Abstract

Conformal compactification of some space-times is studied, with a particular focus on the asymptotically anti-de Sitter space-times, such as Schwarzschild-AdS and Reissner-Nordström-AdS. We will depict Penrose diagrams of such spaces compactified in all of their coordinates, investigating the nature of the time-like infinities i^+ and i^- that connect to the spatial conformal boundary $\mathscr{I}$ and presenting the only non-vanishing Weyl spinor Ψ_2. We delve into the Reissner-Nordström-AdS case, and discuss some results in already published papers. We believe there is a singularity at each end of the time-like infinity (and provide arguments in favour of this conjecure), but unfortunately cannot give a formal proof of our conjecture. The thesis work contains some of the formalism of the conformal techniques, and elaborate analysis of the criterions leading to the choice of the conformal factors Ω for some of the easier and more intuitive spaces i.e. euclidean and hyperbolic planes.

"I could be bounded in a nutshell,
and count myself king of infinite space."
Hamlet

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Preface

Very often when we look at some mathematical models of nature we expect from them not an exact description of the phenomenon, but a prediction of its asymptotic behavior. For example in order to analyze the electromagnetic radiation generated by a charge moving in free space it is very useful to study the asymptotic behavior of its radiation spectrum. In our case we are interested in asymptotical properties of space-times. Let us consider the issue of the gravitational radiation. Take two massive bodies such as planets. If they are close to each other, then there will be a negative gravitational potential energy contribution that makes the total energy smaller than it would be if they were far apart. Although the actual energy-momentum tensor does not take into account the gravitational binding energy, yet the total mass/energy will differ in the two cases and the difference (negative contribution) will be attributed to the energy in the gravitational field itself. If the bodies were in orbit about one another, as a consequence of Einstein equations *gravitational waves* will emanate from the system and carry some positive energy (gravitational radiation) away from it. Most scientists describe gravitational waves as "ripples in spacetime". There are now several projects for the direct detection of such waves but of course the crucial point is to show that the ripples actually carry energy away. It is a very thorny issue then to understand what is going on even "asymptotically" but fortunately Penrose invented a technique called *conformal compactification* of space-times which essentially defines an equivalence class of metrics, g_{ab} being equivalent to $\hat{g}_{ab} = \Omega^2 g_{ab}$ where Ω is a positive scalar function of the space-time that modifies the distance scale making the asymptotics of the physical metric accessible to study.

The conformal techniques, discussed in the first chapter of this thesis work, were invented expressly to understand the aymptotic behavior in general relativity, including those issues like the one of the gravitational energy carried away by gravitational waves. By conformally compactifying a space-time we can not only understand its asymptotics (which we name by the script letter $\mathscr{I}$, to be pronounced "scri") but also study in detail the global structure of different models for simpler space-times. Often it is useful to employ the so called *conformal diagrams* (Penrose diagrams), 2-dimensional representations of highly (spherically in our case) symmetric spaces drawn so that the null directions slope 45°

to the vertical, and where infinity is also represented as part of the boundary of the diagram and identified with $\mathscr{I}$. This implies that the metric has to be conformally compactified in ALL of its coordinates. We show thus the conformally compactified Minkowski and Schwarzschild space-times, and a partially compactified anti-de Sitter space-time.

In the second chapter, we will depict a Penrose diagram of anti-de Sitter space compactified in all of its coordinates, which seems to be somewhat an obscure point in the literature. This is done by going through a comparison with the more straightforward conformal compactification of the $3+1$ de Sitter space-time, where the cosmological constant takes the positive value of $+3$. There we will show how its conformal boundary has the topology of a 3-sphere. In this thesis we mostly treat space-times with a negative cosmological constant, set to be -3 by convention. The value of Λ gives a pivotal contribution to the geometry of the space-time and to the structure of its infinity. Einstein's original cosmological model was a static (at the time the universe was not known to be expanding), homogeneous model with spherical geometry, with a non-zero cosmological constant Λ to balance the gravitational effect of matter, solution of the following:

$$R_{ab} = -\frac{1}{2}Rg_{ab} + \Lambda g_{ab} = -8\pi G T_{ab},$$

Such a model happened to be spatially closed in the cosmological scale. Nevertheless from Hubble's observations in 1929 it became clear that the universe is expanding, and therefore not static. The idea of the cosmological constant hasn't been abandoned and very recent observations of distant supernovae led theorists to reintroduce a positive Λ to make these observations consistent with other requirements as a parameter describing the energy density of the vacuum, a property of spacetime itself.

Third chapter finally is dedicated to the study of the black hole cases, where the space-times of our interest are *asymptotically anti-de Sitter*, the rough idea being that they should look like anti-de Sitter space-time "far away" from any mass concentration or black hole. They all have constant negative curvature at infinity and are maximally symmetric solutions of Einstein's equations. The examples we will discuss are asymptotically anti-de Sitter Schwarzschild and Reissner-Nordström (with a particular focus on the extremal case). The main and original aim of this thesis work is to present a detailed analysis of the conformal boundary of the asymptotically anti-de Sitter Reissner-Nordström solution, presenting the non-vanishing Weyl spinor Ψ_2 and drawing an accurate Penrose diagram of the space-time. A nice proof of the conservation of the parameter M (based on the papers of Ashtekar and Das [1] on the conserved quantities in asymptotically anti-de Sitter space-times) is shown to justify the noun "gravitational mass" we use for it.

Even though they presumably cannot be employed as models for the real universe, the interest in anti-de Sitter space-times increased lately since they are ground states in some supergravity theories that were under consideration in the eighties. During the last years higher dimensional anti-de Sitter space-times have been revived because of a conjectured equivalence between a string theory defined on a space which is assumed to be asymptotically product of anti de Sitter space (AdS) with some closed manifold like a 5-sphere, and a quantum field theory without gravity defined on the conformal boundary of this space, whose dimension is lower by one (AdS/CFT, anti-de Sitter/conformal field theory correspondence). Hence superstring theory on the whole space-time asymptotically $\mathrm{AdS}_5 \times S^5$ is to be equivalent to a certain super-symmetric Yang-Mills theory on $\mathscr{I}$. Now, the metric $\overline{g}_{\mu\nu}$ on the boundary is not uniquely specified by the metric $g_{\mu\nu}$ inside, but there is an equivalence class of metrics defined from $g_{\mu\nu}$ with different conformal factors Ω. The super Yang-Mills theory we apply on $\mathscr{I}$ that gives the correspondence AdS/CFT is conformally invariant though.

This argument shows thus how important might be a complete understanding of the conformally compactified anti-de Sitter space-time, since there are important theories and corroborated conjectures that use such a space-time as their starting point. It might turn out to be much more than a quibble...

Chapter 1

Conformal compactification of space-times

A d-dimensional space-time is a manifold equipped with a metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{1.1}$$

which is Lorentzian in the sense that its signature is $(p, 1)$ with $p = d - 1$ number of positive eigenvalues of the quadratic form $g_{\alpha\beta}$. We will treat mostly manifolds in $(3 + 1)$-dimensions. The idea of *conformally compactifying* a space-time is intuitively to bring the infinite far-away of the physical metric to a finite distance, analyzing thus a new metric $d\hat{s}^2$ that is related to ds^2 through a *conformal factor* Ω in the following way:

$$d\hat{s}^2 = \Omega^2 ds. \tag{1.2}$$

A conformal compactification is a map (that does not distort angles) of an infinite manifold onto a finite one that may make the far away parts of the former accessible to study. Let us consider an infinite plane (that is not a space-time, since its metric doesn't have the required $(p, 1)$ signature), which fig. 1.1 shows a portion of. We can draw a sphere and arrange it in such a way that they would intersect along the equator of the sphere. Chosen a pole on the sphere, say the south pole (point P in the figure), we can project any points on the plane (ex. point B) onto points on the sphere (point b) by intersecting the sphere with the straight line between B and P. The portion of plane outside the sphere is mapped onto the southern hemisphere, while the northern one represents points inside the equator (point A is mapped onto a). One may notice that this correspondence is $1-1$ for all of the points, but the south pole itself. We think of P as representing the infinitely far of the plane.

In the case discussed above, we see how *infinity is a point* for a 2-dimensional flat plane. Of course one may wonder whether this projection is unique or not. It can be proved that the requirements that make the compactification conformal

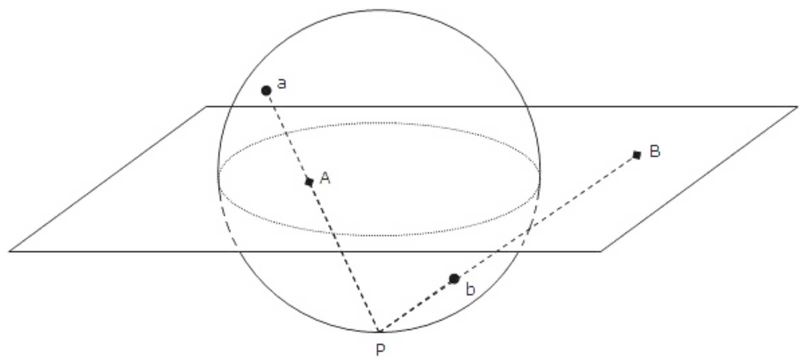

Figure 1.1: Conformal compactification of a plane onto a sphere.

(cf. [9] and [13]) imply such a structure for the infinity. We will discuss this a bit more in detail in the next sections.

1.1 The choice of the conformal factor

Suppose that we are interested in studying the structure of the infinity of a spacetime $\mathscr{S}$. By choosing appropriately the factor Ω, it is possible to adjoin more points to the manifold $\mathscr{S}$ in such a way that the new metric is smooth at those points. Through such a conformal map Ω (suitably smooth diffeomorphism everywhere positive), we take all the points $p \in \mathscr{S}$ into points p' that form a finite portion of a second spacetime $\mathscr{S}'$. The boundary of the image of $\mathscr{S}$ through Ω can be considered infinity of the space $\mathscr{S}$ (see fig. 1.2).

