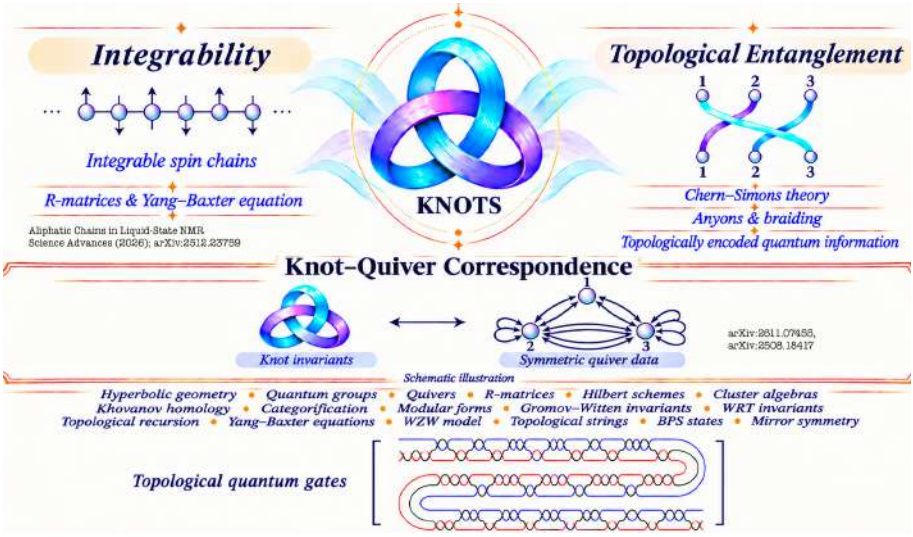
Topological Inspired Entanglement Signals and Quiver Decomposition

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Research Interests



Multipartite Entanglement:Beyond Bipartite Correlations

Entanglement is a purely quantum correlation between subsystems, and entanglement measures quantify its strength.

What is Multipartite Entanglement?

Quantum entanglement shared among N subsystems: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$.

- **Many-party correlations:** Quantum entanglement shared among three or more subsystems.
- **Genuine multipartite entanglement (GME):** Entanglement involving all parties, not reducible to any bipartition.*Examples: GHZ- and W-type entanglement.*
- **A fundamental challenge:** No unique, universally accepted measure of multipartite entanglement exists.

W. K. Wootters(1998), M. Walter, D. Gross and J. Eisert, arXiv:1612.02437 (2017).

Examples: separate, partly entangled, genuinely shared

Fully separable

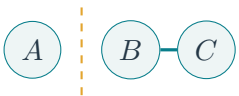
$|000\rangle$



All three parties have their own state.

Biseparable

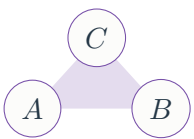
$|0\rangle_A \otimes |\Phi^+\rangle_{BC}$



B and C are entangled, but A is a spectator.

Genuinely tripartite

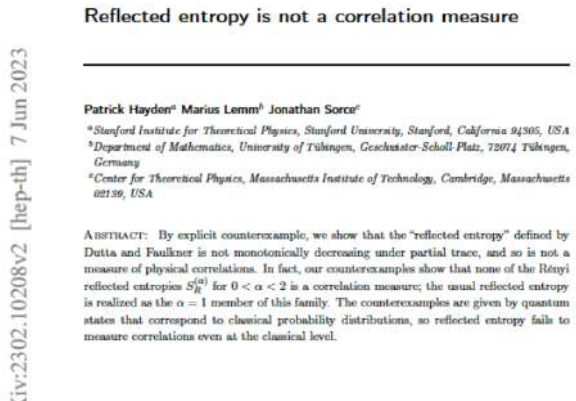
$$|\mathrm{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$



No split separates this pure state into independent groups.

Does an Entanglement Measure Really Capture Quantum Entanglement?

Recent work highlights that not every quantity called an "entanglement measure" necessarily captures genuinely quantum, multipartite correlations.



arXiv:2302.10208v2 [hep-th] 7 Jun 2023

Reflected entropy is not a correlation measure

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ABSTRACT: By explicit counterexample, we show that the "reflected entropy" defined by Dutta and Faulkner is not monotonically decreasing under partial trace, and so is not a measure of physical correlations. In fact, our counterexamples show that none of the Rényi reflected entropies $S_R^{(\alpha)}$ for $0 < \alpha < 2$ is a correlation measure; the usual reflected entropy is realized as the $\alpha = 1$ member of this family. The counterexamples are given by quantum states that correspond to classical probability distributions, so reflected entropy fails to measure correlations even at the classical level.

