathbb{C}$ a functor with an initial algebra (which we denote $(\mu F, \mathsf{in}_F)$) and $K \in \mathbb{C} \to$ **Set** a functor (whose padding into a mixed-variant functor we denote also by K). Then*

$$[\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), K) \cong K(\mu F)$$

(so $[\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), K)$ is, in fact, a set too). This isomorphism is natural in F (to the extent that initial algebras exist in $\mathbb{C}$).

A strongly dinatural transformation between $\mathbb{C}(F-,-)$ and K is given by, for any X, a map $\Theta_X \in \mathbb{C}(FX,X) \to KX$, such that, for any X, X', $\phi \in \mathbb{C}(FX,X)$, $\phi' \in \mathbb{C}(FX',X')$, $f \in \mathbb{C}(X,X')$, $f \circ \phi = \phi' \circ F f$ (i.e., f being an F-algebra map from between (X,ϕ) and (X',ϕ')) implies $\Theta_{X'} = K f\, \Theta_X \in K X'$.

We denote the natural isomorphism by i_F. It is defined as follows: for $\Theta \in [\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), K)$, $\mathsf{i}_F\, \Theta =_{\mathrm{df}} \Theta_{\mu F}\, \mathsf{in}_F \in K(\mu F)$; and, for $x \in K(\mu F)$, $X \in |\mathbb{C}|$, $k \in \mathbb{C}(FX,X)$, $(\mathsf{i}_F^{-1}\, x)_X\, k =_{\mathrm{df}} K(\mathsf{fold}_{F,X}\, k)\, x \in K X$, where $\mathsf{fold}_{F,X}$ denotes the unique algebra map from $(\mu F, \mathsf{in}_F)$ to (X,k).

An important special case is when $KX =_{\mathrm{df}} \mathbb{C}(1,X)$. We get that

$$[\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), \mathbb{C}(1,-)) \cong \mathbb{C}(1, \mu F)$$

This is closely related to Church representations of inductive types. Remember that, in System F, we represent μF by $\forall X.\ (F X \Rightarrow X) \Rightarrow X)$.

Needless to say, for final coalgebras, a dual proposition is true; I refrain from spelling it out here.

3 Uniform parameterized fixpoint operators

We now turn to parameterized fixpoint-like operators and uniformity. The dependencies between different axiomatiozations and sufficient conditions for existence have been studied by Bloom and Ésik [1], Freyd [3, 4], Mulry [7], Simpson and Plotkin [10] etc. Hyland and Hasegawa [6] showed that (uniform) Conway operators are equivalent to (uniform) traces (definable in general monoidal categories, not just categories with finite products).

We will closely follow the account of Simpson and Plotkin [10]. First we recall the definitions of parameterized fixpoint operators, parameterized Conway operators and uniformity.

We assume given a category $\mathbb{D}$ with finite products.

Definition 3 (Parameterized fixpoint-like operator)**.** *A* parameterized fixpoint-like operator *on* $\mathbb{D}$ *is given by, for any* $X,Y \in |\mathbb{D}|$*, a function* $\mathsf{fix}_{X,Y} \in \mathbb{D}(X \times Y, Y) \to \mathbb{D}(X,Y)$.

Definition 4 (Parameterized fixpoint operator)**.** *A* parameterized fixpoint operator *on* $\mathbb{D}$ *is a parameterized fixpoint-like operator* fix *on* $\mathbb{D}$ *such that*

- *for any* $f \in \mathbb{D}(X,X')$ *and* $k' \in \mathbb{D}(X' \times Y, Y)$*,* $\mathsf{fix}\,(k' \circ (f \times \mathsf{id}_Y)) = \mathsf{fix}\,k' \circ f$ (naturality)*;*
- *for any* $k \in \mathbb{D}(X \times Y, Y)$*,* $\mathsf{fix}\,k = k \circ \langle \mathsf{id}_X, \mathsf{fix}\,k\rangle$ (parameterized fixpoint property).

Definition 5 (Conway operator)**.** *A* Conway *operator on* $\mathbb{D}$ *is a parameterized fixpoint operator* fix *on* $\mathbb{D}$ *with the further properties that*

- *for any* $f \in \mathbb{D}(X \times Y, Y')$ *and* $h \in \mathbb{D}(X \times Y', Y)$*,* $f \circ \langle \mathsf{id}_X, \mathsf{fix}\,(h \circ \langle \mathsf{fst}, f\rangle)\rangle = \mathsf{fix}\,(f \circ \langle \mathsf{fst}, h\rangle)$ (parameterized dinaturality)*;*
- *for any* $k \in \mathbb{D}((X \times Y) \times Y, Y)$*,* $\mathsf{fix}\,(k \circ \langle \mathsf{id}_{X\times Y}, \mathsf{snd}_{X,Y}\rangle) = \mathsf{fix}\,(\mathsf{fix}\,k)$ (diagonal property).