Suppose we want to study the nature of the infinity of some simple 2 dimensional spaces, for example those for which there is a Killing symmetry along one of the coordinates, say φ. The metric for such a space can be given in the form

$$ds^2 = dr^2 + f^2(r)d\varphi^2, \tag{1.3}$$

where $0 < r < +\infty$, $0 < \varphi < 2\pi$ and $f(r)$ is a function that doesn't necessary well-behave at the points at the boundary of its domain. What we aim at is to get rid of such irregularities by multiplying the physical metric by a conformal factor Ω and examine an unphysical metric which satisfies some requirements for *any* $0 < r < +\infty$:

1. we want it to be regular on its domain of definition and especially at those points that were ill-defined for the original metric. Thus the curvature (which in 2 dimensions coincides with the scalar curvature $\hat{R}$) of the compactified metric $d\hat{s}^2$ must be finite;

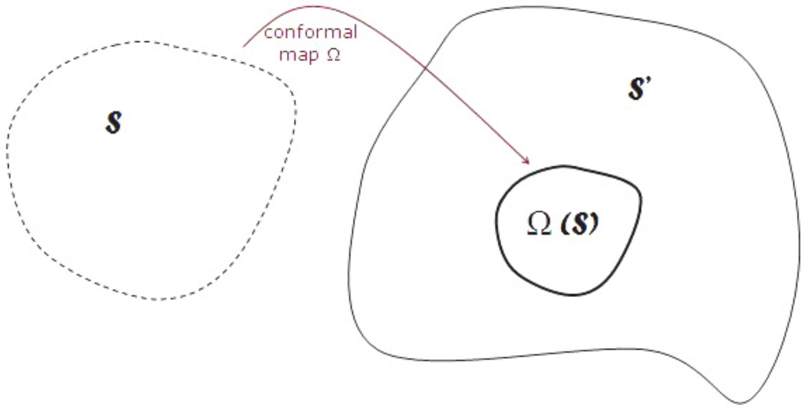

Figure 1.2: The mapping of the spacetime $\mathscr{S}$ into $\Omega(S)$. $\mathscr{S}$ on the left side of the figure is drawn in dashed line to stress the fact that the boundary of the space is not well-defined before the conformal compactification, whereas it is such after the mapping in the subset $\Omega(\mathscr{S})$ of the embedding space $\mathscr{S}'$.

2. the conformal infinity we add is at a finite distance $\int^{\infty} d\hat{s} = \int_0^{+\infty} \Omega(r)dr < +\infty$;

3. we want to be sure that the conformal boundary we add to the space is a compact set, therefore $\hat{C} = \oint d\hat{s} = \int_0^{2\pi} \Omega(r)f(r)d\varphi < +\infty$.

$\hat{C}$ is the circumference of a circle around the origin as measured from the unphysical metric. The conditions 2 and 3 ensure that the unphysical manifold is smooth at the infinity of the physical one.

The conformal factor has to satify some regularity requirements itself, namely:

- $\Omega > 0$ on the original coordinate domain;
- if it is possible to "add" points at infinity to the physical metric (which is, the new manifold is smooth at ∞ and the hatted scalar curvature is everywhere finite), then it must be $\Omega = 0$ at infinity;
- Ω is differentiable as many time as it is needed.

Euclidean plane

An easy example of a space whose metric may be written in the form (1.3) is the flat plane. To see this, we can start from the usual way of writing the metric on a plane, which is $ds^2 = dx^2 + dy^2$. Changing now coordinates into the two polar r and φ such that $x = r\cos\varphi$ and $y = r\sin\varphi$, we find:

$$ds^2 = dr^2 + r^2 d\varphi^2. \tag{1.4}$$

where $0 < r < +\infty$ and $0 < \varphi < 2\pi$. The physical metric ds^2 is ill-defined as $r \to +\infty$. We want to choose a conformal factor that spawns a well-behaved unphysical metric for all the r in the interval. By looking at (1.4) we understand we'd need an Ω that goes to zero at infinity as $1/r^\alpha$, with $\alpha > 1$. Let us analyze the metric we obtain by using such a Ω, recalling that we want to make sure that the hatted metric describes a space-time which is smooth at all of its points, especially those that were irregular for the physiscal metric. The conformal metric would be

$$d\hat{s}^2 = \Omega^2 ds^2 = \frac{1}{r^{2\alpha}}(dr^2 + r^2 d\varphi^2). \tag{1.5}$$

We may change coordinates and write the metric above in a nicer form by defining $dz = dr/r^\alpha$ which means

$$d\hat{s}^2 = dz^2 + (1-\alpha)^2 z^2 d\varphi^2. \tag{1.6}$$

This hatted metric looks somewhat similar to the euclidean metric again, but contains an extra factor of $(1-\alpha)^2$. Define thus a new coordinate $\theta = |1-\alpha|\varphi$ and see how the metric reads

$$d\hat{s}^2 = dz^2 + z^2 d\theta^2, \tag{1.7}$$

which is different from the flat plane metric only for the fact that the θ coordinate this time doesn't have the same periodicity of the old φ, but runs between 0 and $|1-\alpha|2\pi$. This makes an important difference, because it gives rise to a *conical singularity*: since we are not covering now the whole plane but identifying some points in the plane due to the periodicity of θ (points along $\theta = 0$ are equivalent to points along $\theta = |1-\alpha|2\pi$), there is some point (at the tip of the cone) in which the structure of the manifold breaks down: the tangent space at that point is ill-defined thus the manifold is not differentiable. Of course there are two exceptions to the case, which occur at $\alpha = 0$ or $\alpha = 2$, where the period of θ turns out to be $|1-\alpha|2\pi = 2\pi$ as in the flat plane. Consider the requirements on the conformal metric listed above:

1. the requirement on the curvature is satisfied since the scalar curvature is identically zero, hence finite.
2. $\int_0^{+\infty} dz/z^\alpha = z^{1-\alpha}/(1-\alpha)\,\big|_0^{+\infty}$.
3. $\int_0^{|1-\alpha|2\pi} d\theta/z^{\alpha-1} = |1-\alpha|2\pi/z^{\alpha-1}$.

Requirements 2 and 3 imply the following cases

- case $\alpha > 1$: $z = +\infty$ sits at a finite distance, but so does not point $z = 0$; there is a conical singularity at ∞, except if $\alpha = 2$.

- case $\alpha < 1$: $z = +\infty$ is infinitely far, but the point at $z = 0$ is at a finite distance; there is a conical singularity at 0, except if $\alpha = 0$.

- case $\alpha = 1$: both $z = +\infty$ and $z = 0$ are infinitely far away;

the strategy could now be of define a conformal factor on patches of the z domain in a smooth way so that $\alpha = 0$ (the conformal factor is simply 1) in a neighborhood of the origin and $\alpha = 2$ at infinity. In this way, we ensure that the hatted manifold is perfectly smooth in all of the points (it would in fact have the topological stucture of a sphere).

A simpler way to avoid conical singularities could be the use of a more sophisticated conformal factor, to obtain by adding some constant $k^2 > 0$ in the denominator

$$\Omega(r) = \frac{1}{k^2 + r^\alpha}. \tag{1.8}$$

Let us check that the hatted metric satisfies the regularity conditions:

1. $\hat{R}$ for the hatted metric can be calculated (see [13], appendix D) as:

   $$\hat{R} = \Omega^{-2}[R - 2(n-1)g^{ac}\nabla_a\nabla_c \ln\Omega - (n-2)(n-1)g^{ac}(\nabla_a \ln\Omega)\nabla_c \ln\Omega \tag{1.9}$$

   where n is the dimension of the manifold, and R is the scalar curvature of the unhatted metric. For the flat plane, we have of course that $R = 0$, and $n = 2$. After some calculations (see Appendix A), equation (1.9) reads:

   $$\hat{R} = 2k^2\alpha^2 r^{\alpha-2}. \tag{1.10}$$

   Given that both α and k are constants, for $\hat{R}$ to be finite at $r = +\infty$ we require $\alpha \leq 2$, while we would require $\alpha \geq 2$ for $\hat{R}$ to be finite at the origin.

2. $\int^\infty d\hat{s} = \int^\infty \Omega(r)ds = \int_0^\infty \frac{1}{k^2+r^\alpha}dr < +\infty \Leftrightarrow \alpha > 1.$

3. $\hat{C} = \oint d\hat{s} = \int_0^{2\pi} \Omega(r)\ r\ d\varphi = 2\pi\frac{r}{k^2+r^\alpha} < +\infty \forall r \Leftrightarrow \alpha \geq 1.$

The first is a very strict condition: since we want the curvature to be finite for any value of r, we are forced to choose $\alpha = 2$. This is consistent with requirements 2 and 3. The fact that we must set α to 2 is an important feature, as it tells us something about the structure of the hatted manifold and its asymptotics. See in fact how the equation (1.10) for $\alpha = 2$ reads $\hat{R} = 8k^2$, which means that the curvature of the conformally compactified manifold is a positive constant, i.e. the hatted manifold has the geometry of a round sphere. Further, by setting $\alpha = 2$ in the espression for the circle $\hat{C}$ we find

$$\hat{C} = 2\pi\frac{r}{k^2 + r^2} \tag{1.11}$$

which in the limit for $r \to +\infty$ is zero. This means that the conformal boundary of the flat plane is necessarily a point.