Entanglement measures: what should a good measure satisfy?

Refs.: Vidal, J. Mod. Opt. **47**, 355 (2000); Plenio & Virmani, Quant. Inf. Comput. **7**, 1 (2007); Bennett et al., Phys. Rev. A **63**, 012307 (2001).

Definition

An entanglement measure is a map $E : \mathcal{S}(\mathcal{H}) \to \mathbb{R}_{\geq 0}$ quantifying the entanglement in a quantum state.

1. Faithfulness
$E(\rho) = 0 \iff \rho$ is separable.

2. LOCC monotonicity
$E(\Lambda_{\mathrm{LOCC}}[\rho]) \leq E(\rho), \quad \Lambda \in$ LOCC. Entanglement cannot be created by means of LOCC transformations

3. Local-unitary invariance
$E(\rho) = E(U\rho U^{\dagger})$.

4. Convexity:
$E(p\rho + (1-p)\sigma) \leq pE(\rho) + (1-p)E(\sigma)$.
Randomization of quantum states decreases the entanglement

5. Additivity
$E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$ (entanglement present in two uncorrelated quantum systems).

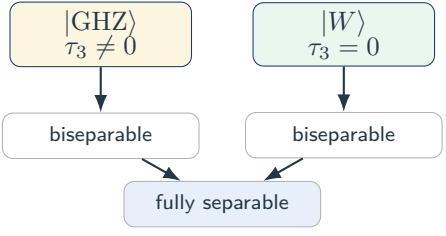
6.Superadditivity
Alice holds the system A_1A_2 and Bob holds B_1B_2 $E(\varrho_{A_1A_2B_1B_2}) \geq E(\varrho_{A_1B_1}) + E(\varrho_{A_2B_2})$.

Core operational idea: local processing cannot create entanglement, while local basis changes cannot alter it.

Multipartite structures

Three qubits: inequivalent genuine entanglement

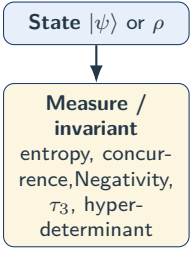
SLOCC distinguishes two genuine multipartite classes: GHZ and W. They distribute correlations differently, so a single scalar does not capture the whole structure.



Invariant theory: $\tau_3(\psi) = 4|\,\mathrm{hdet}_{2\times2\times2}(\psi)|$,
$\mathrm{hdet}_{2\times2\times2\times2}(GHZ_4) = 0$.

What the measures see

- **GHZ:** maximal 3-body correlation; fragile to particle loss.
- **W:** robust pairwise entanglement survives tracing out one party.



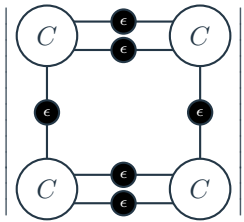
Refs.: Dür, Vidal & Cirac (2000); Coffman, Kundu & Wootters(2000), Dobes & Jing, arXiv:2504.15511 (2025).

Hyperdeterminant and Three-tangle

The hyperdeterminant is the generalization of the determinant defined for the matrices, to higher order tensors(I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, 1994). For three qubits, Cayley's hyperdeterminant

$$\tau_3(|\psi\rangle) = 4|\Delta_{222}(c)|. \tag{1}$$

Equivalently,Coffman–Kundu–Wootters(CKW) construction(closed diagram formed):

$\tau = 2$ 

(2)

Refs.: Coffman, Kundu & Wootters(2000), Dobes & Jing, arXiv:2504.15511 (2025).

Can Hyperdeterminants Detect All Genuine Entanglement?

Three qutrits: the degree-36 hyperdeterminant Δ_{333} provides a higher-dimensional generalization. On the semisimple family, the degree-36 hyperdeterminant takes the factorized form :

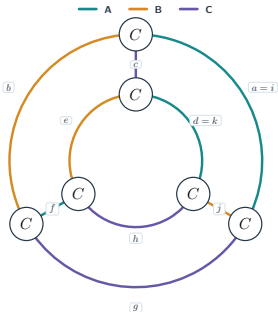
$$\begin{aligned}\Delta_{333}(|\psi_S\rangle) = & -\tfrac{4}{3^{18}}\, a^3b^3c^3(a+b+c)^3 \\ & \times (a^2+2ab-ac+b^2-bc+c^2)^3 \times (a^2-ab+2ac+b^2-bc+c^2)^3 \\ & \times (a^2-ab-ac+b^2+2bc+c^2)^3 \times (a^2-ab-ac+b^2-bc+c^2)^3. \end{aligned} \tag{3}$$

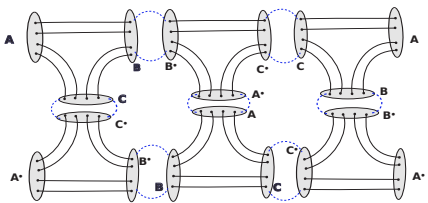
Thus the zero locus of the hyperdeterminant is not a criterion for the absence of genuine tripartite entanglement.

