# Differential Cohomology and Quantum Gauge Fields

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July 11th, 2008

arXiv:hep-th/0605200
arXiv:hep-th/0605198
arXiv:math/0211216
arXiv:0710.2773

- From Maxwell equations to Abelian gauge theories
- Quantization of Abelian gauge theories
- RR-fields and differential K-theory
- RR-fields on orbifolds: a proposal

- Maxwell equations on $\mathrm{M} = \mathbb{R}_t \times \mathbb{R}^3$

$$\mathrm{d}F = 0$$
$$\mathrm{d} \star F = j_e$$

where $F \in \Omega^2(\mathrm{M}; \mathbb{R})$, and $j_e \in \Omega^3_{cpt}(\mathrm{M}; \mathbb{R})$.
$\mathrm{M}$ is contractible, hence $\exists\, A \in \Omega^1(\mathrm{M}; \mathbb{R})$ such that

$$F = \mathrm{d}A$$

This is automatically equivalent to

$$[F]_{\mathrm{dR}} = 0 \text{ in } \mathrm{H}^2(\mathrm{M}; \mathbb{R})$$

The total electric charge is given by

$$[j_e|_{\mathbb{R}^3}]_{\mathrm{dR}} \in \mathrm{H}^3_{cpt}(\mathbb{R}^3; \mathbb{R}) \simeq \mathbb{R}$$

- The space of classical fields modulo gauge transformations is

$$\mathcal{F}_{classical} := \Omega^1(\mathrm{M};\mathbb{R})/\Omega^1_{\mathrm{cl}}(\mathrm{M};\mathbb{R})$$

  with action

$$S(A) \sim \int_{\mathrm{M}} dA \wedge \star \mathrm{d}A + A \wedge j_e$$

- At the quantum level, the gauge potential $A$ is the relevant degree of freedom: Aharonov-Bohm effect, etc.
- For $j_e$ the current of a charged particle $e$

$$\int_{\mathrm{M}} A \wedge j_e = \int_{\gamma} A$$

  with $\gamma$ the worldline of $e$.

- Consider Maxwell equations on $\mathrm{M} = \mathbb{R}_t \times \mathrm{N}$.
  If $\mathrm{H}^2(\mathrm{N};\mathbb{R}) \neq 0$, we may have

$$[F]_{\mathrm{dR}} \neq 0 \text{ in } \mathrm{H}^2(\mathrm{M};\mathbb{R})$$

- The gauge potential $A$ exists only locally.
  The coupling term

$$\int_{\mathrm{M}} A \wedge j_e$$

  is only defined up to a constant.

- Dirac quantization condition:

$$[F]_{\mathrm{dR}} \in \Lambda \subset \mathrm{H}^2(\mathrm{M};\mathbb{R})$$

  where $\Lambda$ is a lattice given by

$$\mathrm{H}^2(\mathrm{M};\mathbb{Z}) \hookrightarrow \mathrm{H}^2(\mathrm{M};\mathbb{R})$$

- The space of "quantum" fields modulo gauge transformations is

$$\mathcal{F}_{\mathrm{quantum}} := \text{equiv. classes of line bundles with connection}$$

$$\bigcup_{\mathrm{c}_1(\mathcal{L})\in \mathrm{H}^2(\mathrm{M};\mathbb{Z})} \mathcal{A}\left(\mathcal{L}_{\mathrm{c}_1}\right)$$

- $\mathcal{F}_{\mathrm{quantum}}$ is equivalent to the group of holonomies

$$\chi : \Sigma \to \mathrm{U}(1), \quad \Sigma \in \mathrm{Z}_1(\mathrm{M})$$

such that $\exists\, F \in \Omega^2(\mathrm{M};\mathbb{R})$

$$\chi(\partial B) = \exp\, 2\pi i \int_{\mathrm{B}} F, \quad B \in \mathrm{C}_2(\mathrm{M})$$

- Generalized Maxwell equations on $\mathrm{M} = \mathbb{R}_t \times \mathrm{N}$, $n-1 = \dim \mathrm{N}$

$$\mathrm{d}F = 0$$
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where $F \in \Omega^p(\mathrm{M}; \mathbb{R})$, and $j_e \in \Omega^{n-p+1}(\mathrm{M}; \mathbb{R})$.
Ex. B-field, supergravity fields, etc.

- No geometric description.