The hyperbolic plane

We may consider now the hyperbolic space, where the function $f(r)$ in formula (1.3) is $\sinh(r)$. This is a space of constant negative curvature, with metric

$$ds^2 = dr^2 + \sinh^2 r\ d\varphi^2. \tag{1.12}$$

We notice that the metric is again ill-defined at $r \to \infty$ where the term $\sinh^2 r$ diverges. A possible choice of conformal factor that would work well on the metric (1.12) is for example e^{-r}. The unphysical metric $d\hat{s}^2$ would in fact be perfectly regular for $0 \le r \le +\infty$. The scalar curvature would be

$$\hat{R} = 2e^{2r}\big(-1 + \coth(r)\big) \tag{1.13}$$

that goes to a finite value (4) as $r \to +\infty$, $\int_0^\infty e^{-r}$ is by all means less than infinity, and $\hat{C} = 2\pi$ is finite too. What we want to investigate now is whether it is possible to choose a different conformal factor such that $\hat{C} = 0$: this would mean the conformal boundary of the hyperbolic plane is a point similarly to the flat plane. Up to now, we only proved that for the specific choice of conformal factor

$$\Omega = e^{-r}, \tag{1.14}$$

the infinity of the physical metric is reduced to "something" rather than a point, and that the circle around it measures 2π. Let us analyze better $\hat{C}$: it is 2π times the limit to infinity of the conformal factor multiplied by $\sinh r$. To make $\hat{C}$ go to zero, we could choose an Ω that for large values of r is asymptotical to $\Omega(r) = \frac{1}{r^\alpha}e^{-\beta r}$, where α and β are to be determined in such a way that the requirements 1 and 2 are satisfied and such that

$$\lim \frac{\sinh r}{r^\alpha} e^{-\beta r} = 0. \tag{1.15}$$

1. We require the scalar curvature to be finite for any value of r, also for $r \to +\infty$. By definition, the scalar curvature $\hat{R}$ is equal to $\hat{g}^{ab}\ \hat{g}^{cd}\ \hat{R}_{acbd}$ with $\hat{R}_{abcd}$ Riemann tensor of the unphysical metric. Given the symmetries of such a tensor, we get $\hat{R} = 2\hat{g}^{rr}\hat{g}^{\varphi\varphi}\hat{R}_{r\varphi r\varphi}$. Calculating now the Christoffel symbols, we find that the only nonvanishing algebraically independent components are

$$\hat{\Gamma}^{\varphi}_{r\varphi} = \frac{\Omega\Omega' \sinh^2 r + \Omega^2 \sinh r \cosh r}{\Omega^2 \sinh^2 r},$$
$$\hat{\Gamma}^{r}_{rr} = \frac{\Omega'}{\Omega},$$
$$\hat{\Gamma}_{\varphi\varphi r} = \Omega^2 \sinh^2 r\ \hat{\Gamma}^{\varphi}_{r\varphi}.$$

Consequently the Riemann tensor has only one nonzero component [1]

$$\hat{R}_{r\varphi r\varphi} = [(\Omega')^2 - \Omega\Omega'' - \Omega^2]\sinh^2 r - \Omega\Omega' \sinh r \cosh r. \tag{1.16}$$

The scalar curvature thus is:

$$\begin{aligned}\hat{R} \quad &= \frac{2}{\Omega^4}\big[(\Omega')^2 - \Omega\Omega'\frac{\cosh r}{\sinh r} - \Omega\Omega'' - \Omega^2\big] = \\ &= 2r^{2\alpha}e^{2\beta r}\big[(\frac{\alpha}{r} + \beta)\coth r - 1 - \frac{\alpha}{r^2}\big],\end{aligned} \tag{1.17}$$

from which it follows that we must set $\alpha = 0$ and $\beta = 1$ for $\hat{R}$ to be finite.

2. $\int^{\infty} d\hat{s} = \int_0^{\infty} e^{-r}dr = 1 < +\infty$.

3. $\hat{C} = \int_0^{2\pi} re^{-r}d\varphi = 2\pi re^{-r}$ that is finite not zero for $r \to \infty$.

Hence we cannot choose arbitrarily the exponents α and β. We proved that the conditions on Ω (1 especially) force the conformal boundary to be else than a point, namely the circle surrounding a disk (the Poincaré disk, [3] and [6]).

In general, we understand out of these two simple examples how the choice of the conformal factor is not completely arbitrary, in the sense that the geometry of the original space forces the conformal boundary to be a point in the flat plane, and a circle in the hyperbolic plane.

1.2 Conformal compactification of some important space-times

Minkowski space

Minkowski space has as metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\sigma^2 \tag{1.18}$$

with $d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2$ metric on the 2-sphere. This time we choose to change coordinates and define a retarded time parameter $u = t - r$ and an advanced time parameter $v = t + r$. Metric (1.18) takes the form

$$ds^2 = -du\ dv + \frac{1}{4}(v-u)^2 d\sigma^2.$$

One possible choice of Ω is $1/\sqrt{(1+u^2)(1+v^2)}$ so that we define $d\hat{s}^2 = \Omega^2 ds^2$. In order to assign finite coordinates to the points at infinity, we change coordinates

[1] $\hat{R}_{abcd} = \frac{1}{2}(\hat{g}_{bc,ad} - \hat{g}_{bd,ac} + \hat{g}_{ad,bc} - \hat{g}_{ac,bd}) + \hat{\Gamma}^j_{ad}\hat{\Gamma}_{bjc} - \hat{\Gamma}^j_{ac}\hat{\Gamma}_{bjd}$.

again, by using $u = \tan p$ and $v = \tan q$ (where $-\pi/2 \leq p, q \leq \pi/2$) and define finally $T = \frac{1}{2}(p+q)$ and $\rho = q - p$:

$$d\hat{s}^2 = -dT^2 + \frac{1}{4} \underbrace{\left[d\rho^2 + \sin^2\rho \; d\sigma^2\right]}_{\text{metric of a unit 3-sphere}} \quad ,$$

where $-\pi \leq 2T, \rho \leq +\pi$. If we let the coordinates to vary in some appropriate ranges, $(0 < |T|, \rho < +\infty)$, the resulting space has the structure of a product of a space-like 3-sphere with an infinite time-like line. This means that we embedded Minkowski space in a space that in 2 dimensions (suppressing the θ and φ coordinates that only translate a rotational symmetry) looks like a timelike cylinder (fig. 1.3). The points i^0, i^+ and i^- represent respectively *spatial, future temporal*

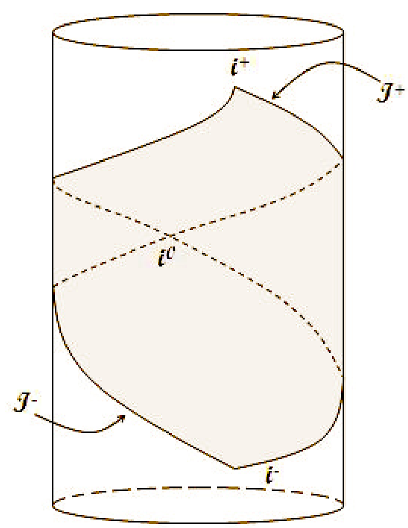

Figure 1.3: Minkowski space wraps around the embedding space to meet back in the point i^0 (which is a single point, as we showed, and not a 2-sphere).

and *past temporal infinity*, while $\mathscr{I}^+$ and $\mathscr{I}^-$ are *future* and *past null infinity*. The Penrose diagram of Minkowski space-time is shown in figure 1.4. We are neglecting the spherical part of the metric, suppressing in this way 2 of the 4 dimensions. Thus every point in the Penrose diagram will represent a 2-sphere (the conformal boundary $\mathscr{I}$ has the topology of a line cross a two sphere), except for i^0, i^+, and i^- that are single points, and the vertical line on the left in the figure: each point on that line corresponds to the origin of Minkowski space, and represents a single point.

Anti-de Sitter space-time

Anti-de Sitter space-time is the maximally symmetric solution to Einstein's vacuum equations once we choose the cosmological constant Λ to be negative (by

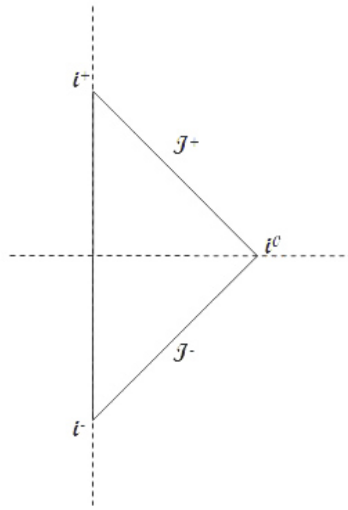

Figure 1.4: Penrose diagram of Minkowski space.

convention -3), whereas it was set to be 0 in the Minkowski solution. The metric for some choice of coordinates can be given in the static form

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\sigma^2, \tag{1.19}$$

where $0 \leq r, |t| < +\infty$. Changing coordinates with $w = \frac{1}{r}$ $(0 < w < +\infty)$, and conformally compactifying in the space coordinate with $\Omega(w) = w$ we get:

$$d\hat{s}^2 = -(1+w^2)dt^2 + \frac{1}{1+w^2}dw^2 + d\sigma^2, \tag{1.20}$$

which is a well-defined metric also on the conformal boundary $\mathscr{I}$ at $w = 0$. The diagram of anti-de Sitter space-time we can draw so far is an infinitely long strip (fig. 1.5), where one can see that we haven't considered the compactification in the time direction yet, since as we said before the rules for drawing Penrose diagrams are that infinity (past and future time-like infinity also!) has to be represented as part of the boundary. We will come back to this important point later on in the next chapter. The conformal boundary of anti-de Sitter space in the picture 1.5 looks like an infinitely long time-like line. Because scri is such, anti-de Sitter is said to be a not "globally hyperbolic" space-time, in the sense that it does not admit a well posed initial data problem.