We need complementary tools to characterize multipartite entanglement.

New Candidate measures:

work in preparation with Hisham Sati, Dmitry Melnikov, and Jefferson Oliveira,





- **Fails monotonicity:** LU invariance, separability but not a valid entanglement measure, despite .
- **GME signals**:capture genuine multipartite quantum entanglement.

Definition: genuine multipartite entanglement signals

A.Gadde, arXiv:2603.07680, 2026 A **signal** is a local-unitary (LU)-invariant function

$$f : \bigotimes_{a=1}^{q} \mathcal{H}_{A_a} \longrightarrow \mathbb{C}$$

that vanishes on every layerwise-separable state.

Local-unitary invariance:

$$f\left[\left(\bigotimes_{a=1}^{q} U_a\right)|\psi\rangle\right] = f(|\psi\rangle).$$

Vanishing condition:

$$\boxed{f(|\psi\rangle) = 0 \qquad \text{for every layerwise-separable } |\psi\rangle.}$$

Therefore,

$$\boxed{f(|\psi\rangle) \neq 0 \quad \Longrightarrow \quad |\psi\rangle \text{ is not layerwise-separable.}}$$

Signal versus entanglement measure

M. Del Zotto, A. Gadde, and P. Putrov,(arXiv:2602.16770)

GME signal

LU invariant
vanishes on all
layerwise-separable states

A diagnostic for GME

GME measure

Quantifies entanglement
satisfies positivity
obeys LOCC monotonicity

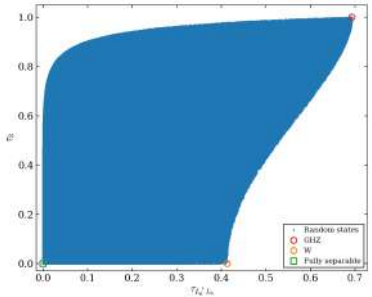
A quantitative measure

Applications

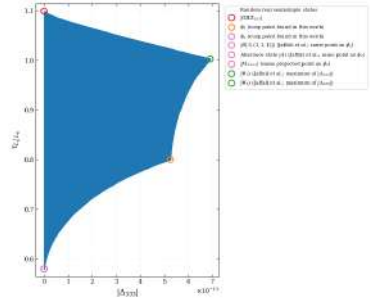
$(2+1)$-dimensional Levin-Wen string-net models: ground states are described as wave functions on the space of string-net configurations.

Signal $\neq$ GME measure

Properties of a GME Signal



Three tangle vs GME signals



Hyperdeterminant vs GME signals

Part B: Knot-Quiver Correspondence

CHERN–SIMONS ACTION (k: level, A: gauge connection)

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Topological theory

Metric-independent; no local propagating bulk modes.

KNOT TYPE



$K \subset S^3$

WILSON LOOP

$$W_R(K) = \mathrm{Tr}_R \; \mathcal{P}\, e^{\, i \oint_K A}$$

Holonomy of A around K.

TOPOLOGICAL OBSERVABLE

$$\langle W_R(K) \rangle_{\mathrm{CS}} = \frac{\displaystyle\int \mathcal{D}A \; e^{iS_{\mathrm{CS}}[A]} W_R(K)}{\displaystyle\int \mathcal{D}A \; e^{iS_{\mathrm{CS}}[A]}}$$

$SU(2)$ fundamental representation $\Longrightarrow$ Jones polynomial (fixed framing; conventional normalization)

E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121**, 351–399 (1989).

TQFT Knot Explorer

Amena Al Rawi, Hisham Sati, Vivek Kumar Singh, arXiv:2607.23551,2026,Live at: (https://tqft-knot-explorer-559830223866.me-central1.run.app/)

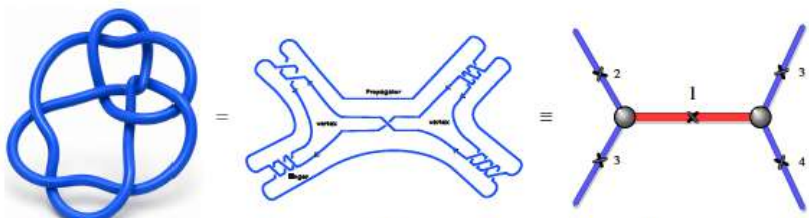


Figure 1: Three complementary views of the knot 10_{93}: its conventional projection, its FRD description, and the corresponding TQFT tangle representation.