**Definition**

The Cheeger-Simons group $\check{\mathrm{H}}^p(\mathrm{M})$ is the subgroup

$$\chi \in \check{\mathrm{H}}^p(\mathrm{M}) \subset \mathrm{Hom}\left(\mathrm{Z}_{\mathrm{p}-1}(\mathrm{M}), \mathrm{U}(1)\right)$$

such that $\exists\, F_\chi \in \Omega^p(\mathrm{M}; \mathbb{R})$

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- $\check{\mathrm{H}}^p(\mathrm{M})$ is an infinite dimensional Lie group, whose components are labelled by $\mathrm{H}^p(\mathrm{M};\mathbb{Z})$.
- $\check{\mathrm{H}}^*(\mathrm{M})$ admits a ring structure, and an *integration* map

$$\int_{\mathrm{M}}^{\check{\mathrm{H}}} : \check{\mathrm{H}}^{n+1}(\mathrm{M}) \to \mathrm{U}(1)$$

- The following exact sequences hold

$$0 \to \mathrm{H}^{p-1}(\mathrm{M};\mathrm{U}(1)) \to \check{\mathrm{H}}^p(\mathrm{M}) \to \Omega^{\mathrm{p}}_{\mathbb{Z}}(\mathrm{M};\mathbb{R}) \to 0$$

$$0 \to \Omega^{p-1}(\mathrm{M};\mathbb{R})/\Omega^{\mathrm{p}-1}_{\mathbb{Z}}(\mathrm{M};\mathbb{R}) \to \check{\mathrm{H}}^{\mathrm{p}}(\mathrm{M}) \to \mathrm{H}^{\mathrm{p}}(\mathrm{M};\mathbb{Z}) \to 0$$

- Pontrjagin Duality

$$\mathrm{Hom}(\check{\mathrm{H}}^p(\mathrm{M}),\mathrm{U}(1)) \simeq \check{\mathrm{H}}^{\mathrm{n}+1-\mathrm{p}}(\mathrm{M})$$

- In Maths: refinement of characteristic classes, obstructions to conformal embeddings, geometric index theory, ...

- The spacetime is $\mathrm{M} = \mathbb{R}_t \times \mathrm{N}$, $\mathrm{N}$ compact $n-1$ manifold.
- The configuration space is $\check{\mathrm{H}}^p(\mathrm{N})$.
- The Hilbert space is $\mathcal{H} = \mathrm{L}^2(\check{\mathrm{H}}^p(\mathrm{N}))\dots$
- $\dots$ but $\check{\mathrm{H}}^p(\mathrm{N})$ is an infinite dimensional manifold, tricky to define measures on it.
- Better try a group theoretic description.

- An abelian Lie group with measure $\mathrm{G}$ and its dual $\mathrm{G}^\vee$ act on $\mathcal{H}_\mathrm{G} := \mathrm{L}^2(\mathrm{G})$

$$\mathrm{T}_a\psi(x) := \psi(x+a) \quad \mathrm{U}_\chi\psi(x) := \chi(x)\psi(x)$$

- $\mathcal{H}_\mathrm{G}$ is *not* a representation of $\tilde{\mathrm{G}} := \mathrm{G}\times\mathrm{G}^\vee$, since

$$\mathrm{U}_\chi\mathrm{T}_a = \chi(a)\mathrm{T}_a\mathrm{U}_\chi$$

- However, $\mathcal{H}_\mathrm{G}$ is an irreducible representation of $\mathrm{Heis}(\tilde{\mathrm{G}})$

$$0 \to \mathrm{U}(1) \to \mathrm{Heis}(\tilde{\mathrm{G}}) \to \tilde{\mathrm{G}} \to 0$$

which is made up of pairs $(g, z) \in \tilde{\mathrm{G}} \times \mathrm{U}(1)$

$$(g_1, z_1)\cdot(g_2, z_2) := (g_1 g_2, c(g_1, g_2) z_1 z_2)$$

where $c : \tilde{\mathrm{G}} \times \tilde{\mathrm{G}} \to \mathrm{U}(1)$ is the *cocyle* map

$$c((a_1,\chi_1),(a_2,\chi_2)) := \frac{1}{\chi_1(a_2)}$$

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**Theorem**

Let $\mathrm{S}$ be an abelian group, and let $c : \mathrm{S} \times \mathrm{S} \to \mathrm{U}(1)$ be a cocycle map such that $s(x,y) := c(x,y)/c(y,x)$ is nondegenerate. Then $\mathrm{Heis}(\mathrm{S})$ has a unique irreducible representation where $(0,z)$ acts by scalar multiplication.

- Ex. Consider $\mathrm{S} = \mathbb{R} \times \mathbb{R}^\vee$. Use that $\mathbb{R}^\vee \simeq \mathbb{R}$, $\chi(p) = e^{i\,xp}$ for some $x \in \mathbb{R}$. The map $s$ gives the canonical symplectic pairing on the phase space $\mathbb{R} \times \mathbb{R}$. Stone-Von Neumann Uniqueness Theorem.
- Fact: Any $\mathrm{Heis}(\mathrm{S})$ is determined up to noncanonical isomorphisms by the map $s$.
- Abelian gauge theories: set $\mathrm{S} = \check{\mathrm{H}}^{\mathrm{p}}(\mathrm{N}) \times \left(\check{\mathrm{H}}^{\mathrm{p}}(\mathrm{N})\right)^\vee \simeq \check{\mathrm{H}}^{\mathrm{p}}(\mathrm{N}) \times \check{\mathrm{H}}^{\mathrm{n-p}}(\mathrm{N})$. The Hilbert space is the unique irrep. of $\mathrm{Heis}(\mathrm{S})$, and the map $s$ is constructed with the ring product and the integration map.
- Self dual fields: set $\mathrm{S} = \check{\mathrm{H}}^{\mathrm{p}}(\mathrm{N})$. The map $s$ is obtained by restriction from $\check{\mathrm{H}}^{p}(\mathrm{N}) \times \check{\mathrm{H}}^{\mathrm{p}}(\mathrm{N})$ to the "diagonal". (Freed,Moore,Segal '06)