Schwarzschild space-time

The metric of Schwarzschild space-time is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\sigma^2, \tag{1.21}$$

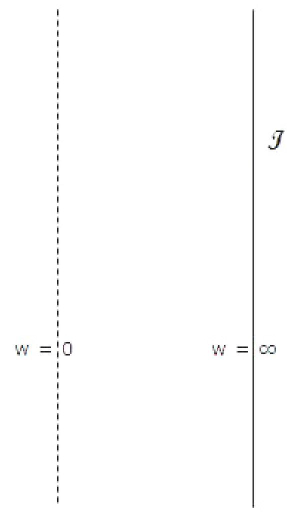

Figure 1.5: Penrose diagram of anti-de Sitter space-time.

where M is the mass parameter. Here is $-\infty < t < +\infty$ and $0 < r < +\infty$ regarded as empty space solution rather than solution outside some spherical body. By using Kruskal coordinates for example [5], we may write the metric as

$$ds^2 = 32\frac{M^3}{r}e^{-\frac{r}{2M}}(-dT^2 + dX^2) + r^2 d\sigma^2 \tag{1.22}$$

with X and T implicitely defined as

$$\begin{cases} X^2 - T^2 = e^{\frac{r}{2M}}(\frac{r}{2M} - 1) \\ 2\tanh^{-1}\frac{T}{X} = \frac{t}{2M} \end{cases}$$

We are interested in the infinite far-away, which occurs at $r \to \infty$. A conformal compactification of the metric with the factor $\Omega(r) = \frac{1}{r}$ produces

$$d\hat{s}^2 = 32\frac{M^3}{r^3}e^{-\frac{r}{2M}}(-dT^2 + dX^2) + d\sigma^2, \tag{1.23}$$

that is well-defined on the conformal boundary $\mathscr{I}$ at $r = \infty$, but is singular in $r = 0$. The invariant $R^{abcd}R_{abcd}$ is equal to $\frac{48M^2}{r^6}$, divergent at $r = 0$: there is a *true* singularity. The Penrose diagram is drawn in figure 1.6.

1.3 Induced metric on $\mathscr{I}$

Let us consider now a generic space-time whose metric is given in the form

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2 d\sigma^2. \tag{1.24}$$

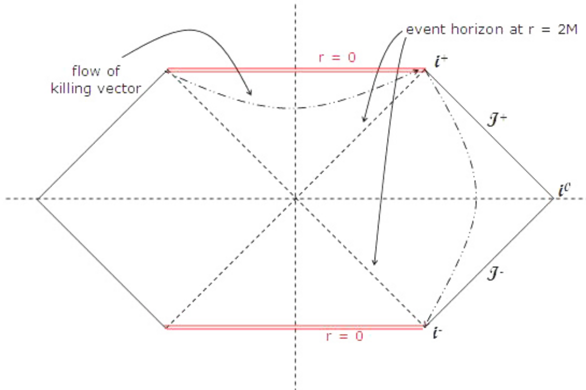

Figure 1.6: Penrose diagram of the Schwarzschild space-time. There is a singularity at $r = 0$ and an event horizon at $r = 2M$. Note that the timelike Killing field (dashed line) becomes spacelike inside the black hole.

We may change coordinates in such a way that

$$du^2 = dt^2 - f^2(r)dr^2 - 2f(r)\ du\ dr, \tag{1.25}$$

where $f(r)$ will be chosen in an suitable way later on. Equation (1.24) becomes

$$ds^2 = -V(r)du^2 - V(r)f^2(r)dr^2 + \frac{1}{V(r)}dr^2 - 2V(r)f(r)\ du\ dr + r^2 d\sigma^2. \tag{1.26}$$

By using again $w = \frac{1}{r}$, and defining $f(r) = \frac{1}{V(r)}$, we get

$$ds^2 = -V(\frac{1}{w})du^2 + \frac{2}{w^2}\ du\ dw + \frac{1}{w^2}d\sigma^2. \tag{1.27}$$

A conformal compactification with a factor $\Omega = w$ leads us to an unphysical metric

$$d\hat{s}^2 = -w^2 V(\frac{1}{w})du^2 - 2\ du\ dw + d\sigma^2. \tag{1.28}$$

Assuming that the function $V(r)$ is a $\mathscr{O}(r)$, on $\mathscr{I}$ $(w = 0)$ the induced metric is:

$$d\hat{s}\,\Big|_{w=0} = 0 \cdot du^2 + 0 \cdot du + d\sigma^2 \tag{1.29}$$

This shows that the direction u is *null* and that $\mathscr{I}$ is a null surface. We can thus find the equation of the outgoing geodesics:

$$0 = du = dt - \frac{1}{V(r)}dr \Rightarrow \int dt = \int \frac{dr}{V(r)} \Rightarrow t - \int \frac{dr}{V(r)} = \text{const.} \tag{1.30}$$

The constant above distinguishes the different geodesics. It is important to see how u is a natural coordinate for our space-time. Note that this arguments does not apply to the aymptotically anti-de Sitter cases for example, since there the $V(r)$ is in the order of r^2 which is not allowed by the assumptions we made. Schwarzschild (metric in (1.21)) and Reissner-Nordström space-times are in the form (1.24).

Chapter 2

Conformal compactification of anti-de Sitter space-time

2.1 The de-Sitter case

We will first analyze the conformal compactification of a space-time whose metric looks somewhat similar to the anti-de Sitter case, but with a positive cosmological constant Λ. It is believed nowadays that $\Lambda > 0$ in our universe. The line element is

$$ds^2 = (R^2 - 1)dT^2 + \frac{dR^2}{1 - R^2} + R^2 d\sigma^2. \tag{2.1}$$

Notice an important feature of de-Sitter space: time and space coordinates switch place at $R \geq 1$. We say that at $R = 1$ there is a *cosmological event horizon*. Look at the metric in (2.1); the R is meant to be "large": this makes the time coefficient be positive -spacelike- and the spatial one be negative - timelike. Namely R can be regarded as the time coordinate and T as the spatial one. Performing a change of coordinates into the usual $w = \frac{1}{R}$, we get

$$ds^2 = (\frac{1}{w^2} - 1)dT^2 + \frac{1}{w^2}\frac{dw^2}{w^2 - 1} + \frac{1}{w^2}d\sigma^2 \tag{2.2}$$

that is again ill-defined on $\mathscr{I}$ surface $w = 0$. The conformal compactification in the space coordinate may be done by using for example the factor $\Omega(w) = w$, obtaining

$$d\hat{s}^2 = \Omega^2(w)ds^2 = (1 - w^2)dT^2 + \frac{dw^2}{w^2 - 1} + d\sigma^2. \tag{2.3}$$

We could "add" the points at $w = 0$ which become the conformal infinity of the de-Sitter space-time. Notice how on $\mathscr{I}$ the metric is that of a *spacelike cylinder*. By looking carefully at the metric, we may see how the T coordinate can still run between $-\infty$ and $+\infty$, which makes our space-time not compact. What we would like to do now is to "add" two points at the spatial ends of that cylinder.

By changing coordinates again into $\rho = e^T$ and conformally compactifying with the factor $\Omega(\rho) = \frac{2\rho}{1+\rho^2}$ we can write

$$d\hat{\hat{s}}^2 = \frac{4\rho^2}{(1+\rho^2)^2} d\hat{s}^2 = \frac{4(1-w^2)}{(1+\rho^2)^2} d\rho^2 + \frac{4\rho^2}{(1+\rho^2)^2} (\frac{dw^2}{w^2-1} + d\sigma^2). \tag{2.4}$$

Let us now investigate the topology allowed by the metric induced on $w = 0$. We have

$$d\hat{\hat{s}}^2 = \frac{4}{(1+\rho^2)^2} (d\rho^2 + \rho^2 d\sigma^2). \tag{2.5}$$

We claim that this is a sphere. Consider thus the equation that describes a 3-sphere, namely

$$\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2 + \hat{U}^2 = 1 \tag{2.6}$$

with line element $ds^2 = d\hat{X}^2 + d\hat{Y}^2 + d\hat{Z}^2 + d\hat{U}^2$. Define now

$$x = \frac{\hat{X}}{1+\hat{U}}, \quad y = \frac{\hat{Y}}{1+\hat{U}}, \quad z = \frac{\hat{Z}}{1+\hat{U}}, \quad \rho^2 = \frac{1-\hat{U}}{1+\hat{U}} \tag{2.7}$$

and write the metric as

$$ds^2 = \frac{4}{(1+\rho^2)^2} (dx^2 + dy^2 + dz^2) \tag{2.8}$$

which in polar coordinates is exactly the metric in (2.5). This means that by looking at the boundary of the compactified de-Sitter space-time, on $\mathscr{I}$ we turned the infinte long cylinder into a perfectly smooth surface, a sphere indeed (fig. 2.1). Notice also how this argument is related to the stereographic projection described in chapter 1 for the flat plane, only in one dimension higher (see fig. 1.1). In

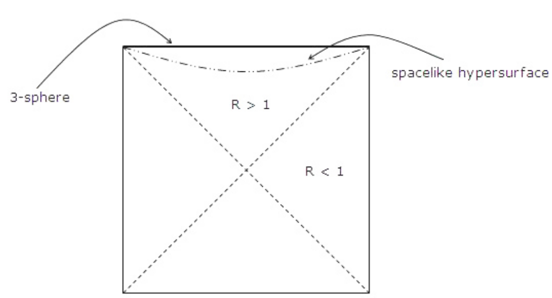

Figure 2.1: Penrose diagram of de-Sitter space. On top of the diagram there's the surface we turned into a smooth 3-sphere.

the Penrose diagram of figure 2.1, every point is meant to represent a 2-sphere whereas the points on the vertical lines represent single points.

2.2 Anti-de Sitter space-time

Let us try to apply the same manoeuvre used in the de-Sitter case to the metric

$$ds^2 = -(R^2+1)dT^2 + \frac{dR^2}{1+R^2} + R^2 d\sigma^2. \tag{2.9}$$

Again now, if we change coordinates ($w = \frac{1}{R}$ and $\rho = e^T$) and conformally compactify in time and space ($\Omega = \frac{2w}{1+\rho^2}$) as done before, we end up with the line element

$$d\hat{\hat{s}}^2 = -\frac{4(1+w^2)}{(1+\rho^2)^2} d\rho^2 + \frac{4\rho^2}{(1+\rho^2)^2}(\frac{dw^2}{w^2+1} + d\sigma^2), \tag{2.10}$$

and on $\mathscr{I}$

$$d\hat{\hat{s}}^2 = \frac{4}{(1+\rho^2)^2}(-d\rho^2 + \rho^2 d\sigma^2) \tag{2.11}$$

that looks very similar to (2.5), except for a "$-$ sign in front of $d\rho^2$.