Knot–quiver proposal

Kucharski–Reineke–Stošić–Sułkowski, Adv. Theor. Math. Phys. 23 (2019) 1849–1902.

Knot–Quiver proposal(KQC)

For every knot K, (refined) symmetric-colored HOMFLY–PT generating series admits a finite symmetric-quiver partition-function representation.

$$\underbrace{\mathcal{P}_K(x;a,q,t)}_{\text{all symmetric colours}} = \underbrace{Z_Q(\boldsymbol{x},q)|_{x_i=x^{m_i}a^{\ell_i}q^{k_i}}}_{\text{one weighted quiver}} = \sum_{d_1,\ldots,d_n\geq 0} \underbrace{(-q)^{\boldsymbol{d}^{\mathsf{T}}C\boldsymbol{d}}}_{\text{interactions}} \prod_i \underbrace{\frac{x_i^{d_i}}{(q^2,q^2)_{d_i}}}_{\text{copies of node } i}$$

Ooguri–Vafa origin((2000)).

$U(N)$ Chern–Simons theory on S^3	$\longleftrightarrow$	topological string theory on the resolved conifold

The Ooguri–Vafa operator generates the colored HOMFLY–PT series $\mathcal{P}_K(x;a,q,t) = \sum_{r\geq 0} P_{S^r}(K;a,q,t)\,x^r$.

Finite input

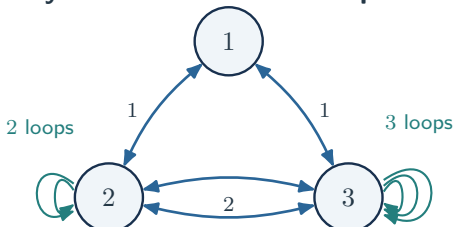
A matrix and a list of weights.

Infinite output

The full series of coloured invariants.

Example: the trefoil knot 3_1

Symmetric three-vertex quiver



Edge label k: k arrows in each direction.

Quiver data

$$C_{3_1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} xa^2q^{-2} \\ xa^2(-t)^2 \\ xa^4q^{-3}(-t)^3 \end{pmatrix}.$$

$$\underbrace{\mathcal{P}_{3_1}(x;a,q,t)}_{\text{all symmetric colours}} = \sum_{d_1,\ldots,d_n \geq 0} \underbrace{(-q)^{\boldsymbol{d}^{\mathsf{T}} C_{3_1} \boldsymbol{d}}}_{\text{interactions}} \prod_i \underbrace{\frac{x_i^{d_i}}{(q^2,q^2)_{d_i}}}_{\text{copies of node } i}$$

Recent works: arXiv:2508.18417(2025), arXiv:2611.07455(2026),*Joint work with* M. Stošić, P. Kucharski, R. Pichai, D. Noshchenko, L. Nakamura, and S. Chauhan.

From a HOMFLYPT skein move to a quiver picture

Three links differ only inside one small ball.



$$P(L_+) - P(L_-) = z\,P(L_0) \qquad \text{with } z = q - q^{-1}$$

Skein relation Outside the ball, the link is unchanged.

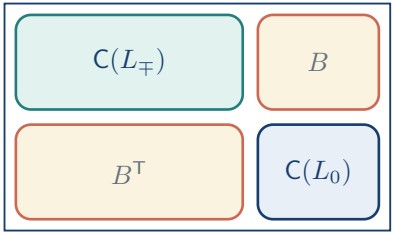
Question Can the move be lifted to quiver data?

Conjecture : the quiver splits into two blocks

After a suitable quiver equivalence, the resolved links appear on the diagonal.

ADJACENCY-MATRIX FORM

$$\mathsf{C}^{\,g}_{L_\pm} \;\sim_{\text{quiv.}}\; \begin{pmatrix} \mathsf{C}^{\,g,f_1}_{L_\mp} & B \\ B^{\mathsf{T}} & \mathsf{C}^{\,g',f_2}_{L_0} \end{pmatrix}$$



The off-diagonal blocks couple the two sectors.