Parameterized dinaturality implies the parameterized fixpoint property, so for Conway operators the latter condition is redundant.

For our final definition, we assume we also have a category $\mathbb{C}$ with finite products and the same objects as $\mathbb{D}$ together with an identity-on-objects functor $J \in \mathbb{C} \to \mathbb{D}$ preserving the finite products of $\mathbb{C}$ strictly. We call the maps of $\mathbb{D}$ in the image of J *pure*.

Definition 6 (Uniformity)**.** *A parameterized fixpoint-like operator* fix *on* $\mathbb{D}$ *is said to be* uniform *wrt.* J*, if*

- *for any* $f \in \mathbb{C}(Y,Y')$*,* $k \in \mathbb{D}(X \times Y, Y)$ *and* $k' \in \mathbb{D}(X \times Y', Y')$*,* $J\,f \circ k = k' \circ (\mathsf{id}_X \times J\,f)$ *implies* $J\,f \circ \mathsf{fix}\,k = \mathsf{fix}\,k'$.

(In the terminology of iteration theories [1], naturality is parameter identity, parameterized fixpoint property is fixpoint identity, parameterized dinaturality is composition identity and diagonal property is double dagger identity. Finally, uniformity corresponds to the functoriality condition.)

We now focus on the special case of $\mathbb{D}$ arising as the coKleisli category of a comonad $(D, \varepsilon, (-)^\dagger)$[1] on $\mathbb{C}$ with J the right adjoint in its coKleisli splitting. The prototypical well-behaved situation here has $\mathbb{C} =_{\mathrm{df}} \mathbf{Cppo}_\perp$, $D =_{\mathrm{df}} (-)_\perp$ and $\mathbb{D} \cong \mathbf{Cppo}$, where **Cppo** stands for the category of ω-complete pointed partial orders and ω-continuous functions, $\mathbf{Cppo}_\perp$ is as **Cppo** but has as maps only the strict (bottom-preserving) maps of **Cppo** and $(-)_\perp$ is the lifting endofunctor. More generally, the lifting comonad can be replaced with any comonad on $\mathbf{Cppo}_\perp$ that has its underlying functor **Cppo**-enriched.

In terms of the "base" category $\mathbb{C}$, a parameterized fixpoint-like operator is now, for any $X,Y \in |\mathbb{C}|$, a function $\mathsf{fix}_{X,Y} \in \mathbb{C}(D\,(X \times Y), Y) \to \mathbb{C}(DX, Y)$. The various optional additional conditions specialize into the following:

[1] We write $(-)^\dagger$ for the coKleisli extension operation of a comonad.

- for any $f \in \mathbb{C}(DX, X')$ and $k' \in \mathbb{C}(D(X' \times Y), Y)$, $\mathsf{fix}\,(k' \circ \langle f \circ D\,\mathsf{fst}, \varepsilon_Y \circ D\,\mathsf{snd}\rangle^\dagger) = \mathsf{fix}\,k' \circ f^\dagger$ (naturality);
- for any $k \in \mathbb{C}(D\,(X \times Y), Y)$, $\mathsf{fix}\,k = k \circ \langle \varepsilon_X, \mathsf{fix}\,k\rangle^\dagger$ (parameterized fixpoint property);
- for any $f \in \mathbb{C}(D\,(X \times Y), Y')$ and $h \in \mathbb{C}(D\,(X \times Y'), Y)$, $f \circ \langle \varepsilon_X, \mathsf{fix}\,(h \circ \langle \varepsilon_X \circ D\,\mathsf{fst}, f\rangle^\dagger)\rangle^\dagger = \mathsf{fix}\,(f \circ \langle \varepsilon_X \circ D\,\mathsf{fst}, h\rangle^\dagger)$ (parameterized dinaturality);
- for any $k \in \mathbb{C}(D\,((X \times Y) \times Y), Y)$, $\mathsf{fix}\,(k \circ D\,\langle \mathsf{id}_{X \times Y}, \mathsf{snd}_{X,Y}\rangle) = \mathsf{fix}\,(\mathsf{fix}\,k)$ (diagonal property);
- for any $f \in \mathbb{C}(Y, Y')$, $k \in \mathbb{C}(D\,(X \times Y), Y)$ and $k' \in \mathbb{C}(D\,(X \times Y'), Y')$, $f \circ k = k' \circ D\,(\mathsf{id}_X \times f)$ implies $f \circ \mathsf{fix}\,k = \mathsf{fix}\,k'$ (uniformity).