Let us focus on the study of the line element $dl^2 = (-d\rho^2 + \rho^2 d\sigma^2)$. If we were in $(2+1)$ dimensions, we would write the metric on the conformal boundary as

$$dl^2 = (-d\rho^2 + \rho^2 d\varphi^2) \tag{2.12}$$

with $0 \leq \varphi < 2\pi$ periodically, that has the topology of $S^1 \times R^1$ and is an example that has been discussed by Misner (cf. [5] and his original paper [8]). This metric has a singularity at $\rho = 0$, which is not a curvature singularity. It is the mere structure of the manifold that collapses in those points. It is somewhat similar to the conical singularity we described when analyzing the flat plane: if the coordinate φ had an unrestricted domain ($-\infty < \varphi < +\infty$), there wouldn't be any singluarity.

In $(3+1)$ dimensions the situation is more complicated. A computation of the scalar curvature shows that

$$\hat{\hat{R}} = \frac{w^2}{4(1+w^2)\rho^2}(6 - 8\rho^2 + 6\rho^4 + 7w^2(2+w^2)(1+\rho^2)^2),$$

which is divergent at $\rho = 0$. This might depend on the choice of conformal factor, so we try now a different approach. Anti-de Sitter space is conformal to the part of the Einstein universe whose metric is

$$ds'^2 = -dt'^2 + R_0^2(d\chi^2 + \sin^2\chi d\sigma^2) \tag{2.13}$$

where R_0 is a constant, $-\infty < t' < +\infty$, $0 < \chi < \pi$, $d\sigma^2$ being the metric on the 2-sphere. In order to draw the Penrose diagram of the Einstein universe, we need to map it into a finite subset where $R_0 = 1$, and given the manifest spherical symmetry ($d\sigma$ term) we shall only focus on the t and χ coordinates. From metric (2.13), we see that on the (t', χ) plane the line element will be

$$ds^2 = -dt'^2 + d\chi^2 \tag{2.14}$$

which means that the coordinates t' and χ represent proper time and distance respectively. If we wish to change the time coordinate into

$$t = \arctan \frac{2a}{R_0} t', \tag{2.15}$$

where a is a constant to be determined, the metric will be

$$ds^2 = -dt^2 + (2a)^2 \cos^4(t)(d\chi^2 + \sin^2 \chi d\sigma^2); \tag{2.16}$$

with $-\frac{\pi}{2} < t < \frac{\pi}{2}$, which has spatial topology S^3, but some odd behavior at $t = -\pi/2$ and $t = \pi/2$ respectively where the $\cos^4(t)$ is zero. The Penrose diagram can be obtained by considering the proper time and distance. The coordinate t is already proper time. The proper distance is

$$\int dr = \int 2a \cos^2(t) d\chi. \tag{2.17}$$

Integrating in χ and recalling that $\cos^2 t = (1 + \cos(2t))/2$, we get that

$$r = a\chi(1 + \cos(2t)). \tag{2.18}$$

If, as suggested by Penrose, the fundamental observer is chosen to be the maximum area 2-sphere in a t =constant surface, their paths in the (t, r) plane are given by

$$r = a(1 + \cos(2t))(\chi - \frac{\pi}{2}). \tag{2.19}$$

Now comes the restriction on a. We require the paths of the fundamental observers at χ =constant to be timelike. Thus $|dr/dt| < 1$ that implies $a < 2/\pi$. This issue was argued by Tipler (cf. [12], see also the paper from García-Parrado and Senovilla in [11]) who drew the Penrose diagram for the Einstein universe like the one in fig. 2.2. Notice that by calculating the derivative of r with respect to t, we get

$$\frac{\partial r}{\partial t} = -2a(\chi - \frac{\pi}{2}) \sin(2t), \tag{2.20}$$

which goes to zero at $t = \pm\frac{\pi}{2}$. This allows light rays to travel back and forth an infinite number of times at the points i^+ and i^-. In figure 2.2 every point represents a 2-sphere, except for i^+ and i^- that are just points. Since any attempt to extend the conformal metric to include those points led to a singularity (Misner singularity in (2+1) dimensions and curvature singularity in (3+1) dimensions), it seems to be impossible to attach smoothly these points to the conformal boundary $\mathscr{I}$.

Coming back now to our original purpouse of studying anti-de Sitter space, we restrict the coordinate χ to vary between 0 and $\pi/2$ only. Thus the Penrose diagram of anti-de Sitter space-time must look like in figure 2.3. Every point

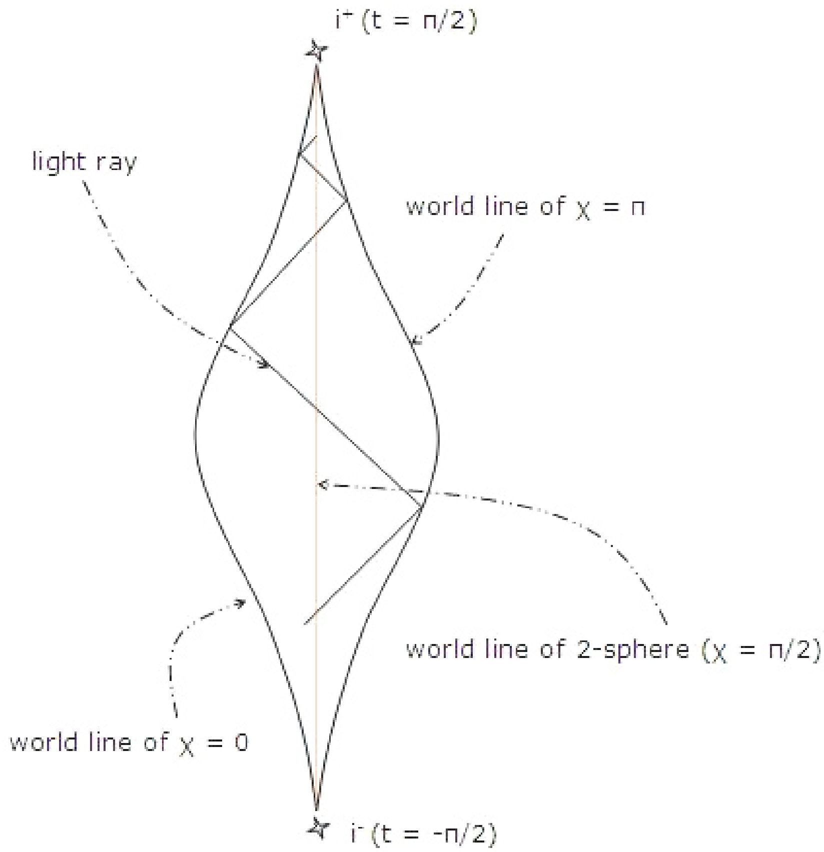

Figure 2.2: Penrose diagram of the Einstein universe. On the x-axis we represented r, while t is on the y-axis.

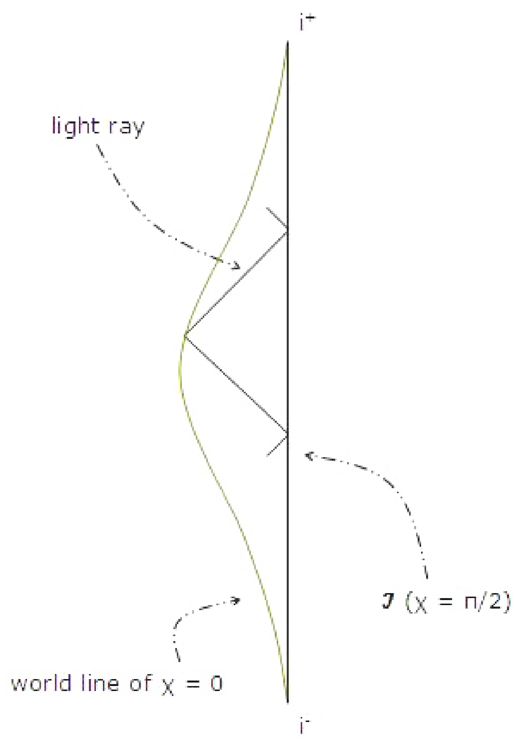

Figure 2.3: Penrose diagram of the compactified anti-de Sitter spacetime.

in the figure represent a 2-sphere, except for those with $\chi = 0$ (that are just points at the origin of coordinates of the three-sphere at t =constant), i^+ and i^-. See how the conformal boundary consists of the points i^+ and i^-, and a smooth time-like surface $\mathscr{I}$ that *begins* at i^+ and *ends* at i^-. What's left is to discuss the nature of those two tips of the Penrose diagram, namely i^+ and i^-. Are those pathologies of the metric? Or is the divergence only due to a bad choice of the conformal factor? We cannot find a proof to that. And it seems that a formal discussion of this point does not exist in the literature. Nevertheless, by private communication with Helmut Friedrich, we know there is some reasons (related to the covering of anti-de Sitter space-time by a congruence of conformal geodesics) why we couldn't find such a change of coordinates that would allow us to write anti-de Sitter space-time as embedded in a conformal expansion of it in which i^+ and i^- are regular points that connect smoothly to the conformal boundary $\mathscr{I}$. On the other hand, we did find such a smooth embedding for Minkowski space-time (Einstein universe indeed, as in fig. 1.3) and regarded the resulting compactified space (with past and future infinity added) as a regular portion of it.