The claim

- Diagonal blocks are the quivers of $L_\mp$ and L_0.
- Genus, framing and node weights shift as prescribed by the skein move.
- $g' = g$ for a self-crossing; otherwise $g' = g + 1$.

Caveat

B is not determined in general. Redundant node pairs may be needed; the general statement remains a

Example I: the $(2, 2p+1)$ torus-knot family

A recurring skein relation with a visible submatrix structure.

$$T_{2,2p+1} = T_{2,2p-1} - zT_{2,2p}$$

A two-strand braid picture



$2p+1$ crossings (shown schematically for $p = 2$)

The family comes from closing a two-strand braid.

Expected block picture

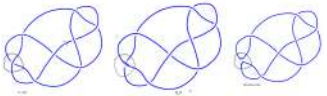
$$\mathsf{C}(T_{2,2p+1}) \sim_{\text{quiv.}} \begin{pmatrix} \mathsf{C}(T_{2,2p-1}) & * \\ * & \mathsf{C}(T_{2,2p}) \end{pmatrix}$$

The diagonal blocks encode the two resolved links.

Example II: 8_{13} and the Whitehead link

A subtler example: the block structure appears after quiver equivalences.

$$8_{13} = 6_3 - z\,\mathrm{Wh}$$



Augmented representative 42 nodes → Quiver equivalences and linking moves → Cancel 13 pairs $\Rightarrow$ 16 nodes

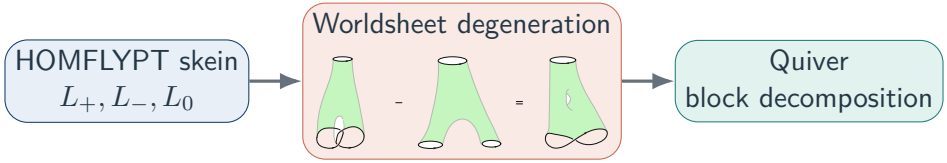
Block interpretation

$$\widetilde{Q}_{8_{13}} \sim_{\mathrm{quiv.}} \begin{pmatrix} Q_{6_3} & * \\ * & Q_{\mathrm{Wh}} \end{pmatrix}.$$

Why it matters The block form depends on the representative: an equivalence move can reveal it or destroy it. This hints at structure among equivalent quivers.

Summary, open questions and recent connections

A quiver-level lift of a local HOMFLYPT skein move.



Take-home message

- connection between skein triangles in link homology and block decompositions of quivers.
- Generalizes the standard knot–quiver correspondence from genus-zero to higher-genus worldsheets.

Recent related studies

2026 — Ekholm–Shende, *Skeins on branes*, Publ. Math. IHÉS: holomorphic-curve foundation for HOMFLYPT skeins.

2026 — Ekholm–Longhi–Nakamura, *Worldsheet Skein D-module*, CMP 407: skein recursions and quiver-like Hopf-link

Definition I: separability and genuine multipartite entanglement

Consider a pure q-partite state

$$|\psi\rangle \in \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_q}.$$

A partition of the parties is written as

$$\pi = \{B_1, B_2, \ldots, B_m\}.$$

The state is called π-**separable** if it factorizes across the blocks of the partition:

$$\boxed{|\psi\rangle = \bigotimes_{j=1}^{m} |\psi_{B_j}\rangle.}$$

For three parties, examples include

$$A|B|C: \quad |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle,$$

Definition II: layer decomposition

The central idea is to allow each party's Hilbert space to factorize into smaller tensor factors.

For each party A_a, write

$$\boxed{\mathcal{H}_{A_a} = \bigotimes_{i=1}^{n} \mathcal{H}_{A_a^{(i)}}.}$$

A state admits a **layer decomposition** if

$$\boxed{|\psi\rangle = \bigotimes_{i=1}^{n} |\psi^{(i)}\rangle,}$$

where each layer belongs to

$$|\psi^{(i)}\rangle \in \bigotimes_{a=1}^{q} \mathcal{H}_{A_a^{(i)}}.$$

Definition III: layerwise-separable states

A pure state is called **layerwise-separable** if it admits a layer decomposition

$$|\psi\rangle = \bigotimes_{i=1}^{n} |\psi^{(i)}\rangle$$

such that **every layer is separable**.

In other words,

$$\text{Layerwise-separable} \iff \begin{cases} |\psi\rangle = \bigotimes_i |\psi^{(i)}\rangle, \\ \text{each } |\psi^{(i)}\rangle \text{ is separable.} \end{cases}$$

Fully separable

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle.$$

No entanglement.

Layerwise-separable

$$|\psi\rangle = |\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle.$$

Each layer is separable.