(Notice that here, id and $\circ$ refer to identity and composition in $\mathbb{C}$ rather than in $\mathbb{D}$, differently from what they meant above.)

Crucially for us, the uniformity condition asserts nothing else than strong dinaturality of $\mathsf{fix}_{X,Y}$ in Y, i.e., that $\mathsf{fix}_{X,-} \in [\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(D\,(X \times -), -), \mathbb{C}(DX, -))$—an observation first made by Mulry [7].

From Proposition 1, we immediately get:

Corollary 1. *If every functor $D(X \times -) \in \mathbb{C} \to \mathbb{C}$ has an initial algebra, then a uniform wrt. J parameterized fixpoint-like operator* fix *on $\mathbb{D}$ is the same as, for any $X \in |\mathbb{C}|$, a map $\underline{\mathsf{fix}}_X \in \mathbb{C}(DX, \mu(D\,(X \times -)))$.*

The bijection is given by $\underline{\mathsf{fix}}_X =_{\mathrm{df}} \mathsf{fix}_{X,\mu(D\,(X\times -))}\,\mathsf{in}_{D\,(X\times -)}$ and, for $k \in \mathbb{C}(D(X \times Y), Y)$, $\mathsf{fix}_{X,Y}\,k =_{\mathrm{df}} \mathsf{fold}_{D\,(X\times -),Y}\,k \circ \underline{\mathsf{fix}}_X$.

It is not difficult to strengthen this corollary to the following characterization of uniform parameterized fixpoint operators (one has to verify that the conditions are pairwise equivalent):

Proposition 2. *If every functor $D\,(X \times -) \in \mathbb{C} \to \mathbb{C}$ has an initial algebra, then a uniform wrt. J parameterized fixpoint operator* fix *on $\mathbb{D}$ is the same as, for any $X \in |\mathbb{C}|$, a map $\underline{\mathsf{fix}}_X \in \mathbb{C}(DX, \mu(D\,(X \times -)))$ such that*

- *for any $f \in \mathbb{C}(DX, X')$, $\mu(\langle f \circ D\,\mathsf{fst}, \varepsilon_- \circ D\,\mathsf{snd}\rangle^\dagger) \circ \underline{\mathsf{fix}}_X = \underline{\mathsf{fix}}_{X'} \circ f^\dagger$ ("naturality");*
- *for any $X \in |\mathbb{C}|$, $\underline{\mathsf{fix}}_X = \mathsf{in}_{D\,(X\times -)} \circ \langle \varepsilon_X, \underline{\mathsf{fix}}_X\rangle^\dagger$ ("parameterized fixpoint property").*

Recall that a bifree algebra is an initial algebra that is at the same time also a final coalgebra. The following is nearly immediate from the proposition we just stated.

Proposition 3 ([10, Proposition 6.5])**.** *If every functor $D(X \times -) \in \mathbb{C} \to \mathbb{C}$ has a bifree algebra, then $\mathbb{D}$ has a unique uniform wrt. J parameterized fixpoint operator.*

Proof. Just observe that the parameterized fixpoint property can be rewritten as $\mathsf{in}^{-1}_{D\,(X\times -)} \circ \underline{\mathsf{fix}}_X = D\,(X \times \underline{\mathsf{fix}}_X) \circ \langle \varepsilon_X, \mathsf{id}_{DX}\rangle^\dagger$, which stipulates that $\underline{\mathsf{fix}}_X$ (if existing) must be a coalgebra map between $(DX, \langle \varepsilon_X, \mathsf{id}_{DX}\rangle^\dagger)$ and $(\mu(D\,(X \times -)), \mathsf{in}^{-1}_{D\,(X\times -)})$. As the latter is a final coalgebra, there is exactly one such map. This map turns out to also satisfy the required naturality condition. □

Uniform Conway operators can be analyzed similarly. Here we need the existence of further initial algebras to replace conditions on fix with conditions on $\underline{\mathsf{fix}}$. When these initial algebras are also final coalgebras, we have a unique uniform Conway operator.