Chapter 3

The black hole cases

The Weyl tensor

As we said before, the conformal techniques invented by Penrose allow us to define an equivalence class of metrics, g_{ab} being equivalent to $\hat{g}_{ab} = \Omega^2 g_{ab}$ with Ω conformal factor. In order to study the properties and the structure of a certain space-time we cannot therefore consider the Riemann tensor R_{abcd} since it will depend on the choice of Ω, in a very non-trivial way (similarly to the scalar curvature, for which we gave an explicit formula in the appendix A). We need to define a new tensor that has to be invariant under conformal changes to the metric (a conformal tensor). The $(1,3)$ Weyl tensor is such. It is defined to be the traceless component of the Riemann curvature tensor. It has the same symmetries as the Riemann curvature tensor with the further condition that its Ricci curvature tensor must vanish. Its components are given by

$$C_{abcd} = R_{abcd} - \left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{1}{3}R\ g_{a[c}g_{d]b}, \tag{3.1}$$

and are in the number of 10 algebraically independent real components. These may be seen also as 5 complex components Ψ_i. Further, stationary black holes happen to belong to a special class of metrics ("Petrov type D") for which some clever choices will ensure that there is only one non-zero complex component of the Weyl tensor, namely Ψ_2. The Weyl tensor is hence conformally invariant, that is if $\hat{g}_{ab} = \Omega g_{ab}$ for some positive scalar function Ω then the (1,3) valent Weyl tensor satisfies

$$\hat{C}^d{}_{abc} = C^d{}_{abc}. \tag{3.2}$$

In $(2+1)$ dimensions, the Weyl tensor is identically zero, but it is generally not so in higher dimensions. As a matter of fact, it has the nice property of characterizing conformally flat spaces. A space described by the metric ds^2 is said to be *conformally flat* if one can find a conformal factor Ω positive function of the coordinates chosen in the space-time such that $d\hat{s}^2 = \Omega^2 ds^2$ is the flat metric. Hence, the Weyl tensor C_{abcd} vanishes if and only if the metric describes a conformally flat space-time. It is so in the de-Sitter and anti-de Sitter space-times.

Definition of asymptotically anti-de Sitter space-times

In this chapter we will treat some of the black hole examples whose metrics are *asymptotically anti-de Sitter*, namely Schwarzschild-anti-de Sitter and Reissner-Nordström-anti-de Sitter (in particular, we will analyze the extremal case). The idea is that such a space-time should look like anti-de Sitter space-time "far away" from any mass concentration or black hole that may be present. The conformal compactification is very useful in order to give a precise definition of asymptotically anti-de Sitter space-times (cf. [1] and [2]):

Definition: A d-dimensional space-time $(\mathscr{M}, g_{ab})$ is said to be *asymptotically anti-de Sitter* if there exists a manifold $\hat{\mathscr{M}}$ with boundary $\mathscr{I}$, equipped with a metric $\hat{g}_{ab}$ and a diffeomorphism from $\mathscr{M}$ onto $\hat{\mathscr{M}} - \mathscr{I}$ of $\hat{\mathscr{M}}$ (with which we identify $\mathscr{M}$ and $\hat{\mathscr{M}} - \mathscr{I}$) and the interior of $\hat{\mathscr{M}}$ such that:

- there exists a function Ω on $\hat{\mathscr{M}}$ for which $\hat{g}_{ab} = \Omega^2 g_{ab}$ on $\mathscr{M}$
- $\mathscr{I}$ has the topology of S^{d-2}, Ω vanishes on $\mathscr{I}$ but $\nabla_a\Omega$ normal vector of $\mathscr{I}$ is nowhere vanishing on $\mathscr{I}$
- on $\mathscr{M}$, g_{ab} satisfies

$$R_{ab} = -\frac{1}{2}Rg_{ab} + \Lambda g_{ab} = -8\pi G_{(d)}T_{ab}, \tag{3.3}$$

  with Λ constant smaller than zero, $G_{(d)}$ Newton constant in d dimensions, T_{ab} stress-matter tensor such that $\Omega^{2-d}T_{ab}$ admits a smooth limit to scri
- the Weyl tensor of $\hat{g}_{ab}$ is such that $\Omega^{4-d}C_{abcd}$ is smooth on $\hat{\mathscr{M}}$ and vanishes at $\mathscr{I}$.

The first condition ensures that the new metric $\hat{g}_{ab}$ is conformally related to the physical metric g_{ab}; the second instead requires that the topology of the boundary is the one suggested by the geometry of anti-de Sitter space-time, and that the boundary itself is attached at infinity with respect to the physical metric. The reqirement on the normal of scri implies that Ω is a good radial coordinate in a neighborhood of scri in the hatted space-time. Third condition is a restriction to the asymptotic behavior of matter fields which ensures that the fluxes of some conserved quantities across $\mathscr{I}$ are well-defined. We also expect that the limit of $\Omega^{4-d}C_{abcd}$ on the conformal boundary vanishes since as a property of the Weyl tensor, $C_{abcd} = 0$ in anti-de Sitter space-times.

3.1 Schwarzschild-anti-de Sitter black hole

The Schwarzschild-anti-de Sitter solution is the one we obtain once the cosmological constant is set to be $\Lambda = -3$; therefore the metric in its static form is given by

$$ds^2 = -\big(1 + r^2 - \frac{2M}{r}\big)dt^2 + \frac{1}{1 + r^2 - \frac{2M}{r}}dr^2 + r^2 d\sigma^2, \tag{3.4}$$

where $0 < r, |t| < +\infty$, and $d\sigma^2$ is the metric on the 2-sphere. Such a metric is ill-defined as the r coordinate tends to $+\infty$. We can conformally compactify it in the space coordinate by multiplying by $\Omega^2 = \frac{1}{r^2}$ and in another set of coordinates where $r = \frac{1}{w}$:

$$d\hat{s}^2 = -\big(1 + w^2 - 2Mw^3\big)dt^2 + \frac{1}{1 + w^2 - 2Mw^3}dw^2 + d\sigma^2 \tag{3.5}$$

with $0 < w < +\infty$. The new hatted metric is now perfectly regular on $w = 0$. The Penrose diagram is depicted in figure 3.1. Notice how the diagram differs

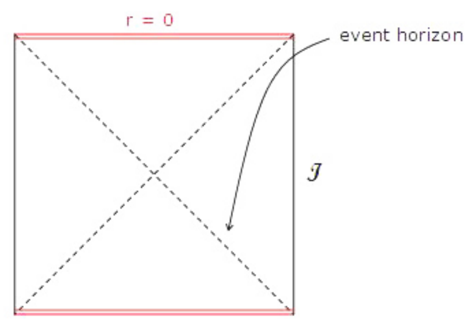

Figure 3.1: The dashed lines are event horizons, the solid lines are the conformal boundaries $\mathscr{I}$ and the double lines are the singularities at $r = 0$.

from the one given in the asymptotically flat case, in figure 1.6. The conformal boundary $\mathscr{I}$ now looks like the $\mathscr{I}$ for an anti-de Sitter space-time (fig. 1.5) rather than the Minkowski $\mathscr{I}$ (fig. 1.4).

We may want now to change coordinates again and use

$$t = \ln \rho \tag{3.6}$$

with $0 < \rho < +\infty$, so that the metric $d\hat{s}^2$ in (3.5) reads

$$d\hat{s}^2 = -\frac{1 + w^2 - 2Mw^3}{\rho^2}d\rho^2 + \frac{1}{1 + w^2 - 2Mw^3}dw^2 + d\sigma^2. \tag{3.7}$$

We see now clearly how this metric is ill-defined on the past time-like infinity $\rho = 0$, so we conformally compactify it:

$$\begin{aligned} d\hat{\hat{s}}^2 &= \big(\frac{2\rho}{1+\rho^2}\big)^2 d\hat{s}^2 = \\ &= \frac{4}{(1+\rho^2)^2}\big[-(1-2Mw^3+w^2)d\rho^2 + \\ &+\rho^2(\frac{1}{w^2(1-2Mw^3+w^2)}dw^2 + d\sigma^2)\big]. \end{aligned} \tag{3.8}$$

On the conformal boundary $w = 0$, the induced metric is

$$d\hat{\hat{s}}^2 = \frac{4}{(1+\rho^2)^2}\big(-d\rho^2 + \rho^2 d\sigma^2\big) \tag{3.9}$$

that is exactly the same we had in the anti-de Sitter case. It seems that *the boundary doesn't feel the mass*. Let us go deeper in this issue and analyze the components of the conformally invariant Weyl tensor ${C^d}_{abc}$. They all are in the form of some functions of the mass (that do not depend on w) multiplied by w. This makes the tensor go to zero as $w \to 0$, as required by the definition given at the beginning of this chapter, but also gives a structure to $\mathscr{I}$ that the anti-de Sitter space-time didn't have (where the Weyl tensor was simply identically zero). We can thus expand the components in a power series in w and analyze a "rescaled" Weyl tensor

$$K^d_{abc} \stackrel{\text{def}}{=} \frac{1}{w} C^d_{abc}. \tag{3.10}$$

On the conformal boundary, some of its components are just numbers (or numbers times well-behaved functions of θ) times mass, but some others contain factors like

$$\frac{1}{\rho^2} \tag{3.11}$$

and hence diverge at $\rho = 0$. The rescaled Weyl tensor seems to diverge for any choice of coordinates on the time boundaries, which led us to suspect that there's something odd happening in the tips of the conformal boundary $\mathscr{I}$, in addition to what happens in the anti-de Sitter case discussed in the last chapter. We will study this peculiarity more carefully in the aymptotically anti-de Sitter Reissner-Nordström case.

3.2 Reissner-Nordström-anti-de Sitter black hole

The Reissner-Nordström solution represents the space-time outside a spherically symmetric charged body or black hole carrying an electrical charge (with no spin or magnetic dipole). It is the spherically symmetric asymptotically anti-de Sitter solution that has the form

$$ds^2 = -\left(1 + r^2 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 + r^2 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\sigma^2, \tag{3.12}$$

where M represents the gravitational mass (later in this chapter we will prove that this parameter is conserved, as expected) and Q the electric charge. It is somewhat rather similar to the Schwarzschild metric locally. This solution would normally only be considered outside the body, but it is anyway interesting to see what happens if we look at it as a function for $0 < r < +\infty$.