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- RR-fields appear in the low-energy limit of type IIA/B superstring. Total RR-fieldstrength

$$\begin{array}{ll} \text{IIA} & G_{\mathrm{A}} := G_0 + G_2 + G_4 + G_6 + G_8 + G_{10} \\ \text{IIB} & G_{\mathrm{B}} := G_1 + G_3 + G_5 + G_7 + G_9 \end{array}$$

where $G_i \in \Omega^i(\mathrm{M}; \mathbb{R})$.
Equations of motion in absence of sources (D-branes)

$$\mathrm{d} G_{\mathrm{A,B}} = 0$$
$$\mathrm{d} \star G_{\mathrm{A,B}} = 0$$

- The Dirac quantization condition is dictated by K-theory (Moore, Witten '00).

$$[G_{\mathrm{A,B}}]_{\mathrm{dR}} \in \Lambda_{\mathrm{K}^0,\mathrm{K}^{-1}} \subset \mathrm{H}^{ev,odd}(\mathrm{M}; \mathbb{R})$$

where $\Lambda_{\mathrm{K}^0,\mathrm{K}^{-1}}$ is the image of

$$\mathrm{ch} : \mathrm{K}^{0,-1}(\mathrm{M}) \to \mathrm{H}^{ev,odd}(\mathrm{M}; \mathbb{R})$$

- The total RR-fieldstrength is self dual.

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**Result (Hopkins,Singer '05)**

Any generalized cohomology theory $\Gamma^*$ admits a "differential" extension $\check{\Gamma}^*$.

- $\check{\mathrm{K}}^*(\mathrm{M})$ is an infinite dimensional Lie group, with components labelled by $\mathrm{K}^*(\mathrm{M})$.
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$$0 \to \frac{\Omega^{ev-1,odd-1}(\mathrm{M};\mathbb{R})}{\Omega_{\mathrm{K}}^{ev-1,odd-1}(\mathrm{M};\mathbb{R})} \to \check{\mathrm{K}}^{0,-1}(\mathrm{M}) \to \mathrm{K}^{0,-1}(\mathrm{M}) \to 0$$

- $\check{\mathrm{K}}^*(\mathrm{M})$ has a ring structure and an integration map.
- Many models: Hopkins-Singer, Freed, Bunke-Schick...
- The Hilbert space for type IIA RR-fields is the unique irrep. of $\mathrm{Heis}(\check{\mathrm{K}}^0(\mathrm{N}))$, where the map $s$ is given "roughly" by

$$\check{\mathrm{K}}^0(\mathrm{N}) \times \check{\mathrm{K}}^0(\mathrm{N}) \xrightarrow{\cup} \check{\mathrm{K}}^0(\mathrm{N}) \xrightarrow{\int_{\mathrm{N}}^{\check{\mathrm{K}}}} \mathrm{U}(1)$$

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- Let $\mathrm{M}$ be a manifold, and $\mathrm{G}$ a finite group acting via isom. of $\mathrm{M}$.
- The quotient $[\mathrm{M}/\mathrm{G}]$ is in general an orbifold.
- String theory "behaves" well on $[\mathrm{M}/\mathrm{G}]$.
  To preserve modular invariance, the Hilbert space is

$$\mathcal{H}_{\mathrm{orb}} := \bigoplus_{[h]} \mathcal{H}_h^{Z_G(h)}$$

  where $\mathcal{H}_h$ is the Hilbert space of closed strings on $\mathrm{M}^h$
- We suggested that RR-fieldstrength takes value in

$$\Omega^*_{\mathrm{G}}(\mathrm{M};\mathbb{C}) := \bigoplus_{[h]} \Omega^*(\mathrm{M}^h;\mathbb{C})^{Z_G(h)}$$

  and that the Dirac quantization condition is dictated by $\mathrm{K}^*_{\mathrm{G}}(\mathrm{M})$ via the equivariant Chern character

$$\mathrm{ch}_{\mathrm{G}} : \mathrm{K}^*_{\mathrm{G}}(\mathrm{M}) \to \mathrm{H}^*_{\mathrm{G}}(\mathrm{M};\mathbb{C})$$

  (Szabo,V. '07)

- We proposed a model for the "equivariant" differential K-theory $\check{K}_G(M)$.
- It satifies the expected exact sequences, and reduces to ordinary differential K-theory when $G = \{e\}$.
- Work in progress with Bunke, Schick, Szabo.
  Apply equivariant (or orbifold) differential K-theory to construct the Hilbert space of orbifold RR-fields.
- Technical issues: construct a cup product, integration, etc.
- Main advantage: independence of the orbifold presentation, "natural" orbifold approach.

Thanks!!