Proposition 4. *If all functors $D\,(X \times -), D\,(X \times D\,(X \times -)), D\,((X \times -) \times -) \in \mathbb{C} \to \mathbb{C}$ have initial algebras, then a uniform wrt. J Conway operator on $\mathbb{D}$ is the same as, for any $X \in |\mathbb{C}|$, a map $\underline{\mathsf{fix}}_X \in \mathbb{C}(DX, \mu(D\,(X \times -)))$ satisfying the conditions of Proposition 2, but also the following conditions:*

- *for any* $X \in |\mathbb{C}|$, $\mathsf{in} \circ \langle \varepsilon_X, \mathsf{fold}\,(\langle \varepsilon_X \circ D\,\mathsf{fst}, \mathsf{in}\rangle^\dagger) \circ \underline{\mathsf{fix}}\rangle^\dagger = \mathsf{fold}\,(\mathsf{in} \circ \langle \varepsilon_X \circ D\,\mathsf{fst}, \mathsf{id}\rangle^\dagger) \circ \underline{\mathsf{fix}} \in \mathbb{C}(DX, \mu(D\,(X \times D\,(X \times -))))$ *("parameterized dinaturality");*
- *for any* $X \in |\mathbb{C}|$, $\mathsf{fold}\,(\mathsf{in} \circ D\,\langle \mathsf{id}, \mathsf{snd}\rangle) \circ \underline{\mathsf{fix}} = \mathsf{fold}\,(\mathsf{fold}\,\mathsf{in} \circ \underline{\mathsf{fix}}) \circ \underline{\mathsf{fix}} \in \mathbb{C}(DX, \mu(D\,((X \times -) \times -)))$ *"diagonal property").*

Proposition 5 ([10, Theorem 3])**.** *If all functors* $D\,(X \times -), D\,(X \times D\,(X \times -)), D\,((X \times -) \times -) \in \mathbb{C} \to \mathbb{C}$ *have bifree algebras, then* $\mathbb{D}$ *has a unique uniform wrt. J Conway operator.*

4 Uniform guarded recursion operators

A similar treatment is possible for guarded recursion operators (we have previously considered some aspects for the dual situation of guarded iteration [13]). Here, the prototypical example is given by cofree recursive comonads on endofunctors on **Set**, such as the nonempty list comonad defined by $DX =_{\mathrm{df}} \mu(X \times (1 + (-)))$.

An ideal comonad on a category $\mathbb{C}$ with finite products is a comonad given by $DX =_{\mathrm{df}} X \times D_0 X$, $\varepsilon_X =_{\mathrm{df}} \mathsf{fst} \in \mathbb{C}(DX, X)$, for any $k \in \mathbb{C}(DX, Y)$, $k^\dagger =_{\mathrm{df}} \langle k, k^\ddagger \circ \mathsf{snd}\rangle \in \mathbb{C}(DX, DY)$ where D_0 is an endofunctor on $\mathbb{C}$ and, for any $X, Y \in |\mathbb{C}|$, $(-)^\ddagger_{X,Y} \in \mathbb{C}(DX, Y) \to \mathbb{C}(D_0 X, D_0 Y)$.

A guarded recursion operator for an ideal comonad is, for any $X, Y \in |\mathbb{C}|$, a unique function $\mathsf{rec}_{X,Y} \in \mathbb{C}(X \times D_0\,(X \times Y), Y) \to \mathbb{C}(DX, Y)$ satisfying the guarded recursion equation $\mathsf{rec}\,k = k \circ (\mathsf{fst} \times \mathsf{id}) \circ \langle \varepsilon, \mathsf{rec}\,k\rangle^\dagger$ and possibly further properties.

As soon as all functors $X \times D_0\,(X \times -)$ have initial algebras, having a uniform guarded recursion operator rec becomes equivalent to having, for any $X \in |\mathbb{C}|$, a map $\underline{\mathsf{rec}}_X \in \mathbb{C}(DX, \mu\,(X \times D_0\,(X \times -))$ such that $\underline{\mathsf{rec}} = \mathsf{in} \circ (\mathsf{fst} \times \mathsf{id}) \circ \langle \varepsilon, \underline{\mathsf{rec}}\rangle^\dagger$.

A uniform guarded recursion operator exists uniquely, e.g., whenever D is the cofree recursive comonad on an endofunctor H on $\mathbb{C}$, in which case $DX \cong \mu\,(X \times H\,(-)) \cong X \times \mu\,(H\,(X \times -))$.

5 Conclusion

I find it intruiging that the use of the Yoneda-like lemma stages the invocations of the initial algebra resp. bifree algebra existence assumptions: the initial algebra existence assumptions ensure the possibility of reducing the existence of a parameterized fixpoint operator to the existence of a family of maps to initial algebras; the bifree algebra existence assumptions ensure that such a family of maps exists uniquely.

I would like to learn more about the relationship of strong dinaturality and models of parametric polymorphism.

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