We find more convenient to write the metric in different coordinates, namely

$$w = \frac{1}{r} \quad \text{and} \quad t = \ln \rho \tag{3.13}$$

where again $0 < w < +\infty$ and $0 < \rho < +\infty$. The metric becomes

$$ds^2 = -\frac{1 - 2Mw + w^{-2} + Q^2 w^2}{\rho^2} d\rho^2 + \frac{dw^2}{w^4(1 - 2Mw + w^{-2} + Q^2 w^2)} + w^{-2} d\sigma^2. \tag{3.14}$$

We choose the unphysical metric to be

$$d\hat{s} = \left(w \frac{2\rho}{1 + \rho^2}\right)^2 ds^2 \tag{3.15}$$

and study the induced one on the conformal boundary $\mathscr{I}$:

$$d\hat{\hat{s}}^2 \Big|_{w=0} = \frac{4}{(1 + \rho^2)^2}[-d\rho^2 + \rho^2 d\sigma^2]. \tag{3.16}$$

The line element in (3.16) presents the same "irregularity" that the anti-de Sitter and schwarzschild-anti-de Sitter space-times had on $w = 0$. Let us analyze the structure of our space-time with the rescaled Weyl tensor K^d_{abc} defined as above. Similarly to the Schwarzschild-anti-de Sitter case, we find that all the components depend on the mass M on the conformal boundary, but some of them diverge with some power of ρ on the time-like infinity i^-. A CLASSI computation of the only non-vanishing Weyl spinor Ψ_2 by Jan E. Åman (private communication) gives for the unhatted metric in (3.14)

$$\Psi_2 = w^4 (Q^2 - \frac{M}{w}) \tag{3.17}$$

which leads to a $\hat{\Psi}_2$

$$\hat{\Psi}_2 = \frac{1}{\Omega^2}\Psi_2 = \frac{(1 + \rho^2)^2}{4w^2\rho^2}\Psi_2 = w\frac{Q^2 w - M}{4}\frac{(1 + \rho^2)^2}{\rho^2}. \tag{3.18}$$

The spinor thus diverges on both time-like infinities. The structure of the space-time must have something special in those points: it seems that the metric itself singles them out.

3.2.1 Asymptotically anti-de Sitter extremal Reissner-Nordström black hole

Let us focus on a particular case of Reissner-Nordström-anti-de Sitter solution, the one called *extremal*. The metric is again

$$ds^2 = -\big(1 + r^2 - \frac{2M}{r} + \frac{Q^2}{r^2}\big)dt^2 + \frac{dr^2}{1 + r^2 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\sigma^2, \tag{3.19}$$

but this time we want to find that particular value of $M = M(Q)$ such that a degenerate Killing horizon appears. The vector $\xi = \partial/\partial t$ is a Killing vector. We calculate its norm

$$\|\xi\|^2 = g_{ab}\xi^a\xi^b = g_{tt} = -\big(r^2 + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\big). \tag{3.20}$$

The vector is thus time-like when $g_{tt} < 0$ and spacelike when $g_{tt} > 0$. In order to study the roots of this polynomial, let us define a function $F(r)$ such that

$$F(r) = r^2 + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{3.21}$$

The polynomial $-r^2F(r)$ is a fourth-order polynomial in r that has therefore four complex roots. Physically we care only about real positive roots. We can factorize the polynomial into

$$r^2F(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4) \tag{3.22}$$

where the r_i-s are the roots. Our aim is to find that value of $M = M(Q)$ such that $F(r)$ has two coincident roots: in that way the space-time will have a degenerate Killing horizon (the Killing flows are timelike in both sides of the horizon). A geometrical study of the equation $-r^2F(r) = r^4 + r^2 - 2Mr + Q^2$ shows that the function must have at most two real roots, which we call r_+ and r_-. It is a convex curve and hence has minimum (look at fig. 3.2). We require thus that the minimum and the root occur at the same $r = r_*$. After some calculations we find that

$$r_* = \frac{3}{2}M + \sqrt{\frac{9}{4}M^2 - 2Q^2} \tag{3.23}$$

and the critical value of the mass is

$$M_* = \frac{1}{3}\sqrt{-\frac{1}{6} + 6Q^2 + \frac{1}{6}(1 + 12Q^2)^{3/2}}. \tag{3.24}$$

The Penrose diagram of the extremal Reissner-Nordström-anti-de Sitter black hole is shown in figure 3.3. The conformal boundary $\mathscr{I}$ is a chain of disconnected components, each one of them similar to the anti-de Sitter case where

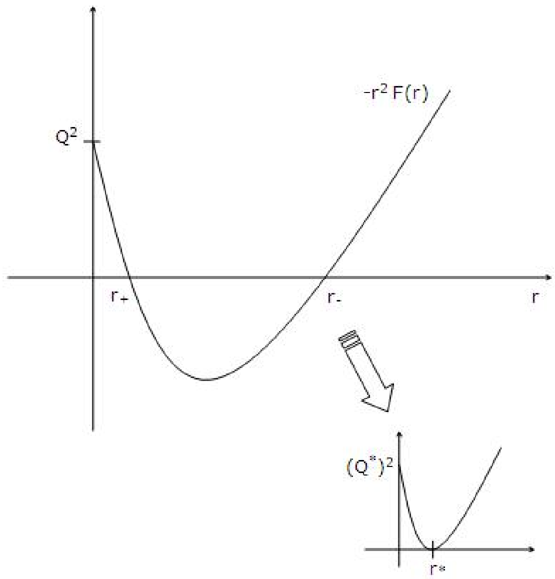

Figure 3.2: The function $-r^2F(r)$ has at most two real roots r_+ and r_-. What we want is to find that value of $M(Q)$ such that the curve looks like the one on the right hand side, the one with one double root r_* where the minimum is on the horizontal axis.

smooth time-like surfaces $\mathscr{I}$ begin at i^+ and end at i^-. A Penrose diagram of the conformal boundary is given in figure 3.4. The points at past and future time-like infinity are NOT attached in a smooth way to the rest of the boundary. We couldn't in fact find any coordinate change that would allow us to attach them smoothly (avoiding a curvature singularity), and have reasons to believe that this is not possible as it was in the anti-de Sitter case discussed in the chapter 2. It would take an observer infinite time to travel from any point to i^+ or backwards to i^-, and at those points the only non-vanishing Weyl spinor Ψ_2 is divergent: it is the metric that singles out i^+ and i^- and makes them exceptional. Unlike in the paper from Brill et al. (cf. [4] and [7]) where it is stated that "the infinity consists of a single connected component" and its "past consists of all of the space-time", we assert that the asymptotically anti-de Sitter Reissner-Nordström space-time has indeed an interpretation as a black hole, and the Killing horizons are in fact event horizons in the space-time since there are points on the conformal boundary that are singled out. Every disconnected component of scri looks as it is depicted in the diagram 3.5 besides the dimension, that is one less in the figure.

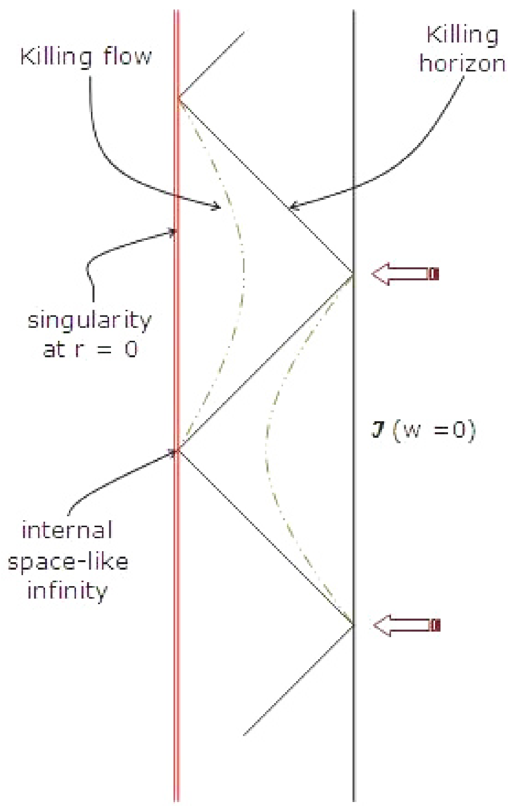

Figure 3.3: Penrose diagram of the extremal Reissner-Nordström-anti-de Sitter black hole. The double line represents the singularity at $r = 0$, the dashed lines are the flows of the Killing vectors on both sides of the Killing horizon (solid line). The conformal boundary is the vertical solid line. The points in which the Weyl spinor Ψ_2 is divergent are those on the boundary where the arrows are pointing.

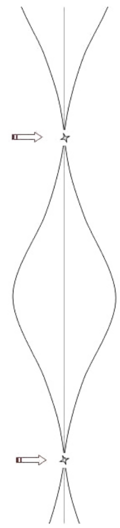

Figure 3.4: Penrose diagram of the conformal boundary of a RN-AdS space-time.

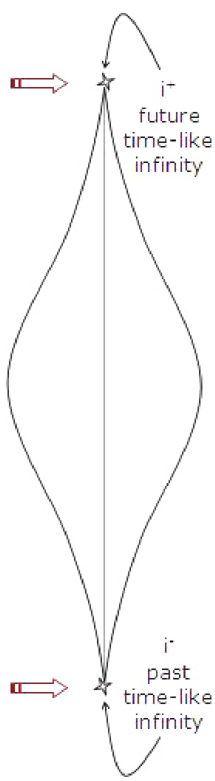

Figure 3.5: Penrose diagram each component of the conformal boundary of the Reissner-Nordström-anti-de Sitter black hole.

Conserved quantities: gravitational mass M

For the sake of completeness, we will present in this section an interesting interpretation of the parameter M, following the arguments in the paper from Ashtekar and Das (cf. [1]) where they discuss how, from the analysis of the asymptotic field equations, one may define conserved quantities as 2-sphere integrals involving the electric part of the Weyl tensor and conformal Killing fields on $\mathscr{I}$. If the gradient of the conformal factor is

$$\hat{n}_a = \nabla_a \Omega, \tag{3.25}$$

the electric part of the rescaled Weyl tensor $\hat{K}_{abcd}$ at $\mathscr{I}$ can be written as

$$\hat{\epsilon}_{ab} = \hat{K}_{ambn}\hat{n}^m \hat{n}^n. \tag{3.26}$$

Given any asymptotic symmetry (i.e. a Killing field ξ^a on $\mathscr{I}$) and a 2-sphere cross section C on the conformal boundary, we can define a conserved quantity

$$Q_\xi[C] = -\frac{1}{8\pi}\oint_C \hat{\epsilon}_{ab}\xi^a dS^b \tag{3.27}$$

that satisfies a balance equation in the sense that $Q_\xi[C_2] - Q_\xi[C_1] = 0$ for any cross sections C_1 and C_2 of $\mathscr{I}$, if the flux of the matter stress-energy tensor across the portion $\Delta\mathscr{I}$ of $\mathscr{I}$ whose boundaries are C_1 and C_2 vanishes.

Let us consider the case of extremal Reissner-Nordström-anti-de Sitter black hole. There is no physical matter field anywhere, so all the Q_ξ are absolutely conserved. In particular, if the conformal Killing field corresponds to the time translation Killing field $\partial/\partial t$ on the original space-time (whose metric is given in the static form) it is straightforward to calculate the conserved quantity Q_t. The only nonzero component of the tensor $\hat{\epsilon}_{ab}$ are

$$\hat{\epsilon}_{11} = -2M, \quad \hat{\epsilon}_{33} = -M, \quad \hat{\epsilon}_{44} = -M\sin^2\theta, \tag{3.28}$$

thus sustituting back in equation (3.27) and integrating on the unit 2-sphere, we find

$$Q_t = -\frac{1}{8\pi}\oint -2M dS = -\frac{1}{8\pi}(-2M)4\pi = M. \tag{3.29}$$

This simple calculation shows how M is a conserved quantity, as we would expect from a parameter which should describe mass.

This proof leads also to consider another important issue, that we mentioned already in the preface, namely what the correct defintion of "gravitational energy-mass" would be. We showed how it can be written as integral on a 2-sphere cross section on the conformal boundary involving the electric part of the rescaled $(0, 4)$ Weyl tensor and a Killing field ξ^a on $\mathscr{I}$ (which in our case was time translation).

In an analogy with electromagnetism again, we recall that the total electric charge (expressed as volume integral of a charge density ρ) can be written (by Gauss' theorem) as an integral of the electric field on a surface enclosing the charges. Gravity is slightly more insidious because the surface we wish to compute the integral on, has to enclose all of the gravitational field, i.e. it has to be "at infinity".

Chapter 4

Conclusions

The aim of the whole thesis work was to understand the features of a particular family of black holes, namely the asymptotically anti-de Sitter solutions, and draw correct Penrose diagrams that could give a 2 dimensional representation of such space-times. To do this, we had to get ahead through small steps and study the theory of conformal compactification, fundamental in order to be able to analyze the asymptotics of the metric associated to the space-time of interest. Most of this work thus is a review of pre-existing material (especially chapter 1), logically stucturated with the particular purpose of simplifying the main discussion. Some of the issues examined though are quite difficult to find in the literature, such as the detailed discussion regarding the appropriate conformal factor to be chosen in the hyperbolic plane example.

The most important part of the work has been trying to conformally compactify asymptotically anti-de Sitter space-times in all of the coordinates, since it seems to be a not very well debated issue in the existing literature. The Penrose diagram of anti-de Sitter space-time for example is usually depicted as an infinitely long strip in the time-direction, which is obviously wrong since the space is not compact in the first place. When trying to find a suitable conformal factor that in some coordinate system compactifies correctly the AdS space, we ended up every time in a singularity (Misner-type in $(2+1)$ dimensions and curvature singularity in $(3+1)$ dimensions) and made the conjecture that it is impossible to add smoothly two points i^+ and i^- at the conformal boundary $\mathscr{I}$. A formal proof of this conjecture couldn't be found from us therefore it is not presented here, but there are some arguments in favour of it such as the divergence of the only non-vanishing Weyl spinor Ψ_2 at i^+ and i^-. Later a sketch of a proof has been provided by Helmuth Friedrich (private communication) who produced arguments based on the impossibility of a covering of anti-de Sitter space-time by a congruence of conformal geodesics. This is probably a not very primary issue in the theory of compactification of space-times, but a deeper understanding of it allowed us to characterize the structure of the asymptotically AdS-Reissner-Nordström black hole and appropriately study its horizons.

Appendix A

Covariant derivative

The covariant derivative is a differential operator that generalizes the notion of directional derivative from vector calculus. It is a linear operator that is required to transform, under a change of coordinates, in the same way as a vector does according to a change of basis formula (i.e. covariant transformation). We will compute covariant derivatives of symmetric $(0, 2)$ tensors T_{jk}, for which the formal definition reads

$$\nabla_i T_{jk} = \frac{\partial T_{jk}}{\partial x^i} - 2\Gamma^l_{ij} T_{lk}, \tag{A.1}$$

where the Christoffel symbols Γ^l_{ij} are defined by applying the definition above on the metric tensor, for which we know $\nabla_i g_{jk} = 0$. Hence

$$\Gamma^l_{ij} = \frac{1}{2} g^{lm} \Big(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \Big). \tag{A.2}$$

The computation of covariant derivatives may be quite tedious since the Christoffel symbols are in the number of n^3 $(i, j, l = 1, 2, \ldots, n)$ where $n^2(n+1)/2$ are independent, but in some special cases it is possible to avoid it. An example is given in chapter 1, when we calculated the hatted curvature tensor in the Euclidean plane. What we had was

$$\hat{R} = \Omega^{-2}[R - 2(n-1) g^{ac} \nabla_a \nabla_c \ln \Omega - (n-2)(n-1) g^{ac} (\nabla_a \ln \Omega) \nabla_c \ln \Omega, \tag{A.3}$$

where Ω was the conformal factor and function of only one of the coordinates, namely r. Let us define the term $f(r, \varphi) = g^{ac} \nabla_a \nabla_c \ln \Omega(r)$ and analyze it separately. We may take g^{ac} inside the derivative sign and write

$$f(r, \varphi) = \nabla_a (g^{ac} \nabla_c \ln \Omega(r)). \tag{A.4}$$

Defined g the determinant of the metric tensor g_{ij}, we may multiply and divide the equation above by the scalar $\sqrt{-g}$ and get

$$f(r, \varphi) = \frac{1}{\sqrt{-g}} \nabla_a (\sqrt{-g}\ g^{ac} \nabla_c \ln \Omega(r)) \tag{A.5}$$

The term $\sqrt{-g}\ g^{ac}\nabla_c \ln f(r)$ is a vector density of weight one, which means that it transforms as a vector except that it is additionally multiplied or "weighted" by the Jacobian determinant. Vector densities have the property that the covariant derivative ∇_a coincides with the partial derivative ∂_a, thus equation (A.5) reads

$$f(r,\varphi) = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ar}\partial_r \ln \Omega(r)). \tag{A.6}$$

Finally we may simplify equation (A.3) into:

$$\begin{aligned} \hat{R} = \quad & \Omega^{-2}[R - 2(n-1)\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}\ g^{ar}\partial_r \ln \Omega(r)) + \\ & -(n-2)(n-1)g^{ac}(\nabla_a \ln \Omega)\nabla_c \ln \Omega. \end{aligned} \tag{A.7}$$

Bibliography

[1] Abhay Ashtekar, Saurya Das. *Asymptotically Anti-de Sitter Space-times: Conserved Quantities*. Class. Quantum Grav., Vol. 17, number 2, p. L17-L30(1) (2000).

[2] Abhay Ashtekar, Anne Magnon. *Asymptotically anti-de Sitter Spacetimes*. Class. Quantum Grav., Vol. 1, number 4, p. L39-L44 (1984)

[3] Ingemar Bengtsson. *Anti-de Sitter Space*. (1998) www.physto.se/∼ingemar.

[4] Dieter R. Brill, Jorma Louko, Peter Peldán. *Thermodynamics of (3+1)-dimensional Black Holes with Toroidal or Higher Genus Horizons*. Phys. Rev. D56, p. 3600-3610 (1997).

[5] Stephen W. Hawking, George F. R. Ellis. *The Large Scale Structure of Space-time*. Cambridge Univ. Press (1973).

[6] Sören Holst. *Horizons and Time Machines - Global Structures in Locally Trivial Spacetimes*. Doctoral thesis, Department of Physics, Stockholm University (2000).

[7] Sören Holst and Peter Peldán. *Black Holes and Causal Structure in anti-de Sitter Isometric Spacetimes*. Class. Quantum Grav., Vol. 14, number 12, p. 3433-3452(20) (1997).

[8] Charles W. Misner. *Taub-NUT Space as a Counterexample to almost Anything* in *Lectures in Applied Mathematics vol. 8, Relativity Theory and Astrophysics 1: Relativity and Cosmology.* Jürgen Ehlers ed. (1967).

[9] Roger Penrose. *Relativistic Symmetry Groups*, in *Group theory in non-linear problems*. A. O. Barut ed. D.Reidel Publishing Company (1974).

[10] Felix A. E. Pirani. *Spinors* in *Lectures on General Relativity, Vol. 1.* Prentice-Hall, Inc. (1965).

[11] Alfonso García-Parrado and José M. M. Senovilla. *Causal Structures and Causal Boundaries*. Class. Quantum Grav. 22, R1 - R84 (2005)

[12] Frank J. Tipler. *Penrose Diagrams for the Einstein, Eddington-Lemaitre, Eddington-Lamaitre-Bondi, and Anti-de Sitter Universes*. Journal of Mathematical Physics, Vol. 27, Issue 2, p.559-561 (1986).

[13] Robert M. Wald. *General Relativity*. University of Chicago Press (1984).

[14] Stefan Åminneborg, Ingemar Bengtsson, Sören Holst, Peter Peldán. *Making anti-de Sitter Black Holes*. Class. Quantum Grav., Vol. 13, p.2707 - 2714 (1996).